Dual analysis of the process $\bar{p}n \to 3\pi$ at rest and in flight

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A comparison of Veneziano 5-point-function fits to the $\bar{p}n \to 3\pi$ at-rest Dalitz-plot data is made using the maximum-likelihood method. A sum of 4-point functions is similarly fitted to the in-flight data at 1.2 GeV/c.

I. INTRODUCTION

In a previous publication[1] the maximum-likelihood (ML) method was used in making a direct fit by sums of Veneziano 4-point functions to the $\bar{p}n \to 3\pi$ at-rest data of Anninos *et al.*[2] and in comparing with other fits current in the literature. Following Lovelace[3] this data fit assumed that the initial state had the quantum numbers of the pion and that an imaginary part could be added to the ρ trajectory. The first assumption may, however, not be justified since recently there has been evidence against complete S-state capture of the $\bar{p}$ reported by Devons *et al.*[4] in the process $\bar{p}p \to 2\pi^0$ at rest. A later analysis by Pokorski *et al.*[5] overcame the second drawback by decomposing the amplitude into a convergent sum of resonant terms in which daughter-resonance total widths (ϵ, ϵ^1, and ρ^1) were determined by a ML fit to the data using the same procedure as in Ref. 1. Gaskell,[6] Cohen-Tannoudji *et al.*,[7] Hicks *et al.*,[8] and Franzen and Roemer[9] have recently looked at this $\bar{p}n \to 3\pi$ problem using different types of amplitudes and good fits to the data have been made.

II. FIVE-POINT-FUNCTION FIT

Sivers[10] has suggested that using five-point functions offers a consistent way to combine the final-state interaction picture, in which the main features of the data are due to a final-state interaction in the pion system, with the t-channel exchange picture. Berger[11] also recommends the use of B_5 function fits to Dalitz plots for $2 \to 3$ body processes and in particular for the $\bar{p}n \to 3\pi$ process in which one might have expected some contribution from baryon graphs. Pokorski *et al.*[5] point out that when evaluating the Rubinstein-Squires-Chaichian[12] B_5 model at a pole in the $\bar{p}n$ channel the resulting B_4 four-point-function fit does not give a reasonable description of the data, and suggest that such five-point functions may have unsolved problems. Boguta[13] also criticized this amplitude and the idea of going from a five-point function to a four-point function by using the residue of B_5.

In this section we investigate the quality of the fits to the $\bar{p}n \to 3\pi$ at-rest data made by the five-point-function amplitudes of Rubinstein *et al.*,[12] Pokorski, Szeptycka, and Zieminski[14] and Bender and Rothe.[15] In the case of the amplitude of Rubinstein *et al.* we also look at the crossed process $\pi^- p \to \pi^- \pi^+ n$.

Using the notation of Fig. 1 with each α_{ij} corresponding to s_{ij} these three amplitudes, respectively A_1, A_2, and A_3, are given by

$$A_1 = R_1 + CR_2, \quad (1)$$

where

$$R_1 = \alpha_{12}^{\rho} B_5(-\alpha_{12}^{\rho}, 1-\alpha_{23}^{\rho}, \tfrac{1}{2}-\alpha_{34}^{B}, -\alpha_{45}^{\pi}, \tfrac{3}{2}-\alpha_{15}^{B}) + (1 \leftrightarrow 3), \quad (2)$$

$$R_2 = (\alpha_{34}^{B} - \tfrac{1}{2}) B_5(1-\alpha_{12}^{\rho}, 1-\alpha_{23}^{\rho}, \tfrac{1}{2}-\alpha_{34}^{B}, 1-\alpha_{45}^{\pi}, \tfrac{1}{2}-\alpha_{15}^{B}) + (1 \leftrightarrow 3),$$

$$A_2 = (1-\alpha_{23}^{\rho}-\alpha_{12}^{\rho}) \times B_5(1-\alpha_{23}^{\rho}, 1-\alpha_{12}^{\rho}, \tfrac{3}{2}-\alpha_{15}^{B}, -\alpha_{45}^{\pi}, \tfrac{3}{2}-\alpha_{34}^{B}) + (1 \leftrightarrow 3), \quad (3)$$

$$A_3 = A_2 + KB_1,$$

where

$$B_1 = [S_{23} + S_{12} - (2M)^2 - m^2] \times B_5(1-\alpha_{23}^{\rho}, 1-\alpha_{12}^{\rho}, \tfrac{1}{2}-\alpha_{15}^{B}, 2-\alpha_{45}^{\pi}, \tfrac{1}{2}-\alpha_{34}^{B}) + (1 \leftrightarrow 3), \quad (4)$$

where C and K were taken to be parameters, M = nucleon mass, and m = pion mass. For the case of decay from rest, if $S_{12} = s$, $S_{23} = t$, and $S_{45} = (2M)^2$, then

$$S_{14} = S_{15} = \frac{t + m^2 - 2M^2}{2}$$

and

$$S_{34} = S_{35} = \frac{s + m^2 - 2M^2}{2}$$

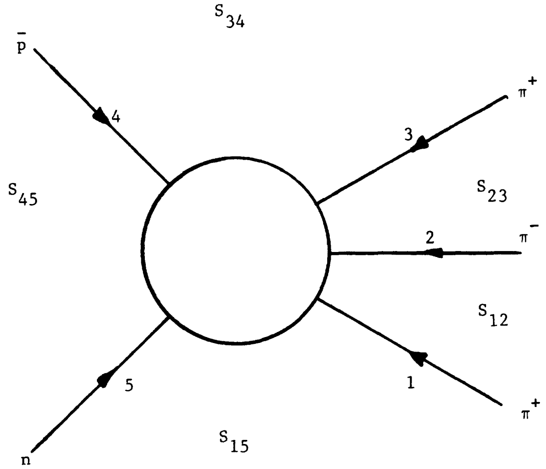

FIG. 1. Definition of the kinematical variables.

and

$$s+t+u=(2M)^2+3m^2 .$$

In Eqs. (2)–(4)

$$\alpha_x^\rho = 0.483+0.885x+i0.33(x-4m^2)^{1/2}$$

is the ρ Regge trajectory form as used in Ref. 1: α^B refers to the nucleon ($B=N$) or $\Delta(1238)$ ($B=\Delta$) trajectory with the same slope of 0.885, and

$$\alpha^\pi = \alpha'(S_{45}-m^2)+iA$$

or

$$\alpha'(S_{45}-(3m)^2)+iB$$

refers to the pion trajectory or a "3π" daughter trajectory again with the same slope, α'. Using the same procedure as in Ref. 1 we perform a ML fit to the (s, t) data of Anninos *et al.* using for the likelihood function

$$L=\prod_{i=1}^{N} F(s_i, t_i), \quad \mathcal{L}=\ln L, \tag{5}$$

where $N=2902$ and

$$F(s_i, t_i)=\frac{|A(s_i, t_i)|^2}{\int_{\sigma_i}|A(s, t)|^2 ds dt}, \tag{6}$$

where (s_i, t_i) are the data points of the Dalitz plot and the integration is taken over this plot. $\mathcal{L}$ was maximized by applying the CERN routine MINUIT to $-\mathcal{L}$ and the imaginary parts A and B were optimized for each amplitude. The B_5 terms were evaluated using the routine of Hopkinson.[16]

This method seems to be preferable to, for example, the χ^2 method where large variations occur from bin to bin due to the hole-enhancement structure of the plot.[8]

The results are summarized in Table I where the values of $\mathcal{L}$ for the one-, three-, and five-term 4-point-function fits[1] are also given. It was found that a slight improvement in the values of $\mathcal{L}$ could be obtained by using the "3π" daughter trajectory instead of the pion trajectory for the initial state, i.e.,

$$B=\text{Im}\,\alpha_\pi(x)=\mu(x-9m^2)$$

(Ref. 12), but even so the likelihood values in most cases do not come near (within a difference of about 3.3 at the 99% level) to those of the 4-point-function fits. The value of C obtained in the fit for A_1 was 0.5, again showing that the calculated value of Rubinstein *et al.*, -1.25,[17] did not lead to a good data fit and that the particular second term R_2 was not suitable. However, a change in the argument of the B_5 function for B_1 of $\alpha_{34} \to \alpha_{15}$ produced much better results so that although the amplitudes given by Rubinstein *et al.*, Pokorski *et al.*, and Bender and Rothe do not produce good fits to the Dalitz-plot data, it should be possible to give an amplitude in terms of 5-point B_5 functions which produces a fit to the decay data of the order of the five-term one given by Nicholas in Ref. 1.

III. THE REACTION $\pi^- p \to \pi^+\pi^- n$ IN THE ρ- AND f^0-MASS REGIONS

For completeness we show the differential cross sections for the related process $\pi^- p \to \pi^+\pi^- n$ in the ρ- and f^0-mass regions for the amplitude of Rubinstein *et al.*[12] Although a fitting procedure was not used, there is considerable latitude in any curve presented due to the wide choices of trajectory function, resonance widths, and coupling constants.

TABLE I. Values of the ln likelihood function $\mathcal{L}$.

Amplitude	$-\mathcal{L}$
A_2 $(B=N)$	5006
R_1 $(B=N)$	5002
A_1 $(B=N)$	4855
A_2 $(B=\Delta)$	4679
A_3 $(B=N)$	4619
A_3 $(B=\Delta)$	4576
Lovelace	4531
$A_2(B=\Delta)+B_1(B=N)$	4568
$R_1(B=N)+B_1(B=N)$	4485
For $\alpha_{34} \leftrightarrow \alpha_{15}$ in B_1	
A_3 $(B=\Delta)$	4548
$A_2(B=\Delta)+B_1(B=N)$	4470
A_3 $(B=N)$	4415
Altarelli and Rubinstein	4409
$R_1(B=N)+B_1(B=N)$	4355
Nicholas	4213

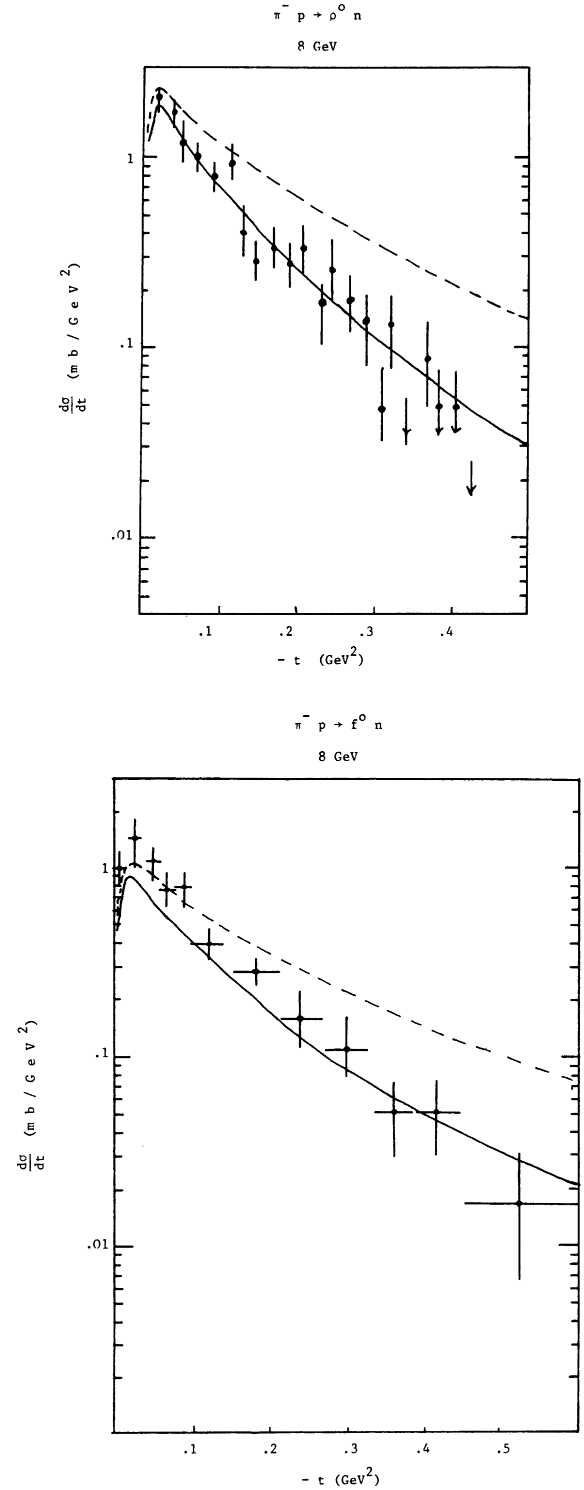

FIG. 2. Differential cross section for the process $\pi^- p \to \pi^-\pi^+ n$ in the ρ- and f^0-mass regions at 8 GeV. The experimental points are taken from Ref. 20. The solid curve has $C=0$ in the Rubinstein *et al.* amplitude $A_1 = R_1 + CR_2$, and the dashed curve has $C=-1.25$.

Bender *et al.*,[18] for example, take $\Gamma_\rho = 90$ MeV, and $g^2/4\pi = 14.7$. The rather lengthy kinematics is omitted and the graphs presented in Fig. 2, where we take the coupling constants[19] and widths to be

$$\frac{f_{\rho\pi\pi}^{\;2}}{4\pi} = 2.4\,, \quad \Gamma_\rho = 90 \text{ MeV}$$
$$\frac{g_{NN\pi}^{\;2}}{4\pi} = 14.4\,, \quad \Gamma_f = 90 \text{ MeV}\,, \tag{7}$$

and the ρ-trajectory function as $\mathrm{Re}\,\alpha(t) = 0.9t + 0.56$. The data are taken from Poirier *et al.*[20] Again it is seen that in the amplitude expression $R_1 + CR_2$ of Rubinstein *et al.*, the given value of C is not favored by the data and in fact with $C=0$ the fit is not too far out considering the likelihood values in Sec. II.

IV. FOUR-POINT-FUNCTION FIT TO THE $\bar{p}n \to 3\pi$ 1.2-GeV IN-FLIGHT DATA

The more recent data of Bettini *et al.*[21] for $\bar{p}n \to \pi^+\pi^-\pi^-$ at 1.2 GeV/c also show a striking pattern of zeros in the experimental Dalitz plot, although the features are not quite the same as those for annihilation at rest as there are not zeros corresponding to

$$M_{\pi^+\pi_1^-}^{\;2} = 2 \text{ GeV}^2, \quad M_{\pi^+\pi_2^-}^{\;2} = 1$$

or $\alpha_s = 2.5$, $\alpha_t = 1.5$ and $\alpha_s = 1.5$, $\alpha_t = 2.5$, i.e., $\alpha_s + \alpha_t \simeq 4$. The relative simplicity of the five-term Veneziano four-point-function-type fit to the at-rest data as given in Ref. 1 suggests that the zeros may still allow a similar type of fit in this case, although the initial state cannot now be represented as a heavy pion.

An alternative but analogous formula for the amplitude was proposed by Odorico,[22] which had the form

$$A(s,t) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)\Gamma(\tfrac{1}{2}(\alpha_s+\alpha_t))}{\Gamma(\tfrac{1}{2}(\alpha_s-\alpha_t))\,\Gamma(\tfrac{1}{2}(\alpha_t-\alpha_s)+1)\,\Gamma(\tfrac{1}{2}(3-\alpha_s-\alpha_t))}$$
$$= \frac{2^{1-\alpha_s-\alpha_t}}{\sqrt{\pi}}\,\frac{\sin\tfrac{1}{2}\pi(\alpha_s-\alpha_t)}{\sin\tfrac{1}{2}\pi(\alpha_s+\alpha_t)}\,\frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)}\,, \tag{8}$$

where $\alpha_x = \frac{1}{2} + x$. This explicitly gives zeros at $\alpha_s - \alpha_t = 2m$ and removes them for $\alpha_s + \alpha_t = 2n$ (m and n arbitrary integers) so the zeros are real straight lines, but this expression implies the existence of exotic meson resonances with $I=2$ and alternating signs of the residues of successive towers of poles. Bugrij, Jenkovsky, and Kobylinsky[23] suggested that the most economical amplitude of the Veneziano type giving absence of zeros at $\alpha_s + \alpha_t = 4$ was of the form

$$A(s,t) = V_{10} + 2V_{11} + C(V_{30} - V_{31} + V_{32}), \tag{9}$$

with

$$V_{nm} = \frac{\Gamma(n-\alpha_s)\Gamma(n-\alpha_t)}{\Gamma(m+n-\alpha_s-\alpha_t)} \tag{10}$$

and

$$\alpha_x = 0.483 + 0.885x + 0.28(4m^2 - x)^{1/2} \tag{11}$$

from Lovelace, which like the Odorico amplitude was equivalent to multiplying the Veneziano-type amplitude by a rational function of α_s and α_t.

Table II shows the ln likelihood values $\mathcal{L}$ for the amplitudes of Odorico and Bugrij *et al.* with a fitted C value and the values C_{nm} obtained in the amplitude expression

$$A(s,t) = \sum_{n,m} C_{nm} V_{nm}$$

found by maximizing $\mathcal{L}$ in the expression

$$L = \prod_{i=1}^{N} |F(s_i, t_i)|^2, \quad \mathcal{L} = \ln L, \tag{12}$$

where $N = 818$ and

TABLE II. Values of the ln likelihood function $\mathcal{L}$.

C_{nm} values used	$-\mathcal{L}$
$C_{10} = -0.131$	
$C_{11} = 1$	
$C_{20} = -0.915$	
$C_{21} = -0.280$	
$C_{22} = 1.826$	
$C_{30} = -2.005$	
$C_{31} = 0.979$	
$C_{32} = 0.870$	
Putting $C_{40} = 0$	1740
$C_{10} = -0.129$	
$C_{11} = 1$	
$C_{20} = -0.787$	
$C_{22} = 1.859$	
$C_{30} = -2.238$	
$C_{32} = 0.878$	
Putting $C_{21} = C_{31} = C_{40} = 0$	1741
$C_{10} = -0.130$	
$C_{11} = 1$	
$C_{20} = -0.847$	
$C_{22} = 1.862$	
$C_{30} = -2.084$	
Putting $C_{21} = C_{31} = C_{32} = C_{40} = 0$	1743
Odorico	2766
Odorico (Lovelace trajectory)	2685
Bugrij *et al.*	
$C_{10} = 1$	
$C_{11} = 2$	
$C_{20} = C_{21} = C_{22} = 0$	
$C_{30} = -C_{31} = C_{32} = -19.19$	2200

$$F(s_i, t_i) = \frac{|A(s_i, t_i)|^2}{\int_\sigma |A(s,t)|^2 ds\,dt}, \tag{13}$$

and the trajectory function was taken to be

$$\alpha_x = 0.483 + 0.885x + 0.33(4m^2 - x)^{1/2}. \tag{14}$$

The data points (s_i, t_i) are taken from the Dalitz-plot events given by Bettini at 1.2 GeV/c incident momenta and the integration is taken over this plot. $\mathcal{L}$ was then maximized by applying the CERN routines MINUETS for $-\mathcal{L}$ and ZFACT for the Γ functions. The 95% and 99% confidence intervals on the coefficients C_{nm} imply changes of the order of $\lambda^2/2$ in $\mathcal{L}$, where λ is 1.96 and 2.576, respectively. The results indicate that the Odorico amplitude is at best like the Lovelace amplitude for the data at rest, being one of several similar terms. The Bugrij *et al.* suggestion also appears to need extra terms even when optimizing the value of C. In contrast the sum of 4-point functions data fit seems to have simple values for the coefficients C_{nm} and there is almost the simple form of fit

$$A(s,t) = V_{11} - V_{20} + 2(V_{22} - V_{30}) + V_{32}. \tag{15}$$

We note that the addition of an imaginary part to the trajectory function had the effect of making uncertain the position of the zeros which might not lie on straight lines as prescribed by Odorico and Lovelace. The V_{nm} terms were taken in the symmetric form above in order to make a comparison with the at-rest case, but nonsymmetric terms could be used in making a fit like that presented by Bettini *et al.*[21] Similarly the trajectory function was taken to have the same form as in Ref. 1 following Lovelace although this could have been optimized in the fit.

One would wish to see a 5-point function fitted to the in-flight data that would suitably extrapolate to that at rest but the lack of quantitative agreement by existing 5-point functions to the at-rest case suggests that this may not be a simple task. Perhaps the differences in the two Dalitz plots are indicative of important dynamical effects in the initial $N\bar{N}$ state which would advise against such a treatment.[8]

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