nances and their interactions.

Note added in proof. It was pointed out to the author that the concept of inclusive decays has been discussed in the following papers: V. Rittenberg and H. R. Rubinstein, Phys. Lett. 40B, 257 (1972); H. Satz, Nuovo Cimento 12A, 205 (1972). These authors discussed applications to $\bar{p}n$ and e^+e^- annihilations, respectively.

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[1]Particle Data Group, Rev. Mod. Phys. 45, S1 (1973).

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ρ^0 shape in photoproduction*

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The theoretical and practical difficulties involved in extracting ρ^0-photoproduction cross sections from measured dipion mass distributions are discussed. To circumvent the theoretical ambiguities, a standard definition of the ρ^0 cross section is suggested. The definition requires that the mass and width of the ρ^0 be externally specified; however, we have also attempted to analyze the data to determine these parameters. Because of the theoretical ambiguities, it is not feasible to determine the mass and width from individual curves, and even a simultaneous fit of all the curves gives results which depend on the assumed fitting function. Thus the width has a theoretical ambiguity of order 15 MeV in addition to any statistical uncertainty.

I. INTRODUCTION

Unfortunately there is no universally accepted procedure for extracting ρ^0 photoproduction cross sections from measured dipion mass spectra. Several, apparently equally plausible, definitions have been used by various experimental groups. Since the final numbers quoted for the cross section usually involve considerable processing of the raw data, this has often made a meaningful comparison of different experiments virtually impossible. The purpose of this paper is to outline the theoretical and practical difficulties which make interpretation of ρ^0-photoproduction experiments uncertain and to propose a standard definition of the cross section which does permit such comparisons. It should be emphasized that while our immediate interest is the analysis of ρ^0 photoproduction from complex nuclei,[1] the present discussion applies equally well to production from individual nucleons.

If we look at some experimental data (see figures in Sec. IV), the reasons for the difficulties are quite apparent. The ρ^0 peak is very broad and is badly skewed by an interfering background. Without an adequate theory of the shape of such a spectrum it is impossible to decide, *in principle*, what fraction of the events is to be attributed to the ρ^0 meson. Further, the shape is found to change as a function of t, becoming less skewed for larger values of $|t|$. It is clear from a study of the shapes of the dipion mass spectra produced from a variety of nuclei that the skewing is a coherent effect, although the data could contain an incoherent component as well. A further practical difficulty is the ρ-ω interference, which seriously distorts the spectrum near its peak.

The physical origin of the skewing is generally well understood, although it has been expressed in a variety of theoretical forms. Basically the dipions are produced by a diffractive mechanism, and the skewing is believed to be due to the interference of a nonresonant background with the ρ^0 resonance. We shall describe the process intuitively in Sec. II, where we also give a qualitative discussion of the theoretical ambiguities which make a unique data analysis impossible. We also propose there the standard definition of the cross section. These ambiguities will be elaborated more concretely in Sec. III, in the context of the Söding model[2] which we favor. This more technical discussion can be skipped without prejudice to the remainder of the paper. Section IV proposes a practical procedure for implementing the standard definition and describes how difficult it is to obtain unique results, even from very-high-quality data on mass distributions. Our conclusions are given in Sec. V.

II. QUALITATIVE DESCRIPTION OF THE THEORETICAL AMBIGUITIES

The intuitive picture which we prefer is that the dipions are mainly diffractively produced. This is suggested by the experimental fact that $d\sigma/dt\,dm_{\pi\pi}$ is not strongly energy dependent in production from hydrogen. There is of course evidence for other mechanisms in the bubble-chamber data.[3] The simplest intuitive picture of diffractive production is that it is a consequence of a kind of shadow scattering associated with the absorption of the ρ^0 and nonresonant dipion constituents of the physical photon. This shows up most clearly in the model proposed by Söding[2] in which the production amplitude is the sum of a diffractive ρ^0 amplitude and a Drell term[4] which may be regarded as coming from nonresonant pion pairs in the physical photon.

If one studies the complete dipion structure of the physical photon, it turns out that the interplay between resonant and nonresonant contributions leads to an enchancement of the spectrum below the resonance and a depletion above the resonance. Further, the ρ^0 constituent of the photon is found to be spatially confined, while the nonresonant dipions have a rather loose structure. This picture will be elaborated in a forthcoming paper by one of the present authors (DRY).

If one accepts this general picture, then the theoretical difficulties begin to become obvious. The two constituents are intimately related and there is no unique way to distinguish them experimentally. We would expect the nonresonant part to have the larger absorption cross section which, however, would have a shadowing correction similar to the Glauber correction for scattering from deuterium, as has been discussed by Bauer.[5] Thus the relative absorption of the two constituents is rather uncertain and would have to be introduced as a free parameter in an analysis. Further, one would expect (gauge invariance demands) a contribution to the production amplitude which cannot be interpreted in terms of shadow scattering. In this last contribution, the photon interacts directly with the nucleon in some manner and produces a dipion state which may interfere with the one which is diffractively produced. Although there are prescriptions for dealing with the problem of gauge invariance, to our knowledge no real understanding of this contribution exists.

There are three or four different theoretical formulations of this physical situation. In the past they seemed to be competing models, but it now appears to us that they are really trying to represent the same basic physics. One of these is the Söding formulation, which has already been mentioned and which will be elaborated in the present paper. Another is the Ross-Stodolsky formulation,[6] which appears to ignore the nonresonant background and instead introduces a strong mass variation in the production amplitude of the ρ^0. While their derivation of this mass variation is unconvincing, it does seem to represent the correct physics (approximately). In fact, if one assumes that the observed dipion mass spectrum should be proportional to that occurring in the physical photon, one obtains their result. This approach may now be criticized on the grounds that the nonresonant dipion constituent should have a greater cross section than the ρ^0, which would affect the amount of skewing. Another approach is that of Kramer and Uretsky[7] in which the basic mechanism was taken to be the Drell amplitude, which then fed into a ρ^0 resonance in a rather *ad hoc* fashion. This led to too high a cross section at the ρ^0 mass since it corresponded, in effect, to taking a rather large value for the ρ^0 absorption cross section. This model has subsequently been modified[8] so that it fits into the general phenomenological framework to be described in the present paper. Finally, there are the dual resonance models[9] which treat the resonant and nonresonant contributions in a unified way and also give a good qualitative description of the ρ^0 shape. Their virtue is that they have a smaller number of free parameters, but we feel that they do not adequately take into account the probable variation of production amplitude with mass which is due to the different nature of the photon's constituents.

For the reader already persuaded by the preceding arguments about the theoretical ambiguities in separating out the ρ^0 cross section, we now present our standard definition as an expedient until these problems are overcome. It seems most plausible to us to try to define the cross section which would result if the ρ^0 were nearly stable but with its other interactions (such as total cross section) unchanged. This is admittedly a dubious concept, but it does suggest a definition. If the ρ^0 were sufficiently narrow, one could simply measure the spectrum at the ρ^0 mass and obtain the cross section from the formula (assuming background and ρ-ω interference are first eliminated)

$$\frac{d\sigma}{dt} = \frac{\pi\Gamma_\rho}{2} \left. \frac{d\sigma}{dt\, dm_{\pi\pi}} \right|_{m_{\pi\pi}=m_\rho} . \tag{2.1}$$

We now propose this definition even for the case of a broad resonance. In support of this proposal, we make the following points:

1. There are arguments by Bauer[5] and Pumplin[10] that the nonresonant contribution is precisely compensated at the ρ^0 mass.
2. This definition eliminates the uncertainty about how much of the skewing is due to the variation of the ρ^0 amplitude with the observed mass and how much is due to the nonresonant background (in our opinion, this is a meaningless question anyway).
3. Aside from any theoretical justification, the definition does permit the comparison of different experiments. There are practical difficulties in applying the definition, but we leave these until later.

III. A CRITIQUE OF THE SÖDING MODEL

Let us now examine the theoretical uncertainties in the Söding model.[2] We first analyze the nonresonant background (the "Drell" terms[4]) and then the photoproduction of the ρ^0 resonance, including the contribution in which the nonresonant background feeds into the resonance. For uniqueness we treat the ρ^0 as an elementary particle (associated with an elementary field) rather than as a dynamical resonance.

We consider coherent 2π photoproduction from a nucleus or nucleon. Since the ρ^0 is treated as an elementary particle, there is no difficulty in separating out unique classes of diagrams and avoiding "double counting."[11] All diagrams in which the π pair has not been a ρ^0 which decayed will be called "nonresonant". The principal ones are the Drell amplitudes [Fig. 1(a)] but there is also a generalized contribution [Fig. 1(b)]. Not very much is known about this last contribution except that it is required by gauge invariance. For nuclei, Bauer[5] has estimated a part of it in which the photon virtually dissociates into a pion pair, *both* members of which diffract through the nucleus. His contribution is similar to the Glauber correction in deuterium, and it partially compensates the usual Drell contribution.

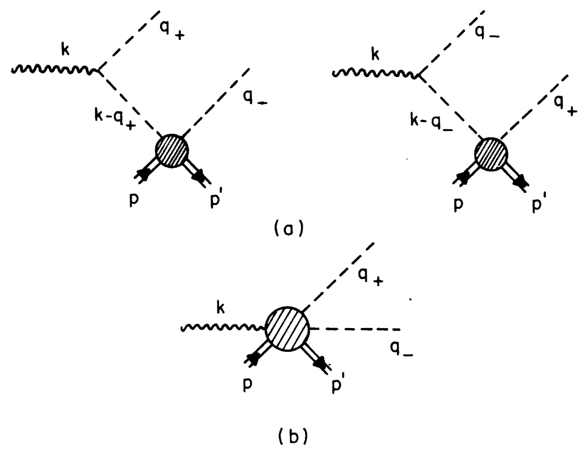

FIG. 1. Diagrams for production of nonresonant pion pairs. Part (a) shows the usual Drell terms while part (b) shows a generalized contribution whose presence is demanded by gauge invariance.

The total nonresonant contribution is[12]

$$F_{\rm NR} = e\epsilon\cdot\left[\frac{q_+}{k\cdot q_+}\, T_-(s_-, t, k\cdot q_+) - \frac{q_-}{k\cdot q_-}\, T_+(s_+, t, k\cdot q_-) + M\right], \tag{3.1}$$

where ϵ^μ is the photon polarization. The first two terms are pictured in Fig. 1(a). T_- (T_+) is the amplitude for scattering of a negative (positive) pion from the nucleon or nucleus; the third argument is the off-mass-shell dependence of this amplitude. All contributions from Fig. 1(b) are lumped together in M.

It seems reasonable to assume that the nonresonant contribution is gauge-invariant separately from the ρ terms. That is,

$$T_-(s_-, t, k\cdot q_+) - T_+(s_+, t, k\cdot q_-) + k\cdot M = 0 . \tag{3.2}$$

In this equation

$$s_\pm \equiv (p' + q_\pm)^2 = (p + k - q_\mp)^2 .$$

We noted that the scattering amplitudes $T_\pm$ depend on the off-mass-shell parameter $[(k+q_\mp)^2 - m_\pi^2 = 2k\cdot q_\mp]$ of the incident pion. Not much is known about this off-mass-shell dependence. We want to emphasize that it is somewhat illogical to try to take it into account while ignoring the contribution from M, since the two are related by gauge invariance. To illustrate the ambiguities associated with this off-mass-shell dependence, we write the amplitude as a mass-shell part plus a correction

$$T_\pm(s_\pm, t, k\cdot q_\mp) = T_\pm(s_\pm, t) + k\cdot q_\mp\, \tau_\pm(s_\pm, t, k\cdot q_\mp) \tag{3.3}$$

and also reexpress M as

$$M = M_1 - q_+ \tau_- + q_- \tau_+ \ .$$

Then

$$F_{NR} = e\epsilon \cdot \left[\frac{q_+}{k \cdot q_+} T_-(s_-, t) - \frac{q_-}{k \cdot q_-} T_+(s_+, t) + M_1 \right] \tag{3.4}$$

with

$$T_-(s_-, t) - T_+(s_+, t) + k \cdot M_1 = 0 \ . \tag{3.5}$$

Nothing has been lost in this redefinition, but it does illustrate our point that it is rather meaningless to take a particular off-mass-shell dependence seriously while neglecting a related contribution from M. In any case, we also note that the off-mass-shell contribution does not have the pion pole as a factor, and therefore there is no reason to expect it to be more important than M.

Our lack of precise theoretical knowledge of the nonresonant background is the first of several ambiguities to which we draw attention. In a potential model treated in lowest order, T_- and T_+ would cancel and it would be possible for M_1 to vanish. If the scattering is primarily diffractive, however,

$$T_- - T_+ \propto i(\omega_- - \omega_+)\sigma_\pi^{\rm tot} \ , \tag{3.6}$$

which is clearly nonvanishing away from symmetry. There is obviously no unique prescription for choosing M_1 to satisfy (3.5). To obtain information about M_1 it is necessary to carry out a dynamical calculation based on a physical model. The part of M_1 estimated by Bauer[5] tends to reduce the Drell amplitude by approximately replacing $\sigma_\pi^{\rm tot}$ in (3.6) by $\sigma_\pi^{\rm abs} = \sigma_\pi^{\rm tot} - \sigma_\pi^{\rm elastic}$. For heavy nuclei, this is a sizable modification. For lighter nuclei, the physics is less reliable, but qualitatively there should be such an effect. However, it should be mentioned that this contribution does not produce gauge invariance, so there must be others as well. Pumplin[10] gives a similar estimate as the correction for the proton. Because so little is known about M_1, we do not attempt to construct a general phenomenological expression for it; we shall simply lump it together with several other unknown contributions in the final data analysis. Intuitively, we expect these corrections to be somewhat smaller than the Drell terms which contain the poles and lead to the skewing of the ρ peak.

Having discussed the nonresonant background, let us consider all contributions in which the π pair results from a ρ decay. Collectively these are represented in Fig. 2(a). However, we wish to draw attention to a particular subclass of diagrams

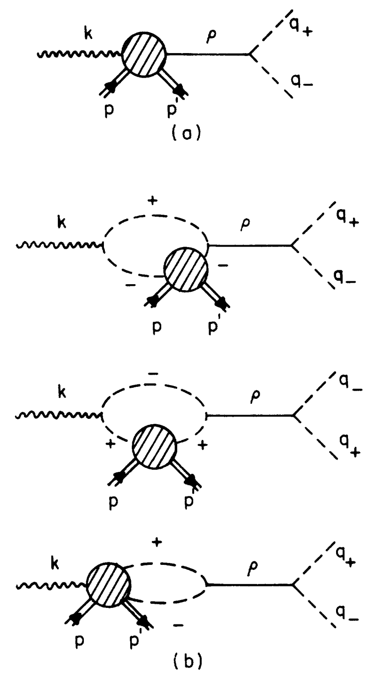

FIG. 2. Diagrams for production of pion pairs from ρ^0 decay. Part (a) shows the general term. Part (b) singles out the contributions from pion pairs feeding into the ρ^0. This is called the "nonresonant" part of Fig. 2(a).

corresponding to the nonresonant amplitude feeding into the resonance as shown in Fig. 2(b). (Here the ρ^0 propagator includes vacuum polarization bubbles.) The total contribution from the ρ is

$$F_\rho = e\epsilon^\mu I_{\mu\nu}(q_+ - q_-)^\nu \frac{1}{m^2 - m_\rho^2 + \pi(m^2)} f_\rho,$$
$$m^2 = (q_+ + q_-)^2, \tag{3.7}$$
$$e\epsilon^\mu I_{\mu\nu}(q_+ - q_-)^\nu \equiv I_\rho^{(B)} + I_\rho^{(D)} \ ,$$

where

$$k^\mu I_{\mu\nu} = 0, \quad (q_+ + q_-)^\nu I_{\mu\nu} = 0 \ .$$

$I_\rho^{(B)}$ is the part of I associated with the subclass of diagrams shown in Fig. 2(b) (bubble graphs) and $I_\rho^{(D)}$ is the remainder, which is usually thought of as the diffracting ρ^0 contribution. However, the diffracting ρ^0 would also obtain contributions from the subclass. We draw attention to the subclass primarily to emphasize the fact that since the ρ^0 is unstable to π-pair decay, we should expect a very significant variation of the ρ^0 scattering amplitude with respect to the external mass. Before discussing this, we wish to draw attention to some features of the ρ^0 propagator which will also affect our phenomenological analysis.

The function $\pi(m^2)$ in the denominator of the propagator is important primarily because its imaginary part is related to the total width. However, the entire denominator is analytic in the cut plane (cut: $4m_\pi^2 < m^2 < \infty$) and the existence of an imaginary part implies that $\mathrm{Re}\,\pi$ is nonvanishing. It is conventional to define the ρ^0 mass as the point at which the real part of the denominator vanishes.[13] Hence

$$\mathrm{Re}\,\pi(m_\rho^2) = 0.$$

It is also convenient to define the pole to have unit strength at this point. Therefore, we set

$$\mathrm{Re}\,\pi'(m_\rho^2) = 0.$$

This is like the definition of charge renormalization in the case of a stable particle. (We note that it is the value of π at $m^2 = 0$ which gives the Gounaris-Sakurai[14] finite-width correction in the electron-positron colliding-beam analysis.) We then expect the ρ propagator to have a structure

$$D_\rho(m^2) = \frac{1}{(m^2 - m_\rho^2) + (m^2 - m_\rho^2)^2 f(m^2) + i m_\rho \Gamma_\rho(m)} , \tag{3.8}$$

where f is related to Γ_ρ by a twice-subtracted dispersion relation. When Γ_ρ is taken to be a relativistic p-wave width, f is easily calculated and is given explicitly in the work of Gounaris and Sakurai. Since we expect that Γ_ρ may not be that simple, we will treat f phenomenologically. Let

$$\frac{1}{1 + (m^2 - m_\rho^2) f(m^2)} = 1 + (m^2 - m_\rho^2)\, g(m^2) ;$$

then

$$D_\rho(m^2) = \frac{1 + (m^2 - m_\rho^2)\, g(m^2)}{(m^2 - m_\rho^2) + i m_\rho \Gamma'_\rho(m)} ,$$

where

$$\Gamma'_\rho \equiv \Gamma_\rho [1 + (m^2 - m_\rho^2) g] .$$

Since we expect f to be of order Γ_ρ / m_ρ^3 from the dispersion relation, these changes from the form usually used are rather small. However, it is necessary to be aware of their presence, particularly if one attempts to fit the data well out on the tail of the peak.

Now we return to the subclass of terms in which the nonresonant amplitude feeds into the ρ^0. Proceeding in analogy to our discussion of the ρ^0 propagator, we note that the loop integral has an imaginary part because of the instability of ρ^0. (Note, however, that the net contribution is nearly real since F_{NR} is nearly imaginary.) This is the analog of the width in the propagator. In fact, Bauer[5] and Pumplin[10] noticed that if the two pions are put on the mass shell before forming the ρ^0, $I^{(B)}$ reduces to the p-wave projection of F_{NR} times just the right phase-space integral to give the width of the ρ^0. The effect of adding this to the p-wave part of F_{NR} is to make the replacement

$$F_{\mathrm{NR}} \to F_{\mathrm{NR}}^{(p)} \frac{m^2 - m_\rho^2 + \mathrm{Re}\,\pi(m^2)}{m^2 - m_\rho^2 + \pi(m^2)} + F_{\mathrm{NR}}^{(\mathrm{non}-p)} . \tag{3.9}$$

Just as in the case of the propagator, we expect that associated with this imaginary contribution there will be a real contribution which will vary significantly with m^2. With plausible assumptions about the mass dependence of the imaginary part (i.e., the m^2 dependence of F_{NR}), we could attempt to calculate this real part of the loop integral (hence imaginary contribution to the amplitude). Such attempts have been made by Kramer and Quinn,[8] and Kramer,[8] using dispersion relations, but we shall not follow that path here. Suffice it to say that the resulting m^2 variation in the amplitude would give a violation of vector-meson dominance (VMD) in the comparison of Compton scattering and ρ^0 photoproduction.

The total amplitude for π-pair production may now be written

$$F_{2\pi} = F_{\mathrm{NR}}^{(\mathrm{non}-p)} + \frac{1}{m^2 - m_\rho^2 + i m_\rho \Gamma'_\rho} \times \{F_{\mathrm{NR}}^{(p)} [m^2 - m_\rho^2 + \mathrm{Re}\,\pi(m^2)] + f_\rho I_\rho^{(D')}(m^2)\} \times [1 + (m^2 - m_\rho^2) g] , \tag{3.10}$$

where $I_\rho^{(D')}$ includes the contribution from the real part of the loop integral as well as that from $I^{(D)}$. We may think of $I_\rho^{(D')}$ as the amplitude for a photon to produce a stable ρ meson of mass m; it includes the polarization sums of (3.7). The other factors then give its probability of propagating as an unstable particle and finally decaying. Excluding the imaginary part of the loop integral from the definition of $I_\rho^{(D')}$ is somewhat a matter of taste. Gutbrod has argued (privately) in favor of including it in the amplitude since it represents a contribution where the π pair did propagate as a ρ^0 before decaying. In the nuclear case it seems clear that it should not be included if one wishes to compare experiment with the usual optical-model calculation. On the other hand, the usual optical model calculation is obviously oversimplified. A more correct treatment would treat the ρ^0 and π-pair states as a coupled-channel problem. Since it does not appear feasible to give such a treatment (in fact, it may be inconsistent in view of all the other approximations made), there is an inherent ambiguity in the physical interpretation of the data. However, we feel that this ambiguity does not invalidate the general physical picture and conclusions. Some discussion of these subtleties may be found in the work of Gottfried and Julius[15] and Bauer.[5]

IV. FITTING THE DIPION MASS SPECTRA

We have seen enough of the theory of the ρ^0 shape to recognize that it is somewhat ambiguous except at the ρ mass. This theoretical ambiguity should be taken into account in fitting the data and will be reflected in the resulting uncertainties in the cross section, in the principal fitting parameters (m_ρ, Γ_ρ), and in the strength and structure of the Drell term. Omitting the non-p-wave background term (which, however, could influence experimental results in some circumstances), we may rewrite the π-pair amplitude as

$$F_{2\pi} = \frac{f_\rho I_\rho^{(D')}(m_\rho{}^2) + (m^2 - m_\rho{}^2)C_1 + (m^2 - m_\rho{}^2)^2 C_2 + \cdots}{m^2 - m_\rho{}^2 + i m_\rho \Gamma_\rho'}.$$

At the present time we cannot hope to calculate the coefficients C_i. For example, in Sec. III we have identified several contributions to C_1: the Drell amplitude (most important, but with strength somewhat uncertain), propagator correction, and the variation of F_ρ with m^2. Undoubtedly there are others. We are relatively confident only of the sign and approximate size of this term, which has the Drell amplitude as its most important constituent. (Incidentally, we may want to have C_1 include the mass dependence expected for the Drell term, rather than have that dependence taken care of by higher terms.) The other terms are corrections to these corrections, and we have no confidence even in estimating their sign. One might hope that their importance will be small if a narrow mass region is used for analysis. If a larger mass region is used, we would expect these terms to be important and take into account our theoretical ignorance, deemphasizing the influence of experimental points far out on the tail. Unfortunately, as we shall see later, correlations among the parameters make things much more complicated.

Alternatively, we may write the cross section as

$$\frac{1}{m\Gamma_\rho'}\frac{d\sigma}{dt\,dm} = A_0 \left| \frac{1}{m^2 - m_\rho{}^2 + i m_\rho \Gamma_\rho'} + \frac{\xi e^{i\phi}}{m^2 - m_\omega{}^2 + i m_\omega \Gamma_\omega} \right|^2 + \left[A_1 \frac{m^2 - m_\rho{}^2}{m^2} + A_2 \left(\frac{m^2 - m_\rho{}^2}{m^2} \right)^2 + \sum_{i=3}^{N-1} A_i \left(\frac{m^2 - m_\rho{}^2}{m_\rho{}^2} \right)^i \right] [(m^2 - m_\rho{}^2)^2 + m_\rho{}^2 \Gamma_\rho'^2]^{-1} . \tag{4.1}$$

In writing this expression, we have made use of our prejudice that A_1 and A_2 are approximately proportional to $1/m^2$ and $1/m^4$, respectively, from the Drell amplitude. We see no reason to continue such dependence for higher terms. We have no idea what the radius of convergence of such an expansion is, but we intend to use it only in the mass range reasonably close to the ρ^0 mass, so that only a finite number of terms should be important.

We have also taken into account the ρ-ω interference by modifying the A_0 term as shown. The complex number $\xi e^{i\phi}$ is called the "mixing parameter" or "mixing strength". It has a value close to $0.01\, e^{i95°}$ according to the DESY-MIT experiment.[16] The parameters to be determined by least-squares fitting are thus m_ρ, Γ_ρ,[17] $\mathrm{Re}(\xi e^{i\phi})$, $\mathrm{Im}(\xi e^{i\phi})$, $A_0, \ldots, A_{N-1}$.

Now suppose we have a set of data, possibly including an incoherent background, from which we wish to determine the cross section. Let us also suppose that m_ρ and Γ_ρ are specified and are not to be obtained from the data. (We will discuss later the difficulties of determining m_ρ and Γ_ρ from an experimental mass distribution.) According to (2.1) we have only to determine $d\sigma/dt\,dm|_{m=m_\rho}$, with its error. This may be done by fitting any convenient interpolating formula to the data (we shall discuss the complications of ρ-ω interference later). Thus all the data will participate in the determination of the cross section, not simply the value measured at m_ρ. It is to be emphasized, however, that this is *not* an area method. The difficulty with an area method is that we simply do not know what portion of the area is to be attributed to the ρ^0, i.e., we do not know *a priori* the shape of the ρ^0 in photoproduction.

Next we consider the effect of a smooth, incoherent background. Suppose this can be adequately represented in the region of interest by

$$B = a_0 + a_1(m^2 - m_\rho{}^2) + \cdots + a_i(m^2 - m_\rho{}^2)^i .$$

We see immediately by comparison with (4.1) that the observed cross section could be fitted with the form (4.1) if we modify the A_j through $j = i + 2$ (ignoring the slight difference in form for the first few terms as well as the small mass dependence of Γ_ρ). Thus, if we include the possibility of a numerator polynomial in (4.1), it is inherently impossible to separate background out of the data by a mass-distribution analysis alone. There is of course also the possibility of an incoherent background having the basic shape of the ρ^0, say in the case of inelastic ρ^0 production from hydrogen or with nuclear excitation. These must be allowed for by experimental measurement or estimated theoretically. We feel that because of coherence, smooth incoherent backgrounds should be completely unimportant in most nuclei.

We now describe some of our experiences in applying (4.1) to experimental data. Our first concern was with the extraction of ρ^0-photoproduction cross sections from the dipion mass spectra of the two experiments analyzed in Ref. 1. The Cornell group[18] had already analyzed their data with a procedure similar to the one we propose here, but with a three-parameter numerator polynomial. It was therefore decided not to reanalyze their mass distributions. Incidentally, in their analysis they obtained $\Gamma_\rho = 124 \pm 4$ MeV with m_ρ fixed at 775 MeV. On the other hand, the 195 mass spectra of the DESY-MIT group[19] had been analyzed in a rather different way and it was decided to redo the analysis using our prescription.

According to the definition of the cross section in Sec. II, this requires knowledge (or arbitrary choice) of the ρ^0 mass and width. In order to find these parameters, one could fit the data of Ref. 18 directly, and indeed this has been done and will be discussed below. However, the data of Ref. 19 are low-resolution data with at most 18 points (spaced 30 MeV apart) in each mass spectrum. Since the parameters m_ρ and Γ_ρ are of such great importance, we thought it best to look at more-accurate higher-resolution data.

The DESY-MIT,[16] Cornell-Rochester,[20] and Daresbury[21] groups have performed very precise measurements of the $\pi\pi$ mass spectrum in order to observe the interference of ρ and ω final states. The mass resolution in these experiments is about 5 MeV and there are nearly 60 data points on many mass distributions. From the wealth of these magnificent data generously supplied to us by these groups, we have made a serious attempt to determine a value of Γ_ρ to be input to the optical model analysis of Ref. 1. As for m_ρ, it is our position that due to possible uncertainties in the mass scale, it is best to let each experiment choose its own ρ^0 mass. The low-resolution data of Ref. 19 (LRD), choose $m_\rho = 770$ MeV and that is the mass used in Ref. 1. However, it is comforting to note that when the DESY-MIT high-resolution data are fitted in the same way as LRD, both sets of data agree on $m_\rho = 770$ MeV (see Tables IV and VI, below).

Naturally $\xi e^{i\phi}$ also emerges from fits to the high-resolution data. But our primary purpose was to compare LRD with the optical model. In LRD the experimental aperture largely "washed out" the ρ-ω interference and indeed fits to LRD do not demand the presence of ω decays; in fact, for some values of the mass and width, even very small mixing strength can worsen the fit. Thus we did not consider the mixing parameter to be a goal of our fitting efforts. For this reason we felt justified in not bothering to fold experimental mass resolution into our theoretical fitting functions. In the same spirit we arbitrarily fixed Γ_ω at 11.4 MeV and m_ω at 783.9 MeV throughout, making no attempt to determine these from the data. Such simplifications may well affect $\xi e^{i\phi}$ which depends greatly on the narrow "cliff" region of the mass spectrum (see Figs. 3 and 4), but should not affect the ρ^0 width significantly. (If they do, the task of determining Γ_ρ from complex-nuclei mass spectra is even more uncertain than we believe it to be.)

We now proceed to describe some of our exper-

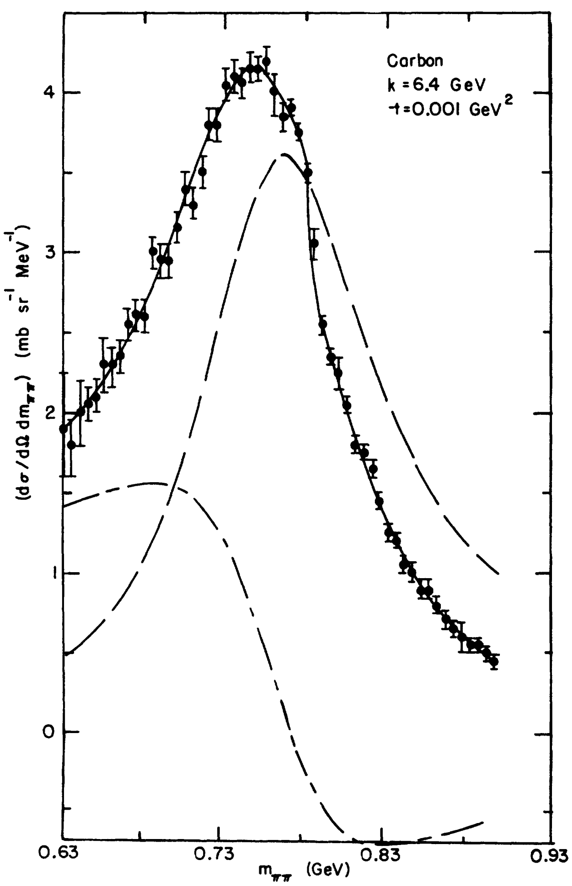

FIG. 3. A typical nonrunaway fit to a typical mass spectrum. Values of m_ρ, Γ_ρ, and $\xi e^{i\phi}$ are given in the $N=5$ line of Table III. The long-dashed curve is the contribution of the ρ only $[A_0 m \Gamma_\rho'/((m^2 - m_\rho^2)^2 + m_\rho^2 \Gamma_\rho'^2)]$. The curve with alternate long and short dashes is the background contribution (A_1 term $+ \cdots + A_4$ term). The solid curve is the total fit, $\rho + \omega$ + background.

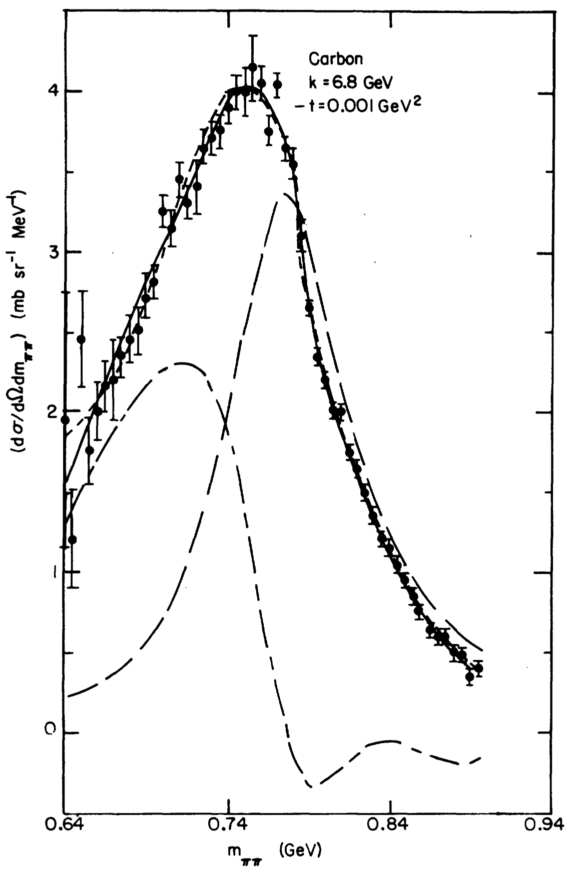

FIG. 4. A typical runaway fit to a typical mass spectrum. The values of m_ρ, Γ_ρ, and $\xi e^{i\phi}$ are given in the $N=6$ line of Table I. The long-dashed curve is the contribution of the ρ only. The curve with alternate long and short dashes is the background contribution (A_1 term $+\cdots+A_5$ term). The solid curve is the total fit, $\rho+\omega+$background. The short-dashed curve is an $N=5$ fit to the same data with m_ρ fixed at 770 MeV, Γ_ρ fixed at 139 MeV, and $\chi^2=59.9$.

iences in applying the fitting procedure of (4.1) to the high-resolution data of the various groups. While this account may simply reflect the naiveté of the authors, who are novices at such statistical analysis, we believe it does point up some genuine problems in fitting broad resonances which may have implications beyond the present example.

Our first approach was to try to analyze each curve independently of the others, with the aim of taking an average of the results at the end. However, if one allows all parameters to vary freely to minimize the χ^2 of each curve, chaos reigns supreme.[22] The fits are excellent (as is clear from Figs. 3 and 4), but no sense emerges from the results because of serious difficulties:

The form of (4.1) clearly admits simultaneous changes of parameters which cause little change in the function through the region of interest. It had been expected that this would merely lead to larger and larger errors in m_ρ and Γ_ρ as the number of parameters increased. Instead, in some cases, these large correlations among parameters led to ridiculous "runaway solutions" where the values of m_ρ and Γ_ρ were quite unreasonable. A typical example is given in Table I. This runaway behavior corresponds to a long multidimensional tunnel in parameter space in which χ^2 varies only slowly. Except in cases where the iteration procedure for minimizing χ^2 failed to converge, we could find no bias-free criterion for eliminating these runaway solutions. Nor could we find any such criterion for choosing the number of parameters to be used. Some of the ridiculous values of m_ρ and Γ_ρ have errors which are deceptively small. That is, they indicate that a fit with reasonable values would have a very large χ^2. But this turns out not to be the case; the quadratic fit at the minimum is valid over too limited a region and cannot begin to describe the complicated tunnel connecting the reasonable and the ridiculous fits. Some examples of values of m_ρ and Γ_ρ from individual curves are given in Table II. We were unable to find any consistent pattern in such results; in particular, the spread in values seems to be somewhat larger than statistical. Table III

TABLE I. A typical runaway fit to a DESY-MIT spectrum (carbon, $k=6.8$ GeV, $t=-0.001$ GeV^2). For each value of N, the data were fitted with Eq. (4.1) leaving m_ρ, Γ_ρ, $\mathrm{Re}(\xi e^{i\phi})$, and $\mathrm{Im}(\xi e^{i\phi})$ free. The $N=6$ fit is drawn in Fig. 4. The errors given in the tables depend on the goodness of a quadratic fit to the region near the χ^2 minimum. They are therefore unreliable when very large.

N	χ^2	m_ρ (MeV)	Γ_ρ (MeV)	$\mathrm{Re}(\xi e^{i\phi})$	$\mathrm{Im}(\xi e^{i\phi})$
3	59.6	775 ± 2.2	158 ± 4.9	0.0025 ± 0.0009	0.0069 ± 0.0008
4	59.1	773 ± 2.8	151 ± 10.4	0.0025 ± 0.0010	0.0069 ± 0.0008
5	59.0	771 ± 8.5	149 ± 12.7	0.0024 ± 0.0010	0.0069 ± 0.0008
6	50.0	779 ± 6.3	90.7 ± 12.4	0.0052 ± 0.0022	0.0082 ± 0.0017
7	48.1	773 ± 7.6	70.7 ± 14.6	0.0066 ± 0.0037	0.0103 ± 0.0039

TABLE II. Typical results of $N=5$ fits to the DESY-MIT data using Eq. (4.1) leaving m_ρ, Γ_ρ, $\mathrm{Re}(\xi e^{i\phi})$, and $\mathrm{Im}(\xi e^{i\phi})$ free. In general, the curves disagree with each other, but the results are not inconsistent with the ρ^0 having a definite mass and width.

Element	k (GeV)	$-t$ (GeV^2)	χ^2	m_ρ	Γ_ρ
C	6.4	0.001	28.8	773 ± 6	137 ± 8
C	6.0	0.001	38.0	758 ± 7	115 ± 9
C	6.0	0.003	44.1	772 ± 46	153 ± 106
H	6.0	0.001	49.1	758 ± 25	136 ± 76
H	6.4	0.005	19.0	709 ± 39	84.9 ± 71
Pb	6.4	0.001	32.0	782 ± 11	95 ± 25

TABLE III. A typical nonrunaway fit of a DESY-MIT spectrum (carbon, $k=6.4$ GeV, $-t=0.001$ GeV^2). The procedure was the same as for Table I. The $N=5$ fit is drawn in Fig. 3.

N	χ^2	m_ρ (MeV)	Γ_ρ (MeV)	$\mathrm{Re}(\xi e^{i\phi})$	$\mathrm{Im}(\xi e^{i\phi})$
3	29.9	775 ± 1.6	145 ± 4.0	0.0010 ± 0.0009	0.0067 ± 0.0008
4	28.8	773 ± 2.5	137 ± 7.7	0.0012 ± 0.0009	0.0067 ± 0.0008
5	28.8	773 ± 6.0	137 ± 8.0	0.0012 ± 0.0010	0.0067 ± 0.0008
6	28.8	773 ± 5.9	133 ± 18.7	0.0014 ± 0.0011	0.0068 ± 0.0010
7	26.9	757 ± 12.5	116 ± 18.3	-0.00003 ± 0.0019	0.0081 ± 0.0020

shows a typical nonrunaway fit.

In order to further illustrate these difficulties, we call the reader's attention to Figs. 3 and 4. Figure 3 shows a fit to a typical mass spectrum for a nonrunaway case. That is, the parameters for this fit (see Table III) are reasonable; and as N varies from 3 to 7, no drastic changes in the parameters occur. Figure 4 shows a runaway fit to a different spectrum, with parameters given in Table I. Also shown is a fit to the same data with m_ρ fixed at 770 MeV and Γ_ρ fixed at 139 MeV. The two curves look almost identical to the eye, although their χ^2's differ by 9. Note also the great similarity between the data and curves of Figs. 3 and 4.

Finally, we were forced to conclude that no procedure was likely to permit analysis of individual spectra for the values of m_ρ and Γ_ρ. It seemed clear that all curves must be fitted simultaneously to obtain the best values of m_ρ and Γ_ρ. Since the ρ-ω interference and other shape factors might well vary as a function of A, k, and t, no effort was made to fit them simultaneously for all curves. In retrospect, it might have been more desirable to put a smoothness criterion on these parameters, but we were not prepared to undertake the massive computing program which that would have entailed.

We then resorted to an "over-all" fit. Our procedure was to evaluate χ^2_{tot} for several pairs of fixed m_ρ and Γ_ρ values as the sum of χ^2 from individual curves. In each individual fit the mixing and $A_0, \ldots, A_{N-1}$ were permitted to vary freely. For $N=3,4,5$ the best combination of m_ρ and Γ_ρ was determined by interpolation to minimize χ^2_{tot}.

The results of this procedure for 32 DESY-MIT high-resolution spectra are shown in Table IV. Note the startling variation of Γ_ρ with N. We do not see how to choose between the different values of N on any statistical basis. Incidentally, with this procedure, we were able to include most of the data curves. This is in contrast to the attempt to fit individual curves where we generally obtained runaway solutions for curves with too small a number of data points.

The Cornell-Rochester and Daresbury groups did not supply us with a sufficient number of high-resolution spectra to make an over-all fit worthwhile. Instead, we fitted their high-resolution spectra with parameters close to those resulting from the DESY-MIT over-all fit. The results of these fits are presented in Table V. The fits are very good in 5 of the 7 cases and we have no reason to believe that the problems with the other two cases are due to anything other than statistical fluctuations.

TABLE IV. Results of the over-all fit to the DESY-MIT data for $N=3$, 4, and 5 (Ref. 16). Note the dependence of the parameters on N.

N	Γ_ρ (MeV)	m_ρ (MeV)	χ^2	Degrees of freedom
3	153.9 ± 2.2	776.1 ± 0.9	1083.0	1103
4	145.0 ± 3.7	773.6 ± 1.3	1059.1	1071
5	138.4 ± 3.9	769.7 ± 2.7	1015.9	1039

After this experience, we tried the simultaneous fitting procedure on LRD. We briefly investigated the effect of $\xi\neq0$ on LRD and found that $\xi>0.003$ increased χ^2_{tot} for the 195 curves for most values of m_ρ and Γ_ρ in the range 760–780 MeV and 120–150 MeV. With $N=5$, the best mass for LRD is undeniably 770 ± 2 MeV (see Table VI), and the best width is 124 MeV. It is interesting to note that this agrees with the width determined by the Cornell group.[18] The DESY-MIT group had analyzed their curves under different assumptions and preferred a width of 140 MeV. These differences are quite compatible with the ambiguities we encountered in fitting the high-resolution data. The lower width in LRD may be due to some subtle effect of the ρ-ω interference on the data. Although the resolution is too broad to detect the interference, the data might retain some systematic behavior from the interference which is now manifest in a change in the apparent width.

The analysis of the accompanying paper[1] was carried out before the present one and we had to

TABLE V. Fits to the Cornell-Rochester and Daresbury data with $m_\rho=770$ MeV and $\Gamma_\rho=137$ MeV.

Element	Energy (GeV)	Degrees of freedom	χ^2	Group
C	7–8	20	22.2	Cornell-Rochester
C	8–9	25	72.5	Cornell-Rochester
Al	7–8	27	25.7	Cornell-Rochester
Al	8–9	32	27.3	Cornell-Rochester
Pb	7–8	32	34.8	Cornell-Rochester
Pb	8–9	38	54.9	Cornell-Rochester
C	4.2	49	55.2	Daresbury

TABLE VI. Best mass as a function of assumed width for the DESY-MIT low-resolution data (LRD). There are 698 degrees of freedom in these over-all fits. (At the time these fits were made, 20 mass distributions were omitted. These were at high $-t$ values where the data are most uncertain and they could not have had a significant effect on the results. Had they been included, there would have been 778 degrees of freedom.)

Γ_ρ (MeV)	m_ρ (MeV)	χ^2
120	768.8 ± 2.3	665.7
127	769.6 ± 2.5	664.0
135	770.3 ± 2.8	672.7
145	771.2 ± 3.2	695.4

select arbitrary values of Γ_ρ and m_ρ as "standard". These were taken to be 127 MeV and 770 MeV, quite close to the values later obtained from LRD. However, the most important feature of the LRD results for the cross sections is their almost precise linearity in Γ_ρ as illustrated in Table VII. [This is not totally unexpected. The basic idea of our procedure is to use (4.1) to fit a smooth curve to $d\sigma/dt\,dm$ in order to read off the value at m_ρ. All reasonable curves that fit the data well should give roughly the same value of $d\sigma/dt\,dm|_{m_\rho}$.] This makes it trivial to convert our results for any width in the range 120–150 MeV and allows us to extract information from the A dependence of the cross sections independently of Γ_ρ. See Ref. 1.

In retrospect, we feel that our analysis of the low-resolution data probably gives a misleading value for Γ_ρ because of its neglect of the ρ-ω interference. One of the values from the high-resolution data is probably more reasonable. But which one? Not only do we have the range of values given in Table IV, but additional parameters or other assumptions could have led to values well outside this range. The ambiguity for theoretical reasons is accordingly considerably larger than the statistical error in fitting these fine data. Our present prejudice is that the width (from photoproduction) is about 140 MeV with a (conservative) theoretical ambiguity of 15 MeV.

TABLE VII. An illustration of the linearity in Γ_ρ of the cross sections obtained from LRD with our procedure. $m_\rho = 770$ MeV was used throughout. Examples were selected at random. 195 mass spectra were fitted in all.

A	t (GeV^2)	Γ_ρ (MeV)	$\frac{d\sigma}{dt}\Big/\Gamma_\rho$
	$k = 5.8$ GeV		
9	−0.001	120	5.21
		125	5.20
		130	5.19
		135	5.18
		140	5.17
64	−0.003	120	66.9
		125	67.0
		130	67.1
		135	67.2
		140	67.3
115	−0.007	120	37.7
		125	37.9
		130	38.0
		135	38.2
		140	38.3
183	−0.001	120	346
		125	345
		130	345
		135	344
		140	344
	$k = 6.2$ GeV		
12	−0.009	120	5.83
		125	5.82
		130	5.81
		135	5.80
		140	5.80
197	−0.001	120	424
		125	423
		130	421
		135	420
		140	419
	$k = 6.6$ GeV		
108	−0.003	120	141
		125	140
		130	140
		135	139
		140	139

V. SUMMARY AND CONCLUSIONS

(1) We have emphasized the theoretical ambiguities which make a unique decomposition of the data into ρ^0 production plus background inherently impossible at the present time. While we elaborated these ambiguities for the Söding model, they would be true in any model when one goes beyond the leading approximation. At the same time, all the current models appear to describe well the qualitative features of the data, and to incorporate the underlying physics which accounts for these features.

(2) We proposed a standard definition of the cross section, Eq. (2.1), which minimized the consequences of the theoretical ambiguities, *provided the mass m_ρ and width Γ_ρ are externally provided.*

(3) We studied the problem of determining m_ρ and Γ_ρ directly from measured mass distributions. We were unable to develop a reliable procedure for analyzing individual curves for this purpose, but found that some sense could be made out of

over-all fits to a large number of curves. However, it still turned out that the value of Γ_ρ was very sensitive to the number of free parameters used in fitting the curves. The difficulties were probably aggravated by the ρ-ω interference; but, in our opinion, similar difficulties would be present in fitting any broad resonance.

(4) The DESY-MIT low-resolution data gave a rather different value of Γ_ρ from their high-resolution data, and in agreement with that from the Cornell data fitted by a similar procedure. We had believed that the ρ-ω interference would be washed out by the broad resolution (and the data reject its presence), but it may be leaving its mark somehow on the fitted value of the width.

(5) The large uncertainty in our knowledge of the width does not appreciably affect the A dependence of the photoproduction results. Rather, it is reflected in the values of $|f_0|^2$ and $\gamma_\rho^2/4\pi$.

(6) We have not tried to relate this analysis to other determinations of the ρ^0 width. It is not in conflict with the colliding-beam experiments ($e^+e^- \to \pi^+\pi^-$) which give $\Gamma_\rho = 149 \pm 23$ MeV.[23] Incidentally, that experiment also suffers from theoretical ambiguities (e.g., finite width corrections to the ρ propagator, possible form factor at the $\rho^0\pi\pi$ vertex) which, while perhaps less severe than in the case of photoproduction, should be taken into account if there should ever be a statistical improvement in the data.

Finally, it is interesting to note that in the framework of vector-meson dominance the value of the ρ^0-photon coupling constant, $e/f_\rho \equiv e/2\gamma_\rho$, enters the discussion in two different ways. This apparently makes it possible to find a unique value of f_ρ. Also since in VMD

$$\Gamma_\rho = \frac{f_\rho^2 m_\rho}{6\pi}\left(\frac{\frac{1}{4}m_\rho^2 - m_\pi^2}{m_\rho^2}\right)^{3/2}, \tag{5.1}$$

this would determine the ρ^0 width as well. For on the one hand, the experimental forward photoproduction cross section is proportional to Γ_ρ (see Sec. IV) and hence to f_ρ^2. On the other hand, the theory is proportional to $1/f_\rho^2$ once the ρ^0-nucleon forward elastic scattering amplitude is specified.

When theoretical and experimental uncertainties are taken into account the overlap of these two relations is indicted by the shaded region of Fig. 5. The center of the region has $f_\rho^2/4\pi \approx 2.5$ (or $\gamma_\rho^2/4\pi \approx 0.63$) whence $\Gamma_\rho \approx 130$ MeV. Unfortunately, finite-width corrections (expected to be of order Γ_ρ/m_ρ and indicated by the 30% theoretical band of Fig. 5) destroy the quantitative usefulness of this approach to Γ_ρ. Nevertheless, the reasonableness of the result indicates that VMD gives a good qualitative description of ρ^0 photoproduction.

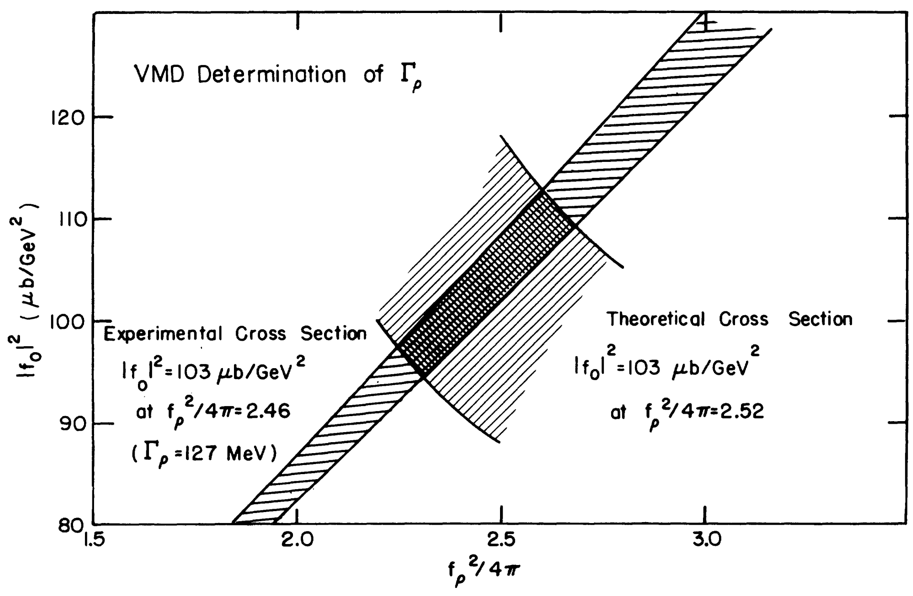

FIG. 5. Experimental and theoretical one-nucleon forward photoproduction cross sections ($|f_0|^2$) as a function of f_ρ^2. We write the ρ^0-nucleon forward elastic scattering amplitude as $f_{\rho\rho} \equiv (ik/4\pi)\sigma_\rho(1-i\alpha_\rho)$, where k is the ρ^0 momentum. The value of σ_ρ is taken to be 25.9 mb to agree with the fixed $\alpha_\rho = -0.2$, A-dependence fit (Ref. 1) of the 8.8-GeV data of Ref. 18. (See Ref. 1 for more details.) The theoretical value of $|f_0|^2$ is then (e^2/f_ρ^2) (π/k^2) $(k^2/16\pi^2)$ $\sigma_\rho^2(1+\alpha_\rho^2)$ and the band is centered at this value with an uncertainty of 15% added in both directions to indicate the possible effect of finite-width corrections. (A very naive calculation with a p-wave width indicates an increase of $|f_0|^2$ above the VMD value.) The experimental band conforms to $|f_0|^2 = 103$ μb/GeV$^2 \pm 3\%$ for $\Gamma_\rho = 127$ MeV in accordance with the result of the A-dependence fit.

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$$\Gamma_\rho'(m) \equiv \Gamma_\rho \frac{m_\rho}{m}\left(\frac{m^2-4m_\pi^2}{m_\rho^2-4m_\pi^2}\right)^{3/2}.$$

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