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# Deviation from power law at a Callan-Symanzik eigenvalue and the non-negligibility of mass-insertion terms

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The mass-insertion term in the Callan-Symanzik equation may not be negligible, even in the deep Euclidean region, because the asymptotic behavior of vertex functions may be different from that of individual dominant graphs. When contributions from different orders of perturbation are related by a constraint, such as the condition that the physical coupling constant is at a nontrivial Callan-Symanzik eigenvalue, cancellations between dominant graphs may occur. This and other circumstances under which vertex functions may fail to satisfy homogeneous Callan-Symanzik equations asymptotically are illustrated with a very simple example, thus emphasizing that the question of asymptotic scale invariance may not be always decided by studying the properties of the $\beta$ function alone.

## I. ASYMPTOTIC BEHAVIOR AT A CALLAN-SYMANZIK EIGENVALUE

The Callan-Symanzik equations have been very useful in studying the asymptotic behavior of Green's functions. They are equations of the form

$$\left( m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} - N\gamma \right) \Gamma^{(N)}(p_i, m^2, g) = \alpha m^2 \Delta\Gamma^{(N)}(p_i, m^2, g) , \quad (1)$$

where $\Gamma^{(N)}$ denotes an $N$-point proper vertex,[1] and $\Delta\Gamma^{(N)}$ the corresponding vertex with the insertion of a zero 4-momentum mass operator. If $g$ is dimensionless, as in the $g\phi^4$ theory, for instance, dimensional analysis gives

$$\Gamma^{(N)}(\lambda p_i, m^2, g) = \lambda^{4-N} \tilde{\Gamma}^{(N)}(p_i, m^2/\lambda^2, g) \quad (2)$$

and $\beta, \gamma, \alpha$ are only functions of $g$.

For Euclidean, nonexceptional momenta,[2] Weinberg's theorem applied to each dominant graph to any order in $g$ leads one to expect that

$$\frac{\Delta\Gamma^{(N)}(\lambda p_i, m^2, g)}{\Gamma^{(N)}(\lambda p_i, m^2, g)} \underset{\lambda\to\infty}{\sim} \frac{A}{\lambda^2} (\ln\lambda)^M , \quad (3)$$

where $A$ is independent of $\lambda$. Hence it is commonly assumed that the asymptotic behavior of $\Gamma^{(N)}$, when all the momenta are large but nonexceptional and Euclidean, can be deduced from the approximate equation

$$\left( -\lambda^2 \frac{\partial}{\partial \lambda^2} + \beta(g) \frac{\partial}{\partial g} - N\gamma(g) \right) \tilde{\Gamma}_{as}(p_i, m^2/\lambda^2, g) = 0 . \quad (4)$$

Since the behavior of $\tilde{\Gamma}_{as}$ for large $\lambda$ obviously depends on the properties of $\beta(g)$ and $\gamma(g)$, intensive studies have been made about their behavior as a function of $g$, particularly about the existence and the nature of the zeros of $\beta$.

For diverse reasons the possibility has been considered that the physical coupling constant $g$ satisfies an eigenvalue condition.[3] One assumes that $\beta$ has real zeros: $\beta(g_i) = 0$, and that the physical coupling constant $g$ happens to coincide with one of the zeros of $\beta$, i.e., $g = g_i$ for some $i$. (All the quantities $m$, $g$, as well as the vertex functions $\Gamma$, refer to renormalized ones. We avoid any reference to unrenormalized quantities except in Sec. III.)

When $g$ is equal to an eigenvalue, Eq. (4) reduces to

$$\left( -\lambda^2 \frac{\partial}{\partial \lambda^2} - N\gamma \right) \tilde{\Gamma}_{as}\left( p_i, \frac{m^2}{\lambda^2}, g \right) = 0 , \quad (5)$$

corresponding to simple scaling with anomalous dimension $\gamma(g)$, it is often assumed that in this special case one recovers asymptotic scale invariance, probably with anomalous dimensions for the fields. It is also known that it is not necessary for $g$ to be such that $\beta(g) = 0$ identically: If the effective coupling constant $\bar{g}(\ln\lambda)$, to be defined shortly, goes to a zero of $\beta$ for large $\lambda$ [i.e., $\bar{g}(\ln\lambda) \sim_{\lambda\to\infty} g_i$], one may also recover asymptotic scale inveriance. But having the physical coupling constant $g$ to be such that $\beta(g) = 0$ apparently seems to be a clear-cut situation with asymptotic power behavior. The purpose of this note is to emphasize that this is not always true, and to illustrate this with a simple example. More generally, the point is that the equation $\beta(g) = 0$, when satisfied for a nonvanishing $g$, relates terms of different orders in $g$. As a result Eq. (3), true graph by graph in the example, is not true for the sum of graphs. The simple example simultaneously illustrates the known fact that $\bar{g}(\ln\lambda) \sim_{\lambda\to\infty} 0$ in the weak coupling region need not imply asymptotic freedom, if the physical coupling constant is large and outside the range of attraction to the trivial zero of $\beta$ at the origin.

## II. A SIMPLE EXAMPLE

We will consider scalar fields $\Phi$ and $\phi$, corresponding to particles with physical masses $\mu$ and $m$, respectively, with a $g\Phi\phi^2$ interaction.[4] A simple illustration of the phenomenon mentioned in Sec. I is obtained in the so-called chain approximation. In this approximation the connected 2-point function $\langle\Phi\Phi\rangle_+$ and 3-point function $\langle\Phi\phi\phi\rangle_+$ are given by summing chains of $\phi$ loops in the $\Phi$ line only, and the infinite series can be trivially computed. The corresponding $\Gamma^{(2)}$ for the $\Phi$ field is

$$\Gamma^{(2)}(s,m,\mu,G)=(s-\mu^2)\left[1-GdH\left(\frac{m^2}{\mu^2}\right)\right]+Gd\mu^2\left[F\left(\frac{m^2}{s}\right)-F\left(\frac{m^2}{\mu^2}\right)\right], \tag{6}$$

where $G\equiv g^2/\mu^2$ so that it is dimensionless, $s$ is the square of the momentum of the $\Phi$ leg,

$$F(x)=2+(1-4x)^{1/2}\ln\frac{(1-4x)^{1/2}-1}{(1-4x)^{1/2}+1}, \tag{7}$$

$$H(x)=-\frac{d}{dx}F(x), \tag{8}$$

with the logarithm negative and the square root positive for $x<0$, and $d$ is a positive constant independent of $G$ and the masses.

The Callan-Symanzik equation reads

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+m^2\frac{\partial}{\partial m^2}+\beta\left(G,\frac{m^2}{\mu^2}\right)\frac{\partial}{\partial G}-2\gamma\left(G,\frac{m^2}{\mu^2}\right)\right]\times\Gamma^{(2)}(s,m,\mu,G)=\Delta\Gamma^{(2)}, \tag{9}$$

$$\Delta\Gamma^{(2)}\equiv m^2\Delta_1\Gamma^{(2)}+\alpha\left(G,\frac{m^2}{\mu^2}\right)\mu^2\Delta_2\Gamma^{(2)}.$$

One finds, in this chain approximation,

$$\beta=-G+G^2dH\left(\frac{m^2}{\mu^2}\right),\quad \gamma=\tfrac{1}{2}GdH\left(\frac{m^2}{\mu^2}\right), \tag{10}$$

$$\alpha=1-\frac{m^2}{\mu^2}\frac{Gd}{(1-4m^2/\mu^2)^{1/2}}\ln\frac{(1-4m^2/\mu^2)^{1/2}-1}{(1-4m^2/\mu^2)^{1/2}+1},$$

$$\Delta_1\Gamma^{(2)}=-Gd\frac{\mu^2}{s}\frac{1}{(1-4m^2/s)^{1/2}}\ln\frac{(1-4m^2/s)^{1/2}-1}{(1-4m^2/s)^{1/2}+1}, \tag{11}$$

$$\Delta_2\Gamma^{(2)}=-1.$$

We see from Eq. (6) and Eq. (11) that at least for small $G$,

$$\Gamma^{(2)}\underset{s\to-\infty}{\simeq}\text{const}\times s,$$

$$\Delta\Gamma^{(2)}\underset{s\to-\infty}{\simeq}\text{const},$$

so that Eq. (3) is true.

By neglecting $\Delta\Gamma^{(2)}$ in the asymptotic region and defining $\Gamma^{(2)}\equiv s\bar{\Gamma}^{(2)}$, one arrives from Eq. (9) to the asymptotic equation

$$\left(s\frac{\partial}{\partial s}-\beta\frac{\partial}{\partial G}+2\gamma\right)\bar{\Gamma}^{(2)}_{as}(s,m,\mu,G)=0. \tag{12}$$

The solution is

$$\bar{\Gamma}^{(2)}_{as}(s,m,\mu,G)=\bar{\Gamma}^{(2)}_{as}(s_0,m,\mu,\bar{G}(t))\times\exp\left[-2\int_0^t\gamma(\bar{G}(t'))dt'\right], \tag{13}$$

where

$$t=\ln(s/s_0)$$

and the effective coupling constant $\bar{G}(t)$ is a solution to the equation

$$\frac{d}{dt}\bar{G}(t)=\beta(\bar{G}), \tag{14}$$

with the boundary condition $\bar{G}(0)=G$. Equation (14) yields

$$\bar{G}(t)=\frac{Ge^{-t}}{[1-GdH(m^2/\mu^2)+GdH(m^2/\mu^2)e^{-t}]}. \tag{15}$$

So at least for small $G$, $\bar{G}(t)\sim_{t\to\infty}0$ and one may expect asymptotic freedom. Explicitly,

$$\bar{\Gamma}^{(2)}_{as}(s_0,m,\mu,\bar{G}(t))\underset{t\to\infty}{\simeq}\text{const},$$

while

$$\exp\left\{-2\int_0^t\gamma(\bar{G}(t'))dt'\right\}\underset{t\to\infty}{\simeq}\text{const}. \tag{16}$$

Thus $\bar{\Gamma}_{as}(s,m,\mu,G)\sim_{s\to-\infty}$ const., in agreement with asymptotic freedom.

However, suppose that the physical coupling constant is such that $G^{-1}=dH(m^2/\mu^2)$, so that it is at a Callan-Symanzik eigenvalue corresponding to a less trivial zero of $\beta$. Equation (12) reduces to

$$\left(s\frac{\partial}{\partial s}+2\gamma\right)\bar{\Gamma}^{(2)}_{as}(s,m,\mu,G)=0. \tag{17}$$

One might think that one then has a less trivial type of asymptotic scale invariance, with anomalous dimension $2\gamma=1$, so that $\bar{\Gamma}^{(2)}\sim_{s\to-\infty}s^{-1}$, and

$$\Gamma^{(2)}\underset{s\to-\infty}{\simeq}\text{const}. \tag{18}$$

This answer is not quite correct as one sees from Eq. (6); the correct behavior for this case is

$$\Gamma^{(2)}\underset{s\to-\infty}{\simeq}\ln s. \tag{19}$$

The reason for the discrepancy is easy to locate.

From Eq. (11) one finds that at the Callan-Symanzik eigenvalue

$$\frac{\Delta\Gamma^{(2)}}{\Gamma^{(2)}} \underset{\lambda\to\infty}{\simeq} \frac{1}{\ln\lambda} \tag{20}$$

instead of the ratio given by Eq. (3).. The fact that $\Delta\Gamma$ is no longer so relatively small is responsible for the logarithmic deviation from a power behavior even when $\beta$ is identically zero. Symanzik already stressed[5] that the consistency of neglecting $\Delta\Gamma$ always has to be checked by examining the anomalous dimensions of the fields. In this case, apart from logarithms $\Phi$ has effectively dimension 2 and the normal product $N_2(\Phi^2)$ corresponding to one of the mass insertions has effectively dimension 4 at the eigenvalue (in the one $\phi$-loop chain approximation), so $\Delta\Gamma$ is not negligible.

This extremely simple example also serves to illustrate how the weak coupling result $\overline{G}(t)\to_{t\to\infty} 0$ for asymptotic freedom can be upset when the coupling constant becomes big enough to reach the second zero of $\beta$. The interesting results in non-Abelian gauge theories,[6] showing asymptotic freedom for weak coupling, may not be necessarily valid for strong-interaction physics unless one can establish a large domain of attraction in the coupling-constant space for the trivial zero of $\beta$ at the origin.[7]

To be sure, the departure from power behavior in this example is only logarithmic; but the mechanisms brought out very simply by this example, namely, the non-negligible mass-insertion term, in addition to the escape of the effective coupling constant from the trivial zero of $\beta$, which is commonly known but also clearly exhibited in this example, deserve careful study in any consideration of asymptotic scale invariance in the strong coupling domain.

## III. CONNECTION WITH BOUND STATES

A further motivation for looking at this model with the over-simplified chain approximation is to compare with the corresponding $f\phi^4$ theory with a bound state. It is well known[8] that in the $Z_3=0$ limit of the $g\Phi\phi^2$ theory, where $Z_3$ is the wave-function renormalization constant for the $\Phi$ field, the $2\phi$ elastic scattering amplitude in the one-loop chain approximation coincides with that of the $f\phi^4$ theory, also in the chain approximation, if there is a 2-$\phi$ bound state in the $f\phi^4$ theory at the mass $\mu$. We do not know to what extent $Z_3=0$ is a generally meaningful condition for turning a particle composite.[9] But for our purpose, since the equivalence does hold for the amplitude that we have considered in the chain approximation, we can ask what happens if one considers the question of asymptotic scale invariance within the framework of the $f\phi^4$ theory for the corresponding 2-point function of the composite field.

The connected Green's function $G_N^{(2)}$ for the normal product $N_2(\phi^2)$ in the $f\phi^4$ theory, with the existence of a bound state at mass $\mu$ explicitly exhibited, is

$$G_N^{(2)}(s,m,f)=\frac{1}{f}-\frac{1}{bf^2[F(m^2/s)-F(m^2/\mu^2)]}, \tag{21}$$

where $s$ denotes the square of the momentum carried by the composite field, $b$ is a positive constant, and $\mu^2$ is a function of $f$ and $m^2$ through the relation

$$1+fbF(m^2/\mu^2)=0. \tag{22}$$

In the $g\Phi\phi^2$ theory, on the other hand, the connected 2-point Green's function for $\Phi$ is $G^{(2)}(s,m,\mu,g)=-(\Gamma^{(2)})^{-1}$, and $Z_3=1-GdH(m^2/\mu^2)$. So in the limit $Z_3=0$, which in this case coincides with the eigenvalue point studied earlier,

$$G^{(2)}(s,m,\mu,g)=\frac{\text{const.}}{[F(m^2/s)-F(m^2/\mu^2)]}. \tag{23}$$

Thus the 2-point functions are not identical if we use $N_2(\phi^2)$ as the composite particle field.

$G_N^{(2)}$ approaches a constant as $s\to-\infty$; therefore the asymptotic behavior seems compatible with canonical scaling. The Callan-Symanzik equation for $G_N^{(2)}$, however, takes the form

$$\left[m^2\frac{\partial}{\partial m^2}+\beta'(f)\frac{\partial}{\partial f}-2\gamma'(f)\right]G_N^{(2)}=\Delta G_N^{(2)}+b, \tag{24}$$

where $\Delta G_N^{(2)}$ denotes the usual mass-insertion term, and where $\beta'=bf^2$, $\gamma'=-bf$. The presence of the additional term on the right-hand side of Eq. (24) seems to have been first discussed by Coleman and Jackiw.[10] As $s\to-\infty$ the right-hand side of Eq. (24) approaches the constant $b$, hence $G_N^{(2)}$ does not satisfy a homogeneous Callan-Symanzik equation asymptotically. It is easy to see that no further modification of the anomalous dimension term $2\gamma'$ can get rid of the constant $b$ on the right-hand side of Eq. (24), simply from considerations of the behavior at the subtraction point. Although this further anomaly occurs only for the 2-point function in this case and in the case analyzed in Ref. 10, there seems to be no compelling reason why it may not appear in some higher-point functions for more complicated theories, rendering the deduction of asymptotic behavior from a homogeneous Callan-Symanzik equation impossible for such functions.

We may note that without the term $1/f$ in Eq. (21), $G_N^{(2)}$ would have satisfied Eq. (24) without the constant $b$ on the right-hand side. In other words, if one can consistently define an alternative local product $\phi^2$ such that its 2-point function coincides with Eq. (23) of the original $\Phi$-field Green's function, then from the viewpoint of deducing asymptotic behavior from the Callan-Symanzik equations it is advantageous to consider the problem in the $\phi^4$ framework rather than the $\Phi\phi^2$ framework. This is intuitively plausible since the large dimensional mass-insertion term $N_2(\Phi^2)$ does not appear in the $\phi^4$ framework. This interesting possibility is under further investigation.

## IV. CONCLUDING REMARKS

We learned that in a theory with the physical coupling constant at a nontrivial zero of $\beta$ the vertex functions do not necessarily have asymptotic power behavior, because the mass-insertion terms may be non-negligible compared to the vertex functions. This happens in a finite order in the example considered; obviously one has to be more careful if all orders are important in strong-interaction physics. The same problem of non-negligible mass insertions may arise in theories with other constraints, such as $Z=0$ conditions. The inhomogeneous term in the Callan-Symanzik equation for Green's functions involving two or more normal products also may not be asymptotically negligible.

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[1]The sum of connected 1-particle irreducible graphs with amputated external legs.

[2]A set of Euclidean momentum of which no nontrivial partial sum is zero.

[3]The existence of these zeros for strong-interaction physics has been conjectured by K. Wilson, Phys. Rev. D 3, 1818 (1971); for the hypothesis that the physical charge equals a zero of $\beta$ in QED, see S. L. Adler, *ibid.* 5, 3021 (1972); 7, 1948(E) (1973) where references to earlier works can be found; B. Schroer, in Nuovo Cimento Lett. 2, 869 (1971) showed that the satisfaction of this eigenvalue condition in $\phi^4$ theory is necessary and sufficient for the existence of a stress tensor with soft trace.

[4]There are well-known difficulties with this theory in connection with the lack of a stable vacuum; we ignore this difficulty since it does not seem to be directly related to the aspect that we study.

[5]K. Symanzik, Commun. Math. Phys. 23, 49 (1971).

[6]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, *ibid.* 30, 1346 (1973).

[7]Studies of the size of regions of attraction have been made for some simple models, see, e.g., P. K. Mitter and P. Weisz, Phys. Rev. D 8, 4410 (1973), but often only to low orders of perturbation.

[8]See, e.g., Hayashi *et al.*, Fortschr. Phys. 15, 625 (1967).

[9]See, e.g., M. L. W. Whippman, Phys. Rev. D 3, 2372 (1971).

[10]S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971).