By expanding the Bessel function in the power series, and fixing the normalization of the charge density with

$$2\pi \int \rho(|\vec{x}_\perp|)|\vec{x}_\perp| d|\vec{x}_\perp| = 1,$$

we arrive at

$$G(|\vec{q}_\perp|) = 1 - \tfrac{1}{4}\langle \vec{x}_\perp^{\,2} \rangle |\vec{q}_\perp|^2 + \ldots,$$

where

$$\langle \vec{x}_\perp^{\,2} \rangle = 2\pi \int \rho(|\vec{x}_\perp|) \vec{x}_\perp^{\,2} |\vec{x}_\perp| d|\vec{x}_\perp|$$

can be regarded as the mean value of the square of the electromagnetic transversal radius of the particle under consideration. Comparing $G(|\vec{q}_\perp|)$ with the experimental data (proton electric and magnetic form factors:

$$|G(|\vec{q}_\perp|)|^2 = (1 + \vec{q}_\perp^{\,2}/4M_n^2)^{-1}[G_E^2 + (\vec{q}_\perp^{\,2}/4M_n^2)G_M^2])$$

we obtain the quoted number.

[8] J. D. Bjorken and E. A. Paschos, Phys. Rev. 185, 1975 (1969).

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Multiparticle partial-wave amplitudes and inelastic unitarity. IV. A scattering model with production and its characteristic operator function

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A completely solvable quantum-mechanical three-particle system that includes breakup is discussed. The partial-wave amplitudes obtained are written in an abstract form to remove their dependence on potential theory, and unitarity is then used to specify genuine production amplitudes. The characteristic operator function is calculated for the resulting set of partial-wave amplitudes, as a physical illustration of the harmonic-analysis approach to three-body unitarity.

I. INTRODUCTION

In previous papers[1,2] it has been shown that much of the structure of multiparticle partial-wave amplitudes constrained by the unitarity equations can be understood in terms of harmonic analysis of completely nonunitary operators. With a view toward checking and illustrating some of the abstract results obtained previously this paper accomplishes two purposes. First of all, it provides a simple example of a completely solved three-body problem in which production processes are allowed. Using a solution given elsewhere[3] for the problem of three arbitrary-mass spinless particles constrained to move along a line, interacting via a single attractive two-body potential and a superposition of separable three-body potentials, we make a partial-wave expansion of the three-dimensional generalization of the system. Abstracting the results from their foundations in potential theory into a form which most clearly displays their functional structure, we find that the unitarity equations then dictate the form for actual production, all obtained without introducing the machinery of quantum field theory.

There are only a few three-body problems that have been solved exactly, and of these we are aware of only three that include production; also, few can be solved as straightforwardly as this one. McGuire,[4] Yang,[5] and others[6] have discussed the problem of equal-mass particles interacting via equal-strength two-body δ-function potentials. Only elastic and rearrangement scattering occur in this model. Breakup can be obtained in a process with two-body δ-function interactions only if the masses or potential strengths are different. The system then resembles two-dimensional optics, a super "wedge" problem with six pie-shaped regions, mixed boundary conditions, and diffraction. This is a very difficult problem, as anyone familiar with wedges knows,[7] and a solution has never been given, although several tantalizing possibilities exist.

Recently Calogero and Marchioro[8] and Wolfes[9] solved a linear three-body problem in which the particles interact via inverse-square two-body forces and an inverse-square three-body force. No production is obtained in this model.

McGuire and Hurst[10] have given an explicit solution to the linear case of three impenetrable spinless particles and have obtained a set of unitary amplitudes with production allowed. However,

because there are no forces binding the particles they have to resort to complicated limiting procedures to obtain composite systems. Our model is in some respects the analog of that given by Fuda.[11] Using the Faddeev equations he has shown that amplitudes for both elastic and inelastic scattering can be found for particles interacting via a single two-body and separable three-body potential We feel that his approach and results, however, somewhat obscure the underlying simplicity of the problem. The single example of a relativistic three-body problem for which a set of amplitudes has been obtained that includes production is the $(V-\theta)$ sector calculation for the Lee model below the four-body threshold.[12] The solution in this case is given in terms of dispersion integrals.

As discussed by Klink in two recent papers[1,2] (hereafter referred to as paper II and paper III), the $3\to 3$ partial-wave amplitude generates a contraction operator which is defined on a Hilbert space $\mathcal{K}$ in which the production ($2\to 3$ and $3\to 2$) partial-wave amplitudes lie as vectors. On a certain subspace of $\mathcal{K}$ the contraction is completely nonunitary, allowing the characterization of S and $\mathcal{K}$ to be realized by certain methods from harmonic analysis. In particular, the characteristic operator function Θ_S plays an important role in this construction.

The second purpose of this paper is to use our solution to illustrate the structure of the characteristic operator function in its simplest form for a physical problem. Certain results will emerge; it will be shown, for example, that the canonical form for the characteristic operator function depends on the behavior of the phase shifts below the three-body threshold. This means, for example, that experimental constraints may be put on Θ_S for realistic systems.

In Sec. II the scattering model is presented. Section III A is a brief review of some of the relevant results of paper III, and in Sec. III B the structure of Θ_S for the scattering model is discussed. The meaning of partial-wave analysis for one-dimensional systems is explained in Appendix A, and in Appendix B the criterion that the $3\to 3$ partial-wave amplitude be non-normal is examined.

II. THE MODEL

A. Potential-theory foundation

In this section the solution for the one-dimensional scattering model is generalized to three dimensions and evaluated in a partial-wave basis. After the results are abstracted from their origins in potential theory, partial-wave amplitudes emerge which can be interpreted as production amplitudes, the form of which are dictated by the requirements of unitarity.

In Ref. 3 the three-particle Schrödinger equation with center-of-mass coordinates separated out,

$$(E+\nabla^2)\,\psi(x,y)=[U_2(x)+U_3]\,\psi(x,y)\,,$$
$$U_3(\vec{x},\vec{x}')=\sum_{i=1}^{N}\omega_i^*(\vec{x})\,\omega_i(\vec{x}')\,, \tag{2.1}$$

was solved and the complete set of scattering amplitudes obtained. Here x is the relative coordinate between particles 1 and 2, and y is the relative coordinate between the center of mass of the 1-2 subsystem and particle 3. Using standard two-potential theory, the S matrix was evaluated between scattering states which satisfy the intermediate Lippmann-Schwinger equations

$$\phi^{\pm}=\phi_0+G\,U_2\,\phi^{\pm}\,,$$
$$\phi^{b}=G\,U_2\,\phi^{b}\,, \tag{2.2}$$

and

$$G=\frac{1}{E+\nabla^2}\,,$$

where ϕ_0 is the solution to the homogeneous version of Eq. (2.1), that is, the solution for three free particles. The S matrix, which can be shown to be unitary by direct calculation or by operator techniques,[13] was shown to be

$$S_{3\to 3}=(\phi_f^-,\phi_i^+)+\sum_{l,k}\hat{\omega}_l^{-*}(f)M^{-1}{}_{lk}\,\hat{\omega}_k^{+}(i)\,,$$
$$S_{3\to 2}=\sum_{l,k}\hat{\omega}_l^{b*}(f)M^{-1}{}_{lk}\,\hat{\omega}_k^{+}(i)\,,$$
$$S_{2\to 3}=\sum_{l,k}\hat{\omega}_l^{-*}(f)M^{-1}{}_{lk}\,\hat{\omega}_k^{b}(i)\,, \tag{2.3}$$

and

$$S_{2\to 2}=(\phi_f^b,\phi_i^b)+\sum_{l,k}\hat{\omega}_l^{b*}(f)M^{-1}{}_{lk}\,\hat{\omega}_k^{b}(i)\,.$$

Here

$$M_{ij}=-2\pi i\,(\delta_{ij}-G_{ij})\,,$$

where δ_{ij} is the Kronecker delta and

$$G_{ij}=\int\omega_i\,G^{+}\,\omega_j^*\,.$$

The $\hat{\omega}_i$ are generalized Fourier transforms with respect to the complete set of solutions to Eq. (2.2). That is,

$$\hat{\omega}_i^{+}=(\omega_i^*,\phi^{+})\,.$$

Notice that in none of the equations above has the dimension of x or y been specified. Hence the result (2.3) is as valid for three dimensions as for one. Furthermore the inner products, Fou-

rier transforms, and traces may be evaluated in a partial-wave basis as well as position or momentum basis. Equation (2.3) can therefore be regarded as one-dimensional partial-wave S-matrix elements labeled by π (parity) or as three-dimensional partial-wave elements labeled by J (angular momentum), and written with these labels suppressed.

It was shown in Ref. 3 that (ϕ^-, ϕ^+) is a one-line disconnected element in the $3 \to 3$ amplitude. As expected, it satisfies two-body unitarity below the three-body threshold. For partial-wave amplitudes in one dimension $\hat{\omega}^\pm$ are functions of the total energy E and a subenergy and $\hat{\omega}^b$ are functions of E alone. In three dimensions $\hat{\omega}^\pm$ are functions of E and two subenergies and $\hat{\omega}^b$ is a function of E alone.

B. Generalization and partial-wave unitarity

In the following we consider only three-dimensional partial-wave amplitudes (PWA) unless explicitly stated and continue to suppress J. As in paper II, if only one two-body channel is open, the partial-wave S matrix for three-body scattering is written

$$\mathcal{S} = \begin{pmatrix} s & A^* \\ B & S \end{pmatrix},$$

where s is a complex number of magnitude less than or equal to one and S is a contraction operator defined on a Hilbert space $\mathcal{K}$ in which the vectors B and A lie. The unitarity relations are preserved if the inner product on $\mathcal{K}$ is defined as integration over subenergies together with summation over spin-component indices, and S is an integral operator on $\mathcal{K}$ whose kernel is the $3 \to 3$ PWA. That is, if the subenergies are s_q, the total energy E, and

$$S = S^{3\to 3}_{JMM'}(E, s_q, s_q') ,$$

$$A^* = S^{3\to 2}_{JM}(E, s_q) ,$$

$$B = S^{2\to 3}_{JM}(E, s_q) ,$$

and

$$s = S^{2\to 2}_{J}(E) ,$$

then

$$\mathcal{S}^\dagger \mathcal{S} = \mathcal{S}\mathcal{S}^\dagger = \mathcal{I}$$

reproduces the partial-wave unitarity relations

$$\|A\|^2 = \|B\|^2 = 1 - \eta^2 , \tag{2.4a}$$

$$0 = \eta e^{-2i\delta} B + SA , \tag{2.4b}$$

$$A \otimes A^* = I - S^\dagger S , \tag{2.4c}$$

and

$$B \otimes B^* = I - SS^\dagger , \tag{2.4d}$$

where s has been written in terms of the phase-shift δ and inelasticity parameter η:

$$s = \eta e^{2i\delta} .$$

The symbol $\phi \otimes \psi^*$ represents an operator on $\mathcal{K}$ defined by[14]

$$(\phi \otimes \psi^*) f = (\psi, f)\, \phi ,$$

for $\phi, \psi, f \in \mathcal{K}$. Consequently, from the definitions, the inner product $\|B\|^2$, for example, means

$$\|B\|^2 = \sum_{M=-J}^{J} \int ds_q \, |S^{2\to 3}_{JM}(E, s_q)|^2 ,$$

and the action of S on A is written

$$SA = \sum_{M=-J}^{J} \int ds_q S^{3\to 3}_{JMM'}(E, s_q, s_q') S^{3\to 2 *}_{JM}(E, s_q) .$$

That S is a contraction (i.e., $\|Sf\| \leq \|f\|$ for any $f \in \mathcal{K}$) is demonstrated in paper III.

Motivated by the results (2.3) from potential scattering theory, choose for the partial-wave amplitudes the following:

$$s = 1 + \sum_{ij} \epsilon_i \lambda_{ij} \epsilon_j^* ,$$

$$A^* = \sum_{ij} \epsilon_i \lambda_{ij} w_j^* ,$$

$$B = \sum_{ij} v_i \lambda_{ij} \epsilon_j^* , \tag{2.5}$$

and

$$S = S^d + \sum_{ij} v_i \lambda_{ij} \otimes w_j^* ,$$

where $\{v_i\}$ and $\{w_i\}$ are each a set of N vectors in our Hilbert space $\mathcal{K}$, $\{\epsilon_i\}$ is a set of N complex numbers, and S^d, generated by the disconnected element of the $3 \to 3$ PWA, is a unitary operator on $\mathcal{K}$. In order that these amplitudes satisfy unitarity it is necessary that the following auxiliary conditions hold:

$$S^{d\dagger} S^d = I , \tag{2.6a}$$

$$(\lambda^{-1} + \lambda^{-1\dagger})_{ij} = -(v_i, v_j) - \epsilon_i^* \epsilon_j , \tag{2.6b}$$

and

$$S^{d\dagger} v_i = w_i . \tag{2.6c}$$

That they must be true can be seen by writing out Eq. (2.4c) and forcing it to hold:

$$
\begin{aligned}
I - S^\dagger S &= I - \Big(S^{d\dagger} + \sum_{ij} w_j \otimes \lambda_{ij}^* v_i^*\Big)\Big(S^d + \sum_{ij} v_i \lambda_{ij} \otimes w_j^*\Big) \\
&= -\Big(\sum_{ij} w_i \otimes \lambda_{ji}^* v_i^*\Big) S^d - S^{d\dagger}\Big(\sum_{ij} v_i \lambda_{ij} \otimes w_j^*\Big) \\
&\quad - \sum_{ijkl} w_i \otimes \lambda_{ki}^* (v_k, v_l) \lambda_{lj} w_j^* \\
&= -\sum_{ij} w_i \otimes \Big[\lambda_{ji}^* + \lambda_{ij} + \sum_{kl} \lambda_{ki}^* (v_k, v_l) \lambda_{lj}\Big] w_j^* \\
&= \sum_{ijkl} w_i \otimes \lambda_{ki}^* \epsilon_k^* \epsilon_l \lambda_{lj} w_j^* \\
&= A \otimes A^* ,
\end{aligned}
$$

where all three conditions have been used. The other unitarity relationships, Eqs. (2.4), are also satisfied by Eqs. (2.5) and (2.6).

To make concrete the correspondence between the partial-wave amplitudes abstractly defined in Eqs. (2.5) and those derived from the potential model [Eq. (2.3)], set

$$v_i = \hat{\omega}_i^{-*} ,$$

$$w_i = \hat{\omega}_i^{+*} ,$$

$$\epsilon_i = \hat{\omega}_i^{b*} ,$$

and

$$S^d(f, i) = (\phi_f^-, \phi_i^+) .$$

Then Eqs. (2.5) reproduce Eqs. (2.3).

The auxiliary conditions, Eqs. (2.6), also have their analogs in the potential model and are discussed in one and three dimensions in the following. The first condition, Eq. (2.6a), expresses the unitarity of the disconnected element S^d. In one dimension S^d has the form

$$S^d = s_f I + s_b I' ,$$

where s_f and s_b are the forward and backward two-body S-matrix amplitudes below the three-body threshold, I is the identity operator, and I' an involution:

$$II' = I' ,$$

$$I'I' = I ,$$

and I' is self-adjoint. This can be seen by writing off the energy shell

$$
\begin{aligned}
S^d(\vec{k}, \vec{k}') &= (\phi^-(\vec{k}), \phi^+(\vec{k}')) \\
&= \int \phi^{-*}(\vec{k}, \vec{p})\, \phi^+(\vec{k}', \vec{p})\, d\vec{p} \\
&= (\tfrac{1}{2})^{1/2}\, \delta(k_y - k_y')\, (\xi^-(k_x), \xi^+(k_x')) \\
&= (\tfrac{1}{2})^{1/2}\, \delta(k_y - k_y')\, [s_f\, \delta(k_x - k_x') + s_b\, \delta(k_x + k_x')] \\
&= \delta(E - E')\, [s_f\, \delta(\hat{k} - \hat{k}') + s_b\, \delta(\pi - \hat{k} - \hat{k}')] ,
\end{aligned}
$$

where we have used the fact that Eq. (2.2) separates into $\phi(\vec{k}, \vec{x}) = \xi(k_x, x)\, \eta(k_y, y)$, as explained in Ref. 3. Because x is the relative coordinate between particles 1 and 2, $\xi^\pm(x)$ are the two-particle wave functions below the three-body threshold. Putting (2.8) on the energy shell and identifying

$$I(\hat{k}, \hat{k}') = \delta(\hat{k} - \hat{k}')$$

and

$$I'(\hat{k}, \hat{k}') = \delta(\pi - \hat{k} - \hat{k}')$$

gives Eq. (2.7).

The form for S^d in three dimensions is fundamentally different from what it is in one dimension. As shown in paper II, the disconnected $3-3$ PWA can be written

$$
\begin{aligned}
S^d_{JMM'}(E, \theta, s_3, \theta', s_3') &= 4\pi \sum_{j=0}^{\infty} Y_{jM}(\theta)\, Y^*_{jM'}(\theta') \\
&\quad \times e^{2i\delta_j(s_3)} \delta(s_3 - s_3')\, \delta_{MM'} ,
\end{aligned}
$$

where δ_j is the two-body phase shift below the three-body threshold and s_3 is the subenergy

$$s_3 = (p_1 + p_2)^2 .$$

The angle θ is measured between the momentum direction of particle 3 and that of particle 1, as seen in the 1-2 center-of-mass system. It can be seen that the three-dimensional case offers much richer structure than is true in one dimension. If only the first few phase shifts below the three-body threshold are known from experiment to be nonzero, however, an intermediate case is obtained in which S^d can be written as a finite sum of operators.

It should be emphasized that in our model only one of the three possible disconnected terms appears. This, of course, comes about because only two of the three particles interact via a two-body potential. Such was assumed in order to obtain an easily solvable system, and resulted in S^d being unitary. If more than one disconnected term appears, it is easily seen that S^d will no longer be unitary.

The second auxiliary condition, Eq. (2.6b), replaces the Green's function relation

$$G^- - G^+ = 2\pi i (E - H) ,$$

where $H = -\nabla^2 + U_2$. In particular, from the potential model,

$$
\begin{aligned}
(\lambda^{-1} + \lambda^{-1\dagger})_{ij} &= \frac{1}{2\pi i} (G_{ij} - G_{ji}^*) \\
&= \frac{1}{2\pi i} \Big(\int \omega_i\, G^+\, \omega_j^* - \int \omega_j^*\, G^{+*}\, \omega_i \Big) \\
&= \frac{1}{2\pi i} \Big(\int \omega_i\, (G^+ - G^{+\dagger})\, \omega_j^* \Big) \\
&= \frac{1}{2\pi i} \Big(\int \omega_i\, (G^+ - G^-)\, \omega_j^* \Big) \\
&= -\int \omega_i\, \omega_j^* .
\end{aligned}
$$

But

$$(v_i, v_j) + \epsilon_i^* \epsilon_j = \int [(\omega_i^*, \phi_\alpha^-)(\phi_\alpha^-, \omega_i^*)]\, d\alpha + (\omega_i^*, \phi^b)(\phi^b, \omega_j^*) = (\omega_i^*, \omega_j^*) = \int \omega_i \omega_j^*$$

by completeness. Furthermore, time-reversal invariance implies that ω_i is real in the position basis. Hence $(\lambda^{-1} + \lambda^{-1\dagger})$ is a real symmetric matrix.

The final auxiliary condition, $w_i = S^{a\dagger} v_i$, has interesting consequences for the form of the operator S. For example, it is shown in Appendix B that if the subspace spanned by w_i is identical to that spanned by v_i, then S is normal ($S^\dagger S = SS^\dagger$) and can be diagonalized and described in terms of its spectral properties. The problem for S normal has been solved in general in paper II. For S non-normal (the usual case), diagonalization is no longer possible but, because S is a contraction, certain methods from harmonic analysis can be used to characterize S, and within this framework the relation $w_i = S^{a\dagger} v_i$ will still be important. The case of S non-normal is the concern of Sec. III.

III. Θ_S FOR THE SCATTERING MODEL

A. General considerations

The usual operator-valued analytic function for describing the structure of a normal operator S defined on a Hilbert space $\mathcal{H}$ is the resolvent $(S - zI)^{-1}$, which for self-adjoint differential operators is just the usual Green's function common in physics. If S is non-normal, however, the resolvent yields little information about S. If S is a non-normal contraction, a more useful function is the characteristic operator function defined by

$$\Theta_S(z) = -S + zD_{S\dagger}(I - zS^\dagger)^{-1} D_S | \mathfrak{D}_S \tag{3.1}$$

for all complex numbers z such that $(I - zS^\dagger)$ is boundedly invertible. The defect operator D_S is defined by

$$D_S = (I - S^\dagger S)^{1/2} \tag{3.2}$$

and the defect space $\mathfrak{D}_S$ is the domain of D_S. The range of Θ_S is $\mathfrak{D}_{S\dagger}$, a fact used in Sec. III B.

The characteristic operator function is used in the construction of a functional model for the non-normal contraction S. That is, the action of S on $\mathcal{H}$ is realized by a unitarily equivalent operator $\tilde{S}$ acting on a particular subspace $\mathfrak{M}^\perp$ of vector-valued analytic functions. Whereas S may be very complicated on $\mathcal{H}$, $\tilde{S}$ is quite simple on $\mathfrak{M}^\perp$, essentially multiplication by z. Θ_S is used in the construction of the subspace $\mathfrak{M}^\perp$. (See, for example, paper III or Ref. 15, page 248.) The characteristic operator function itself specifies the complete spectrum of S. For example, the zeros of $\Theta_S(z)$ are the point spectra (discrete eigenvalues). (See Ref. 15, page 259.)

When thought of as a function on the unit disk [we will write $\theta_S(z)$ in this case] the canonical factorization of $\theta_S(z)$ gives information about S. For example, if θ_S is inner [that is, $|\theta_S(z)| \leq 1$ and $|\theta_S(e^{i\theta})| = 1$ almost everywhere on the unit circle], it can be decomposed into a product of a Blaschke factor and a singular factor[16]:

$$\theta_S(z) = \kappa B(z) S(z), \tag{3.3}$$

where κ is of modulus 1. The Blaschke factor $B(z)$ includes all the zeros of θ_S and $S(z)$ is a function without zeros which is positive at the origin. In this case it is true that the defect spaces are one-dimensional, that $(S^\dagger)^n h \to 0$ for any $h \in \mathcal{H}$, and that $S | \mathfrak{D}_S$ is completely nonunitary with the modulus of its eigenvalues strictly less than 1. This case is explicitly mentioned because our S (in one dimension) falls into this category.

It follows from the unitarity relations that if the defect spaces are one-dimensional (which correspond to one two-body channel open above the three-body threshold), then as was shown in paper III

$$|\theta_S(0)| = \eta \tag{3.4}$$

and

$$\|\Theta_S(0)\hat{A}\| = \eta, \tag{3.5}$$

where $A \in \mathfrak{D}_S$ and $B \in \mathfrak{D}_{S\dagger}$ and η is the inelasticity parameter. This follows from $\Theta_S(0) = -S$ and Eq. (2.4b).

Since in our scattering model the defect spaces are one-dimensional, it is natural to calculate the inner product $(\hat{B}, \Theta_S \hat{A})$ in order to investigate the structure of Θ_S as a function of z. This is the subject of Sec. III B.

B. Structure of $\Theta_S(z)$

In this section $\theta_S(z)$, defined by

$$\theta_S(z) = (\hat{B}, \Theta_S(z)\hat{A}) = \frac{(B, \Theta_S(z)A)}{\|A\|^2},$$

is calculated for our one-dimensional scattering model, first with one separable potential because then the structure is most transparent, and then with a superposition of N separable potentials.

Noting that

$$D_S = \frac{A \otimes A^*}{\|A\|} = |\epsilon\lambda| \frac{w \otimes w^*}{\|w\|},$$

$$D_{S\dagger} = \frac{B \otimes B^*}{\|B\|} = |\epsilon\lambda| \frac{v \otimes v^*}{\|v\|},$$

and that $\|v\| = \|w\|$ because S^d is unitary allows $\theta_S(z)$ to be written

$$\theta_S(z) = \frac{\lambda^*}{\lambda}\left[-1 - \lambda\|v\|^2 + z|\epsilon|\,|\lambda|^2 (v, (I - zS^\dagger)^{-1} w)\right]. \quad (3.6)$$

The operator inverse $(I - zS^\dagger)^{-1}$ can be expanded in powers of $S^\dagger$ or, as will be done here, in operators of the form $v \otimes v'$, where v and v' span the range of $S^{d\dagger} v$. Writing

$$S^d = s_f I + s_b I'$$

and defining

$$v' = I' v ,$$

it can be shown that the operator inverse is given by

$$(I - zS^\dagger)^{-1} = \frac{1}{P_1}\left(Q_1 I - Q_2 I' - \frac{Q_1 Q_3}{P_2} v \otimes v^* - \frac{Q_1 Q_4}{P_2} v' \otimes v^* + \frac{Q_2 Q_3}{P_2} v \otimes v'^* + \frac{Q_2 Q_4}{P_2} v' \otimes v'^*\right) \quad (3.7)$$

where the polynomials $P(z)$ and $Q(z)$ are given by

$$P_1(z) = z^2(s_b^{*2} - s_f^{*2}) + 2z s_f^* - 1 ,$$

$$P_2(z) = z^2(s_b^{*2} - s_f^{*2})(1 + \|v\|^2 \lambda^*) + z[2s_f^* + \lambda^* s_f^* \|v\|^2 + \lambda^* s_b^* (v', v)] - 1 ,$$

$$Q_1(z) = z s_f^* - 1 ,$$

$$Q_2(z) = z s_b^* ,$$

$$Q_3(z) = z^2 \lambda^*(s_b^{*2} - s_f^{*2}) + z\lambda^* s_f^* ,$$

and

$$Q_4(z) = z\lambda^* s_b^* .$$

Substituting Eq. (3.7) into Eq. (3.6) and making heavy use of the identity (2.6b) gives

$$\theta_S(z) = -\frac{\lambda^*}{\lambda}\frac{1}{P_2(z)}\left(z^2(s_b^{*2} - s_f^{*2}) + z\{2s_f^* + \lambda[\|v\|^2 s_f^* - (v', v) s_b^*]\} - (1 + \lambda\|v\|^2)\right) \quad (3.8)$$

a ratio of second-order polynomials in z. It should be noted that it is straightforward to extract the $2 \to 2$ inelasticity parameter from this, in accord with Eq. (3.4). Again using Eq. (2.6b) one can show that $\theta_S(z)$ can be factored into the form (3.3) where the singular function is a constant:

$$\theta_S(z) = \frac{\lambda^*}{\lambda}\frac{1}{s_b^{*2} - s_f^{*2}} \prod_{n=1,2} \frac{z - a_n}{1 - a_n^* z} . \quad (3.9)$$

The zeros a_1, a_2 of the Blaschke product (and therefore the eigenvalues of S) are given by

$$a_{1,2} = -\frac{\{2s_f^* + \lambda[\|v\|^2 s_f^* - (v', v) s_b^*]\} \pm \sqrt{Q}}{2(s_b^{*2} - s_f^{*2})} , \quad (3.10)$$

where

$$Q = 4s_b^*\{s_b^* + \lambda[s_b^* \|v\|^2 - s_f^*(v', v)]\} + \lambda^2[s_f^* \|v\|^2 - s_b^*(v', v)] .$$

For a superposition of N separable potentials a similar, although more tedious, calculation shows that

$$\theta_S(z) = -(\epsilon^T \lambda^\dagger O \lambda \epsilon^*)^{-1} \epsilon^T \lambda^\dagger \{z^2 hI + z[2s_f^* I + O\lambda O^{-1}(s_f^* - s_b^* O')] - O(I + \lambda O) O^{-1}\} \lambda^{-1\dagger} P_2^{-1}(z) \lambda^\dagger O \lambda^\dagger \epsilon^* , \quad (3.11)$$

where λ is the $N \times N$ matrix given in Eq. (2.6b), O is an $N \times N$ matrix whose elements are (v_i, v_j), O' is the matrix (v_i', v_j), and ϵ is the column vector whose components are ϵ_i. I is the unit $N \times N$ matrix, and in this case $P_2(z)$ is a matrix polynomial:

$$P_2(z) = z^2 h(I + \lambda^\dagger O) + z(2s_f^* I + s_f^* \lambda^\dagger O + s_b^* \lambda^\dagger O') - I ,$$

where

$$h \equiv (s_b^{*2} - s_f^{*2}) .$$

Because of the complicated nature of $\theta_S(z)$ in Eq. (3.11) it appears to be very difficult to factor it into the canonical form (3.3), although general arguments about the spectrum of S^d show that it must factor into a Blaschke product with a finite number of terms.

Although in one dimension $\theta_S(z)$ is always a Blaschke product, it is possible to show that in three dimensions, with S^d as given in Sec. II B, θ_S will have a nontrivial singular function and possibly an outer function in its factorization. This follows because $(S^d)^n$ maps vectors from the span of $\{v_i\} \cup \{w_i\}$ into vectors outside this subspace, resulting in the lack of strong convergence $S^n \to 0$. By a theorem in Ref. 15 the factorization therefore cannot consist only of inner functions.

IV. CONCLUSION

A solution to a three-body problem has been constructed for which the production (partial-wave) amplitudes lie in a Hilbert space $\mathcal{H}$, the two-body

amplitudes are complex numbers, and the $3 \to 3$ amplitudes generate an operator on $\mathcal{H}$ of the form

$$S = S^d + \sum_{i=1}^{N} v_i \otimes \lambda_{ij} w_j^* ,$$

where S^d is unitary, λ is a complex matrix, and $\{v_i\}$ and $\{w_i\}$ each span a finite subspace of $\mathcal{H}$. The amplitudes satisfy unitarity and have been used to construct the characteristic operator function for the completely nonunitary part of S.

Several further results have emerged. First of all, although in this model there is a large subspace of $\mathcal{H}$ on which S is unitary (that is, the space orthogonal to the span of $\{v_i\} \cup \{w_i\}$), the production amplitudes lie only in that subspace on which S is completely nonunitary, as predicted in paper III. The model also shows that although it is necessary that there be more than one two-particle bound state or that spin be included in order that the defect spaces have dimension greater than one, it is not sufficient. Thus, if there are several bound states in the model generated by a superposition of separable potentials, it is possible to have various rearrangement and production amplitudes. On the other hand, for a single separable potential the production amplitudes obtained from the various two-body bound states differ only in norm, not "direction". That is, for the various two-body bound states from which production can occur, the production amplitudes vary only if the factor ϵ which has no vector character. Stated in the language of the potential model, this means that the production amplitudes vary only in their dependence on the total energy E, and not at all on the subenergy s_q. We have also seen that in this very simple model it is possible that the characteristic operator function may have a nontrivial factorization, with possible singular and outer functions as well as Blaschke products. The model we have is obviously deficient in that it allows a two-body force between only two of the three particles. A more interesting case would be one in which all three particles feel two-body forces only. In this case S^d would be nonunitary, and Θ_S would presumably be more complex. Such a model awaits solution.

Finally, it should be emphasized that no claim is made that we have learned more about S with the characteristic operator function than we knew already from the model. Quite the contrary in fact is the case, since all information about the $2 \to 2$ phase shift is lost, as explained in detail in paper III. This example was solely a first attempt to illustrate the calculation and meaning of Θ_S for three-body scattering. The real interest in the harmonic-analysis approach to multiparticle unitarity lies in constraints that may be placed on Θ_S by other physical ideas, crossing or resonance production, for example. Then, once a choice is made for Θ_S, the predictions made about the production amplitudes and inelasticity parameters can be checked against experiment.

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APPENDIX A: PARTIAL-WAVE EXPANSION FOR ONE-DIMENSIONAL SYSTEMS

The analog in one dimension of a three-dimensional partial-wave expansion is discussed in this appendix. Just as the three-dimensional partial-wave amplitudes are expansion coefficients in the rotation group, so the one-dimensional PWA are expansion coefficients in the parity group. For three particles in the over-all center-of-mass system, the one-dimensional scattering amplitudes are functions of the total energy E and an angle $\hat{k}$, where $0 \leq \hat{k} < 2\pi$.

For $0 \leq \hat{k} < \pi$, $\hat{p}_3$ is positive, where $\hat{p}_3$ is the momentum direction of particle 3 in the over-all center of mass. For $\pi \leq \hat{k} \leq 2\pi$, $\hat{p}_3$ is negative. In three dimensions a rotation R is defined which relates the directed plane formed by the final-state particles to that formed by the initial-state particles.[17] The amplitude in that case is a function of the total energy, some subenergies, and R. Similarly, in one dimension one can define a parameter η which takes on two values, $\eta = +$ or $-$, describing whether $\hat{p}_3$ final is $+\hat{p}_3$ initial or $-\hat{p}_3$ initial.

Then letting α be the subenergy, $0 \leq \alpha < \pi$, the scattering amplitudes can be expanded in D functions of η, just as the three-dimensional amplitudes are expanded in D functions of R. That is,

$$S_\pi(E, \alpha) = \sum_{\eta = \pm} \mathfrak{D}^{(\pi)}(\eta) S(E, \alpha, \eta) ,$$

with inverse

$$S(E, \alpha, \eta) = \sum_{\pi = \pm} N_\pi \mathfrak{D}^{(\pi)*}(\eta) S_\pi(E, \alpha) .$$

The $S_\pm$ are the one-dimensional PWA. One can write[18]

$$S_\pm = S(+) \pm S(-)$$

and

$$S(\pm) = \tfrac{1}{2}(S_+ \pm S_-) .$$

APPENDIX B: CONDITION FOR NORMAL S

Let $H' \subseteq \mathcal{H}$ be the subspace spanned by v_i, and S and S^d be defined as in the text. Then a nec-

essary and sufficient condition that S be normal is that H' coincide with the range of $S^{d\dagger}$. From Eq. (2.4), if S is normal, it must be true that $A\otimes A^*=B\otimes B^*$. But

$$A\otimes A^*=\sum_{ijkl} w_i\,\lambda^*_{ki}\otimes\epsilon^*_k\,\epsilon_l\,\lambda_{lj}w^*_j$$

and

$$B\otimes B^*=\sum_{ijkl} v_i\,\lambda_{ik}\otimes\epsilon^*_k\,\epsilon_l\,\lambda^*_{jl}\,v^*_j\ .$$

Consequently for

$$v=\begin{pmatrix} v_1\\ v_2\\ \vdots\\ \vdots\\ v_n\end{pmatrix}$$

and

$$w=\begin{pmatrix} w_1\\ w_2\\ \vdots\\ \vdots\\ w_n\end{pmatrix}$$

it is true that $v^T\lambda=w^T\lambda^\dagger$, which implies that $w=\alpha v$, where $\alpha=\lambda^{-1*}\lambda^T$. The auxiliary condition (2.5c), however, states that $w_i=S^{d\dagger}v_i$, and it therefore follows that

$$S^{d\dagger}v_i=\sum_j \alpha_{ij}v_j$$

and that $w_i\in H'$, proving necessity.

Now suppose that $w_i\in H'$. Then $w=\beta v$ for some matrix β. For

$$\epsilon=\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \vdots\\ \vdots\\ \epsilon_n\end{pmatrix}$$

one can write

$$\|A\|^2=\epsilon^T\lambda w^* w^T\lambda^\dagger\epsilon^*$$

and

$$\|B\|^2=\epsilon^T\lambda^\dagger v^* v^T\lambda\epsilon^*\ .$$

But Eq. (2.4) requires that $\|A\|^2=\|B\|^2$. This implies that $\beta^T\lambda^\dagger=\lambda$ or $\beta=\lambda^{-1*}\lambda^T$. Consequently $\beta=\alpha$, which implies $A\otimes A^*=B\otimes B^*$, and S is normal.

In the potential model this condition is quite restrictive. Using the above and $\psi^\pm$, the scattering eigenfunctions of the complete Hamiltonian, one can show that the condition that S is normal is equivalent to

$$0=(\omega_i^*,(\psi^+-\psi^-))$$
$$=\left(\omega_i^*,\left(U_2-\sum_j\omega_j^*\omega_j\right)\phi_0\right)$$

for each ω_i, the three-body potentials from Eq. (2.1). The integration is over the wave-function basis.

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