Approaching the conformal window of $O(n) \times O(m)$ symmetric Landau-Ginzburg models using the conformal bootstrap

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$O(n) \times O(m)$ symmetric Landau-Ginzburg models in $d = 3$ dimensions possess a rich structure of the renormalization group, which offers a theoretical prediction of the phase diagram in frustrated spin models with noncollinear order. Depending on n and m, they may show chiral/antichiral/Heisenberg/Gaussian fixed points within the same universality class. We approach all the fixed points in the conformal bootstrap program by examining the bound on the conformal dimensions for scalar operators as well as nonconserved current operators with consistency cross-checks. For large n/m, we show strong evidence for the existence of four fixed points by comparing the operator spectrum obtained from the conformal bootstrap program with that from the large-n/m analysis. We propose a novel nonperturbative approach to the determination of the conformal window in these models based on the conformal bootstrap program. From our numerical results, we predict that for $m = 3$, $n = 7 \sim 8$ is the edge of the conformal window for the antichiral fixed points.

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I. INTRODUCTION

Conformal field theories (CFTs) have played central roles in theoretical physics as they lie at the end points of generic renormalization group (RG) flows. In the study of critical phenomena, it is crucial to understand what kind of fixed points are present with the desired symmetries (see, e.g., Refs. [1,2]). Within Hamiltonian (Lagrangian) approaches there have been several methods proposed to investigate the problem both perturbatively (such as ε and $1/N$ expansions; see, e.g., Ref. [3]) and nonperturbatively (such as Monte Carlo simulations), but they may suffer from questions of validity and numerical costs.

Recently we have seen substantial progress in understanding higher-dimensional CFTs via the conformal bootstrap program [4,5]. Without using an explicit form of the Hamiltonian, it provides us with rigorous but nontrivial constraints on the spectra of conformal dimensions and operator product expansion (OPE) coefficients [6–9]. The information encoded in the conformal bootstrap allows us to "rediscover" nontrivial CFTs and reproduces known critical exponents [10–16]. From our experience, we are convinced that there exists a certain class of CFTs that occupy the extreme corners of the conformal bootstrap constraints, although the fundamental reason is not obvious to us yet.

In this paper we study $O(n) \times O(m)$-symmetric CFTs in space-time dimension $d = 3$ with a particular emphasis on the $m = 3$ cases. The motivation for this choice of symmetry group is twofold. Firstly, these symmetries are realized in condensed matter systems like frustrated spin models with noncollinear order [17,18]. Secondly, $O(n) \times O(m)$-symmetric Landau-Ginzburg models have richer dynamical structures than $O(n)$ vector models.

There is a long-standing debate over whether the frustrated spin systems in noncollinear order show the first-order phase transition or the second-order phase transition. If the $O(n) \times O(m)$-symmetric Landau-Ginzburg models do not possess nontrivial fixed points other than the $O(nm)$-symmetric Heisernberg fixed point, the second-order phase transition is impossible. On the other hand, the existence of the stable $O(n) \times O(m)$-symmetric fixed point is strong evidence for the second-order phase transition if the system is inside the attraction domain of the RG flow.

When we study the RG flow of the $O(n) \times O(m)$-symmetric Landau-Ginzburg models with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(\partial_\mu \phi_a^\alpha)(\partial_\mu \phi_a^\alpha) + \frac{u}{4!}(\phi_a^\alpha \phi_a^\alpha)^2 + \frac{v}{4!}(\phi_a^\alpha \phi_b^\alpha \phi_a^\beta \phi_b^\beta - \phi_a^\alpha \phi_a^\alpha \phi_b^\beta \phi_b^\beta), \quad (1)$$

where $a = 1, \cdots n$ and $\alpha = 1 \cdots m$, we find that for a sufficiently large ratio n/m, there are two extra fixed points in addition to the Gaussian ($u = v = 0$) and $O(nm)$ Heisenberg ($v = 0$) fixed points, which are named the chiral (stable) and antichiral (unstable) fixed points [17,19] (see Fig. 1). In perturbative approaches, the existence of these additional fixed points was established for the large-n limit with fixed m [19], but the situations for smaller values of n are still unsettled and controversial [18,20,21], so it is desirable to see if we can find any indications of these additional fixed points in the conformal bootstrap program without using the explicit form of the Hamiltonian. In this paper, we will indeed show strong evidence that for sufficiently large n/m, the theories at these fixed points solve the conformal bootstrap constraints at their extreme

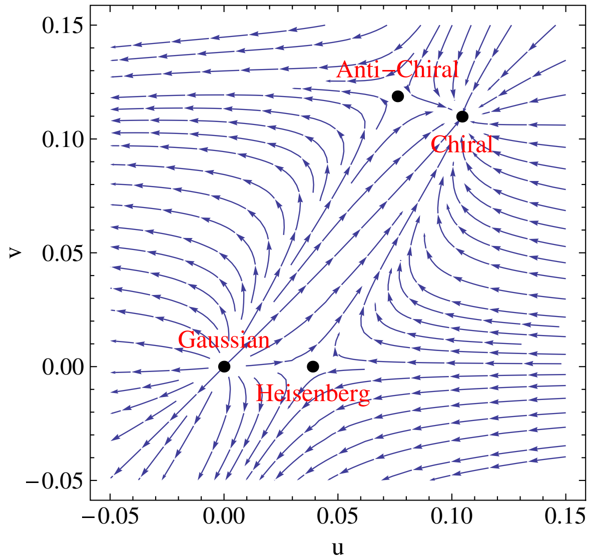

FIG. 1 (color online). The RG flow of $O(n) \times O(m)$ Landau-Ginzburg models with sufficiently large n. In the present figure, $(n, m) = (50, 3)$ and the β functions are taken from Sec. 3 of Ref. [19].

corners (as in Refs. [11,16]) but in different OPE sectors. Furthermore, we will predict the edge of the conformal window for $O(n) \times O(3)$ antichiral fixed points.

Our careful choice of $m = 3$ is used to establish the usefulness of the conformal bootstrap program to determine the edge of the conformal window in less controversial situations [22] while still finding the conjecture from the other methods before we venture into the most controversial cases of $m = 2$ with smaller n. Note that $m = 3$ cases are not necessarily unrealistic (compared, e.g., with $m > 3$). As reviewed in Ref. [17], when the frustrated spins are three-dimensionally aligned, the system admits the $O(n) \times O(3)$ symmetry.

II. CONFORMAL BOOTSTRAP

Let us begin with CFTs with no continuous global symmetry. The strategy of the conformal bootstrap program is to constrain CFTs from their fundamental properties, such as conformal invariance, unitarity, and crossing symmetry. Specifically, we demand these properties in the four-point function of identical scalar primary operators,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle.$$

The crossing relation gives a sum rule in terms of OPE coefficients and conformal blocks. After truncation to a finite-dimensional convex optimization problem, we find an upper bound for the conformal dimension of the lowest-dimensional operator (other than the unit operator) of definite spin appearing in the $\phi \times \phi$ OPE. Repeating this procedure by changing the conformal dimension δ of the external scalar operator, we obtain a critical curve $\Delta_c^l(\delta)$, with l being the spin of the operator. The behavior of $\Delta_c^0(\delta)$ is summarized as follows: for space-time dimensions $d < 4$, it shows a discontinuity in its slope, and the location of such a "kink" corresponds to the Wilson-Fisher fixed point. The singular behavior of $\Delta_c^0(\delta)$ knows the existence of a nontrivial CFT.

Let us now consider the CFT with an additional symmetry group G. We study the four-point function of scalar operators in a certain real representation R of G,

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle,$$

where i is the index for the representation R. In these cases, intermediate states are labelled by their spin and the representation appearing in $R \otimes R$, and the crossing relation gives a vectorial sum rule [13,23–25]. Correspondingly, we find a bound for the conformal dimension of the lowest operator appearing in the $\phi_i \times \phi_j$ OPE for each irreducible representation contained in $R \otimes R$ to obtain a critical curve $\Delta_c^{I,l}(\delta)$ (with I and l denoting the representation and spin, repsectively). Note that the possible representation I and parity of l are correlated by the Bose symmetry of the operator.

In Ref. [13], the case of $G = O(n)$ in $d = 3$ with R given by a fundamental representation was studied in detail. In this case we have three irreducible representations: a singlet (which we denote S), a second-rank symmetric tensor (T), and an antisymmetric tensor representation (A). In the S sector, they found kinks (though the changes of the slope were milder) in $\Delta_c^{S,0}(\delta)$, the location of which exhibits an excellent agreement with that of $O(n)$ Heisenberg fixed points proposed using other methods (e.g., large-n expansion). They also obtained the bounds for scalar operators in the T sector. The resulting $\Delta_c^{T,0}(\delta)$ reveals intriguing (but not as prominent) features: at the point where the $O(n)$ model sits, it starts to grow approximately linearly.

Our focus is $G = O(n) \times O(m)$ (with $n, m \geq 3$ for simplicity) in $d = 3$ under the presence of a scalar operator in the bifundamental representation R corresponding to ϕ_a^α in Eq. (1). In this case, $R \otimes R$ contains nine irreducible representations, which are product representations formed by S, T, and A. The sum rule encoding this information is conveniently expressed in terms of a 9×9 matrix, which we have derived following the lines of Ref. [25].

To compute numerical bounds, we follow the methods described in Ref. [13] though the details are somewhat distinct. For example, we include intermediate operators with spin $l \leq 20$. Since the sum-rule matrix is larger and the computational task is much heavier than in the $O(n)$ case, we work in lower-dimensional search spaces, namely, 36×9-dimensional ones (or $k = 8$ in Ref. [13], $N_{\max} = 7$ in Ref. [11]). In Ref. [13], bounds were obtained by assuming that the conformal dimension of the intermediate scalar operator in the T sector was greater than one for technical reasons, but we only impose the unitarity bound in every sector. We use the ρ-series expansions in Ref. [26] to generate the residues of the conformal blocks. Our normalization condition of the linear functional Λ is such that it gives the value 1 when it acts on the vector for a dimension-five conformal block in the spin 0 TT sector. Our implementation for SDPA-GMP [27,28] is the same as in Ref. [13] except that the parameter PRECISION is 350.

III. RESULTS

We first performed the numerical computation for the $O(15) \times O(3)$ model since the value $n = 15$ is well above the existence bound on additional chiral/antichiral fixed points obtained from the large-n analysis of Ref. [19].

We computed the bounds for the first scalar operator in the SS sector, $\Delta_c^{\mathrm{SS},0}(\delta)$. We present the results in Fig. 2. They turned out to be identical within the precision of 10^{-4} to those of the S sector in the $O(45)$ model. Such "symmetry enhancement" behavior was reported in Ref. [25] between $SU(N)$ and $O(2N)$. As stressed there, we can prove the inequality $\Delta_{c,O(45)}^{\mathrm{S},0}(\delta) \leq \Delta_{c,O(15)\times O(3)}^{\mathrm{SS},0}(\delta)$ immediately because we can regard every CFT with $O(45)$ symmetry as a CFT with $O(15) \times O(3)$ symmetry upon decomposing $O(45)$ representations into those of $O(15) \times O(3)$. We therefore conclude that the kink shown in the SS sector corresponds to the Heisenberg fixed point. However, the reason for the actual equality here is still mysterious. Meanwhile, chiral/antichiral fixed points lie well below the bound, which confirms the consistency of our analysis, but simultaneously implies that we will not be able to approach these fixed points in this manner. We propose an alternative way to spot them in the rest of the present section.

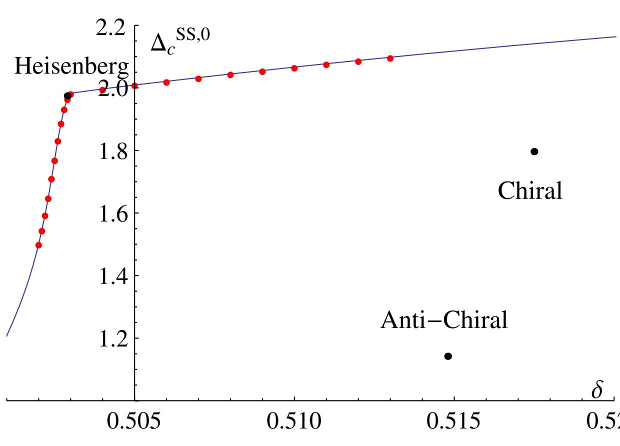

FIG. 2 (color online). The plot for $\Delta_c^{\mathrm{SS},0}(\delta)$ for $O(15) \times O(3)$ (red dots) and $\Delta_c^{\mathrm{S},0}$ for $O(45)$ (blue line). The vertical precision is 10^{-3}. The location of Heisenberg fixed point is from the large-nm analysis reviewed in Ref. [3]. The location of chiral/antichiral fixed points are from the large-n predictions of Ref. [19].

We next computed the bounds for spin-1 operators in the $O(15)$-symmetric, $O(3)$-antisymmetric tensor representation, $\Delta_c^{\mathrm{TA},1}(\delta)$. Such an operator would be conserved if $O(15) \times O(3)$ were enhanced to $O(45)$. The result in Fig. 3 shows a kink behavior around $\delta \sim 0.515(1)$. This value is quite close to the value $\delta = 0.5148$ predicted by the large-n analysis [19] of the antichiral fixed point (c.f. $\delta = 1/2 + \eta/2$). For a further check of this identification, we have derived low-lying spectra at the kink from the SDPA-GMP output, following the strategy of Ref. [12]. The first operator in the SS sector has the conformal dimension 1.16(3). On the other hand, the large-n analysis predicts that it is 1.142, in close agreement with ours (c.f. $\Delta^{\mathrm{SS},0} = 3 - 1/\nu$). We estimate the systematic errors conservatively from three sources; the vertical bisection precision, the horizontal impreciseness to locate the kink, and the convergence with respect to the number of derivatives in our search space. For reference, we also computed the low-lying spectra in the other sectors from the same output. For the SA sector the first operator has the conformal dimension 2.02, while it should be exactly 2 since it must be a conserved current. Hence the error in this analysis could be as large as 0.02, and our prediction for the SS scalar does not seem to contradict the large-n analysis.

We also computed the numerical bounds for the other sectors with the lowest spins.

We present the results for the ST spin-0 sector in Fig. 4. While the change of the slope is not as sharp as that in the TA sector, the shape resembles the bound for the spin-0 operator in the two-dimensional Ising model reported in Ref. [10] with $k = 8$. (More recently the bound has been improved to give a sharper kink behavior; see Ref. [12].) It might be feasible to sharpen the bound so that this becomes an actual kink. Reading off the spectra at $\delta = 0.515$, we find that the conformal dimension of the SS operator is 1.16 (3), which is again close to the large-n prediction for the antichiral fixed point.

We present the results for the TS spin-0 sector in Fig. 5. We see no sudden change of the slope, but its behavior looks similar to the T-sector bounds in Ref. [13], so we conjecture that there is a nontrivial CFT saturating the

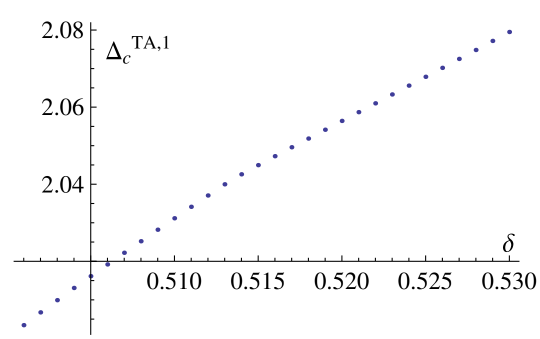

FIG. 3 (color online). The plot for $\Delta_c^{\mathrm{TA},1}(\delta)$. The vertical precision is 4×10^{-4}. See also Fig. 7 for the differentiated plot.

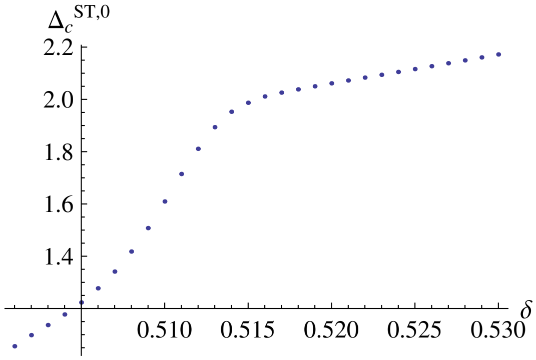

FIG. 4 (color online). The plot for $\Delta_c^{ST,0}(\delta)$. The vertical precision is 4×10^{-4}.

bound at the point where $\Delta_c^{TS,0}(\delta)$ starts to behave linearly, that is, at $\delta = 0.517(1)$. Reading off the spectra at $\delta = 0.517$, we obtain the conformal dimension of the SS scalar as 1.81(3). The large-n analysis at the chiral fixed point predicts that δ is 0.5175 and the conformal dimension of the first SS scalar is 1.796, which is once again close to our present estimate.

On the other hand, the TT and AA spin-0 sectors indicate a weak kink-like behavior near the Heisenberg fixed point with similar spectra, but the AT spin-1 sector shows no interesting behavior at all. It is not clear to us why such a preference among different sectors exists.

Now that we have demonstrated that antichiral fixed points show up in the TA spin-1 operator bound for $n = 15$, we want to apply the same method to determine the edge of the conformal window under which the antichiral fixed point disappears. To do this we computed TA bounds for $n = 20, 8, 7, 6, 5$. We present the results in Fig. 6. For convenience we complement the plot of its first derivatives (generated with the "Interpolation" function of MATHEMATICA) in Fig. 7. As we decrease n, the change of the slope decreases and disappears at $n = 6 \sim 7$. Thus we predict that the edge of the conformal window for antichiral fixed points is $7 \sim 8$. In comparison, we quote that the predicted value from the large-n analysis in Ref. [19] was 7.3. See also Table I of Ref. [20] for a summary of the predictions from the other approaches. We observed that there is a wider discrepancy between the location of the kink and the large-n prediction of δ for antichiral fixed points toward smaller n around 8. It is interesting to see if the analysis with a higher-dimensional search space would resolve this gap.

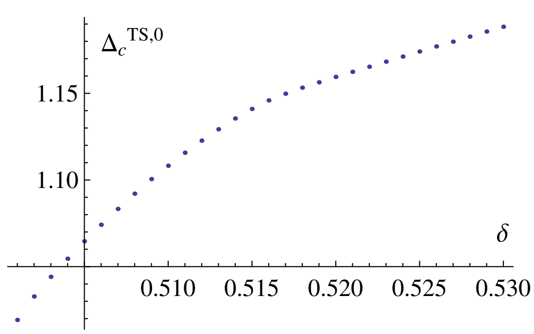

FIG. 5 (color online). The plot for $\Delta_c^{TS,0}(\delta)$. The vertical precision is 2×10^{-4}.

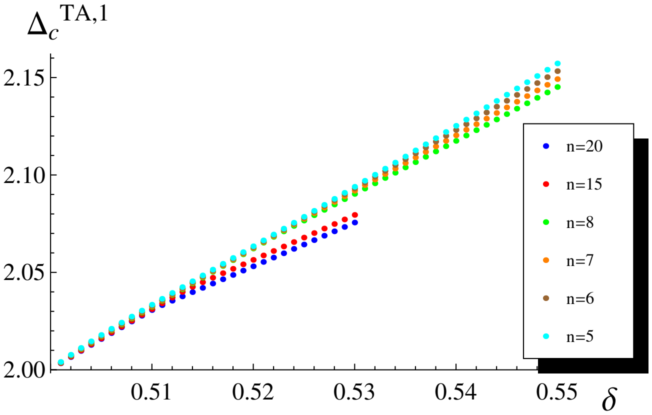

FIG. 6 (color online). The plot for $\Delta_c^{TA,1}(\delta)$ with $n = 20, 8, 7, 6, 5$. The vertical precision with $n = 20, 15, 8$ is 10^{-4}, while that with $n = 7, 6, 5$ is 2×10^{-5}.

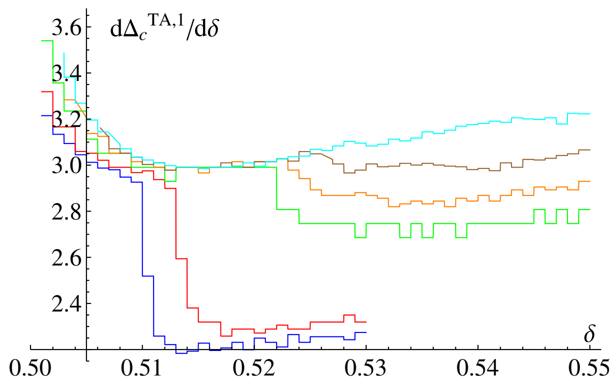

FIG. 7 (color online). The slopes of Fig. 6. The legend is same as in Fig. 6.

IV. CONCLUDING REMARKS

In this paper we have accomplished the conformal bootstrap program for $O(n) \times O(3)$-symmetric CFTs with various values of n. In particular, the detailed study for $n = 15$ has revealed the existence of singular behaviors in numerical bounds corresponding to Heisenberg as well as chiral and antichiral fixed points of the RG flow, depending on the sector for which we computed the bound. Moreover, we have predicted the edge of the conformal window for antichiral fixed points. We emphasize that not only can our results serve as a novel tool for the study of critical phenomena, but they are also encouraging for the conformal bootstrap program itself because our result makes it evident that *different* kinds of CFTs are hidden behind a *single* (vectorial) bootstrap equation.

There are several future directions to pursue. The obvious one is to study other symmetry groups. Among them, $O(n) \times O(2)$ will be particularly important in the context of condensed matter physics. For QCD applications in mind, a similar analysis for $U(n) \times U(m)$ groups is under way [29]. We could also refine our results by extending the search space to include a larger number of derivatives. This might fill the observed gap between our analysis and the large-n one. Our prediction of the conformal window is based on the antichiral fixed point. If the RG picture shows that the disappearance of the antichiral fixed point is induced by the annihilation with the chiral fixed point, the same conformal window should have been obtained from the chiral fixed point seen in the TS spin-0 sector by changing n. This may not be the case, as was proposed in Ref. [21] for $m = 2$ cases, and it should be interesting to see it directly in our approach. Since the signal is weaker, however, it may require the use of wider search spaces of the conformal bootstrap program, and more CPU power is needed.

Finally, it has been recently conjectured that the three-dimensional Ising model can be characterized as the CFT which minimizes the central charge in the entire space of CFTs [16]. In analogy, we speculate on the possibility of characterizing the CFTs at chiral/antichiral fixed points as those extremizing some quantities (e.g., current central charges).

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