[11]The conclusion that clusters of galaxies are bound is reached by observing that clusters are found much more commonly than could come about by statistical fluctuations. Further, the observed velocities of the component galaxies would have greatly dispersed them over the age of the universe, if there were no binding. However, a long-standing problem has been that the virial masses are greater than the observed masses for many clusters. [One does a virial theorem on the observed parameters of a particular system and hopes to find $V_{tot} = -2$(total kinetic energy).] It is generally agreed that there are two possible solutions to this problem: (1) Either at various times in the past, up to quite recently, the clusters lost some of their existing masses, for instance, by quasar explosions, or (2) there are intergalactic "missing masses," probably in the form of ionized hydrogen, which account for the discrepancies. (There is preliminary evidence for this solution.) But the important point to observe is that even under solution (1) we are not prevented from establishing our conservative mass limit because the dispersion that may have occurred in the possibly recently disrupted clusters would have to be relatively small. Finally, we note that another solution would be to postulate some unknown long-distance force other than gravity. If one assumes this undefined *ad hoc* hypothesis then, of course, nothing at all can be said about large-scale dynamics. We should mention that Y. Yamaguchi (private communication) has suggested that a magnetically contained plasma between galaxies might conceivably yield such a force. Although this in an interesting speculation, it is doubtful that such a mechanism could mimic accurately the effects of gravitation. See H. J. Rood, V. C. A. Rothman, and B. E. Turnrose, Astrophys. J. 162, 411 (1970); G. B. Field and W. C. Saslaw, *ibid.* 170, 199 (1971); D. S. De Young, *ibid.* 173, L7 (1972), and references therein.

[12]E. Holmberg, Ark. Astron. 5, 305 (1969). See p. 309.

[13]A. Sandage, Astrophys. J. 178, 1 (1972). See p. 22.

[14]G. de Vaucouleurs, Publ. Astron. Soc. Pac. 83, 113 (1971).

[15]Somewhat weaker limits on the graviton mass, using a variety of less sensitive methods, have been obtained by M. G. Hare, Can. J. Phys. 51, 431 (1973).

[16]It is interesting to note that as long ago as 1957 Zwicky suggested that clustering up to certain sizes might imply a gravitational cutoff at distances of around 3×10^6 pc. Since the Hubble scale has changed by roughly a factor of 10 in the meantime, this implies that Zwicky was speculating on a cutoff limit roughly 50 times our own more conservative estimate. Among the possibilities suggested for the origin of this speculated cutoff was the existence of a cosmological term in Einstein's equations. See F. Zwicky, Publ. Astron. Soc. Pac. 69, 518 (1957).

[17]Earlier K. Hiida and Y. Yamaguchi [Prog. Theor. Phys. Suppl., extra number, 262 (1965)] suggested that the analysis of the dynamics of clusters of galaxies could yield a limit on the graviton mass as low as 5×10^{-62} g. (See p. 264 of the above reference.)

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Addendum to Wilson's theory of critical phenomena and Callan-Symanzik equations in $4-\epsilon$ dimensions

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In a previous work, it was shown how to derive the scaling laws near a critical point using renormalized perturbation theory. The calculations of the Callan-Symanzik functions β and γ which lead to the critical exponents are extended to next order in ϵ. The existence of a solution to the eigenvalue conditions $\beta(g)=0$ in four dimensions, at fourth order in the coupling constant, is shown to be renormalization-dependent.

In the framework of Wilson's theory of critical phenomena,[1] we have discussed in a recent article[2] how scaling laws for the correlation functions near the critical point may be derived from the Callan-Symanzik equations in $4-\epsilon$ dimensions applied to a $g(\vec{\varphi}^2)^2$ interaction, where $\vec{\varphi}(x)$ is an n-component order parameter. Higher-order corrections in ϵ have now been computed.[3] The purpose of this addendum is to give various quantities which are useful in these calculations,[2,4] like the expansions of the renormalization constants, of the Callan-Symanzik β and γ functions, and of the solution of the eigenvalue condition. The notations are identical to those of Ref. 2.

For simplicity we have done the calculations in the massless theory[4] with the following convenient renormalization conditions for the vertex functions:

$$\Gamma^{(2)}(p,-p;u)|_{p^2=0}=0\,, \tag{1a}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}(p,-p;u)\Big|_{p^2=e^l}=1\,, \tag{1b}$$

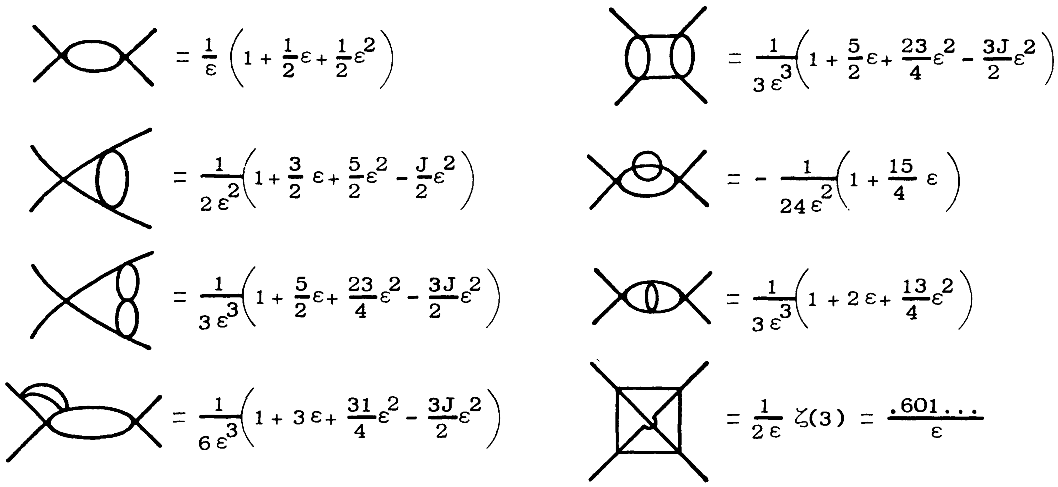

FIG. 1. Contributions to Z_1 and Z_4 expanded in powers of ϵ.

$$\Gamma^{(4)}(p_1, \ldots, p_4; u)|_{SP} = u\,, \tag{1c}$$

where SP is the symmetry point $p_i \cdot p_j = \frac{1}{4}(4\delta_{ij} - 1)$, u is the dimensionless coupling constant, and l is an arbitrary dimensionless parameter that we have introduced in order to exhibit the dependence on the subtraction scheme.

In this work, we had to compute the values of the diagrams given in Figs. 1 and 2. A diagram with L loops is still to be multiplied by S^L, where $S = 2\pi^{d/2}(2\pi)^{-d}/\Gamma(d/2)$, but we shall absorb this normalization factor into a redefinition of the coupling constant $u \to u/S$. Each diagram has been expanded up to the relevant order in ϵ.

Each graph of Fig. 2 stands for the derivative of the corresponding Feynman diagram, taken at the point $p^2 = e^l$.

Then, the renormalization constants Z_1, Z_3, Z_4 of the vertex, wave function, and φ^2 insertion, respectively, are expanded in powers of u:

$$Z_1^{-1} = 1 + A_1 u + B_1 u^2 + C_1 u^3 + O(u^4)\,, \tag{2a}$$

$$Z_3 = 1 + B_3 u^2 + C_3 u^3 + D_3 u^4 + O(u^5)\,, \tag{2b}$$

$$Z_4^{-1} = 1 + A_4 u + B_4 u^2 + C_4 u^3 + O(u^4)\,, \tag{2c}$$

where the coefficients A_i, B_i, and C_i are given in Fig. 3 as functions of the diagrams.

Consequently, the function

$$\beta(u) = -\epsilon \Big/ \frac{d}{du} \ln \frac{u Z_1(u)}{Z_3^2(u)} \tag{3}$$

may be determined up to order $u^p \epsilon^q$ with $p + q \leq 4$ as

$$\begin{aligned}\beta(u) = {} & -\epsilon u + \frac{(n+8)}{6}\left(1 + \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon^2\right)u^2 \\ & - \tfrac{1}{36}\{9n + 42 + \epsilon[\tfrac{5}{4}(19n + 86) - 2(5n+22)J \\ & + (n+2)l]\}u^3 \\ & + \tfrac{1}{432}\{23n^2 + 414n + 1264 \\ & - (n+8)[2(5n+22)J - (n+2)l] \\ & + 24(5n+22)\zeta(3)\}u^4 \\ & + O(u^5, u^4\epsilon, u^3\epsilon^2, u^2\epsilon^3)\,,\end{aligned} \tag{4}$$

where

$$J = \int_0^1 \frac{dx}{x}\int_0^1 dy \ln\left[1 - x + \frac{3}{4}\frac{x}{y}(1 - x + xy)\right] = 1.7494\ldots.$$

The zero u_∞ of order ϵ of this function $\beta(u)$ is then

$$\begin{aligned}u_\infty = \frac{6}{n+8}\epsilon\Bigg[1 + \epsilon\left(\frac{3(3n+14)}{(n+8)^2} - \frac{1}{2}\right) + \epsilon^2\bigg(&\frac{18(3n+14)^2}{(n+8)^4} - \frac{1}{4} - \frac{(5n^2 + 322n + 1104)}{4(n+8)^3} \\ & - \frac{12(5n+22)}{(n+8)^3}\zeta(3) - \frac{[2(5n+22)J - (n+2)l]}{2(n+8)^2}\bigg)\Bigg] + O(\epsilon^4)\,.\end{aligned} \tag{5}$$

The critical exponents η and γ are given in terms of the functions

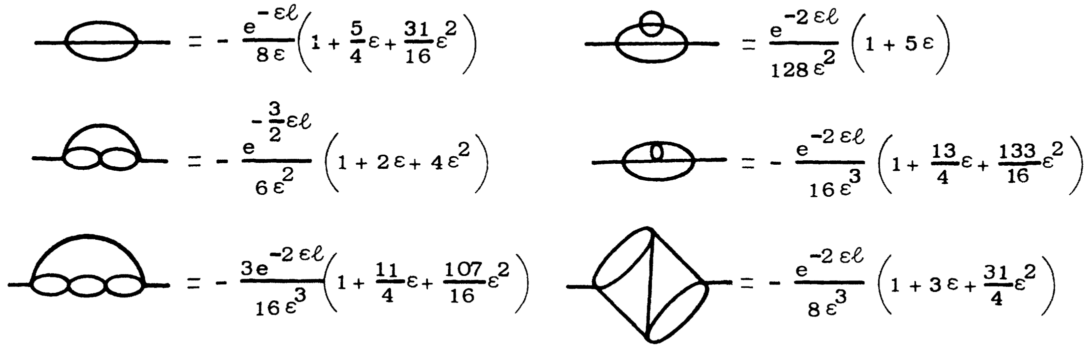

FIG. 2. Contributions to the wave-function renormalization Z_3 expanded in powers of ϵ.

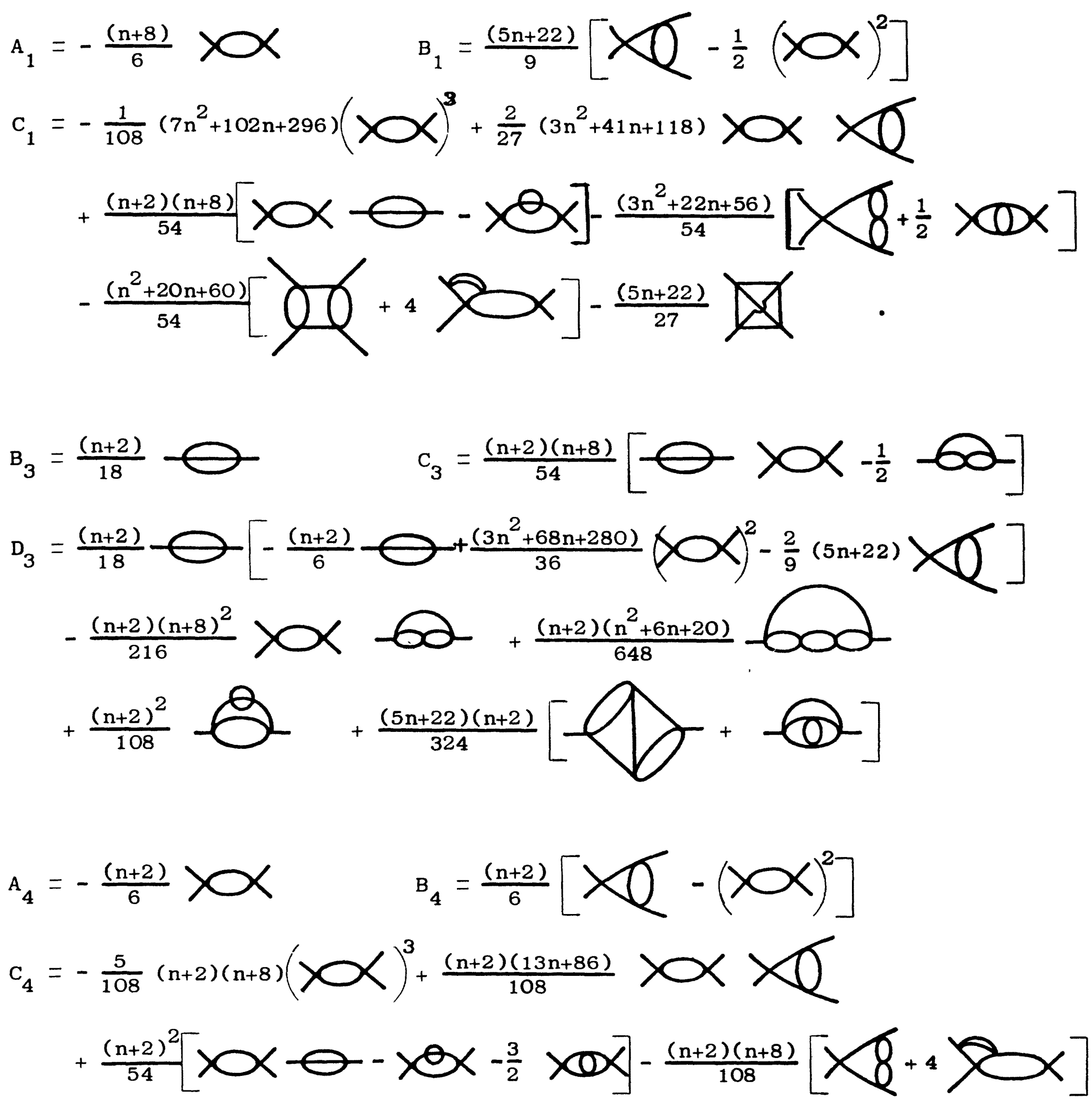

FIG. 3. The coefficients of Eq. (2) as functions of the diagrams.

$$\gamma_3(u) = \beta(u)\frac{d\ln Z_3(u)}{du}$$

and

$$\gamma_4(u) = \beta(u)\frac{d\ln Z_4(u)}{du} \tag{6}$$

by the relations

$$\eta = \gamma_3(u_\infty)$$

and

$$\frac{2-\eta}{\gamma} = 2-\eta+\gamma_4(u_\infty)\,. \tag{7}$$

We first obtained at the needed order

$$\gamma_3(u) = \frac{(n+2)}{72}u^2\Big(1+(\tfrac{5}{4}-l)\epsilon+\tfrac{1}{2}(l^2-\tfrac{5}{2}l+\tfrac{31}{8})\epsilon^2 - \frac{(n+8)}{12}u[1-2l+\epsilon(\tfrac{5}{2}l^2-5l+\tfrac{15}{4})]$$

$$+\frac{u^2}{288}[3n^2(2l^2-2l+1)+n(96l^2-176l+33)+384l^2-736l+162+16(5n+22)J]\Big)\,, \tag{8}$$

$$\gamma_4(u) = -\frac{(n+2)}{6}u\left(1+\tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon^2-\tfrac{1}{2}u(1+\tfrac{5}{2}\epsilon-J\epsilon)+\frac{u^2}{36}[n(1-\tfrac{1}{2}l)+32-l]\right). \tag{9}$$

Then we found

$$\eta = \frac{\epsilon^2(n+2)}{2(n+8)^2}\Big[1+\epsilon\Big(\frac{6(3n+14)}{(n+8)^2}-\tfrac{1}{4}\Big)$$

$$+\epsilon^2\Big(\frac{1}{(n+8)^4}(-\tfrac{5}{16}n^4-\tfrac{115}{8}n^3+\tfrac{281}{4}n^2+1120n+2884)-\frac{24(5n+22)}{(n+8)^3}\zeta(3)\Big)\Big]+O(\epsilon^5)\,, \tag{10}$$

$$\gamma_4(u_\infty) = -\epsilon\frac{(n+2)}{(n+8)}\Big[1+\frac{6(n+3)}{(n+8)^2}\epsilon+\epsilon^2\Big(-\frac{12(5n+22)}{(n+8)^3}\zeta(3)+\frac{(n^2+222n+560)}{4(n+8)^3}+\frac{36(3n+14)(n+3)}{(n+8)^4}\Big)\Big]+O(\epsilon^4)\,, \tag{11}$$

$$\gamma = 1+\frac{(n+2)}{2(n+8)}\epsilon+\frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2$$

$$+(n+2)\epsilon^3\Big[\frac{1}{(n+8)^5}(\tfrac{1}{8}n^4+\tfrac{11}{2}n^3+83n^2+312n+388)-\frac{6(5n+22)}{(n+8)^4}\zeta(3)\Big]+O(\epsilon^4)\,. \tag{12}$$

Notice that in these last results the renormalization-dependent terms l and J cancel as expected.

It is instructive to look for the solution of the equation $\beta(u)=0$ in four dimensions. If we choose for simplicity $n=1$, $\beta(u)$ becomes

$$\beta(u) = \tfrac{3}{2}u^2-\tfrac{17}{12}u^3+\tfrac{1}{16}[63+24\zeta(3)+l-18J]u^4+O(u^5)\,.$$

We note again that the coefficient of u^4 depends on the subtraction scheme through J and l.[4] The renormalization condition (1) fixes here J to an explicit value but l is still an independent parameter. Thus the coefficient of u^4 has a totally arbitrary sign and magnitude. Therefore the problem of a perturbative determination of a nontrivial solution of $\beta(u)=0$ is somewhat ambiguous.

[1]A thorough and extensive review is given in the work of K. G. Wilson and J. Kogut, Phys. Rep. (to be published).

[2]E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D 8, 434 (1973).

[3]E. Brezin, J. C. Le Guillou, J. Zinn-Justin, and B. Nickel, Phys. Lett. 44A, 227 (1973).

[4]E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D 8, 2418 (1973).