[27]Wick rotation of the solutions to Bethe-Salpeter equations has been stated to be justified in renormalizable theories by G. Domokos, P. Suranyi, and A. Vancura, Nucl. Phys. 60, 1 (1964).

[28]Note that

$$p^2 = \vec{p}^2 - p_0^2 = \vec{p}^2 + p_4^2 = |p|^2 > 0$$

in the four-dimensional Euclidean space. Thus in going to the point $p^2 = -m^2$, one has to make an analytic continuation to negative values of p^2; this can, of course, be done.

[29]ϵ is an infrared cutoff which comes in because the exchanged particle is assumed to have zero mass.

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Theory of the electromagnetic structure functions of the proton

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The knowledge of the wave function of a relativistic composite system constitutes a complete description of its intrinsic properties such as mass spectrum, elastic and inelastic form factors, and structure functions. The wave functions of the proton obtained from the O(4,2) infinite-multiplet model, which were used previously to calculate mass spectra and form factors, are applied to reevaluate in a more complete manner the structure functions in closed form. The resultant scaling functions obey the Drell-Yan relation $F_2(\xi) \sim (1-\xi)^3$ and, under certain conditions, the Callan-Cross relation $F_2(\xi) \approx 2\xi F_1(\xi)$.

I. INTRODUCTION

A good model of the nucleon as a relativistic composite object must account for all its properties which are usually associated with the internal structure of the nucleon. It must give a good description of processes which are determined by the nucleon structure alone. Only then can we say that we have a good over-all picture of the nucleon. These properties are the elastic form factors, the spectrum of the excited states, the inelastic transition form factors, the decay rates of the excited states, and the structure functions. In ordinary quantum theory these properties are all determined by the wave function of the system. Hence the exact knowledge of the wave function constitutes a complete description of the system. Experimentally the intrinsic properties of the system are measured by probes which are themselves structureless. For the nucleons, the above-mentioned properties have been and are being analyzed from γN, eN, and νN scattering processes, and considerable information has been obtained, in particular through the inelastic electron-nucleon scattering.[1]

The purpose of this paper is to apply the explicit wave function of the proton and its excited states obtained from an infinite-component wave equation to evaluate the structure functions in inelastic electron-proton scattering. The wave function has previously been used to predict the elastic form factors,[2] the mass spectrum,[2,3] the inelastic form factors,[4] and the partial decay rates.[5] Some aspects of the structure functions have also been reported.[6-9] We present here the details of a more complete calculation, briefly reported earlier,[10] in particular the explicit form of the so-called scaling functions $F_1(\xi)$ and $F_2(\xi)$.

There is an underlying physical picture of the description of the proton by a wave equation. It corresponds to an atomic-type composite system.[11] The relativistic H atom itself, conversely, can be described completely by an infinite-component wave equation.[12] This picture and all the calculations indicated taken together lead to the conclusion that, as far as electromagnetic probes are concerned, the proton, in a very wide range of energy and momentum transfer, behaves like an "atom," the inelastic process proceeding via the excitation of the "atom" (including continuum) and its subsequent decay. The limitations of the picture will come when particle production without the excitation of the proton will be a dominant process.

Of course, the H-atom concept will also be lost in processes where production of other particles becomes dominant. This, however, does not diminish the information in the H-atom concept about the constituents of the atom. Similarly, if we have a convincing picture of the proton and a reliable wave function, we can make more definite statements about the true constituents of the proton, which is perhaps one of the most fundamental questions of particle physics at the present time.

Note that although we discuss mainly the scaling region the model is in principle applicable everywhere, including the resonance region.

II. PRELIMINARIES

We consider the nucleon as a composite system, with its ground and excited states, whose wave function obeys a dynamical wave equation. In a relativistic description it is convenient to use an algebraic formalism for the dynamics. The interparticle relative coordinates and the interparticle interactions are replaced by the global quantum numbers of the system. One then writes an infinite-component wave equation, treating the system as though it were an elementary particle and coupling it minimally to the electromagnetic field. From the theory of the H atom one knows how to pass to the internal dynamics, if necessary.

In our model the nucleon and its excited states are assigned to an irreducible unitary representation of the dynamical group O(4, 2). The states in the rest frame are labeled by $|njm\rangle$, which we shall often write simply as $|n\rangle$. Physical states with momentum p, to be defined below, are labeled by $|\tilde{n}p\rangle$. There is a current operator J_μ satisfying the current conservation equation

$$\langle \tilde{n}p'|(p'-p)^\mu J_\mu|\tilde{n}p\rangle = 0\,, \tag{1}$$

which determines a mass spectrum for the system. The conserved current of the model is given by

$$J_\mu = \alpha_1\Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu S + \alpha_4 q^\nu L_{\mu\nu}\,. \tag{2}$$

The parameters α_i are functions of the masses of the constituents of the system and the strength of their interaction, while Γ_μ, S, and $L_{\mu\nu}$ are the O(4, 2) generators. In Eq. (2) we have $P_\mu = (p+p')_\mu$ and $q_\mu = (p'-p)_\mu$.

Explicit representations of the generators L_{ab} and the basis states on which they act are given in the Appendix. The basis vectors of the Hilbert space are related to the physical state vectors by a tilt operation,

$$|\tilde{n}, 0\rangle \equiv \frac{1}{N}\exp(-i\theta T)|n, 0\rangle\,, \tag{3a}$$

whereas the moving states are obtained from the rest-frame states by a boost operation,

$$|\tilde{n}, \vec{p}\rangle = \exp(-i\vec{M}\cdot\vec{\xi})|\tilde{n}, 0\rangle\,. \tag{3b}$$

Because the last term in Eq. (2) is conserved separately, we see from (1) that it does not contribute to the mass spectrum. The latter is also obtainable from the solution of the following infinite-component wave equation:

$$(J_\mu P^\mu + \beta S + \gamma)\tilde{\psi}(p) = 0\,, \tag{4}$$

where β and γ are parameters similar to α_i and $\tilde{\psi}(p) \equiv |\tilde{n}\vec{p}\rangle$. Consequently, the propagator in the forward Compton scattering amplitude, which we are going to use as a starting point of our calculations, can be written as

$$\tilde{\Omega}(p) = [\alpha_1\Gamma_\mu P^\mu + \alpha_2 P_\mu P^\mu + (\alpha_3 P_\mu P^\mu + \beta)S + \gamma]^{-1}\,. \tag{5}$$

In Sec. III we give a derivation of the matrix elements corresponding to the transition of the nucleon from the ground state to its excited states by the action of the electromagnetic current. In Sec. IV we compute the explicit forms of these matrix elements and the nucleon structure functions W_1 and W_2, which are of interest in deep-inelastic scattering. Finally, we shall show in Sec. V the behavior $MW_1(q^2, \nu) \to F_1(\xi)$ and $\nu W_2(q^2, \nu) \to F_2(\xi)$ in the limit $(-q^2, \nu) \to \infty$ at fixed ξ, where the scaling variable is defined by $1/\xi \equiv 1/x \equiv \omega = -(2M\nu + M^2)/q^2$.

III. DERIVATION OF THE MATRIX ELEMENTS

We start by writing down the explicit form of the forward Compton scattering amplitude on a proton, as follows:

$$T = T_{\mu\nu}\epsilon^\mu\epsilon^\nu = \frac{1}{|N_1|^2}\langle\tilde{1}\,\vec{p}|J_\mu\epsilon^\mu\tilde{\Omega}(p+q)J_\nu\epsilon^\nu|\tilde{1}\,\vec{p}\rangle\,. \tag{6}$$

The propagator can be diagonalized by the following inverse tilt operation:

$$\tilde{\Omega} = \exp(-i\theta_W T)\Omega\exp(i\theta_W T)\,, \tag{7}$$

where the tilting angle θ_W is chosen in such a way that the coefficient of S in the propagator vanishes. If we work in the center-of-mass frame and call the total energy squared $s = W^2$, so that $P_0 = W$, then we obtain from (5) and (7)

$$\Omega(W) = \{\Gamma_0[\alpha_1^2 W^2 - (\alpha_3 W^2 + \beta)^2]^{1/2} + \alpha_2 W^2 + \gamma\}^{-1}\,. \tag{8}$$

The next step we are going to take is to insert the complete set of group basis states behind $\Omega(W)$, operate $\Omega(W)$ on $|n\rangle$, and take the ξ axis along $\vec{p}$.

Then Eq. (6) can be written as

$$T=\frac{1}{|N_1|^2}\sum_n \Omega_n(W)\langle 1|e^{i\theta_1 T}e^{i\zeta M_3}J_\mu\epsilon^\mu e^{-i\theta_W T}|n\rangle \otimes\langle n|e^{i\theta_W T}J_\nu\epsilon^\nu e^{-i\zeta M_3}e^{-i\theta_1 T}|1\rangle . \quad (9)$$

Because $J_\mu\epsilon^\mu$ is a Lorentz invariant, we can move this expression to pass the booster. The components of the tensor $T_{\mu\nu}$ are then

$$T_{\mu\nu}=\frac{1}{|N_1|^2}\sum_n \Omega_n(W)M_\mu^\dagger(n)M_\nu(n), \quad (10)$$

where the matrix element $M_\lambda(n)$ is given by

$$M_\lambda(n)=\langle n|e^{i\theta_W T}e^{-i\zeta M_3}J_\lambda e^{-i\theta_1 T}|1\rangle . \quad (11)$$

If the total center-of-mass energy W is such that $\Omega_n^{-1}(W)=0$, we obtain the intermediate states n of total mass M_n, which form the timelike solutions of the wave equation (4). These are the final hadronic states in the inelastic process, where the proton is raised to its excited states by the energetic electrons. Therefore, the imaginary part of the amplitude (10), i.e.,

$$W_{\mu\nu}=\frac{\pi}{|N_1|^2}\sum_n M_\mu^\dagger(n)M_\nu(n), \quad (12)$$

is related in the usual manner to the inclusive differential cross section of deep-inelastic scattering. To see this, we write the standard decomposition

$$W_{\mu\nu}=\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)W_1 +\frac{1}{M^2}\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right)W_2, \quad (13)$$

where W_1 and W_2 are the well-known structure functions, and recall that the differential cross section for inelastic scattering of an electron of energy E from a proton into the angle θ and energy E' is

$$\frac{d\sigma}{d\Omega dE'}=\frac{\alpha^2}{4E^2\sin^4(\frac{1}{2}\theta)}\left[W_2\cos^2(\tfrac{1}{2}\theta)+2W_1\sin^2(\tfrac{1}{2}\theta)\right]. \quad (14)$$

We can express W_1 and W_2 in terms of $W_{\mu\nu}$ by using Eq. (13). Clearly,

$$W_1=W_{11} \quad (15a)$$

because $p_\mu=(M\cosh\zeta,0,0,M\sinh\zeta)$, and the momentum transferred $(p_n-p)_\mu$ or q_μ equals $(M_n-M\cosh\zeta,0,0,-M\sinh\zeta)$. We use the metric $g_{\mu\nu}=(+\,---)$. The second structure function is given by

$$W_2=-\frac{M^2}{M^2\sinh^2\zeta}\left[\frac{1+M^2\sinh^2\zeta/q^2}{(1+M\nu/q^2)^2}W_{11} -\frac{1}{(1+M\nu/q^2)^2}W_{33}\right], \quad (15b)$$

with

$$M\nu=p\cdot q=M_nM\cosh\zeta-M^2$$

and

$$q^2=M_n^2+M^2-2M_nM\cosh\zeta=M_n^2-2M\nu-M^2 .$$

From Eqs. (12) and (15) we see that we only need to compute W_{11} and W_{33} to obtain the functions W_1 and W_2, where

$$W_{ii}=\frac{\pi}{|N_1|^2}\sum_n |M_i(n)|^2\delta(W^2-M_n^2)\quad (i=1,3). \quad (16)$$

Using Eqs. (11) and (2) and the definitions of P_μ and q_μ given above, we have

$$M_1=\langle n|e^{i\theta_W T}e^{-i\zeta M_3}[\alpha_1L_{16}+\alpha_4(M_n-M\cosh\zeta)L_{10}+\alpha_4M\sinh\zeta L_{13}]e^{-i\theta_1 T}|1\rangle , \quad (17)$$

while

$$M_3=\langle n|e^{i\theta_W T}e^{-i\zeta M_3}[\alpha_1L_{36}+(\alpha_2+\alpha_3L_{46})M\sinh\zeta+\alpha_4(M_n-M\cosh\zeta)L_{30}]e^{-i\theta_1 T}|1\rangle . \quad (18)$$

If we pull the tilt operation on the ground state to the left, we obtain

$$M_1=\langle n|G(\zeta)[\alpha_1L_{16}+\alpha_4(M_n-M\cosh\zeta)(L_{15}\cosh\theta_1-L_{14}\sinh\theta_1)+\alpha_4M\sinh\zeta L_{13}]|1\rangle , \quad (19)$$

$$M_3=\langle n|G(\zeta)[\alpha_1L_{36}+\alpha_2M\sinh\zeta+\alpha_3M\sinh\zeta(L_{46}\cosh\theta_1-L_{56}\sinh\theta_1) +\alpha_4(M_n-M\cosh\zeta)(L_{35}\cosh\theta_1-L_{34}\sinh\theta_1)]|1\rangle , \quad (20)$$

where

$$G(\zeta)=\exp(i\theta_W T)\exp(-i\zeta M_3)\exp(-i\theta_1 T). \quad (21)$$

Because of Eq. (A1) we have $|1\rangle\equiv|000\rangle$. Furthermore, because according to the relations (A6), (A8), and (A9) the operators L_{14}, L_{34}, and L_{13} start with the destruction of the vacuum, they do not contribute to the matrix elements M_i. We can also easily see from (A3), (A4), (A7), and (A10) that the operators L_{15} and L_{35} have the same effect on $|000\rangle$ as L_{16} and L_{36}, respectively, except for the multiple imaginary unit. By this observation the calculation of M_i becomes much simpler, for we have only to evaluate the contributions coming from L_{15}, L_{35}, L_{46}, and L_{56}. We have the following results (see Appendix):

$$L_{56}|000\rangle = |000\rangle\,, \tag{22a}$$

$$L_{46}|000\rangle = \tfrac{1}{2}|010\rangle - \tfrac{1}{2}|100\rangle\,, \tag{22b}$$

$$L_{15}|000\rangle = \tfrac{1}{2}|001\rangle - \tfrac{1}{2}|11,-1\rangle\,, \tag{22c}$$

$$L_{35}|000\rangle = -\tfrac{1}{2}|010\rangle - \tfrac{1}{2}|100\rangle\,. \tag{22d}$$

Now we can proceed to compute the matrix elements explicitly.

IV. EXPLICIT FORMS OF M_i

The techniques for calculating matrix elements of the type (19) and (20) have been used in many applications.[2-5] In what follows we indicate all the steps in a concise form.

Inserting Eqs. (22a)–(22d) into Eqs. (19) and (20), we obtain firstly

$$M_1(n) = \tfrac{1}{2}\{\langle n|G(\zeta)[i\alpha_1 + \alpha_4(M_n - M\cosh\zeta)\cosh\theta_1]|001\rangle - \langle n|G(\zeta)[i\alpha_1 + \alpha_4(M_n - M\cosh\zeta)\cosh\theta_1]|11,-1\rangle\}\,. \tag{23}$$

Because $|001\rangle$ and $|11,-1\rangle$ have different m values, while $G(\zeta)$ does not change m, the $|n\rangle$ states must have either $m=+1$ or $m=-1$. So the two parts of Eq. (23) do not interfere when we take $|M_1|^2$. Secondly, we have

$$\begin{aligned} M_3(n) = {} & \langle n|G(\zeta)(\alpha_2 - \alpha_3\sinh\theta_1)M\sinh\zeta|000\rangle \\ & - \tfrac{1}{2}\{\langle n|G(\zeta)[\alpha_4(M_n - M\cosh\zeta)\cosh\theta_1 + i\alpha_1 - \alpha_3 M\sinh\zeta\cosh\theta_1]|010\rangle \\ & + \langle n|G(\zeta)[\alpha_4(M_n - M\cosh\zeta)\cosh\theta_1 + i\alpha_1 + \alpha_3 M\sinh\zeta\cosh\theta_1]|100\rangle\}\,. \end{aligned} \tag{24}$$

In this case, all three parts will interfere with each other when we take $|M_3|^2$, because all $|n\rangle$ have $m=0$.

In order to obtain $\langle n|G(\zeta)|n'\rangle$, we apply the Euler angle transformation. Using the basis of the $O(2,1)\otimes O(2,1)$ subgroup and writing $|n\rangle \equiv |n_1, n_2, m\rangle$ as $|N_1, N_2\rangle$, where we have suppressed the quantum number m, which must be equal for the initial and final states, we have

$$\langle n|e^{i\theta_W T}e^{-i\zeta M_3}e^{-i\theta_1 T}|n'\rangle = \langle N_1N_2|e^{-i\alpha L_{34}}e^{-i\beta L_{45}}e^{-i\gamma L_{34}}|N_1'N_2'\rangle\,. \tag{25}$$

In Eq. (25) the angles α, β, and γ are related to θ_1, θ_W, and ζ by

$$\sinh^2(\tfrac{1}{2}\beta) = \tfrac{1}{2}(\cosh\theta_W\cosh\theta_1\cosh\zeta - \sinh\theta_W\sinh\theta_1 - 1)\,, \tag{26a}$$

$$\cosh^2(\tfrac{1}{2}\beta) = \tfrac{1}{2}(\cosh\theta_W\cosh\theta_1\cosh\zeta - \sinh\theta_W\sinh\theta_1 + 1), \tag{26b}$$

$$\sin\gamma = \frac{\cosh\theta_W\sinh\zeta}{\sinh\beta}\,, \tag{26c}$$

$$\cos\gamma = \frac{\cosh\theta_W\sinh\theta_1\cosh\zeta - \sinh\theta_W\cosh\theta_1}{\sinh\beta}\,, \tag{26d}$$

$$\sin\alpha = -\frac{\cosh\theta_1\sinh\zeta}{\sinh\beta}\,, \tag{26e}$$

$$\cos\alpha = \frac{\cosh\theta_W\sinh\theta_1 - \sinh\theta_W\cosh\theta_1\cosh\zeta}{\sinh\beta}\,. \tag{26f}$$

Acting on the states $|N_1, N_2\rangle$ the operator L_{34} obeys the following eigenvalue equation:

$$L_{34}|N_1, N_2\rangle = (N_1 - N_2)|N_1, N_2\rangle\,, \tag{27}$$

with $N_{1,2} = n_{1,2} + k$, where $k = \tfrac{1}{2}(|m|+1)$. The reduced matrix element can then be factorized as follows:

$$W^k_{NN'}(\beta) = \langle N_1N_2|\exp(-i\beta L_{45})|N_1'N_2'\rangle = V^k_{N_1N_1'}(\beta)V^k_{N_2N_2'}(-\beta)\,, \tag{28}$$

where

$$V^k_{N_iN_i'}(\beta) = \frac{1}{(N_i - N_i')!}\left[\frac{(N_i-k)!\,(N_i+k-1)!}{(N_i'-k)!\,(N_i'+k-1)!}\right]^{1/2} \times [\cosh(\tfrac{1}{2}\beta)]^{-N_i-N_i'}[\sinh(\tfrac{1}{2}\beta)]^{N_i-N_i'} \times F(\mu,\lambda,\nu;-x),\quad i=1,2 \tag{29}$$

for $N \geq N'$. In Eq. (29) the hypergeometric function has the expansion

$$F(\mu,\lambda,\nu;x) = 1 + \frac{\mu\lambda}{1!\,\nu}x + \frac{\mu(\mu+1)\lambda(\lambda+1)}{2!\,\nu(\nu+1)}x^2 + \cdots\,, \tag{30}$$

where $\mu = k - N_i'$, $\lambda = 1 - N_i' - k$, and $\nu = 1 + N_i - N_i'$, while $x = -\sinh^2(\tfrac{1}{2}\beta)$.

The resulting contributions to M_i in Eqs. (22) and (23) clearly come from

$$\langle n_1n_20|G(\zeta)|000\rangle = (-1)^{n_2}e^{-i\alpha(n_1-n_2)}[\sinh(\tfrac{1}{2}\beta)]^{n-1}[\cosh(\tfrac{1}{2}\beta)]^{-n-1}\,, \tag{31a}$$

$$\langle n_1n_21|G(\zeta)|001\rangle = (-1)^{n_2}e^{-i\alpha(n_1-n_2)}[\sinh(\tfrac{1}{2}\beta)]^{n-2}[\cosh(\tfrac{1}{2}\beta)]^{-n-2}[(n_1+1)(n_2+1)]^{1/2}\,, \tag{31b}$$

$$\langle n_1 n_2, -1|G(\zeta)|11, -1\rangle = (-1)^{n_2} e^{-i\alpha(n_1-n_2)}[\sinh(\tfrac{1}{2}\beta)]^{n-4}[\cosh(\tfrac{1}{2}\beta)]^{-n-4}[(n_1+1)(n_2+1)]^{1/2} \times [n_2 - 2\sinh^2(\tfrac{1}{2}\beta)][n_1 - 2\sinh^2(\tfrac{1}{2}\beta)], \quad (31c)$$

$$\langle n_1 n_2 0|G(\zeta)|010\rangle = (-1)^{n_2} e^{-i\alpha(n_1-n_2)+i\gamma}[\sinh(\tfrac{1}{2}\beta)]^{n-2}[\cosh(\tfrac{1}{2}\beta)]^{-n-2}[n_2 - \sinh^2(\tfrac{1}{2}\beta)], \quad (31d)$$

$$\langle n_1 n_2 0|G(\zeta)|100\rangle = (-1)^{n_2} e^{-i\alpha(n_1-n_2)-i\gamma}[\sinh(\tfrac{1}{2}\beta)]^{n-2}[\cosh(\tfrac{1}{2}\beta)]^{-n-2}[n_1 - \sinh^2(\tfrac{1}{2}\beta)]. \quad (31e)$$

Inserting these relations into $|M_1|^2$, using Eq. (23) and recalling that intermediate states $|n\rangle$ with $m=\pm1$ do not mix, we obtain

$$|M_1|^2 = \tfrac{1}{4}(n_1+1)(n_2+1)[\sinh^2(\tfrac{1}{2}\beta)]^{n-4}[\cosh^2(\tfrac{1}{2}\beta)]^{-n-4}[\alpha_1{}^2 + \alpha_4{}^2(M_n - M\cosh\zeta)^2\cosh^2\theta_1] \times\{[\sinh^2(\tfrac{1}{2}\beta)\cosh^2(\tfrac{1}{2}\beta)]^2 + \tfrac{1}{2}[n_1 - 2\sinh^2(\tfrac{1}{2}\beta)]^2[n_2 - 2\sinh^2(\tfrac{1}{2}\beta)]^2\}, \quad (32)$$

where later we must sum over n_1 and n_2 restricted by $n_1+n_2+2=n$. From Eqs. (31a)–(31e) and (24) we get for the other matrix element

$$\begin{aligned}
|M_3|^2 = {} & (\alpha_2 - \alpha_3\sinh\theta_1)^2 M^2\sinh^2\zeta[\sinh^2(\tfrac{1}{2}\beta)]^{n-1}[\cosh^2(\tfrac{1}{2}\beta)]^{-n-1} \\
& + \tfrac{1}{4}[n_2 - \sinh^2(\tfrac{1}{2}\beta)]^2[\sinh^2(\tfrac{1}{2}\beta)]^{n-2}[\cosh^2(\tfrac{1}{2}\beta)]^{-n-2}[\alpha_1{}^2 + (\alpha_4 M_n - \alpha_4 M\cosh\zeta - \alpha_3 M\sinh\zeta)^2\cosh^2\theta_1] \\
& + \tfrac{1}{4}[n_1 - \sinh^2(\tfrac{1}{2}\beta)]^2[\sinh^2(\tfrac{1}{2}\beta)]^{n-2}[\cosh^2(\tfrac{1}{2}\beta)]^{-n-2}[\alpha_1{}^2 + (\alpha_4 M_n - \alpha_4 M\cosh\zeta + \alpha_3 M\sinh\zeta)^2\cosh^2\theta_1] \\
& - [n_2 - \sinh^2(\tfrac{1}{2}\beta)]\tanh(\tfrac{1}{2}\beta)[\sinh^2(\tfrac{1}{2}\beta)]^{n-2}[\cosh^2(\tfrac{1}{2}\beta)]^{-n-1}(\alpha_2 - \alpha_3\sinh\theta_1) \\
& \times M\sinh\zeta[-\alpha_1\sin\gamma + (\alpha_4 M_n - \alpha_4 M\cosh\zeta - \alpha_3 M\sinh\zeta)\cosh\theta_1\cos\gamma] \\
& - (\alpha_2 - \alpha_3\sinh\theta_1)M\sinh\zeta[\alpha_1\sin\gamma + (\alpha_4 M_n - \alpha_4 M_n\cosh\zeta + \alpha_3 M\sinh\zeta)\cosh\theta_1\cos\gamma] \\
& \times[n_1 - \sinh^2(\tfrac{1}{2}\beta)]\tanh(\tfrac{1}{2}\beta)[\sinh^2(\tfrac{1}{2}\beta)]^{n-2}[\cosh^2(\tfrac{1}{2}\beta)]^{-n-1} \\
& + \tfrac{1}{2}[n_2 - \sinh^2(\tfrac{1}{2}\beta)][n_1 - \sinh^2(\tfrac{1}{2}\beta)] \\
& \times[\sinh^2(\tfrac{1}{2}\beta)]^{n-2}[\cosh^2(\tfrac{1}{2}\beta)]^{-n-2}(\{[(\alpha_4 M_n - \alpha_4 M\cosh\zeta)^2 - (\alpha_3 M\sinh\zeta)^2]\cosh^2\theta_1 + \alpha_1{}^2\} \\
& \times(\cos 2\gamma - 4\alpha_1\alpha_3 M\sinh\zeta\sin\gamma\cos\gamma\cosh\theta_1)),
\end{aligned} \quad (33)$$

where later we must sum over n_1 and n_2 such that $n_1+n_2+1=n$, because both final and intermediate states have $m=0$. The sums over n in Eq. (16) are clearly convergent, because the terms are of the form $n^c(\tanh\frac{1}{2}\beta)^n$, where c is some integer, while $\tanh(\frac{1}{2}\beta)<1$ for definite values of the squared center-of-mass energy $W^2=s$.

V. THE SCALING LIMIT OF THE STRUCTURE FUNCTIONS

We are going to investigate the behavior of W_1 and W_2 in the scaling region $(-q^2, \nu\to\infty)$, and thus we shall have to deal with scattering states. Where the quantum number n of the intermediate states assumes continuous values, we shall not sum the discrete series, but instead use the Sommerfeld-Watson transform to replace the summations and continue the amplitude analytically to the right-hand cut.

First we write down the diagonalized form of the propagator in the center-of-mass frame, as follows:

$$\Omega_n(W) = \{Q_W[n + (\alpha_2 W^2 + \gamma)/Q_W]\}^{-1}, \quad (34)$$

where

$$Q_W = [\alpha_1{}^2 W^2 - (\alpha_3 W^2 + \beta)^2]^{1/2}.$$

It looks as if the ratio $(\alpha_3 W^2 + \gamma)/Q_W$ would become constant if $W^2\to\infty$. In fact, its behavior depends on the values of the parameters α_1, α_2, α_3, β, and γ. If we choose their values equal to those we have used previously[13] for the hydrogen atom, then the vanishing of $\Omega_n{}^{-1}(W)$ will force n to behave like $(1/M_n{}^2)$ in the scaling region, because $Q_n \sim M_n{}^2$. Hence, for $Q_n\to\infty$ we must have $n\to 0$. Other choices of the values of the parameters may yield a similar behavior, although not exactly like that mentioned above.

Applying the Sommerfeld-Watson transform to $T_{\alpha\alpha}$ of Eq. (10), we obtain

$$T_{\alpha\alpha} = \tfrac{1}{2}i\int_C \frac{dn}{|N_1|^2}\frac{|M_\alpha(n)|^2 Q_W{}^{-1}(n)}{[n + (\alpha_2 W^2 + \gamma)/Q_W(n)]\sin\pi n}, \quad (35)$$

where we must take the appropriate contour C, avoiding the poles of the propagator. Now if we evaluate the integral by using the residue of these poles, then we obtain on the right-hand cut for $Q_n\to\infty$ (so $n\to 0$ continuously)

$$W_{\alpha\alpha} = \frac{\text{const}}{|N_1|^2}\lim_{n\to 0}|M_\alpha(n)|^2. \quad (36)$$

The limiting values which interest us here are those associated with $-q^2, \nu \to \infty$ at fixed ξ. They can easily be calculated by inserting the following quantities:

$$\sinh^2(\tfrac{1}{2}\beta) \to -\tfrac{1}{2}(1+iA), \tag{37a}$$

$$\cosh^2(\tfrac{1}{2}\beta) \to \tfrac{1}{2}(1-iA), \tag{37b}$$

where

$$A = \frac{\alpha_1}{2\alpha_3 Q_1}(1-\xi)^{-1} - \sinh\theta_1$$

and

$$\tanh\theta_i = \frac{\alpha_3 M_i^2 + \beta}{\alpha_1 M_i},$$

and

$$\sin\gamma \to \frac{1}{2\alpha_3 M}(1-\xi)^{-1}\frac{1}{(1+A^2)^{1/2}}, \tag{38a}$$

$$\cos\gamma \to \left[\cosh\theta_1 - \frac{\sinh\theta_1}{2\alpha_3 M}(1-\xi)^{-1}\right]\frac{1}{(1+A^2)^{1/2}}. \tag{38b}$$

It can be seen by inspection that the terms in the infinite sum over n in Eqs. (16) have at least a factor n of the first power, so that the convergent sums go at least like n. In the scaling limit we have only to retain terms containing np_3^2, np_3q_0, or nq_0^2, since they remain finite for $n \to 0$ and $p_3, q_0 \to \infty$. Other terms containing n only or $n^2p_3^2$, etc. will vanish.

The scaling limits of the factors in front of W_{11} and W_{33} in Eq. (15) are

$$-\frac{M^2}{p_3^2}\frac{(1+p_3^2/q^2)}{(1+M\nu/q^2)^2} \sim \frac{2M\xi}{\nu}, \tag{39a}$$

$$\frac{M^2}{p_3^2}\frac{1}{(1+M\nu/q^2)^2} \sim -\frac{2M\xi}{\nu}\left[\frac{4\xi(\xi-1)}{(2\xi-1)^2}\right], \tag{39b}$$

so that the limiting scaling functions defined as

$$F_1(\xi) = \lim_{-q^2,\nu\to\infty} MW_1(q^2,\nu),$$
$$F_2(\xi) = \lim_{-q^2,\nu\to\infty} \nu W_2(q^2,\nu) \tag{40}$$

are, apart from the factor $C/|N_1|^2$, given by

$$F_1(\xi) = \tfrac{1}{4}M\alpha_4^2\cosh^2\theta_1 f_1(\xi)(1-\xi)^4, \tag{41a}$$

$$F_2(\xi) = 2M\xi\left[\tfrac{1}{4}\alpha_4^2\cosh^2\theta_1 f_1(\xi)(1-\xi)^4 + 4f_2(\xi)\frac{\xi(1-\xi)^3}{(2\xi-1)^2}\right]. \tag{41b}$$

The constant C, which we omitted, is equal to the parameter γ if we assign to the parameters α_1, α_2, α_3, β, and γ values equal to those we have used for the hydrogen atom.[13]

From Eq. (41b) we see that near threshold ($\xi=1$) we have

$$F_2(\xi) \sim (1-\xi)^3,$$

satisfying the Drell-Yan relation.[14] Furthermore, for $f_2 \ll f_1$ we have

$$F_2(\xi) \approx 2\xi F_1(\xi),$$

which is the Callan-Gross relation.[15,16] We also note here that $F_1(\xi)$ is proportional to α_4, which is the coefficient of the nonminimal current in J_μ, the term which is essential for the anomalous magnetic moment of the proton.[17]

The functions $f_1(\xi)$ and $f_2(\xi)$ are not difficult to calculate, but are rather long expressions, viz.,

$$f_1(\xi) = -2^8\,\frac{\frac{17}{96}[\sinh\theta_1(1-\xi) - (\alpha_1^2/2\alpha_3Q_1)]^4 - \frac{11}{240}[\sinh\theta_1(1-\xi) - \alpha_1^2/2\alpha_3Q_1]^2(1-\xi)^2 + \frac{307}{3360}(1-\xi)^4}{[\cosh^2\theta_1(1-\xi)^2 - (\alpha_1^2/\alpha_3Q_1)\sinh\theta_1(1-\xi) + (\alpha_1^4/4\alpha_3^2Q_1^2)]^4}, \tag{42}$$

$$Q_n = [\alpha_1^2M_n^2 - (\alpha_3M_n^2+\beta)]^{1/2}, \quad \tanh\theta_n = \frac{\alpha_3M_n^2+\beta}{\alpha_1M_n}.$$

In the limit when $\xi=1$ we have $f_1(\xi) = -\frac{45}{3}(\alpha_1^2/2\alpha_3Q_1)^4$.

$$f_2(\xi) = \frac{4[A+B+C+D]}{[\cosh^2\theta_1(1-\xi)^2 - (\alpha_1^2/\alpha_3Q_1)\sinh\theta_1(1-\xi) + (\alpha_1^4/4\alpha_3^2Q_1^2)]^3}, \tag{43}$$

where the functions A, B, C, and D are the following:

$$A(\xi) = (\alpha_2 - \alpha_3\sinh\theta_1)^2\left[\cosh^2\theta_1(1-\xi)^2 - \frac{\alpha_1^2}{\alpha_3Q_1}\sinh\theta_1(1-\xi) + \frac{\alpha_1^4}{4\alpha_3^2Q_1^2}\right]^2, \tag{44a}$$

$$B(\xi) = \alpha_3^2\cosh^2\theta_1\left\{\frac{1}{3}\left[\cosh\theta_1(1-\xi) - \frac{\sinh\theta_1}{2\alpha_3M}\right]^2 + \frac{[\sinh\theta_1(1-\xi)^3 - \alpha_1^2/2\alpha_3\theta_1]^2}{4\alpha_3^2M^2}\right\}, \tag{44b}$$

$$C(\xi) = \alpha_4^2\cosh^2\theta_1\left\{\frac{1}{12\alpha_3^2M^2} + \left[\cosh\theta_1(1-\xi) - \frac{\sinh\theta_1}{2\alpha_3M}\right]^2\left[\sinh\theta_1(1-\xi) - \frac{\alpha_1^2}{2\alpha_3Q_1}\right]^2\right\}, \tag{44c}$$

$$D(\xi)=2\alpha_4\cosh\theta_1(\alpha_2-\alpha_3\sinh\theta_1)\left[\cosh^2\theta_1(1-\xi)^2-\frac{\alpha_1^2}{\alpha_3Q_1}\sinh\theta_1(1-\xi)+\frac{\alpha_1^2}{4\alpha_3^2Q_1^2}\right]$$
$$\times\left[\sinh\theta_1(1-\xi)-\frac{\alpha_1^2}{2\alpha_3Q_1}\right]\left[\cosh\theta_1(1-\xi)-\frac{\sinh\theta_1}{2\alpha_3M}\right]. \quad (44d)$$

VI. CONCLUDING REMARKS

We have used, for simplicity, the boson representation of SO(4, 2). The fermion representation will change the coefficients of the structure functions, but not the general qualitative behavior.

Also, we have used only the physical timelike solutions of the wave equation. Because we are not interpreting the current (2) as a second-quantized *local* current operator, the spacelike solutions do not enter into the present calculations. Rather, our wave equation describes composite relativistic wave functions (first-quantized only) and we sum over all *physical* timelike excited states of the proton in the unitarity relation (9).

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APPENDIX

The basis $|n\rangle$ is defined as follows:

$$|n\rangle\equiv|n_1n_2m\rangle$$
$$=[(n_2+m)!n_1!(n_1+m)!n_2!]^{-1/2}$$
$$\times(a_1^\dagger)^{n_2+m}(a_2^\dagger)^{n_1}(b_1^\dagger)^{n_1+m}(b_2^\dagger)^{n_2}|0\rangle \quad (A1)$$

for $m\geq 0$, where a_i and b_i are boson operators.

The generators which are involved have the following explicit forms:

$$L_{56}\equiv\Gamma_0=\tfrac{1}{2}(a^\dagger a+b^\dagger b+2)$$
$$=\tfrac{1}{2}(a_1^\dagger a_1+a_2^\dagger a_2+b_1^\dagger b_1+b_2^\dagger b_2+2), \quad (A2)$$

$$L_{16}\equiv\Gamma_1=-\tfrac{1}{2}i(a^\dagger\sigma_1Cb^\dagger+aC\sigma_1b)$$
$$=\tfrac{1}{2}i(a_1^\dagger b_1^\dagger-a_2^\dagger b_2^\dagger-a_1b_1+a_2b_2), \quad (A3)$$

$$L_{36}\equiv\Gamma_3=-\tfrac{1}{2}i(a^\dagger\sigma_3Cb^\dagger+aC\sigma_3b)$$
$$=-\tfrac{1}{2}i(a_1^\dagger b_2^\dagger+a_2^\dagger b_1^\dagger-a_1b_2-a_2b_1), \quad (A4)$$

$$L_{46}\equiv S=\tfrac{1}{2}(a^\dagger Cb^\dagger+aCb)$$
$$=\tfrac{1}{2}(a_1^\dagger b_2^\dagger-a_2^\dagger b_1^\dagger+a_1b_2-a_2b_1), \quad (A5)$$

$$L_{34}\equiv A_3=-\tfrac{1}{2}(a^\dagger\sigma_3a-b^\dagger\sigma_3b)$$
$$=-\tfrac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2-b_1^\dagger b_1+b_2^\dagger b_2), \quad (A6)$$

$$L_{35}=M_3=-\tfrac{1}{2}(a^\dagger\sigma_3Cb^\dagger-aC\sigma_3b)$$
$$=-\tfrac{1}{2}(a_1^\dagger b_2^\dagger+a_2^\dagger b_1^\dagger+a_1b_2+a_2b_1), \quad (A7)$$

$$L_{31}=L_2=\tfrac{1}{2}(a^\dagger\sigma_2a+b^\dagger\sigma_2b)$$
$$=-\tfrac{1}{2}i(a_1^\dagger a_2-a_2^\dagger a_1+b_1^\dagger b_2-b_2^\dagger b_1), \quad (A8)$$

$$L_{14}=A_1=-\tfrac{1}{2}(a^\dagger\sigma_1a-b^\dagger\sigma_1b)$$
$$=-\tfrac{1}{2}(a_1^\dagger a_2+a_2^\dagger a_1-b_1^\dagger b_2-b_2^\dagger b_1), \quad (A9)$$

$$L_{15}=M_1=-\tfrac{1}{2}(a^\dagger\sigma_1Cb^\dagger-aC\sigma_1b)$$
$$=\tfrac{1}{2}(a_1^\dagger b_1^\dagger-a_2^\dagger b_2^\dagger+a_1b_1-a_2b_1). \quad (A10)$$

The formulas used for calculating the sums over n_1 and n_2 are

$$\sum_1^n n^1=\tfrac{1}{2}n(n+1),$$
$$\sum_1^n n^2=\tfrac{1}{6}n(n+1)(2n+1),$$
$$\sum_1^n n^3=\tfrac{1}{4}n^2(n+1)^2,$$
$$\sum_1^n n^4=\tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1),$$
$$\sum_1^n n^5=\tfrac{1}{12}n^2(n+1)^2(2n^2+2n-1),$$
$$\sum_1^n n^6=\tfrac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1).$$

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[17]The singular factor $(2\xi - 1)^{-2}$ in (39b) and (41b) is also purely kinematical. It comes from the standard decomposition (13) and the special frame used, namely the rest-frame system of the final state. It is canceled by a corresponding zero of W_{33} or of f_2. The singularity is not there if we use the laboratory frame, for example. Because our current (2) is conserved and the theory is gauge-invariant and covariant for the mass spectrum given by the wave equation, this cancellation is guaranteed.

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Relativistic center-of-mass variables and relativistic corrections to phenomenological Hamiltonians*

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A new method for obtaining the relativistic internal center-of-mass variables without using the singular Gartenhaus-Schwartz transformation is proposed. We have shown that the relativistic c.m. dynamical variables can be obtained from the nonrelativistic ones by a unitary transformation. A general method for determining the relativistic interaction in terms of a given nonrelativistic phenomenological potential is developed. Many different results obtained previously by various approaches have been reproduced.

The relativistic center-of-mass variables for a two-particle system have been discussed recently by Osborn.[1] He has defined the total momentum, position, and spin operators for the system in terms of the generators of the Lorentz group, and has obtained the internal c.m. dynamical variables of the system by applying the singular transformation by Gartenhaus and Schwartz[2] to the single-particle variables. Although exact forms for these variables have been obtained by this method, the use of a singular transformation has been criticized. Here, we present a new method to obtain the internal c.m. variables without using the singular Gartenhaus-Schwartz transformation. We define the total position operator in the same way as Osborn, and we use it to find a unitary transformation (i.e., to construct a unitary operator e^{iu}) which relates the nonrelativistic (lowest-order) total position operator[3] to the relativistic total position operator. The relativistic internal c.m. variables are then obtained by applying this unitary transformation to the nonrelativistic internal c.m. variables. The total momentum and the angular momentum are left invariant by the unitary transformation. One of the main advantages of this approach is that the expressions for the c.m. variables can be written in a compact and useful form. When they are used in actual computation, many complicated manipulations can be simplified. We have reproduced Osborn's results and have also obtained, to order $(\text{mass})^{-2}$, Foldy's operator.[4,5] Thus, our results serve to generalize the approach of Krajcik and Foldy[5] to higher order and to clarify