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# Finite $N$ quiver gauge theory

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At finite $N$, the number of restricted Schur polynomials is greater than or equal to the number of generalized restricted Schur polynomials. In this note, we study this discrepancy and explain its origin. We conclude that for quiver gauge theories, in general, the generalized restricted Shur polynomials correctly account for the complete set of finite $N$ constraints, and they provide a basis, while the restricted Schur polynomials only account for a subset of the finite $N$ constraints and are thus overcomplete. We identify several situations in which the restricted Schur polynomials do in fact account for the complete set of finite $N$ constraints. In these situations, the restricted Schur polynomials and the generalized restricted Schur polynomials both provide good bases for the quiver gauge theory. Finally, we demonstrate situations in which the generalized restricted Schur polynomials reduce to the restricted Schur polynomials.

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## I. SUMMARY AND CONCLUSIONS

Our focus in this article is on free gauge theories whose structure is elegantly summarized in a quiver. By a quiver we mean a set of nodes (or vertices) connected by directed arrows; that is, a quiver is a directed graph. The gauge group of the quiver gauge theory is a direct product of groups, one associated with each node of the quiver so that there is a gauge field associated with each node of the quiver. We are interested in the case that each node corresponds to a unitary group $U(N_a)$. Although our arguments carry over to a general quiver gauge theory, we mostly focus on quivers with two nodes, which correspond to studying a $U(N_1)\times U(N_2)$ gauge group. For each directed arrow, there is a bifundamental scalar. An arrow stretching from node $a$ to node $b$ gives a field that transforms in the fundamental representation of $U(N_a)$ in the antifundamental of $U(N_b)$ and is a singlet of $U(N_c)$, $c\neq a,b$.

Our primary interest is in the finite $N$ physics of these theories. A natural basis for the local gauge-invariant operators of the theory is provided by taking traces of products of fields. At finite $N$, not all trace structures are independent. As a simple example, consider a scalar field $Z$ that is an $N\times N$ matrix transforming in the adjoint representation of $U(N)$. A complete set of operators built using three fields is given by $\{\mathrm{Tr}(Z^3),\mathrm{Tr}(Z^2)\mathrm{Tr}(Z),\mathrm{Tr}(Z)^3\}$, when $N>2$. For $N=2$, this set is overcomplete because we have the identity

$$\mathrm{Tr}(Z^3)=\frac{1}{2}[3\mathrm{Tr}(Z^2)\mathrm{Tr}(Z)-\mathrm{Tr}(Z)^3]. \qquad (1.1)$$

It is a highly nontrivial problem to write a basis of local operators that is not overcomplete at finite $N$. This problem has been solved for multimatrix models with $U(N)$ gauge group in [1–9] and for single matrix models with $SO(N)$ or $Sp(N)$ gauge groups in [10–12]. The result of these studies is a basis of local operators that also diagonalizes the free field two-point function. These bases have been useful for exploring giant gravitons [13–23] and new background geometries [24–35] in AdS/CFT [36], as well as for the computations of anomalous dimensions in large $N$ but nonplanar limits [37–43]. Elements in the basis are labeled by Young diagrams. The finite $N$ relations are encoded in the statement that operators labeled by Young diagrams with more than $N$ rows vanish. To illustrate this point, note that a basis for operators built using a single field are the Schur polynomials. For $N=2$, the constraint (1.1) is the statement

$$\chi_{(1,1,1)}(Z)=\frac{1}{6}\big(\mathrm{Tr}(Z)^3-3\mathrm{Tr}(Z^2)\mathrm{Tr}(Z)+2\mathrm{Tr}(Z^3)\big)=0 \qquad (1.2)$$

For quiver gauge theories, there are two distinct approaches that have been developed to study the finite $N$ physics [44,45].[1] In the remainder of this introduction, we review these two approaches with the goal of exhibiting a tension between them. The primary goal of this article is to clarify the origin of this tension and to explain how it is resolved.

For concreteness, consider a quiver gauge theory with gauge group $U(N_1)\times U(N_2)$ and assume that $N_1>N_2$. We use Roman indices for the $U(N_1)$ gauge group and Greek indices for the $U(N_2)$ gauge group. Consider the problem of building gauge-invariant operators using the bifundamentals $(A^I)^a_\alpha$ and $(B^{I\dagger})^\alpha_a$, where $I=1,2$. It is clear that any gauge-invariant operator must be a product of

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[1]For earlier work, focusing on essentially single matrix dynamics, see [46–49].

 

traces of an alternating product of $A$s and $B^\dagger$s. This motivates the products

$$\phi^{IJ}{}^a_b = (A^I)^a_\alpha (B^{J\dagger})^\alpha_b, \quad (1.3)$$

which transform in the adjoint of $U(N_1)$. Any gauge-invariant single trace operator is the trace of a unique (up to cyclic permutations) product of $\phi^{IJ}$ fields. Thus, we can use the restricted Schur polynomials [4] to build a basis for the local operators of the quiver [44]. The Young diagrams labeling these operators are cut off to have no more than $N_1$ rows. If we had instead chosen to work with the fields

$$\psi^{JI}{}^\alpha_\beta = (B^{J\dagger})^\alpha_a (A^I)^a_\beta, \quad (1.4)$$

we would have constructed restricted Schur polynomials that have Young diagram labels cut off to have no more than $N_2$ rows. These cutoffs are different, and they do not give the same number of gauge-invariant operators; thus, there is a puzzle. To see how this is resolved, restrict attention to a single field $\phi^{11}$ in which case our operators are the Schur polynomials $\chi_R(\phi^{11})$. For $R \vdash d$, we obtain a Schur polynomial of degree $d$. Recall that the degree $d$ Schur polynomials in $N$ variables are a linear basis for the space of homogeneous degree $d$ symmetric polynomials in $N$ variables [50]. Thus, these Schur polynomials are functions of the $N_1$ eigenvalues $\lambda_i$ of $\phi^{11}$. Concretely, we can write the Schur polynomial as a sum of monomials

$$\chi_R(\lambda_1, \lambda_2, \ldots, \lambda_N) = \sum_T \lambda^T = \sum_T \lambda_1^{t_1} \ldots \lambda_n^{t_n}, \quad (1.5)$$

where the summation is over all semistandard Young tableaux $T$ of shape $R$. The powers of the eigenvalues $t_i$ counts the number of times the number $i$ appears in $T$. We have not yet considered the eigenvalues of

$$\phi^{11} = A^1 (B^1)^\dagger. \quad (1.6)$$

$(B^1)^\dagger$ is an $N_2 \times N_1$ matrix, while $A^1$ is an $N_1 \times N_2$ matrix. These matrices are not square, so they do not admit an eigendecomposition. There is, however, the notion of a singular value decomposition (SVD), which can be applied [51]. The SVD decomposition of $(B^1)^\dagger$ is

$$(B^1)^\dagger = U_B \Sigma_B V_B^\dagger, \quad (1.7)$$

where $U_B$ is an $N_2 \times N_2$ unitary matrix, $V_B^\dagger$ is an $N_1 \times N_1$ unitary matrix, and $\Sigma_B$ is an $N_2 \times N_1$ rectangular matrix with nonzero singular values on its diagonal. Since $(B^1)\dagger$ has (at most) $N_2$ nonzero singular values, the generic matrix $(B^1)^\dagger$ has a null space of dimension $N_1 - N_2$. [Nongeneric $(B^1)^\dagger$ can have an even larger null space.] Of course, $\phi^{11}$ and $(B^1)^\dagger$ share the same null space so that $\phi^{11}$ has at least $N_1 - N_2$ zero eigenvalues.

Recall that a semistandard Young tableau is column strict, that is, the entries weakly increase along each row and strictly increase down each column. This implies that if $R$ has more than $N_2$ rows every term in $\chi_R(\phi^{11})$ is a product of at least $N_2 + 1$ distinct eigenvalues. Since only $N_2$ of these can be nonzero, it follows that $\chi_R(\phi^{11})$ actually vanishes as soon as $R$ has more than $N_2$ rows. This proves that the Schur polynomials $\chi_R(\phi^{11})$ and $\chi_R(\psi^{11})$ are both cut off such that $R$ must have at most $N_2$ rows. A very simple generalization of this reasoning allows us to conclude that we can construct restricted Schur polynomials using either $\psi^{IJ}$ or $\phi^{IJ}$. The finite $N$ constraints are encoded in the statements that operators labeled by Young diagrams with more than[2] $\min(N_1, N_2)$ rows vanish. This implies in particular that the number of gauge-invariant operators that can be constructed will depend only on the smallest of $N_1$ and $N_2$. We call this the restricted Schur basis.

A second approach to the finite $N$ physics entails working with the field $A^I$ and $(B^I)^\dagger$ directly [45]. In this case, we organize the $U(N_1)$ indices using Young diagrams that have no more than $N_1$ rows, and we organize the $U(N_2)$ indices using Young diagrams that have no more than $N_2$ rows. Thus, each operator is labeled by two types of Young diagrams that have distinct cutoffs. In this case, both $N_1$ and $N_2$ enter. This dependence is genuine, and one finds, for example, that the number of operators that can be constructed depend on both $N_1$ and $N_2$. This is the generalized restricted Schur basis [45].

At infinite $N$, the counting of restricted Schur polynomials and generalized restricted Schur polynomials agree. At finite $N$, there are more restricted Schur polynomials than there are generalized restricted Schur polynomials. This means that either the restricted Schur polynomials are overcomplete or the generalized restricted Schur polynomials are undercomplete. We show in what follows that the restricted Schur polynomials are overcomplete, for a subtle reason that is peculiar to quiver gauge theories, as we now explain. Given a collection of fields $\{A^I, (B^J)^\dagger\}$, we can form the fields $\phi^{IJ}$. The number $n_{IJ}$ of each type of field is not unique, and it depends on the details of how we pair the $A^I$s and the $(B^J)^\dagger$s. To get the complete set of restricted Schur polynomials, we need to consider each possible pairing with its collections of fields described by the numbers $\{n_{IJ}\}$. For a given pairing $\{n_{IJ}\}$, the restricted Schur polynomials do give the correct finite $N$ constraints. There are, however, extra genuinely new conditions that can be written that involve fields that come from different pairings, pairing $\{n_{IJ}\}$ and pairing $\{n_{IJ}{}'\}$ say. The restricted Schur polynomials do not respect these additional constraints and are thus overcomplete. The generalized restricted Schur basis correctly accounts for the complete set of finite $N$ trace relations. This is an important general lesson: at finite $N$, the physics of quiver

[2] $\min(N_1, N_2)$ is equal to the smallest of $N_1$ or $N_2$.

gauge theories is not correctly captured by contracting fields to construct adjoints of specific gauge groups and then building operators from these adjoints. The adjoints retain knowledge that they are constructed from more basic bifundamental fields in the form of extra finite $N$ relations. To correctly account for the complete set of finite $N$ relations, it seems easiest to work directly with the original bifundamental fields and hence the generalized restricted Schur polynomial basis.

There are exceptions to this general lesson: in certain subsectors of the theory and in specific limits, some of which we identify below, the restricted Schur polynomials do provide a complete basis and do account for all finite $N$ relations. In these cases, it may be simpler to use the restricted Schur polynomials rather than the generalized restricted Schur polynomials.

In Sec. II, we outline in detail, using a specific example, the origin and form of the new constraints. There are situations in which the restricted Schur polynomials do capture the complete set of finite $N$ constraints and are consequently not overcomplete. In these situations, one may use either basis, as dictated by the problem being considered. In Sec. III, we identify and describe these situations. Sec. IV considers the computation of some simple correlators that provide further useful and independent insight into the finite $N$ physics. Finally, in Sec. V, we compare the structure of the restricted Schur polynomials and the generalized restricted Schur polynomials, with the goal of explaining why it may be simpler to use the restricted Schur polynomials rather than the generalized restricted Schur polynomials for certain computations. Section V also demonstrates situations in which the generalized restricted Schur polynomials reduce to the restricted Schur polynomials.

In what follows, we talk of a Young diagram $r$ that has $m$ boxes or of a Young diagram $r$ that is a partition of $m$ or even more simply, $r\vdash m$.

## II. NEW FINITE $N$ RELATIONS

The number of generalized restricted Schur polynomials $\mathcal{N}_g(n_1,n_2,m_1,m_2)$ that can be built in a theory with gauge group $U(N_1)\times U(N_2)$, using $n_1$ copies of the field $A^1$, $n_2$ copies of $A^2$, $m_1$ copies of $(B^1)^\dagger$, and $m_2$ copies of $(B^2)^\dagger$ is given by [$l(R)$ is the length of the first column in $R$ and $l(S)$ is the length of the first column in $S$] [45]

$$\sum_{\substack{R,S\vdash n_1+n_2\\ l(R)\le N_1\, l(S)\le N_2}}\ \sum_{\substack{r_1\vdash n_1\\ r_2\vdash n_2}}\ \sum_{\substack{s_1\vdash m_1\\ s_2\vdash m_2}} g(r_1,r_2,R)g(r_1,r_2,S)g(s_1,s_2,R)g(s_1,s_2,S), \tag{2.1}$$

where we have $n_1+n_2=m_1+m_2$ and where $g(\cdot,\cdot,\cdot)$ is a Littlewood-Richardson coefficient. The finite $N$ relations are accounted for by restricting the above sum so that $R$ has no more than $N_1$ rows and $S$ has no more than $N_2$ rows.

Consider now the counting for the restricted Schur polynomial. The first step in the construction of the resticted Schur polynomials entails pairing the $A$s and $B^\dagger$s to produce $n_{IJ}$ copies of $\phi^{IJ}$. There is one Young diagram for each of these $\phi^{IJ}$ fields. The number of restricted Schur polynomials is now given by [$N_-\equiv\min(N_1,N_2)$]

$$\mathcal{N}_r(n_1,n_2,m_1,m_2)=\sum_{\{n_{IJ}\}}\mathcal{N}_{\{n^{IJ}\}}, \tag{2.2}$$

where the above sum is a sum over all possible distinct ways of pairing; that is, it is a sum over all possible distinct sets $\{n_{IJ}\}$ and [52]

$$\mathcal{N}_{\{n^{IJ}\}}=\sum_{\substack{R\vdash n_1+n_2\\ l(R)\le N_-}}\ \sum_{r_{IJ}\vdash n_{IJ}}(g(r_{11},r_{12},r_{21},r_{22};R))^2, \tag{2.3}$$

In general, (2.1) and (2.2) do not agree. The goal of this section is to explain the origin of the discrepancy.[3]

To make the discussion concrete, we focus on a specific example. Consider $n_1=3$, $n_2=1$, $m_1=m_2=2$, and take $N_1,N_2>4$ so that there are no finite $N$ constraints. In this case, a simple application of (2.1) gives $\mathcal{N}_g(3,1,2,2)=28$ generalized restricted Schur polynomials. For the number of restricted Schur polynomials, we need to consider two cases

$$\begin{aligned}&\text{Case I}: n_{11}=2\quad n_{12}=1\quad n_{21}=0\quad n_{22}=1\\ &\text{Case II}: n_{11}=1\quad n_{12}=2\quad n_{21}=1\quad n_{22}=0.\end{aligned} \tag{2.4}$$

For these cases, (2.2) gives $\mathcal{N}_I=14$, $\mathcal{N}_{II}=14$ so that in total $\mathcal{N}_r(3,1,2,2)=28$. In the next section, we prove that the number of restricted Schur polynomials and generalized restricted Schur polynomials always agree in the absence of finite $N$ constraints.

[3]The Littlewood-Richardson number has three indices $g(r,s,t)$. The number $g(r,s,t)$ gives the number of times irrep $t$ of $\mathrm{GL}_N$ appears in the tensor product of $\mathrm{GL}_N$ representations $r$ and $s$. By $g(r_1,r_2,...,r_n;R)$ we mean the number of times $R$ appears in the tensor product of $r_1$ with $r_2$ with $r_3$ with … with $r_n$. We could write this as $\sum_{s_i}g(r_1,r_2,s_1)g(s_1,r_3,s_2)\dots g(s_{n-1},r_n,R)$.

We will see that it is $\mathcal{N}_r(3,1,2,2)$ that does not correctly count the number of gauge-invariant operators at finite $N$. Since this is one of the main points of our discussion, we give the complete details on how Eq. (2.2) is applied. Toward this end, we have summarized the labels for the relevant restricted Schur polynomials in Appendix A. Consider next the case that $N_1 = N_2 = 2$. A simple application of (2.1) gives $\mathcal{N}_g(3,1,2,2) = 13$ generalized restricted Schur polynomials. Next, consider the complete set of possible restricted Schur polynomial labels given in Appendix A. For Case I, the operators given in (A1), (A2), and (A3) vanish so that we have eight operators. For Case II, the operators given in (A9), (A10), and (A11) vanish so that we have eight operators. This gives a total of $\mathcal{N}_r(3,1,2,2) = 16$ restricted Schur polynomials, which shows a clear discrepancy between (2.1) and (2.2).

To explore the origin of this discrepancy, we have developed a numerical algorithm to determine the number and precise form of the finite $N$ constraints. Consider first the case of a single $N \times N$ matrix $Z$. For $N = 2$, we know one of the finite $N$ constraints is given by (1.1). If we choose a random $2 \times 2$ matrix $Z$ and form the vector

$$\vec{v} = \begin{bmatrix} \mathrm{Tr}(Z^3) \\ \mathrm{Tr}(Z^2)\mathrm{Tr}(Z) \\ \mathrm{Tr}(Z)^3 \end{bmatrix}, \tag{2.5}$$

it will point in a random direction depending on the specific matrix $Z$. However, we know that it must lie in a two-dimensional subspace of the three-dimensional space it belongs to because thanks to (1.1) we know that

$$\vec{v} \cdot \vec{u} = 0 \quad \vec{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}. \tag{2.6}$$

Now imagine preparing an ensemble of random matrices $Z^{(i)}$, $i = 1, \ldots, k$. This ensemble of $Z^{(i)}$ can be used to construct an ensemble $\vec{v}^{(i)}$ using (2.5), and then we can form the matrix

$$M = \frac{1}{k} \sum_{i=1}^{k} v^{(i)T} v^{(i)}. \tag{2.7}$$

Since the $\vec{v}^{(i)}$ are all orthogonal to $\vec{u}$ but otherwise explore the orthogonal two-dimensional subspace, we know that $M$ will have a single null vector, which is $\vec{u}$ itself.

The logic clearly generalizes to multimatrix models. We collect the complete set of multitrace structures into a vector $\vec{v}$. By preparing an ensemble of random matrices, we can prepare an ensemble of random vectors $\vec{v}^{(i)}$ and construct the matrix $M$ as in (2.7). Each null vector of $M$ then corresponds to a finite $N$ constraint. In this way, the finite $N$ constraints are recovered from the null vectors of $M$.

For Case I described above, we find that a total of 14 multitrace structures is possible. Setting $N_1 = N_2 = 2$, we find that $M$ has a total of six null vectors. Thus, there are six finite $N$ constraints leaving eight independent multitrace operators, in perfect agreement with the number of restricted Schur polynomials. For Case II, we again find a total of 14 multitrace structures are possible, and again, for $N_1 = N_2 = 2$, we find that $M$ has six null vectors. Thus, there are six finite $N$ constraints leaving eight independent multitrace operators, again in perfect agreement with the number of restricted Schur polynomials. If we now form the complete set of gauge-invariant operators that we can construct using $n_1 = 3$, $n_2 = 1$, and $m_1 = m_2 = 2$, we find that a total of 28 multitrace structures are possible, given by the operators of Case I and Case II above. In this case, $M$ has a total of 15 null vectors, leaving a total of 13 independent multitrace operators, in perfect agreement with the number of generalized restricted Schur polynomials. At this point, the origin of the discrepancy is clear. The construction of restricted Schur polynomials starts by breaking the complete space of gauge-invariant operators up into two sets, Case I and Case II above. By searching for the finite $N$ constraints within the operators of Case I and Case II separately, we have discovered 12 constraints. This is three short of the complete set of 15 constraints discovered when searching in the complete set of gauge-invariant operators. Clearly, there are some finite $N$ constraints that mix operators from Case I and operators from Case II, and these constraints are not captured in the restricted Schur construction of [44].

To summarize the conclusion of our discussion, the generalized restricted Shur polynomials correctly account for the complete set of finite $N$ constraints, and they provide a basis, while the restricted Schur polynomials only account for a subset of the finite $N$ constraints and are thus overcomplete.

## III. SITUATIONS WITHOUT NEW FINITE $N$ RELATIONS

As our discussion in the introduction suggests, in the absence of finite $N$ constraints, we expect that both the generalized restricted Schur polynomials and the restricted Schur polynomials provide good bases. This implies, in particular, that in the absence of finite $N$ constraints the number of restricted Schur polynomials is equal to the number of generalized restricted Schur polynomials. This is indeed the case as we now explain. For concreteness, we again consider a $U(N_1) \times U(N_2)$ model, building our operators from the fields $(A^I)^a_\alpha$ and $(B^{I\dagger})^\alpha_a$, where $I = 1, 2$. Thus, we can form four adjoint fields $\phi^{IJ}$, and our restricted Schur polynomials are labeled by five Young diagrams, one Young diagram $r_{IJ}$ for each field $\phi^{IJ}$ and one which organizes the complete set of fields. According to [9,52], the number of restricted Schur polynomials at $N = \infty$ is given by expanding

$$Z_r(t_{11},t_{12},t_{21},t_{22}) = \sum_{n_1,n_2,m_1,m_2}\sum_{a,b,c,d}\delta_{a+b,n_1}\delta_{c+d,n_2}\delta_{a+c,m_1}\delta_{b+d,m_2}\mathcal{N}_r(n_1,n_2,m_1,m_2)t_{11}^a t_{12}^b t_{21}^c t_{22}^d$$
$$= \prod_{k=1}^{\infty}\frac{1}{1-t_{11}^k - t_{12}^k - t_{21}^k - t_{22}^k}. \tag{3.1}$$

The coefficient of $t_{11}^{n_{11}}t_{12}^{n_{12}}t_{21}^{n_{21}}t_{22}^{n_{22}}$ tells us the number of restricted Schur polynomials that can be built using $n_{11}$ $\phi^{11}$ fields, $n_{12}$ $\phi^{12}$ fields, $n_{21}$ $\phi^{21}$ fields, and $n_{22}$ $\phi^{22}$ fields. The number of generalized restricted Schur polynomials at $N=\infty$ is given by expanding [45]

$$Z_g(t_{a_1},t_{a_2},t_{b_1},t_{b_2}) = \sum_{n_1,n_2,m_1,m_2}\mathcal{N}_g(n_1,n_2,m_1,m_2)t_{a_1}^{n_1}t_{a_2}^{n_2}t_{b_1}^{m_1}t_{b_2}^{m_2}$$
$$= \prod_{k=1}^{\infty}\frac{1}{1-(t_{a_1}t_{b_1})^k-(t_{a_1}t_{b_2})^k-(t_{a_2}t_{b_1})^k-(t_{a_2}t_{b_2})^k}. \tag{3.2}$$

The coefficient of $t_{a_1}^{n_1}t_{a_2}^{n_2}t_{b_1}^{m_1}t_{b_2}^{m_2}$ tells us how many generalized restricted Schur polynomials can be built using $n_1$ $A_1$ fields, $n_2$ $A_2$ fields, $m_1$ $B_1^\dagger$ fields, and $m_2$ $B_2^\dagger$ fields. We can clearly transform (3.1) into (3.2) by setting $t_{ij}=t_{a_i}t_{b_j}$, which proves that in the absence of finite $N$ constraints the number of restricted Schur polynomials is equal to the number of generalized restricted Schur polynomials. This change of variables provides important insight into how to relate the counting of restricted Schur polynomials and generalized restricted Schur polynomials, even when finite $N$ constraints play a role, as we will see.

### A. A single $n_{IJ}$ sector

Consider next the case that one of $n_1, n_2, m_1, m_2$ is equal to zero. In this case, there is only one possible value for the $n_{IJ}$ so that according to our discussion above the restricted Schur polynomials correctly account for all finite $N$ constraints, and we therefore expect that the number of restricted Schur polynomials matches the number of generalized restricted Schur polynomials. For concreteness, consider the case that $n_1=0$. In this case, the Young diagram appearing in (2.1) is the Young diagram with no boxes, which we denote as $\cdot$. Consequently,

$$g(r_1,r_2,R)=g(\cdot,r_2,R)=\delta_{r_2,R}$$
$$g(r_1,r_2,S)=g(\cdot,r_2,S)=\delta_{r_2,S}$$

so that the number of generalized restricted Schur polynomials (2.1) becomes

$$\sum_{R,S\vdash n_2 l(R)\le N_1 l(S)\le N_2}\sum_{r_2\vdash n_2}\sum_{s_1\vdash m_1 s_2\vdash m_2}\delta_{r_2,R}\delta_{r_2,S}g(s_1,s_2,R)g(s_1,s_2,S) = \sum_{R\vdash n_1 l(R)\le N_-}\sum_{s_1\vdash m_1}\sum_{s_2\vdash m_2}g(s_1,s_2,R)g(s_1,s_2,R). \tag{3.3}$$

To count the number of restricted Schur polynomials, note that now $r_{11}=\cdot$, $r_{12}=\cdot$, $n_{21}=m_1$, and $n_{22}=m_2$ so that (2.2) becomes

$$\sum_{R\vdash n_2 l(R)\le N_-}\sum_{r_{21}\vdash m_1}\sum_{r_{22}\vdash m_2}(g(r_{21},r_{22};R))^2. \tag{3.4}$$

This demonstrates an exact match between the number of restricted Schur polynomials and the number of generalized restricted Schur polynomials as we predicted. We recover this result by showing that in this case the generalized restricted Schur polynomials reduce to the restricted Schur polynomials in Sec. V.

### B. One finite rank

Finally, consider the case that one of the ranks of the two gauge groups goes to infinity. For concreteness, we take $N_2\to\infty$. The counting of restricted Schur polynomials is

$$Z_r(t_{11},t_{12},t_{21},t_{22}) = \sum_{r_{11},r_{12},r_{21},r_{22},R,l(R)\le N_1}(g(r_{11},r_{12},r_{21},r_{22};R))^2 t_{11}^{|r_{11}|}t_{12}^{|r_{12}|}t_{21}^{|r_{21}|}t_{22}^{|r_{22}|}. \tag{3.5}$$

A simple change of variables gives

$$Z_r = \sum_{r_{11},r_{12},r_{21},r_{22},R,l(R)\le N_1} (g(r_{11},r_{12},r_{21},r_{22};R))^2 (t_{a_1}t_{b_1})^{|r_{11}|}(t_{a_1}t_{b_2})^{|r_{12}|}(t_{a_2}t_{b_1})^{|r_{21}|}(t_{a_2}r_{b_2})^{|r_{22}|}.$$

Employing the identities

$$\begin{aligned} g(r_{11},r_{12},r_{21},r_{22};R) &= \sum_{r\vdash n_1}\sum_{s\vdash n_2} g(r_{11},r_{12},r)g(r_{21},r_{22},s)g(r,s,R) \\ &= \sum_{t\vdash m_1}\sum_{u\vdash m_2} g(r_{11},r_{21},t)g(r_{12},r_{22},u)g(t,u,R), \end{aligned} \tag{3.6}$$

we find

$$\begin{aligned} Z_r = &\sum_{r,s,t,u}\sum_{R,l(R)\le N_1} g(r,s,R)g(t,u,R)t_{a_1}^{n_1}t_{a_2}^{n_2}t_{b_1}^{m_1}t_{b_2}^{m_2} \\ &\times \sum_{r_{11},r_{12},r_{21},r_{22}} g(r_{11},r_{12},r)g(r_{21},r_{22},s)g(r_{11},r_{21},t)g(r_{12},r_{22},u). \end{aligned} \tag{3.7}$$

We have used $n_1 = |r_{11}| + |r_{12}|$, $n_2 = |r_{21}| + |r_{22}|$, $m_1 = |r_{11}| + |r_{21}|$, and $m_2 = |r_{12}| + |r_{22}|$ in writing this expression. We now compute the sum

$$S = \sum_{r_{11},r_{12},r_{21},r_{22}} g(r_{11},r_{12},r)g(r_{21},r_{22},s)g(r_{11},r_{21},t)g(r_{12},r_{22},u). \tag{3.8}$$

In the sum above, the number of rows in the $r_{IJ}$ is not restricted. Indeed, to capture the finite $N$ constraints, it is enough to cut the number of rows of $R$ off as we have done in (3.7). Making use of the identity ($r\vdash n$, $s\vdash m$, $t\vdash n+m$)

$$g(r,s,t) = \frac{1}{n!m!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_m}\chi_r(\sigma_1)\chi_s(\sigma_2)\chi_t(\sigma_1\circ\sigma_2) \tag{3.9}$$

and the formula

$$\sum_{R\vdash n}\chi_R(\sigma)\chi_R(\tau) = \sum_{\gamma\in S_n}\delta(\gamma\sigma\gamma^{-1}\tau^{-1}), \tag{3.10}$$

we can write $S$ as

$$\begin{aligned} S &= \sum_{n_{i1}+n_{i2}=n_i}\sum_{n_{1i}+n_{2i}=m_i}\sum_{\psi_1\in S_{n_{11}}}\sum_{\psi_2\in S_{n_{21}}}\sum_{\tau_1\in S_{n_{12}}}\sum_{\tau_2\in S_{n_{22}}}\frac{1}{n_{11}!n_{12}!n_{21}!n_{22}!}\chi_r(\psi_1\circ\tau_1)\chi_s(\psi_2\circ\tau_2)\chi_t(\psi_1\circ\psi_2)\chi_u(\tau_1\circ\tau_2) \\ &= \sum_{\sigma_1\in S_{n_1}}\sum_{\sigma_2\in S_{n_2}}\sum_{\rho_1\in S_{m_1}}\sum_{\rho_2\in S_{m_2}}\sum_{\gamma\in S_{n_1+n_2}}\frac{1}{n_1!n_2!m_1!m_2!}\delta(\sigma_1\circ\sigma_2(\rho_1\circ\rho_2)^{-1})\chi_r(\sigma_1)\chi_s(\sigma_2)\chi_t(\rho_1)\chi_u(\rho_2) \\ &= \sum_{\sigma_1\in S_{n_1}}\sum_{\sigma_2\in S_{n_2}}\sum_{\rho_1\in S_{m_1}}\sum_{\rho_2\in S_{m_2}}\sum_{S\vdash n_1+n_2}\frac{1}{n_1!n_2!m_1!m_2!}\chi_S(\sigma_1\circ\sigma_2)\chi_S(\rho_1\circ\rho_2)\chi_r(\sigma_1)\chi_s(\sigma_2)\chi_t(\rho_1)\chi_u(\rho_2) \\ &= \sum_{S\vdash n_1+n_2} g(r,s,S)g(t,u,S). \end{aligned} \tag{3.11}$$

Plugging this back into (3.7), we find

$$\begin{aligned} Z_r &= \sum_{r,s,t,u}\sum_{R,l(R)\le N_1 S} g(r,s,S)g(t,u,S)g(r,s,R)g(t,u,R)t_{a_1}^{n_1}t_{a_2}^{n_2}t_{b_1}^{m_1}t_{b_2}^{m_2} \\ &= Z_g, \end{aligned} \tag{3.12}$$

proving the equality. See Appendix B for a nontrivial example demonstrating this equality.

## IV. CORRELATORS

In this section, we compute correlation functions of restricted Schur polynomials. There are two things this will teach us. First, we can confirm that the correct cutoff on the number of rows of our Young diagram labels is the smallest of $N_1$ and $N_2$. Second, we want to point out that operators from different $n_{IJ}$ sectors are not orthogonal, which corrects a statement in [44].

The operators we study were given in [44]

$$O_{R,\{r\}\alpha\beta} = \frac{1}{\prod_{IJ} n_{IJ}!} \sum_{\sigma\in S_{n_1+n_2}} \mathrm{Tr}_{\{r\}\alpha\beta}(\Gamma_R(\sigma))\mathrm{Tr}(\sigma(\phi^{11})^{\otimes n_{11}}(\phi^{12})^{\otimes n_{12}}(\phi^{21})^{\otimes n_{21}}(\phi^{22})^{\otimes n_{22}}). \tag{4.1}$$

The irrep $R$ will in general be a reducible representation of the $S_{n_{11}}\times S_{n_{12}}\times S_{n_{21}}\times S_{n_{22}}$ subgroup of $S_{n_1+n_2}$. One of the $S_{n_{11}}\times S_{n_{12}}\times S_{n_{21}}\times S_{n_{22}}$ irreps that $R$ subduces is $\{r\}$. $\{r\}$ may be subduced more than once from $R$. $\alpha$ and $\beta$ label these copies. In the above formula, $\mathrm{Tr}_{\{r\}}$ is an instruction to trace only over the $\{r\}$ subspace of the carrier space of $R$. More precisely, we trace the row label over the $\alpha$ copy of $\{r\}$ and the column label over the $\beta$ copy of $\{r\}$. For simplicity, we set $n_2 = 0$. The two-point function

$$\langle O_{R,\{r\}\alpha\beta} O^{\dagger}_{S,\{s\}\gamma\delta}\rangle = \delta_{RS}\delta_{\{r\},\{s\}}\delta_{\alpha\gamma}\delta_{\beta\delta}\frac{\mathrm{hooks}_R f_R(N_1) f_R(N_2)}{\mathrm{hooks}_{r_{11}}\mathrm{hooks}_{r_{12}}} \tag{4.2}$$

follows immediately after using the results of [44]. When the right-hand side of this correlator vanishes, the operator itself vanishes. Thus, by determining where the right-hand side of this correlation function vanishes, we learn how the rows of the Young diagram labels should be restricted to obtain nonzero operators. Toward this end, recall that $f_R(N)$ is a product of the factors of the Young diagram, one for each box, where the box in row $i$ and column $j$ has factor $N - i + j$. Consequently, $f_R(N)$ vanishes whenever $R$ has more than $N$ rows. Studying (4.2), we see that $R$ can have no more than $N_-$ rows where $N_-$ is the smallest of $N_1$ and $N_2$. This is precisely the conclusion we reached in Sec. I. By studying two-point functions, one can in general conclude that for gauge group $U(N_1)\times U(N_2)\times\cdots\times U(N_p)$, all Young diagram labels must have no more than $N_-$ rows, where $N_-$ is the smallest of $N_1, N_2, \ldots, N_p$ [53].

To consider the case of general $n_1$, $n_2$, $m_1$, $m_2$, it proves convenient to use the operators

$$\begin{aligned} O_{R,\{r\}\alpha\beta} &= \mathrm{Tr}(P_{R,\{r\}\alpha\beta} A^{\otimes n}\tau B^{\dagger\otimes n}) \\ &= \frac{1}{n_{11}!n_{22}!n_{12}!n_{21}!}\sum_{\sigma\in S_n}\mathrm{Tr}_{\{r\}}(\Gamma_R(\sigma))\prod_{i=1}^{n_1}(A_1)^{a_i}_{\alpha_i}\prod_{j=1+n_1}^{n}(A_2)^{a_j}_{\alpha_j}(\tau)^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} \\ &\quad\times\prod_{i=1}^{n_{11}}(B_1^{\dagger})^{\beta_i}_{a_{\sigma(i)}}\prod_{i=1+n_{11}}^{n_1}(B_2^{\dagger})^{\beta_i}_{a_{\sigma(i)}}\prod_{i=1+n_1}^{n_1+n_{21}}(B_1^{\dagger})^{\beta_i}_{a_{\sigma(i)}}\prod_{i=1+n_1+n_{21}}^{n}(B_2^{\dagger})^{\beta_i}_{a_{\sigma(i)}}, \end{aligned} \tag{4.3}$$

where $\tau$ is an element of the group algebra, constructed to obey

$$\mathrm{Tr}(\tau\rho^{-1}\tau\sigma^{-1}) = \delta(\rho^{-1}\sigma^{-1}). \tag{4.4}$$

The two-point function is [44]

$$\langle \mathcal{O}_{R,\{r\}\alpha\beta}\mathcal{O}^{\dagger}_{S,\{s\}\gamma\delta}\rangle = n_{11}!n_{12}!n_{21}!n_{22}!\mathrm{Tr}(P_{R,\{r\}\alpha\beta}P_{S,\{s\}\gamma\delta}).$$

Thus, the two-point function in the subspace of operators with fixed $n_{IJ}$ is diagonal. However, even after fixing $n_I, m_J$, we can change the $n_{IJ}$. Projectors corresponding to different $n_{IJ}$ will not in general be orthogonal. The identity (4.4) also does not help. Operators from different $n_{IJ}$ sectors are not orthogonal, which is again an indication that the restricted Schur basis for quiver gauge theories is, in general, overcomplete. Note, however, that the operators constructed in [45] are a complete basis, and they do diagonalize the two-point function.

## V. POLYNOMIAL STRUCTURE

The key general lesson of this article is that at finite $N$, the physics of quiver gauge theories is not correctly captured by contracting fields to construct adjoints of specific gauge groups. The fact that the adjoints are constructed from more basic bifundamental fields is reflected in extra finite $N$ relations. To correctly account for all finite $N$ relations, it seems easiest to work directly with the original bifundamental fields and hence the generalized restricted Schur polynomial basis. In Sec. III, we have proved that there are exceptions to this general lesson: in certain subsectors and in specific limits, the restricted Schur polynomials correctly account for all finite $N$ relations and hence do provide a suitable basis. In these cases, it may be simpler to use the restricted Schur polynomials rather than the generalized restricted Schur polynomials, as we explain in this section. Finally, we show that when there is a single $n_{IJ}$ sector the generalized restricted Schur polynomials reduce to the restricted Schur polynomials constructed in [44].

The restricted Schur polynomial (4.1) can be written as

$$O_{R,\{r\}\alpha\beta} = \frac{1}{\prod_{IJ} n_{IJ}!} \times \sum_{\sigma\in S_{n_1+n_2}} \sum_{a} \langle R,\{s\},\alpha,a|\Gamma_R(\sigma)|R,\{s\},\beta,a\rangle \times \mathrm{Tr}(\sigma(\phi^{11})^{\otimes n_{11}}(\phi^{12})^{\otimes n_{12}}(\phi^{21})^{\otimes n_{21}}(\phi^{22})^{\otimes n_{22}}).$$

Above, we have explicitly written the restricted trace using the states $|R,\{s\},\gamma,a\rangle$. These states span a subspace of the carrier space of representation $R$ of $S_{n_1+n_2}$. The subspace carries a representation $\{s\}$ of the subgroup $S_{n_{11}}\times S_{n_{12}}\times S_{n_{21}}\times S_{n_{22}}$. Since $\{s\}$ will in general be subduced more than once, we need the multiplicity label $\gamma$. Finally, index $a$ indexes states in the basis that spans the subspace. The key technical challenge is then to develop a good enough working knowledge of the states $|R,r,\gamma,a\rangle$, that one can carry out computations using the restricted Schur polynomials. The group theoretic quantity

$$\sum_{a}\langle R,\{r\},\alpha,a|\Gamma_R(\sigma)|R,\{r\},\beta,a\rangle \tag{5.1}$$

is the "restricted character" introduced in [19].

Using the same notation, the generalized restricted Schur polynomials can be written as

$$O_{R,S;\{t\},\{r\};\alpha\beta\gamma\delta} = \frac{1}{\prod_{IJ} n_{IJ}!}\sum_{\sigma,\rho\in S_{n_1+n_2}}\sum_{a,b}\langle R,\{t\},\alpha,b|\Gamma_R(\sigma)|R,\{r\},\beta,a\rangle \times \langle S,\{r\},\gamma,a|\Gamma_S(\rho)|S,\{t\},\delta,b\rangle \mathrm{Tr}(\sigma A_1^{\otimes n_1}A_2^{\otimes n_2}\rho(B_1^\dagger)^{\otimes m_1}(B_2^\dagger)^{\otimes m_2}).$$

Notice that four collections of states have been introduced: $|R,\{t\},\alpha,b\rangle$, $|R,\{r\},\beta,a\rangle$, $|S,\{t\},\alpha,b\rangle$, and $|S,\{r\},\beta,a\rangle$. The label $\{r\}$ specifies an irrep of $S_{n_1}\times S_{n_2}$ and $\{t\}$ specifies an irrep of $S_{m_1}\times S_{m_2}$. The collections of states introduced provide a basis for the advertised carrier spaces, within the carrier space of $R$ and $S$, which are both irreps of $S_{n_1+n_2}$. Greek labels are multiplicity labels. $a$ labels states within the basis of $\{r\}$ and $b$ labels states within the basis of $\{t\}$. The group theoretic quantity

$$\sum_{a,b}\langle R,\{t\},\alpha,b|\Gamma_R(\sigma)|R,\{r\},\beta,a\rangle\langle S,\{r\},\gamma,a|\Gamma_S(\rho)|S,\{t\},\delta,b\rangle \tag{5.2}$$

is the "quiver character" introduced in [45].

From a group theory point of view, restricted characters seem to be simpler quantities than quiver characters. Efficient methods have been developed in [39] to work with restricted characters. It remains to be seen if these methods can be extended to quiver characters. This investigation is underway [54].

Finally, consider the situation for which (say) $m_2 = 0$ so that there is a single $n_{IJ}$ sector. In this case, we find the generalized restricted Schur polynomial reduces to the restricted Schur polynomial

$$\begin{aligned} O_{R,S;\{t\}\{S\};\alpha\delta} &= \frac{\delta_{RS}}{\prod_{IJ} n_{IJ}!}\sum_{\sigma,\rho\in S_{n_1+n_2}}\sum_{a,b}\langle S,\{t\},\alpha,b|\Gamma_S(\sigma)|S,\{S\},a\rangle\langle S,\{S\},a|\Gamma_S(\rho)|S,\{t\},\delta,b\rangle\mathrm{Tr}(\rho A_1^{\otimes n_1}A_2^{\otimes n_2}\sigma(B_1^\dagger)^{\otimes n_1+n_2}) \\ &= \frac{\delta_{RS}}{\prod_{IJ} n_{IJ}!}\sum_{\sigma,\rho\in S_{n_1+n_2}}\sum_{a,b}\langle S,\{t\},\alpha,b|\Gamma_S(\sigma\rho)|S,\{t\},\delta,b\rangle\mathrm{Tr}(\rho A_1^{\otimes n_1}A_2^{\otimes n_2}\sigma(\sigma^{-1}(B_1^\dagger)^{\otimes n_1+n_2}\sigma)) \\ &= \frac{\delta_{RS}}{\prod_{IJ} n_{IJ}!}\sum_{\sigma,\rho\in S_{n_1+n_2}}\sum_{a,b}\langle S,\{t\},\alpha,b|\Gamma_S(\sigma\rho)|S,\{t\},\delta,b\rangle\mathrm{Tr}(\sigma\rho A_1^{\otimes n_1}A_2^{\otimes n_2}(B_1^\dagger)^{\otimes n_1+n_2})) \\ &= \frac{\delta_{RS}(n_1+n_2)!}{\prod_{IJ} n_{IJ}!}\sum_{\sigma\in S_{n_1+n_2}}\sum_{a,b}\langle S,\{t\},\alpha,b|\Gamma_S(\sigma)|S,\{t\},\delta,b\rangle\mathrm{Tr}(\sigma(\phi^{11})^{\otimes n_1}(\phi^{22})^{\otimes n_2})) \\ &= \frac{\delta_{RS}(n_1+n_2)!}{\prod_{IJ} n_{IJ}!}O_{S,\{t\},\alpha\delta}. \end{aligned} \tag{5.3}$$

In the above computation, $\{t\}$ specifies an irreducible representation of $S_{n_1} \times S_{n_2}$.

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## APPENDIX A: RESTRICTED SCHUR POLYNORMIALS FOR $n_1 = 3$, $n_2 = 1$, $m_1 = m_2 = 2$

The construction of restricted Schur polynomials has been described in full generality in [4]. In this Appendix, we simply list the possible operators that can be defined. This is all that is needed to follow the counting arguments of Sec. II. The notation followed is to list $\chi_{R,(r_{11},r_{12},r_{21},r_{22})\alpha\beta}$ with $\alpha$ and $\beta$ multiplicity labels. When only a single copy of representations appear, there is no need for a multiplicity index, and it is simply omitted.

### 1. Case I

$$\chi_{\yng(4),(\yng(2),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{A.1}$$

$$\chi_{\yng(3,1),(\yng(1,1),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{A.2}$$

$$\chi_{\yng(3,1),(\yng(2),\yng(1),\cdot,\yng(1))\alpha\beta} \qquad \alpha,\beta = 1,2 \qquad \text{Four operators} \tag{A.3}$$

$$\chi_{\yng(2,2),(\yng(2),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{A.4}$$

$$\chi_{\yng(2,2),(\yng(2),\yng(1,1),\cdot,\yng(1))} \qquad \text{One operator} \tag{A.5}$$

$$\chi_{\yng(2,1,1),(\yng(2),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{A.6}$$

$$\chi_{\yng(2,1,1),(\yng(1,1),\yng(1),\cdot,\yng(1))\alpha\beta} \qquad \alpha,\beta = 1,2 \qquad \text{Four operators} \tag{A.7}$$

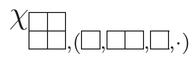
$$\chi_{\yng(1,1,1,1),(\yng(1,1),\yng(1),\cdot,\yng(1))} \qquad \text{One operator} \tag{A.8}$$

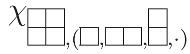
### 2. Case II

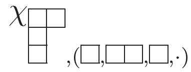
$$\chi_{\yng(4),(\yng(1),\yng(2),\yng(1),\cdot)} \qquad \text{One operator} \tag{A.9}$$

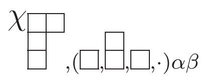
$$\chi_{\yng(3,1),(\yng(1),\yng(1,1),\yng(1),\cdot)} \qquad \text{One operator} \tag{A.10}$$

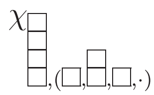
$$\chi_{\yng(3,1),(\yng(1),\yng(2),\yng(1),\cdot)\alpha\beta} \qquad \alpha,\beta = 1,2 \qquad \text{Four operators} \tag{A.11}$$

 One operator (A.12)

 One operator (A.13)

 One operator (A.14)

 $\alpha, \beta = 1, 2$ Four operators (A.15)

 One operator (A.16)

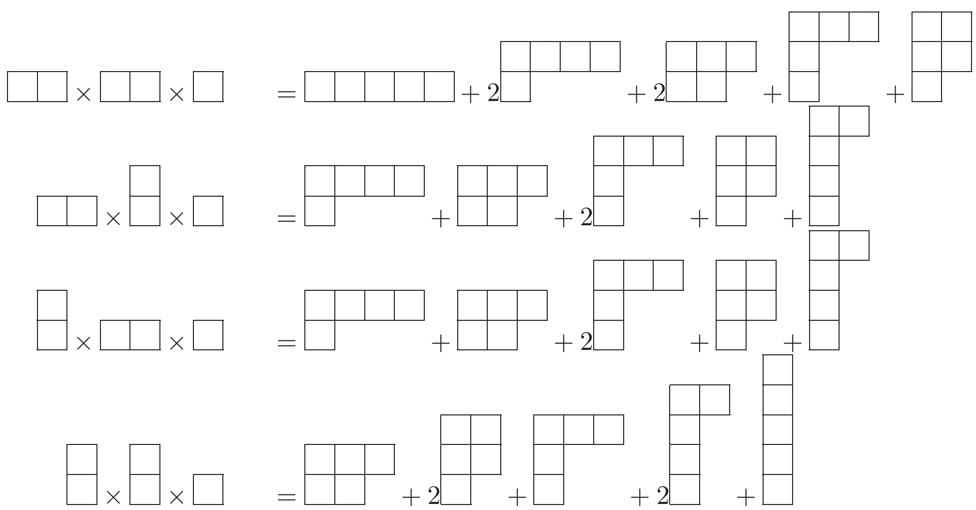
## APPENDIX B: COUNTING WHEN FINITE *N* CONSTRAINTS MATCH

For the counting in this Appendix, we take $n_1 = 1$, $n_2 = 4$, $m_1 = 3$, $m_2 = 2$, $N_1 = \infty$, and $N_2 = 2$. Thus, all restricted Schur polynomials labels have at most two rows. For the generalized restricted Schur polynomials, one of the Young diagrams is unrestricted, and one has at most two rows [see Eq. (2.1)]. In this example, there are two $\{n_{IJ}\}$ sectors of operators:

(1) $\mathrm{tr}(\sigma\phi^{11} \otimes (\phi^{21})^{\otimes 2} \otimes (\phi^{22})^{\otimes 2})$

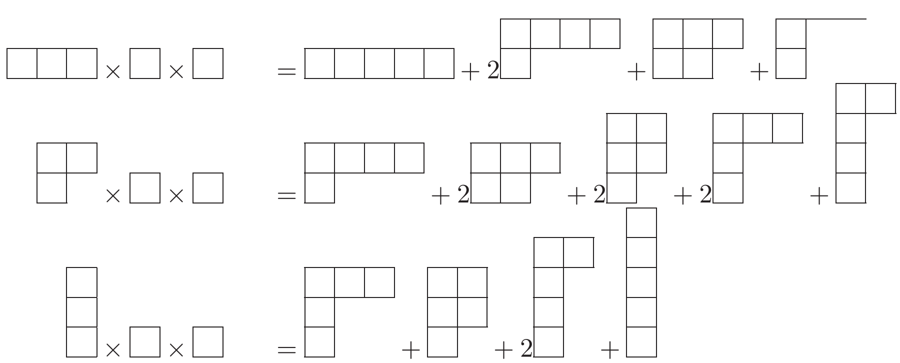
(2) $\mathrm{tr}(\sigma\phi^{12} \otimes (\phi^{21})^{\otimes 3} \otimes \phi^{22})$.

To count the restricted Schur polynomials in sector 1, we use the Littlewood-Richardson numbers appearing in the following products:



(B.1)

To count the restricted Schur polynomials in sector 2, we use the Littlewood-Richardson numbers appearing in the following products:

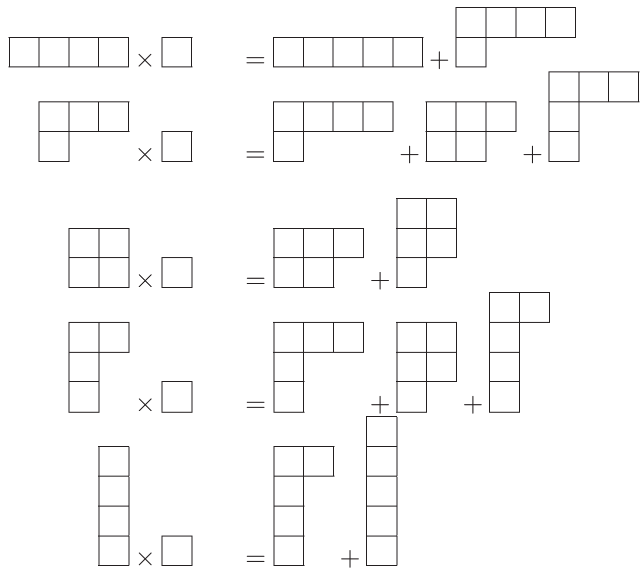


(B.2)

Restricting to Young diagrams with no more than two rows, we find

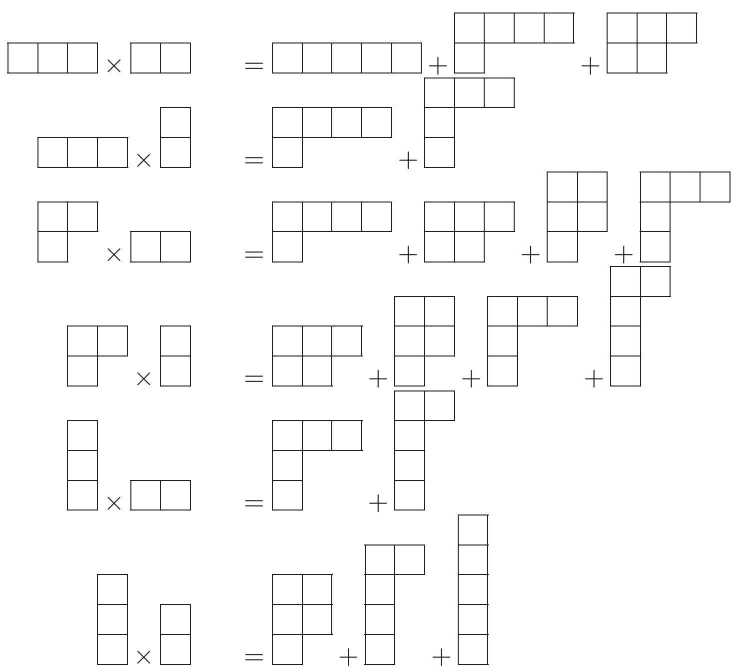
$$\mathcal{N}_{l(R)\leq 2} = \mathcal{N}_1 + \mathcal{N}_2 = 14 + 11 = 25. \tag{B3}$$

The following products appear when counting the number of generalized restricted Schur Polynomials. For $r_1 \vdash 1$ and $r_2 \vdash 4$,



(B.4)

For $s_1 \vdash 3$ and $s_2 \vdash 2$



(B.5)

Using these products of Young diagrams, the number of generalized restricted Schur polynomials after restricting $l(R) \leq 2$ and leaving $S$ unrestricted is $\mathcal{N} = 25$ matching (B3).

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