PHYSICAL REVIEW D **89,** 127502 (2014)

Shear-free axially symmetric dissipative fluids

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(Received 13 April 2014; published 25 June 2014)

We study the general properties of axially symmetric dissipative configurations under the shear-free condition. The link between the magnetic part of the Weyl tensor and the vorticity is clearly exhibited, as well as the role of the dissipative fluxes. As a particular case, we examine the geodesic fluid. In this case, the magnetic part of the Weyl tensor always vanishes, suggesting that no gravitational radiation is produced during the evolution. In addition, for the geodesic case, in the absence of dissipation, the system evolves towards a Friedmann-Roberston-Walker spacetime if the expansion scalar is positive.

DOI: 10.1103/PhysRevD.89.127502 PACS numbers: 04.40.-b, 04.40.Dg, 04.40.Nr

I. INTRODUCTION

In a recent paper [1], using a $1+3$ approach [2–5], we developed a general framework for studying axially symmetric dissipative fluids. In this work we endeavor to apply this approach to the specific case of shear-free fluids.

The relevance of the shear tensor in the evolution of self-gravitating systems, and the consequences emerging from its vanishing, have been discussed by many authors (see [6–12] and references therein).

Furthermore, as it has been recently shown [13], the shear-free flow (in the nondissipative case) appears to be equivalent to the well-known homologous evolution. It should be recalled that homology conditions are of great relevance in astrophysics [14–16].

Thus, in spite of the fact that the shear-free condition appears to be unstable with respect to some important physical phenomena [17], shear-free fluids play an important role in the study of self-gravitating objects.

As we shall see below, the shear-free condition brings out a clear link between the magnetic part of the Weyl tensor ($H_{\alpha\beta}$) and vorticity, even in the general (anisotropic and dissipative) case. It will be shown that for a shear-free fluid that is not necessarily perfect, the necessary and sufficient condition to be irrotational is that the Weyl tensor be purely electric; this thus generalizes a result by Barnes [18,19] and Glass [20].

The subcase represented by the geodesic fluid is analyzed in some detail, the dissipationless case in particular. In this latter case it is shown that if the expansion scalar is positive, the system relaxes asymptotically to a Friedmann-Roberston-Walker (FRW) spacetime. Also, it is shown that, in this case, the magnetic part of the Weyl tensor always vanishes.

In order to avoid rewriting most of the equations, we shall very often refer to [1]. Thus, we suggest that the reader have Ref. [1] at hand when reading this manuscript.

II. THE SHEAR-FREE CONDITION AND ITS CONSEQUENCES

We shall consider axially and reflection-symmetric fluid distributions that are not necessarily bounded. For such a system, the most general line element may be written in "Weyl spherical coordinates" as

$$ds^2 = -A^2dt^2 + B^2(dr^2 + r^2d\theta^2) + C^2d\phi^2 + 2Gd\theta dt, \tag{1}$$

where A, B, C, and G are positive functions of t, r, and θ. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$.

The energy momentum tensor in the "canonical" form reads

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + Pg_{\alpha\beta} + \Pi_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha, \tag{2}$$

where, as usual, $\mu, P, \Pi_{\alpha\beta}, V_\beta$, and q_α denote the energy density, the isotropic pressure, the anisotropic stress tensor, the four-velocity and the heat flow vector, respectively. The anisotropic stress tensor may be written in terms of three scalar functions (Π_I, Π_{II}, and Π_{KL}), whereas the heat flow vector is defined by two scalar functions, q_I and q_{II} [see Eqs. (10)–(16) in [1] for details].

The shear tensor is defined by two scalar functions, σ_I and σ_{II}, which in terms of the metric functions read [see Eqs. (20)–(25) in [1]]

$$2\sigma_I + \sigma_{II} = \frac{3}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \tag{3}$$

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$$2\sigma_{II}+\sigma_I=\frac{3}{A^2B^2r^2+G^2}\left[AB^2r^2\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\right.$$
$$\left.+\frac{G^2}{A}\left(-\frac{\dot{A}}{A}+\frac{\dot{G}}{G}-\frac{\dot{C}}{C}\right)\right]. \tag{4}$$

For the other kinematical variables (the expansion, the four-acceleration, and the vorticity) we have the expansion

$$\Theta=\frac{AB^2}{r^2A^2B^2+G^2}\left[r^2\left(2\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right.$$
$$\left.+\frac{G^2}{A^2B^2}\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}+\frac{\dot{G}}{G}+\frac{\dot{C}}{C}\right)\right], \tag{5}$$

and the four-acceleration

$$a_\alpha=V^\beta V_{\alpha;\beta}=a_IK_\alpha+a_{II}L\alpha, \tag{6}$$

with vectors **K** and **L** having components

$$K_\alpha=(0,B,0,0);\qquad L_\alpha=\left(0,0,\frac{\sqrt{A^2B^2r^2+G^2}}{A},0\right), \tag{7}$$

and where the two scalar functions (a_I and a_{II}) are defined by [see Eq. (17) in [1]]

$$a_I=\frac{A'}{AB}, \tag{8}$$

$$a_{II}=\frac{A}{\sqrt{A^2B^2r^2+G^2}}\left[\frac{G}{A^2}\left(-\frac{\dot{A}}{A}+\frac{\dot{G}}{G}\right)+\frac{A_{,\theta}}{A}\right], \tag{9}$$

whereas the vorticity vector is defined through a single scalar Ω, given by [see Eq. (29) in [1]]

$$\Omega=\frac{(AG'-2GA')}{2AB\sqrt{A^2B^2r^2+G^2}}, \tag{10}$$

where primes and dots denote derivatives with respect to r and t, respectively.

If we assume the evolution to be shear free, i.e.,

$$\sigma_I=\sigma_{II}=0, \tag{11}$$

then from (3) and (4) we have

$$C(t,r,\theta)=R(r,\theta)B(t,r,\theta)$$
$$G(t,r,\theta)=A(t,r,\theta)B(t,r,\theta)\tilde{G}(r,\theta). \tag{12}$$

From regularity conditions at the origin we must require $R(0,\theta)=\tilde{G}(0,\theta)=0$.

Next, from (A.5) in [1] we may write, if $\sigma_{\alpha\beta}=0$,

$$\nabla_{\langle\alpha}\omega_{\beta\rangle}+2\omega_{\langle\alpha}a_{\beta\rangle}=H_{\alpha\beta}, \tag{13}$$

where angled brackets denote the spatially projected, symmetric, and trace-free part, and $\nabla_\alpha\omega_\beta\equiv h^\delta_\alpha\omega_{\beta;\delta}$.

From the above, it follows at once that $\omega_\alpha=0\Rightarrow H_{\alpha\beta}=0$. Furthermore, the inverse is also true. Indeed, assuming $H_{\alpha\beta}=0$ in (13), we obtain

$$\nabla_\alpha\omega^\alpha=-2a_\alpha\omega^\alpha, \tag{14}$$

however, if the shear tensor vanishes, the following identity holds:

$$\nabla_\alpha\omega^\alpha=a_\alpha\omega^\alpha. \tag{15}$$

Equations (14) and (15) imply that $\omega_{\alpha\beta}=0$. Alternatively, the regularity condition at the origin, $\omega_\alpha(r=0)=0$, can be analytically extended to the whole distribution by taking successive ∇_α derivatives of (14), thereby leading to the same result.

Thus, using the notation of [1], we have established that

$$H_1=H_2=0\Leftrightarrow\Omega=0, \tag{16}$$

where H_1 and H_2 are the two scalar functions that define the magnetic part of the Weyl tensor.

It is important to stress the point that in order to arrive at (16), we have used the tensorial equation (A.5) [Eq. (13) above], which is not restricted to the axially symmetric case. In other words, the necessary and sufficient condition for a shear-free fluid to be irrotational is that the Weyl tensor be purely electric. This generalizes a result by Barnes [18,19] and Glass [20] to anisotropic and dissipative fluids. (Observe that in [1] it was incorrectly stated that such a generalization only applies to nondissipative fluids.)

For the heat flow scalars, we obtain in this case (shear free and axially symmetric), using (B.6) and (B.7) from [1],

$$4\pi q_I=\frac{1}{3B}\Theta', \tag{17}$$

$$4\pi q_{II}=\frac{1}{3Br}\Theta_{,\theta}. \tag{18}$$

Thus, in the dissipationless case, the expansion scalar is homogeneous, $\Theta=\Theta(t)$.

III. GEODESIC CONDITION: $a_\alpha=0$

We shall further restrict our system to the case of vanishing four-acceleration. Two important observations are in order at this point:

(i) As it will be shown below, all geodesic and shear-free fluids are necessarily irrotational.

(ii) Shear-free irrotational, geodesic fluids have been analyzed in great detail by Coley and McManus [21,22]. Here we look at the axially symmetric heat-conducting case of these fluid distributions.

Next, the geodesic condition implies that

$$a_I = \frac{A'}{AB} = 0 \Rightarrow A = \tilde{A}(t, \theta) \tag{19}$$

and

$$a_{II} = \frac{1}{B\sqrt{r^2 + \tilde{G}^2}}\left(\frac{\tilde{G}\Theta B}{3} + \frac{A_{,\theta}}{A}\right) = 0 \Rightarrow \tilde{G}\Theta B = F_2(t, \theta). \tag{20}$$

Given that $\Omega(t, 0, \theta) = \tilde{G}(t, 0, \theta) = 0$, from (20) we find that

$$F_2(t, \theta) = 0 \Rightarrow \Omega = 0 \quad \text{or} \quad \Theta = 0. \tag{21}$$

The above results can also be obtained from (A.3) in [1], which reads in this particular case as

$$h^\beta_\alpha V^\delta \omega_{\beta;\delta} = -\frac{2}{3}\Theta\omega_\alpha. \tag{22}$$

Indeed, combining the above equation—or its projection on the **KL** vectors [Eq. (B.5) in [1]]—with (3), (4), and (5), we obtain the same result, i.e., $\Theta\Omega = 0$. This is in agreement with the so-called "shear-free conjecture" for perfect fluids, which suggests that $\sigma_{\alpha\beta} = 0$ implies $\Theta\Omega = 0$ (see [23] and references therein). Here we have not restricted ourselves to the perfect fluid case, although our result only applies to geodesic fluids.

Let us first consider the case $\Omega_{\alpha\beta} = 0$, $\Theta \neq 0$.

A. $\Omega_{\alpha\beta} = 0$, $\Theta \neq 0$

In this case the line element takes the form

$$ds^2 = -dt^2 + B^2(t, r, \theta)[dr^2 + r^2 d\theta^2 + R^2(r, \theta)d\phi^2], \tag{23}$$

and the following equations have to be satisfied: the "continuity" equation [Eq. (A.6) in [1]],

$$\dot{\mu} + (\mu + P)\Theta + q^\alpha_{;\alpha} = 0, \tag{24}$$

and the generalized "Euler" equation [Eq. (A.7) in [1]],

$$h^\beta_\alpha(P_{,\beta} + \Pi^\mu_{\beta;\mu} + q_{\beta;\mu}V^\mu) + \frac{4}{3}\Theta q_\alpha = 0. \tag{25}$$

In the nondissipative case, it is known that the shear-free condition poses restrictions on the equilibrium equation of state (see [6,7]) even in the nongeodesic case. Thus, it is legitimate to ask, in our case, whether or not any admissible equation of state is restricted by the transport equation assumed for the heat transport.

The answer to the above question seems to be affirmative, if we observe that the last term within the round bracket in (25) is related to the thermodynamic variables through the transport equation (57) in [1] (if we assume the Israel-Stewart theory). However, in the general case ($a_\alpha \neq 0$), there is four-acceleration–heat coupling, and the answer is not so evident; this requires a more detailed analysis, which is outside the scope of this manuscript.

Next, from Eqs. (B.2), (B.3), and (B.4) in [1], it follows that

$$Y_I = Y_{KL} = Y_{II} = 0, \tag{26}$$

implying that

$$X_I = -2\mathcal{E}_I, \quad X_{II} = -2\mathcal{E}_{II}, \quad X_{KL} = -2\mathcal{E}_{KL},$$
$$\mathcal{E}_I = 4\pi\Pi_I, \quad \mathcal{E}_{II} = 4\pi\Pi_{II}, \quad \mathcal{E}_{KL} = 4\pi\Pi_{KL}, \tag{27}$$

where $\mathcal{E}_I, \mathcal{E}_{II}$, and $\mathcal{E}_{KL}$ are the three scalar functions defining the electric part of the Weyl tensor (see [1] for details), and $Y_I, Y_{KL}, Y_{II}, X_I, X_{II}$, and X_{KL} are some of the structure scalars obtained from the orthogonal splitting of the Riemann tensor, which are defined in Eqs. (38)–(50) in [1].

Additionally, as stated before, we obtain (17) and (18) from (B.6) and (B.7).

Finally, (B.10)–(B.18) in [1] produce (some of which are redundant)

$$-\frac{1}{3}(X_I - 4\pi\mu)^{\cdot} + \frac{1}{3}\mathcal{E}_I\Theta = -\frac{4\pi}{3}\left(\mu + P + \frac{1}{3}\Pi_I\right)\Theta - \frac{4\pi}{B}q_I' - 4\pi\frac{q_{II}B_\theta}{B^2 r}, \tag{28}$$

$$-\dot{X}_{KL} - \Theta X_{KL} = \frac{8\pi}{3}\Pi_{KL}\Theta - 2\pi(K^\mu L^\nu + K^\nu L^\mu)q_{\nu;\mu}, \tag{29}$$

$$\frac{1}{3}(-X_{II} + 4\pi\mu)^{\cdot} + \frac{\Theta}{3}\mathcal{E}_{II} = -\frac{4\pi}{3}\left(\mu + P + \frac{1}{3}\Pi_{II}\right)\Theta - 4\pi L^\mu L^\nu q_{\nu;\mu}. \tag{30}$$

We shall now turn our attention specifically to the dissipationless case $q_I = q_{II} = 0$; this is similar to the models analyzed in [22], although the anisotropic stress tensor is more general. The Petrov type of each specific model depends on the number of distinct eigenvalues of $\Pi_{\alpha\beta}$ [24,25].

From Eqs. (5), (17), (18), (24), and (26) it follows at once that in the dissipationless case,

$$\Theta = \Theta(t) \Rightarrow B(t,r,\theta) = f(t)b(r,\theta),$$
$$\mu = \mu(t), \quad P = P(t), \quad \Pi_I = \Pi_I(t),$$
$$\Pi_{II} = \Pi_{II}(t), \quad \Pi_{KL} = \Pi_{KL}(t), \tag{31}$$

where we use the fact that $Y_T = 4\pi(\mu + 3P)$ [Eq. (42) in [1]].

Then, Eqs. (28), (29), and (30) may be easily integrated to obtain

$$\mathcal{E}_I = \mathcal{E}_I(0)\exp\left[-\frac{2}{3}\int \Theta dt\right], \; \mathcal{E}_{II} = \mathcal{E}_{II}(0)\exp\left[-\frac{2}{3}\int \Theta dt\right],$$
$$\mathcal{E}_{KL} = \mathcal{E}_{KL}(0)\exp\left[-\frac{2}{3}\int \Theta dt\right], \tag{32}$$

or, feeding the expression of Θ back into (32),

$$\mathcal{E}_I = \frac{\mathcal{E}_I(0)}{B^2}, \qquad \mathcal{E}_{II} = \frac{\mathcal{E}_{II}(0)}{B^2}, \qquad \mathcal{E}_{KL} = \frac{\mathcal{E}_{KL}(0)}{B^2}. \tag{33}$$

From the above it becomes evident that $B = f(t)$, and, in the $\Theta > 0$ case, the system tends to a FRW spacetime.

Let us now analyze the other case.

B. $\Theta = 0$, $\Omega \neq 0$

In this case the system becomes time independent, as it can be easily inferred from (5).

Then, from (B.1), (B.2), (B.3), and (B.4) in [1], we obtain

$$2\Omega^2 = Y_T = 2Y_I = 2Y_{II}, \qquad Y_{KL} = 0, \tag{34}$$

and from (B.6) and (B.7) in [1],

$$-(\Omega B R)_{,\theta} = 8\pi q_I B^2 R\sqrt{r^2 + \tilde{G}^2}, \tag{35}$$

$$(\Omega B R)' = 8\pi q_{II} B^2 R. \tag{36}$$

From the two equations above, it becomes evident that $\Omega = 0$ in the dissipationless case; this implies, because of (34), that $\mu = P = 0$, unless we assume the equation of state $\mu = -3P$. In other words, any model belonging to this class ($\Theta = 0$) must necessarily be dissipative.

However it is a simple matter to check—from (B.8), (B.9), and (B.13) in [1], together with (35), (36) and the regularity conditions on the axis of symmetry—that no such models ($\Theta = 0$) exist.

IV. CONCLUSIONS

Using the framework developed in [1], we have analyzed in some detail the general properties of the shear-free case. We have seen that, for a general dissipative and anisotropic fluid, vanishing vorticity is a necessary and sufficient condition for the magnetic part of the Weyl tensor to vanish, providing a generalization of the same result for perfect fluids obtained in [18–20]. This result, in turn, implies that vorticity should necessarily appear if the system radiates gravitationally. We stress that this result is not restricted to the axially and reflection-symmetric case. This further reinforces the well-established link between radiation and vorticity (see [26] and references therein).

In the geodesic case, the vorticity always vanishes (and therefore the magnetic parts of the Weyl tensor), suggesting that in this case no gravitational radiation is produced during the evolution. This result is in agreement with the shear-free conjecture mentioned above. However, we do not know if it holds for the nongeodesic case. If it does, then it is clear that we should consider shearing fluids when looking for sources of gravitational radiation.

The above result is also similar to the one obtained for the cylindrically symmetric case [27], and suggests (as does the shear-free conjecture) a link between the shear of the source and the generation of gravitational radiation during its evolution.

In the geodesic case, we also observe that, in the nondissipative case, the models do not need to be FRW (as already stressed in [21]); however, the system relaxes to the FRW spacetime if Θ is positive. In presence of dissipative fluxes, such a tendency does not appear, further illustrating the relevance of dissipative processes in the evolution of self-gravitating fluids.

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