Note on explicit form of entanglement entropy in the Russo-Susskind-Thorlacius model

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For an evaporating black hole which is a radiation-black hole combined system, we express the entanglement entropy and the Page time in terms of the conformal time in the Russo-Susskind-Thorlacius model. The entropy change of the black hole is nicely written in terms of Hawking flux. Integrating the first law of thermodynamics, we can obtain the decreasing black hole entropy and the increasing radiation entropy, and the entanglement entropy for this system based on the Page argument. We also obtain analytically the critical temperature to release black hole information, which corresponds to the Page time, and discuss the relation between the conserved total entropy and information recovering of the black hole in this model.

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Bekenstein has suggested that the entropy of a black hole is proportional to the area of the horizon [1–3], and subsequently Hawking's discovery has led to the result that the black hole has thermal radiation with the temperature $T_{\rm H} = \kappa_{\rm H}/2\pi$ [4], where $\kappa_{\rm H}$ is the surface gravity at the event horizon. It has also been claimed that the black hole would eventually disappear completely through thermal radiation, which gives rise to information loss problem [5]. However, if Hawking radiation plays a role of carrier of information, information will come out so slowly until the Page time [6] when the entanglement entropy becomes maximum such that the dimension of radiation equals to that of the black hole in the Hilbert space. When the dimension of radiation is larger than that of the black hole, information is naturally contained in radiation. Moreover, it has been shown that in Ref. [6] the above statistical analysis can be realized in the Callan-Giddings-Harvey-Strominger (CGHS) model [7] by taking into account the classical metric along with the corresponding constant temperature which is independent of black hole mass so that radiation does not reflect the backreaction of the geometry. Actually, the black hole entropy of a two-dimensional black hole with the backreaction was studied for the static case in Refs. [8,9].

In this work, we are going to study the entanglement entropy based on the Page formulation using the Russo-Susskind-Thorlacius (RST) model [10,11] to take into account backreaction of the geometry, which yields naturally the time-dependent geometry. The essential difficulty is to identify the time-dependent temperature which is quite awkward in standard thermodynamics. So we would like to assume that a radiation-black hole combined system is in equilibrium at each time such that the radiation temperature measured by the fixed observer at the future null infinity is identified with the black hole temperature. Then, the thermodynamic first law is also read off from the differential form of the energy conservation law [12], so that the entropy change of the black hole is neatly written in terms of Hawking flux. Integrating the first law of thermodynamics, we can obtain the decreasing black hole entropy and the increasing radiation entropy, and the entanglement entropy based on the Page argument [6]. So, the total entropy is always constant while the total information is not conserved locally because of the time-dependent entanglement entropy; however, it is expected that the total information is recovered after complete evaporation of the black hole.

Now, let us start with the RST model given by the action [10]

$$I = I_{\rm DG} + I_f + I_{\rm PL} + I_{\rm corr}, \tag{1}$$

with

$$I_{\rm DG} = \frac{1}{2\pi}\int d^2x\sqrt{-g}e^{-2\phi}[R + 4(\nabla\phi)^2 + 4\lambda^2], \tag{2}$$

$$I_f = \frac{1}{2\pi}\int d^2x\sqrt{-g}\left[-\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right], \tag{3}$$

$$I_{\rm PL} = \frac{\kappa}{2\pi}\int d^2x\sqrt{-g}\left[-\frac{1}{4}R\frac{1}{\Box}R\right], \tag{4}$$

$$I_{\rm corr} = \frac{\kappa}{2\pi}\int d^2x\sqrt{-g}\left[-\frac{1}{2}\phi R\right], \tag{5}$$

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where $\kappa = (N-24)/12$ which can be positive by taking into account the ghost decoupling term [13] and λ is a cosmological constant. Equation (5) is added to obtain an exact black hole solution and it is reduced to the CGHS model without this term [7]. From the action (1), the equations of motion are given by $\Box f_i = 0$, $e^{-2\phi}[R - 4(\nabla\phi)^2 + 4\Box\phi + 4\lambda^2] + \frac{\kappa}{4}R = 0$, and $G_{\mu\nu} = T^f_{\mu\nu} + T^{\rm qt}_{\mu\nu}$, where $G_{\mu\nu} \equiv \frac{2\pi}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}(I_{\rm DG} + I_{\rm corr}) = 2e^{-2\phi}[\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}((\nabla\phi)^2 - \Box\phi - \lambda^2)] + \frac{\kappa}{2}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi)$. The energy-momentum tensors for matter are defined as

$$T^f_{\mu\nu} \equiv -\frac{2\pi}{\sqrt{-g}}\frac{\delta I_f}{\delta g^{\mu\nu}} = \sum_{i=1}^{N}\left[\frac{1}{2}\nabla_\mu f_i \nabla_\nu f_i - \frac{1}{4}g_{\mu\nu}(\nabla f_i)^2\right], \tag{6}$$

$$T^{\rm qt}_{\mu\nu} \equiv -\frac{2\pi}{\sqrt{-g}}\frac{\delta I_{\rm PL}}{\delta g^{\mu\nu}}. \tag{7}$$

It can be checked that the dilaton improved Bianchi identity for the RST model is satisfied, i.e., $\nabla^\mu G_{\mu\nu} = 0$, which yields covariant conservation relations for classical matter and quantum matter as $\nabla^\mu T^f_{\mu\nu} = 0 = \nabla^\mu T^{\rm qt}_{\mu\nu} = 0$. Thus, the definitions of the classical and the quantum energy-momentum tensors given by Eqs. (6) and (7) in the RST model are compatible with those of the CGHS model as long as covariant conservation relations are concerned. In the conformal gauge given by $ds^2 = e^{2\rho}dx^+dx^-$, the classical energy-momentum tensor (6) is written as $T^f_{\pm\pm} = \frac{1}{2}\sum_{i=1}^N(\partial_\pm f_i)^2$ and $T^f_{\pm\mp} = 0$ and the quantum energy-momentum tensor (7) is given by $T^{\rm qt}_{\pm\pm} = \kappa[\partial^2_\pm\rho - (\partial_\pm\rho)^2 - t_\pm(x^\pm)]$ and $T^{\rm qt}_{\pm\mp} = -\kappa\partial_+\partial_-\rho$, which agrees with the quantum energy-momentum tensor introduced in the CGHS model [7]. The unknown functions $t_\pm(x^\pm)$ reflect the nonlocal property of the effective action. Of course, one may define the energy-momentum tensor for matter in a different way by including the contribution from Eq. (5) because it is actually not unique. However, it can be well defined at asymptotic future null infinity since it can be expressed by only the boundary function as $T^{\rm qt}_{\pm\pm} = -\kappa t_\pm$ when we consider Hawking radiation in that region.

By introducing new variables given by $\chi = \sqrt{\kappa}\rho - (\sqrt{\kappa}/2)\phi + e^{-2\phi}/\sqrt{\kappa}$ and $\Omega = (\sqrt{\kappa}/2)\phi + e^{-2\phi}/\sqrt{\kappa}$ for simplicity, the action (1) can be written as [10,11]

$$I = \frac{1}{\pi}\int d^2x\left[-\partial_+\chi\partial_-\chi + \partial_+\Omega\partial_-\Omega + \lambda^2 e^{(2/\sqrt{\kappa})(\chi-\Omega)} + \frac{1}{2}\sum_{i=1}^N \partial_+ f_i \partial_- f_i\right], \tag{8}$$

and the two constraints are given by $\kappa t_\pm = -\partial_\pm\chi\partial_\pm\chi + \sqrt{\kappa}\partial^2_\pm\chi + \partial_\pm\Omega\partial_\pm\Omega + \frac{1}{2}\sum_{i=1}^N \partial_\pm f_i\partial_\pm f_i$. The equations of motion derived from the action (8) can be exactly solved. In the Kruskal coordinates where $\chi = \Omega$, the evaporating black hole formed by an incoming shock wave of $T^f_{++} = [M/(\lambda x_0^+)]\delta(x^+ - x_0^+)$ is described by the solution of $\Omega(x^+, x^-) = -\lambda^2 x^+x^-/\sqrt{\kappa} - \frac{\sqrt{\kappa}}{4}\ln(-\lambda^2x^+x^-) - \frac{M}{\lambda\sqrt{\kappa}x_0^+}(x^+ - x_0^+)\Theta(x^+ - x_0^+)$, where the linear dilaton vacuum is chosen for $x^+ < x_0^+$. An asymptotically static coordinate can be obtained from the coordinate transformations defined by $x^+ = (1/\lambda)e^{\lambda\sigma^+}$ and $x^- = -(1/\lambda)e^{-\lambda\sigma^-} - (M/\lambda^2)e^{-\lambda\sigma_0^+}\Theta(\sigma^+ - \sigma_0^+)$, where $\sigma_0^+ = \lambda^{-1}\ln(\lambda x_0^+)$.

Note that the RST model is known to be quantum-mechanically inconsistent after appearance of the naked singularity [14]. The curvature singularity and apparent horizon collide in a finite proper time and the singularity is naked after the two have merged [10]. In order to avoid the naked singularity, a vacuum state can be patched at the intersection point (σ_s^+, σ_s^-) of the singularity curve and the apparent horizon as shown in the Penrose diagram of Fig. 1, where the intersection point is given by $\sigma_s^- = \sigma_0^+ + \lambda^{-1}\ln[(\lambda/M)(\exp(4M/(\kappa\lambda)) - 1)]$ and $\sigma_s^+ = \sigma_s^- + \lambda^{-1}\ln(\kappa/4)$ [10–12,14]. However, this patching procedure requires the thunderpop energy which is the negative classical energy emanated from the black hole. So we are going to mainly discuss the entanglement entropy of the RST model before the negative Bondi mass appears.

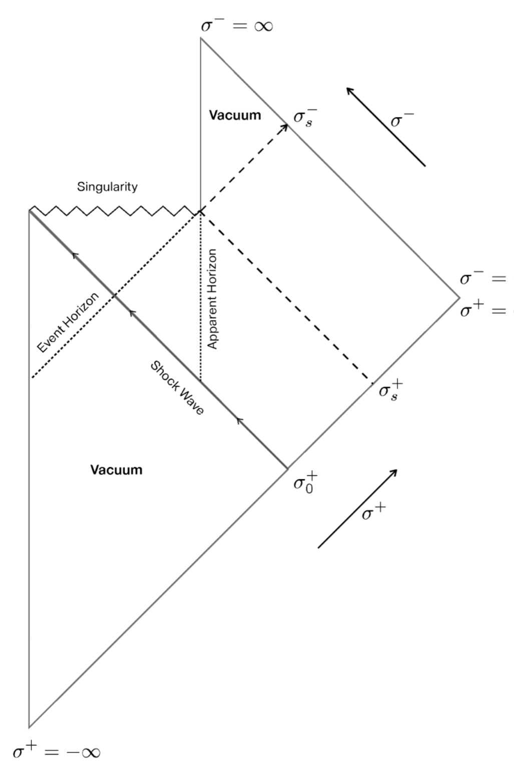

FIG. 1. It shows the Penrose diagram of the black hole formed by a shock wave at $\sigma^+ = \sigma_0^+$.

On the other hand, from the covariant conservation law, one can get the ordinary conserved quantity by expanding the metric and dilaton fields around the linear dilaton vacuum. Then, the linearized equation of motion becomes $G^{(1)}_{\mu\nu} = T^f_{\mu\nu} - G^{(2)}_{\mu\nu}$ [15], where $T^f_{\mu\nu}$ is a classical energy-momentum tensor, $G^{(1)}_{\mu\nu}$ is the linear perturbed part of $G_{\mu\nu}$, and $G^{(2)}_{\mu\nu}$ is the rest. Then, one can choose the time and space coordinate so that it is easy to show that the linearized equation of motion identically satisfies the ordinary conservation law, $\partial_\mu G^{(1)\mu 0} = 0$, by the use of the linearized Bianchi identity [16]. It implies that the current defined as $J^\mu = T^{\mu 0}_f - G^{(2)\mu 0}$ satisfies the ordinary conservation law $\partial_\mu J^\mu = 0$. Thus we can define the Bondi mass $B(\sigma^-)$ which is the energy evaluated along the null line [17], $B(\sigma^-) = (1/2)\int_{-\infty}^{\infty} d\sigma^+ G^{(1)-0}(\sigma^+, \sigma^-)$ while the Anowitt-Deser-Misner (ADM) mass is calculated at the spatial infinity as $E_{\rm ADM}(t) = \int_{-\infty}^{\infty} dq G^{(1)00}(t, q)$ [18]. Using the integrated form of the linearized equation of motion, after some calculations the difference between the ADM mass and the Bondi mass can be obtained as [12]

$$E_{\rm ADM}(t) - B(\sigma^-) = \int_{-\infty}^{\sigma^-} d\sigma^- (T^f_{--} + T^{\rm qt}_{--})|_{\sigma^+\to\infty}. \quad (9)$$

Note that the classical infalling energy-momentum tensor does not exist since it cannot appear in the asymptotic future null infinity. However, in the RST model there may exist the classical outgoing negative energy density called the thunderpop energy at σ_s^-. From now on, we will consider radiation-hole combined system before the thunderpop appears, and then the conformal time is restricted to $-\infty < \sigma < \sigma_s^-$. It means that we can take vanishing outgoing classical energy-momentum tensor in this analysis.

Next, the integrated Hawking flux is given by $H(\sigma^-) = \int_{-\infty}^{\sigma^-} d\sigma^- h(\sigma^-)$, where the Hawking flux is $h(\sigma^-) = T^{\rm qt}_{--}|_{\sigma^+\to\infty}$. The Hawking flux is simply reduced to the boundary function as $h(\sigma^-) = -t_-(\sigma^-)$ since $\sigma^\pm$ is a quasistatic coordinate system at infinity, and so the fields approach the linear dilaton vacuum at $\sigma^+ \to \infty$. In this black hole, the Hawking radiation is written as $h(\sigma^-) = (\kappa\lambda^2/4)[1 - (1 + (M/\lambda)e^{\lambda(\sigma^- - \sigma_0^+)})^{-2}]$ [7]. Note that the Bondi mass is the remaining energy after quantum-mechanical Hawking radiation has been emitted from the system. So it is plausible to regard the Bondi mass as a black hole mass in the quantum backreacted model. From Eq. (9), we can get the conservation law as [12]

$$B(\sigma^-) + H(\sigma^-) = M, \quad (10)$$

where M is the ADM mass. The energy can be conserved in this evaporating black hole system so that the Bondi energy plus the Hawking radiation should be equal to the initial infalling energy by the scalar fields.

Now, we will assume the radiation-black hole combined system as a thermal equilibrium system for each conformal time σ^- in order to apply the thermodynamic first law. Let us first relate the Hawking flux to the Hawking temperature in analogy with the static case [19], then one can read off the black hole temperature $T(\sigma^-)$ from the Hawking radiation by identifying

$$h(\sigma^-) = \kappa\pi^2 T^2(\sigma^-), \quad (11)$$

which yields

$$T(\sigma) = \frac{\lambda}{2\pi}\left[1 - \frac{1}{(1 + \frac{M}{\lambda}e^{\lambda(\sigma^- - \sigma_0^+)})^2}\right]^{1/2}. \quad (12)$$

Note that it vanishes at $\sigma^- \to -\infty$ since the black hole did not radiate yet and the well-known Hawking temperature is recovered as $T_{\rm H} = \lambda/2\pi$ at $\sigma^- \to \infty$ which is compatible with the previous static results [6,19]. Using the differential form of the energy conservation law (10), the change of the black hole entropy can be written as

$$\Delta S_h = S_h(\sigma^-) - S_h^0 = \int \frac{dB}{T} = -\pi\sqrt{\kappa}\int_{-\infty}^{\sigma^-} d\sigma^- \sqrt{h(\sigma^-)}, \quad (13)$$

where S_h^0 denotes the entropy of the black hole at $\sigma^- \to -\infty$. The entropy change is essentially due to Hawking radiation such that the entropy of the black hole is decreasing. From Eq. (13) the entropy is calculated as

$$S_h(\sigma^-) = \frac{2\pi M}{\lambda} - \frac{\pi\kappa}{2}\left[\sec^{-1}\gamma(\sigma^-) + \ln\left(\gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-) - 1}\right)\right], \quad (14)$$

where $\gamma(\sigma^-) = 1 + (M/\lambda)e^{\lambda(\sigma^- - \sigma_0^+)}$ and we employed the fact that the entropy of the black hole is given by $S_h^0 = 2\pi M/\lambda$ at the initial time of $\sigma^- \to -\infty$ since the entropy of the black hole starts with the maximum thermal entropy of the area law, and at the same time the Hawking temperature (12) is zero. As time goes on, the black hole entropy is decreasing according to the increasing Hawking temperature which amounts to $T_{\rm H} = \lambda/2\pi$ at $\sigma^- \to \infty$. Note that in the conventional thermodynamic analysis, the black hole entropy and the temperature are given as $S = 2\pi M/\lambda$, and $T_{\rm H} = \lambda/2\pi$.

On the other hand, for a system consisting of the black hole subsystem and the radiation subsystem, the entanglement entropy for $S_h, S_r \gg 1$ is given by the Page argument as [6]

$$S_{\rm ent} \simeq \begin{cases} S_r - \frac{1}{2}e^{S_r - S_h} & \text{for } S_r \le S_h \\ S_h - \frac{1}{2}e^{S_h - S_r} & \text{for } S_r \ge S_h \end{cases}, \quad (15)$$

where S_h and S_r are the black hole entropy and the radiation entropy, respectively. Note that the total entropy of the system is preserved such that it is given as $S_r + S_h = 2\pi M/\lambda$. The entanglement entropy (15) has a maximum value at the Page time when the black hole emits a half of its initial Bekenstein-Hawking entropy, i.e., $S_r = \pi M/\lambda$. Using Eq. (14), we can write the entanglement entropy explicitly in terms of σ^-, and it becomes

$$S_{\rm ent}(\sigma^-) \simeq \frac{\pi\kappa}{2}\left[\sec^{-1}\gamma(\sigma^-) + \ln\left(\gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-)-1}\right)\right] - \frac{1}{2}\left[\left(\gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-)-1}\right)\exp\left(\sec^{-1}\gamma(\sigma^-) - \frac{2M}{\kappa\lambda}\right)\right]^{\pi\kappa} \quad (16)$$

for $\sigma^- \le \sigma_c^-$ and

$$S_{\rm ent}(\sigma^-) \simeq \frac{2\pi M}{\lambda} - \frac{\pi\kappa}{2}\left[\sec^{-1}\gamma(\sigma^-) + \ln\left(\gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-)-1}\right)\right] - \frac{1}{2}\left[\left(\gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-)-1}\right)\exp\left(\sec^{-1}\gamma(\sigma^-) - \frac{2M}{\kappa\lambda}\right)\right]^{-\pi\kappa} \quad (17)$$

for $\sigma^- \ge \sigma_c^-$. Note that the entanglement entropy becomes maximum at the conformal time of σ_c^- which comes from the maximization of the entanglement entropy formally given in the closed form of $\gamma(\sigma_c^-)\cos[\ln(\gamma(\sigma_c^-) + \sqrt{\gamma^2(\sigma_c^-)-1}) - 2M/(\kappa\lambda)] = 1$. That point is just the Page time expressed by the conformal time since the radiation entropy is the same with the black hole entropy as shown in Fig. 2. The radiation entropy is monotonically increasing while the black hole entropy is monotonically decreasing, and their sum is constant.

By the way, there is a deficiency in this calculation that the black hole entropy is negative for $\sigma^- > \sigma_p^-$ because the Bondi mass in the RST model is negative due to the surplus Hawking radiation after σ_p^- [14] so that the present calculations are meaningful before σ_p^-. Moreover, the expression for the entanglement entropy based on the Page argument is valid only for many degrees of freedom as was noticed below (15) such as $S_h, S_r \gg 1$. So, it seems to be inappropriate to discuss beyond the end point of the entropy in our formulation, and the calculation of the entanglement entropy becomes a good approximation around σ_c^-. One more thing to be mentioned is that we distinguished the definition of the entropy depending on the subsystem: the black hole entropy is defined by employing the Bondi mass, which is plausible in that the entropy change of the black hole should be negative because the black hole radiates while the entropy change of radiation should be positive because Hawking radiation is increased monotonically.

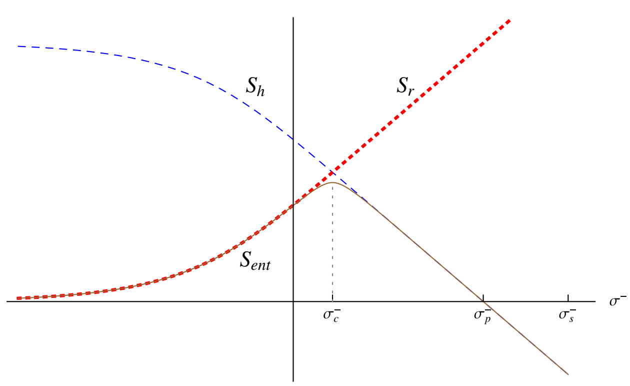

FIG. 2 (color online). The solid, the dashed, and the thick-dotted lines show the behaviors of the entanglement entropy $S_{\rm ent}$, the black hole entropy S_h, and the thermodynamic radiation entropy S_r, respectively. The entanglement entropy has a maximum at σ_c^- and it vanishes at σ_p^-.

As for the naked singularity of the black hole, the black hole can generally form a singularity. However, as seen from the original work [6], the black hole system was assumed to have many degrees of freedom such as $m, n \gg 1$ in order to formulate the system and derive the explicit form of the average value of the entanglement entropy of Eq. (15). It means that even in spite of the small black hole, it should have many degrees of freedom in this formulation so that the black hole does not lose its mass completely and then the naked singularity is no more concerned. Based on this argument, we have employed the same entropy formula in the present RST model so that our result is also valid only for the many degrees of freedom just at the conformal time $\sigma^- \ll \sigma_p^-$ for which $S_h \gg 1$. Therefore, the entanglement entropy turns out to be well defined only around σ_c^- in Fig. 2 except the extreme limits of very small degrees of freedom. Furthermore, the advantage of the RST model in Ref. [10] is that it has been designed to be free from the naked singularity because the flat metric can be patched with the black hole metric when the singularity forms at σ_s^- as seen from Fig. 1.

As a result, we have obtained the decreasing black hole entropy, the increasing radiation entropy, the entanglement entropy, and the Page time in terms of the conformal time in the exactly soluble RST model. Moreover, we can find a Page temperature at the Page time since σ_c^- was identified so that it becomes formally $T(\sigma_c^-)$ from Eq. (12). In other words, information is significantly leased above the critical temperature of $T(\sigma_c^-)$.

In Refs. [20,21], the black hole entropy and increase theorem related to the second law of black hole thermodynamics have been studied for the RST model, and we would like to mention some differences between our work and them. First, the system in our work was divided into two subsystems so that the black hole has the black hole entropy and radiation has the radiation entropy, respectively, while there appears only a single system and the single entropy to define the black hole system in the previous works. Additionally, the entanglement entropy in this work has been defined throughout correlation between the two subsystems, so that the entropy in Refs. [20,21] behaves like not the entanglement entropy but the radiation entropy in our work in the sense that it is always increasing as time goes on and the entropy change is always positive, which guarantees the second law of black hole thermodynamics.

In the original work done by Page in Ref. [6], the system was divided into two subsystems; one is for the black hole with dimension m and the other is for radiation with dimension n. The most important assumption is that these subsystems form a total system in a pure state in a Hilbert space of fixed dimension mn. It means that the total entropy is constant, and consequently $\Delta S = 0$. Following this assumption in Ref. [6], we also assumed that the total entropy should be constant in order to realize the Page argument in the RST model. In particular, for the case of $\Delta S > 0$, the information will be eventually lost like ordinary thermal systems. Therefore, the requirement of the fixed total entropy is a sort of constraint based on the hypothesis that no information is lost in black hole formation and evaporation as was claimed in Ref. [6].

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