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Asymptotic freedom, dimensional transmutation, and an infrared conformal fixed point for the δ-function potential in one-dimensional relativistic quantum mechanics

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We consider the Schrödinger equation for a relativistic point particle in an external one-dimensional δ-function potential. Using dimensional regularization, we investigate both bound and scattering states, and we obtain results that are consistent with the abstract mathematical theory of self-adjoint extensions of the pseudodifferential operator $H = \sqrt{p^2 + m^2}$. Interestingly, this relatively simple system is asymptotically free. In the massless limit, it undergoes dimensional transmutation and it possesses an infrared conformal fixed point. Thus it can be used to illustrate nontrivial concepts of quantum field theory in the simpler framework of relativistic quantum mechanics.

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I. INTRODUCTION

The unification of quantum physics and special relativity is achieved in the framework of relativistic quantum field theories. In particular, in the standard model of particle physics elementary particles are very successfully described as quantized wave excitations of the corresponding quantum fields. As such, they have qualitatively different properties than the point particles of Newtonian mechanics or quantum mechanics. In particular, while the position of a quantum mechanical point particle is in general uncertain, quantized waves do not even have a conceptually well-defined position in space. Unlike in quantum mechanics, in local quantum field theory a "particle" is a nonlocal object [1–5]. It is well known that a unification of point particle mechanics and special relativity is problematic, even at the classical level. In particular, Currie, Jordan, and Sudarshan proved that two point particles cannot interact in such a way that the principles of special relativity are respected, i.e. that the system provides a representation of the Poincaré algebra [6]. Leutwyler has generalized this result to an arbitrary number of particles [7]. His noninteraction theorem states that classical relativistic point particles are necessarily free, as a consequence of Poincaré invariance. The only exception are two particles in one spatial dimension confined to each other by a linearly rising potential. In one dimension, the corresponding confining string has no other degrees of freedom than the positions of its end points, which are represented by the two point particles. While strings can interact relativistically in higher dimensions, according to Leutwyler's noninteraction theorem, point particles cannot. Hence, it is not surprising that particle physics is based on quantum field theory rather than on relativistic point-particle quantum mechanics. It should also be noted that, by including the interaction in the momentum and not in the boost operator, interesting relativistic systems with a fixed number of interacting particles have been constructed and investigated in detail [8–10]. However, in this case, the coordinates and momenta of the particles do not obey canonical commutation relations, and thus do not describe ordinary point particles.

When studying fundamental physics, it is a big step to proceed from nonrelativistic quantum mechanics to relativistic quantum field theory. Not only for pedagogical reasons, it is interesting to ask whether nontrivial systems of relativistic quantum mechanics exist. Even free quantum-mechanical relativistic point particles have some interesting properties [11–14]. Minimal position-velocity wave packets of such particles spread in such a way that probability leaks out of the light cone. While such a quantum mechanical violation of causality does not happen in relativistic quantum field theories, it would arise in a hypothetical world of relativistic point particles [15–24]. While in quantum field theory a local Hamiltonian gives rise to nonlocal field excitations that manifest themselves as "particles," in relativistic quantum mechanics local point particles of mass m follow the dynamics of the nonlocal Hamiltonian $H = \sqrt{p^2 + m^2}$. According to Leutwyler's noninteraction theorem, one cannot add a potential to this Hamiltonian without violating the principles of relativity theory, already at the classical level. This is not surprising, because a potential would describe instantaneous

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interactions at a distance, mediated with infinite speed. The only exception are singular contact interactions, which are not excluded by the classical noninteraction theorem. Hence, there might be a quantum loophole in the theorem, which would be worth exploring, at least for pedagogical reasons, trying to bridge the large gap between nonrelativistic quantum mechanics and relativistic quantum field theory in studying fundamental physics.

In nonrelativistic quantum mechanics, contact interactions have been studied in great detail [25–34], which has been used to illustrate some nontrivial concepts of quantum field theories in the simpler context of nonrelativistic quantum mechanics. In this paper we endow the Hamiltonian $H = \sqrt{p^2 + m^2}$ for a single free relativistic point particle in one spatial dimension with a contact interaction potential $\lambda\delta(x)$. We can imagine that such a potential is generated by a second particle of infinite mass. Once this case is fully understood, as a next step one can then consider two relativistic particles of finite mass, and ask whether a contact interaction leads to a nontrivial representation of the Poincaré group, thus providing a quantum mechanical loophole in the classical noninteraction theorem. In this paper, we do not yet address that question and limit ourselves to a single particle in the external one-dimensional δ-function potential. Remarkably, already this relatively simple problem provides interesting insights into some qualitative differences between relativistic and nonrelativistic quantum mechanics. While the simple δ-function potential provides a standard textbook problem, a nonrelativistic particle moving in one spatial dimension allows more general contact interactions. It can actually distinguish a four-parameter family of such interactions. This follows from the theory of self-adjoint extensions [35,36] of the local free-particle kinetic energy Hamiltonian $H = \frac{p^2}{2m}$ [29,37–41]. There is a four-parameter family of self-adjoint extensions characterized by the boundary condition for the wave function at the contact point,

$$\begin{pmatrix} \Psi(\varepsilon) \\ \partial_x \Psi(\varepsilon) \end{pmatrix} = \exp(i\theta) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Psi(-\varepsilon) \\ \partial_x \Psi(-\varepsilon) \end{pmatrix}. \quad (1.1)$$

Here $\varepsilon \to 0$, $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$, and $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}]$. The five parameters a, b, c, d, θ with the constraint $ad - bc = 1$ provide a four-parameter family of self-adjoint extensions of the nonrelativistic free-particle Hamiltonian, and thus a four-parameter family of quantum-mechanical contact interactions. The standard contact interaction potential $\lambda\delta(x)$ just corresponds to $a = d = 1$, $b = 0$, $c = 2m\lambda$, and $\theta = 0$. The most general contact interaction does not respect parity symmetry, which requires $a = d$ and $\theta = 0$. Still, in the nonrelativistic case, this leaves a two-parameter family of parity-invariant contact interactions. A free particle with a generalized energy-momentum dispersion relation $H = \sum_{n=0}^{N} c_n p^n$ even allows an N^2-parameter family of self-adjoint extensions. For very high momenta p, the energy of such a particle increases as p^N, which for $N > 2$ allows the resolution of further details of a contact point than for the standard nonrelativistic dispersion relation with $N = 2$. If one thinks of the relativistic energy-momentum dispersion relation $H = \sqrt{p^2 + m^2}$ as a power-series expansion in p^2 with $N \to \infty$, in the relativistic case one might perhaps expect an infinite number of self-adjoint extension parameters, and thus an infinite variety of contact interactions, e.g. represented by the δ-function potential and all its derivatives. However, the opposite is true. At large momentum p, the relativistic energy $\sqrt{p^2 + m^2}$ only increases as $|p|$, which provides less short-distance resolution than the nonrelativistic p^2. Indeed, there is just a one-parameter family of self-adjoint extensions of the relativistic free-particle Hamiltonian $H = \sqrt{p^2 + m^2}$, which can be characterized by the parameter λ in the contact interaction potential $\lambda\delta(x)$. This follows from the self-adjoint extension theory of so-called pseudodifferential operators, which includes the nonlocal Hamiltonian $H = \sqrt{p^2 + m^2}$ [42]. This theory also predicts that in higher dimensions, relativistic point particles are completely unaffected by contact interactions and thus remain free. This is again in contrast to the nonrelativistic case, in which there is a one-parameter family of contact interactions both in two and in three spatial dimensions [29].

As a result of Leutwyler's noninteraction theorem as well as of the theory of self-adjoint extensions of the pseudo-differential operator $H = \sqrt{p^2 + m^2}$, relativistic quantum mechanics is a rather narrow subject. In particular, for a single particle one is limited to the simple δ function or to a linear confining potential. In this context, it is important to point out that the Klein-Gordon and Dirac equations do not belong to relativistic quantum mechanics, but to quantum field theory. In particular, it is well known that these equations do not allow a consistent single-particle interpretation, because they address the physics of both particles and antiparticles. The relativistic point particle Hamiltonian $H = \sqrt{p^2 + m^2}$, on the other hand, is concerned just with particles. The problem of the relativistic δ-function potential has already been investigated in the mathematical literature as an application of the theory of self-adjoint extensions of pseudodifferential operators [42]. Here we address the problem using more traditional tools of theoretical physics. Unlike in the nonrelativistic case, the relativistic δ-function potential gives rise to ultraviolet divergences which we regularize and renormalize using dimensional regularization [43–46]. It is reassuring that the results that we obtain are indeed consistent with those obtained by the self-adjoint extension theory of Ref. [42]. Here we study the system in great detail, and address various interesting physics questions, including strong bound states with a binding energy that exceeds the rest mass of the bound particle. Remarkably, this relatively simple quantum mechanical model shares several nontrivial features with relativistic quantum field theories. In particular, just like QCD [47], it is asymptotically free [48,49].

In two spatial dimensions, a nonrelativistic δ-function potential must also be renormalized [25–34]. While this system is classically scale invariant, at the quantum level it dynamically generates a bound state via dimensional transmutation, and it has scattering states which display asymptotic freedom. Hence, it can be used to illustrate these nontrivial features, which are usually encountered in quantum field theory, in the framework of nonrelativistic quantum mechanics. However, this theory cannot be obtained as the nonrelativistic limit of a relativistic theory. In this paper, we show that asymptotic freedom and dimensional transmutation already arise in one-dimensional relativistic point particle quantum mechanics with a δ-function potential. Furthermore, in the massless limit the system is scale invariant, at least at the classical level. However, just like in QCD, scale invariance is anomalously broken at the quantum level. The system then undergoes dimensional transmutation and generates a mass scale nonperturbatively. Unlike QCD, in the massless limit the relativistic quantum mechanical model even has a free infrared conformal fixed point. Although actual elementary particles are quantized waves rather than point-like objects, addressing these topics in relativistic point particle quantum mechanics makes them more easily accessible than just studying them in the standard context of relativistic quantum field theories.

The rest of this paper is organized as follows. In Sec. II, we consider the bound state problem and use the bound-state energy to define a renormalization condition. In Sec. III, we derive the relativistic probability current density and show explicitly that it is conserved. In Sec. IV, we address the scattering states and we show that the energy-dependent running coupling constant is finite after renormalization. Reflection and transmission amplitudes, as well as the scattering phase shift, the scattering length, and the effective range are derived in Sec. V. In Sec. VI, we investigate the energy dependence of the running coupling constant and its β function, and we show that the theory is asymptotically free. In Sec. VII, we study ultra-strong bound states and the corresponding scattering states. Section VIII analyzes the massless limit, in which the system undergoes dimensional transmutation, and develops an infrared conformal fixed point. Finally, Sec. IX contains our conclusions.

II. DIMENSIONAL REGULARIZATION AND RENORMALIZATION OF A BOUND STATE

Let us consider the relativistic time-independent Schrödinger equation

$$\sqrt{p^2+m^2}\Psi(x)+\lambda\delta(x)\Psi(x)=E\Psi(x). \tag{2.1}$$

In momentum space

$$\Psi(x)=\frac{1}{2\pi}\int dp\tilde{\Psi}(p)\exp(ipx),\qquad \Psi(0)=\frac{1}{2\pi}\int dp\tilde{\Psi}(p), \tag{2.2}$$

and the Schrödinger equation takes the form

$$\sqrt{p^2+m^2}\tilde{\Psi}(p)+\frac{\lambda}{2\pi}\int dp'\tilde{\Psi}(p')=E\tilde{\Psi}(p), \tag{2.3}$$

such that for a bound state

$$\tilde{\Psi}_B(p)=\frac{\lambda\Psi_B(0)}{E_B-\sqrt{p^2+m^2}}. \tag{2.4}$$

Integrating this equation over all momenta, we obtain the gap equation

$$\begin{aligned}\Psi_B(0)&=\frac{1}{2\pi}\int dp\tilde{\Psi}_B(p)\\&=\lambda\Psi_B(0)\frac{1}{2\pi}\int dp\frac{1}{E_B-\sqrt{p^2+m^2}}\Rightarrow\\\frac{1}{\lambda}&=\frac{1}{2\pi}\int dp\frac{1}{E_B-\sqrt{p^2+m^2}},\end{aligned} \tag{2.5}$$

which determines the bound state energy E_B. The resulting integral is logarithmically ultraviolet divergent and must hence be regularized. We do this by using dimensional regularization, i.e. by analytically continuing the spatial dimension to $D=1+\varepsilon\in\mathbb{C}$ and by finally taking the limit $\varepsilon\to 0$. While the coupling constant λ is dimensionless in one dimension, in D dimensions the prefactor of the δ function has dimension $(\mathrm{mass})^{1-D}$. In order to renormalize the bare coupling, we let it depend on the cutoff, and we replace λ by $\lambda(\varepsilon)m^{-\varepsilon}$. In order to keep $\lambda(\varepsilon)$ dimensionless, we have factored out the dimensionful term $m^{-\epsilon}=m^{1-D}$, using the particle mass m as the renormalization scale. The regularized gap equation then takes the form

$$\frac{m^{D-1}}{\lambda(\varepsilon)}=\frac{1}{(2\pi)^D}\int d^Dp\frac{1}{E_B-\sqrt{p^2+m^2}}=I(E_B). \tag{2.6}$$

For a bound state $E_B<m$, and we expand the integrand in powers of $E_B/\sqrt{p^2+m^2}$, such that

$$I(E_B)=-\frac{\pi^{D/2}D}{\Gamma(D/2+1)}\times\int_0^\infty dp\frac{p^{D-1}}{\sqrt{p^2+m^2}}\sum_{n=0}^{\infty}\left(\frac{E_B}{\sqrt{p^2+m^2}}\right)^n. \tag{2.7}$$

While all higher-order terms are finite, the leading term (with $n=0$) is logarithmically ultraviolet divergent. All terms can be integrated separately, and then be resummed, which in the limit $\varepsilon \to 0$ yields

$$I(E_B) = m^\varepsilon \left[\frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} - \frac{E_B}{2\pi\sqrt{m^2 - E_B^2}}\left(\pi + 2\arcsin\frac{E_B}{m}\right)\right]. \tag{2.8}$$

Here $\gamma \approx 0.5772$ is Euler's constant. For an ultra-strong bound state with energy $E_B < -m$ the series from above diverges. Still, the result can be obtained by directly integrating the convergent expression

$$\frac{1}{2\pi}\int dp\left(\frac{1}{E_B - \sqrt{p^2+m^2}} + \frac{1}{\sqrt{p^2+m^2}}\right) = \frac{E_B}{\pi\sqrt{E_B^2 - m^2}}\,\mathrm{arctanh}\,\frac{\sqrt{E_B^2 - m^2}}{E_B}. \tag{2.9}$$

As a renormalization condition, we now hold the binding energy E_B fixed in units of the mass m, such that the running bare coupling is given by

$$\frac{1}{\lambda(\varepsilon)} = \frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} - \frac{E_B}{2\pi\sqrt{m^2 - E_B^2}}\left(\pi + 2\arcsin\frac{E_B}{m}\right) = \frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} + \frac{1}{\lambda(E_B)}, \tag{2.10}$$

where $\lambda(E_B)$ is a renormalized coupling defined at the scale E_B. Eliminating the terms $1/\pi\varepsilon + [\gamma - \log(4\pi)]/2\pi$ in the definition of the renormalized coupling corresponds to the modified minimal subtraction scheme that is commonly used in quantum field theory. Let us consider the nonrelativistic limit, in which the binding energy $\Delta E_B = E_B - m$ is small compared to the rest mass. In that case, the renormalized coupling is given by

$$\frac{1}{\lambda(E_B)} = -\frac{E_B}{2\pi\sqrt{m^2 - E_B^2}}\left(\pi + 2\arcsin\frac{E_B}{m}\right) \to -\sqrt{\frac{m}{-2\Delta E_B}}. \tag{2.11}$$

Interestingly, for the nonrelativistic contact interaction $\lambda\delta(x)$, which does not require renormalization, for $\lambda < 0$ the bound state energy is given by $\Delta E_B = -m\lambda^2/2$ such that $1/\lambda = -\sqrt{-m/2\Delta E_B}$. Hence, in the nonrelativistic limit the renormalized coupling reduces to $\lambda(E_B \to m) = \lambda$.

Let us now determine the bound state wave function in coordinate space

$$\Psi_B(x) = \frac{A}{2\pi}\int dp\,\frac{\exp(ipx)}{E_B - \sqrt{p^2+m^2}}. \tag{2.12}$$

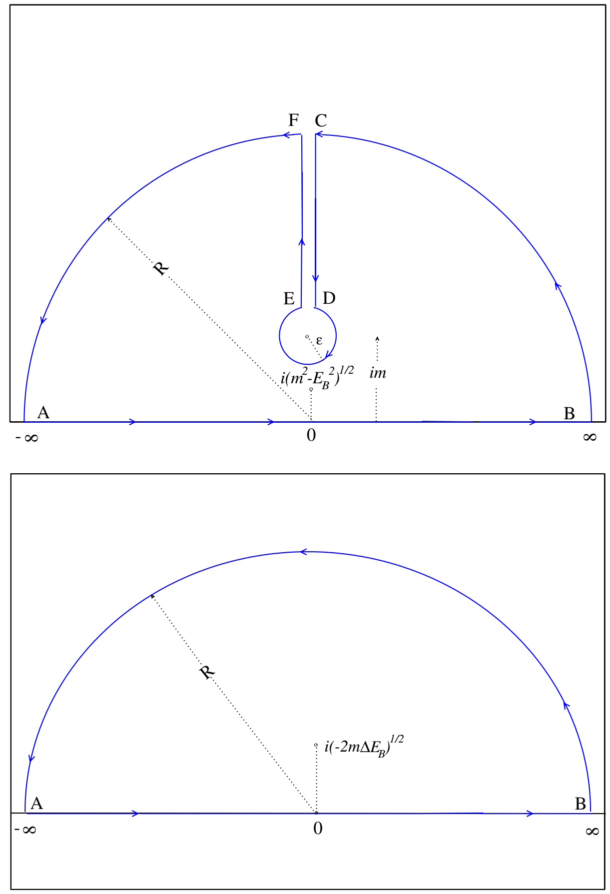

FIG. 1 (color online). Integration contours for the determination of the wave function of the bound state. In the relativistic case (top panel), there is a branch cut along the positive imaginary axis, starting at $p = im$. In addition, for $0 < E_B < m$, there is a pole at $p = i\sqrt{m^2 - E_B^2}$. In the nonrelativistic case (bottom panel), there is still a pole, but no branch cut.

The integration can be extended to the closed contour Γ illustrated in Fig. 1. For $0 < E_B < m$, the integrand has a pole at $p = i\sqrt{m^2 - E_B^2}$, which is enclosed by Γ, as well as a branch cut along the positive imaginary axis starting at $p = im$. The wave function then takes the form

$$\Psi_B(x) = A\left[\frac{1}{\pi}\int_m^\infty d\mu\,\frac{\sqrt{\mu^2 - m^2}}{E_B^2 - m^2 + \mu^2}\exp(-\mu|x|) + \frac{E_B\exp(-\sqrt{m^2 - E_B^2}|x|)}{\sqrt{m^2 - E_B^2}}\right]. \tag{2.13}$$

The integral results from the two contributions along the branch cut, while the last term is the residue of the pole at $p = i\sqrt{m^2 - E_B^2}$. As illustrated in Fig. 2, the wave function is logarithmically divergent at the origin. This short-distance divergence is unaffected by the renormalization. In particular, the singularity of the wave function is integrable and it is thus normalizable in the usual sense.

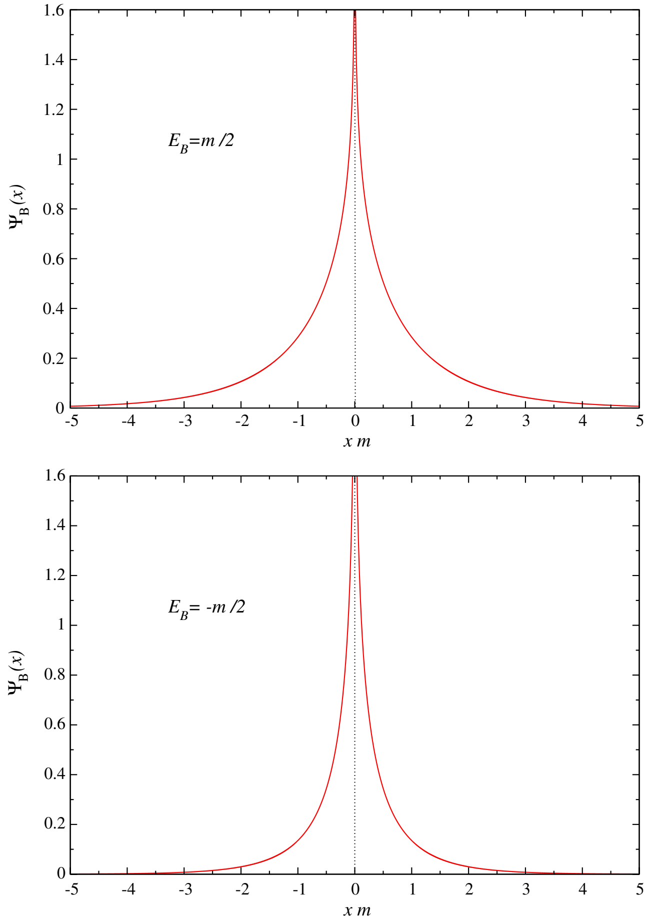

FIG. 2 (color online). Bound state wave function in coordinate space for an ordinary bound state with $E_b = m/2$ (top panel), and for a strong bound state with $E_b = -m/2$ (bottom panel).

Alternatively, the bound state wave function can be expressed in terms of Bessel functions

$$\Psi_B(x) = \frac{A}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{E_B}{m}\right)^n \left(\frac{m|x|}{2}\right)^{n/2} \frac{K_{n/2}(m|x|)}{\Gamma(\frac{n+1}{2})}. \quad (2.14)$$

The normalization constant is most easily determined in momentum space

$$\frac{|A|^2}{2\pi} \int dp \frac{1}{(E_B - \sqrt{p^2 + m^2})^2} = 1 \Rightarrow$$
$$\frac{2\pi}{|A|^2} = \frac{2E_B}{m^2 - E_B^2} + \frac{m^2}{(m^2 - E_B^2)^{3/2}} \left(\pi + 2\arcsin\frac{E_B}{m}\right). \quad (2.15)$$

For the nonrelativistic δ-function potential, the wave function is finite at the origin and given by

$$\Psi_B(x) = \sqrt{\varkappa} \exp(-\varkappa|x|), \qquad \Delta E_B = -\frac{\varkappa^2}{2m}. \quad (2.16)$$

In the nonrelativistic limit, the relativistic wave function of Eq. (2.13) reduces to

$$\Psi_B(x) = \sqrt{\varkappa}\left[\frac{\varkappa}{m\pi} \int_m^{\infty} d\mu \frac{\sqrt{\mu^2 - m^2}}{\mu^2 - \varkappa^2} \exp(-\mu|x|) + \exp(-\varkappa|x|)\right]. \quad (2.17)$$

Since $\varkappa/m \to 0$ in the nonrelativistic limit, it indeed reduces to the nonrelativistic wave function of Eq. (2.16). However, the divergence of the relativistic wave function persists for any nonzero value of $\varkappa/m$. As we discussed in the Introduction, a nonrelativistic contact interaction is characterized by a four-parameter family of self-adjoint extensions, while in the relativistic case there is only a one-parameter family (parametrized by λ). The other nonrelativistic contact interactions cannot be obtained by taking the nonrelativistic limit of a relativistic theory.

Finally, let us also consider the strong bound states, for which the bound state energy $E_B < 0$, i.e. the binding energy $\Delta E_B = E_B - m$ even exceeds the rest mass. In that case, the "pole" at $p = i\sqrt{m^2 - E_B^2}$ has a vanishing residue and hence does not contribute to the result. The wave function of a strong bound state then takes the form

$$\Psi_B(x) = \frac{A}{\pi} \int_m^{\infty} d\mu \frac{\sqrt{\mu^2 - m^2}}{E_B^2 - m^2 + \mu^2} \exp(-\mu|x|), \qquad E_B < 0. \quad (2.18)$$

The wave functions for a bound state with $E_B = m/2$ and for a strong bound state with energy $E_B = -m/2$ are illustrated in Fig. 2. One would think that a relativistic system should not have negative total energy. In fact, the total energy should at least be as large as the positive rest mass of the system. In our case, translation invariance is explicitly broken by the contact potential, which means that the previous argument is not applicable here. One may think of the contact interaction as being generated by an infinitely heavy second particle located at $x = 0$. When the infinite mass of this particle is included in the total energy, it is indeed positive.

III. THE RELATIVISTIC PROBABILITY CURRENT

In the nonrelativistic Schrödinger equation, probability is conserved because of the continuity equation

$$\partial_t \rho(x,t) + \partial_x j(x,t) = 0, \quad (3.1)$$

which relates the probability density $\rho(x,t) = |\Psi(x,t)|^2$ to the probability current density $j(x,t) = \frac{1}{2mi}[\Psi(x,t)^* \partial_x \Psi(x,t) - \partial_x \Psi(x,t)^* \Psi(x,t)]$. While in the Dirac and Klein-Gordon equations, probability conservation is violated due to the presence of antiparticles, in the relativistic Schrödinger equation discussed here, there are no antiparticles and the continuity equation (3.1) still holds with the usual probability density $\rho(x,t) = |\Psi(x,t)|^2$, however, with the modified relativistic probability current density, whose leading terms are

$$j(x,t) = \frac{1}{2mi}[\Psi(x,t)^*\partial_x\Psi(x,t) - \partial_x\Psi(x,t)^*\Psi(x,t)] + \frac{1}{8m^3 i}[\Psi(x,t)^*\partial_x^3\Psi(x,t) - \partial_x\Psi(x,t)^*\partial_x^2\Psi(x,t) + \partial_x^2\Psi(x,t)^*\partial_x\Psi(x,t) - \partial_x^3\Psi(x,t)^*\Psi(x,t)] + \cdots. \quad (3.2)$$

In momentum space the divergence $\partial_x j(x,t)$ takes the compact form

$$p\tilde{j}(p,t) = \frac{1}{2\pi}\int dq \tilde{\Psi}(-q,t)^*\left[\sqrt{(p-q)^2+m^2} - \sqrt{q^2+m^2}\right]\tilde{\Psi}(p-q,t). \quad (3.3)$$

This expression trivially generalizes to an arbitrary energy-momentum dispersion relation $E(p)$ and yields

$$\tilde{j}(p,t) = \frac{1}{2\pi}\int dq \tilde{\Psi}(-q,t)^*\frac{1}{p}[E(p-q) - E(q)]\tilde{\Psi}(p-q,t). \quad (3.4)$$

For a general dispersion relation, the bound state wave function in momentum space takes the form

$$\tilde{\Psi}_B(p) = \frac{A}{E_B - E(p)}. \quad (3.5)$$

The divergence of the probability density then automatically vanishes because

$$p\tilde{j}(p) = \frac{|A|^2}{2\pi}\int dq \frac{1}{E_B - E(q)} \times [E(p-q) - E(q)]\frac{1}{E_B - E(p-q)} = \frac{|A|^2}{2\pi}\int dq\left(\frac{1}{E_B - E(p-q)} - \frac{1}{E_B - E(q)}\right) = 0. \quad (3.6)$$

IV. DIMENSIONAL REGULARIZATION AND RENORMALIZATION OF SCATTERING STATES

Let us now consider the scattering states. First of all, the states of odd parity, which vanish at the origin, are unaffected by the δ-function potential. Hence, we limit ourselves to stationary scattering states of even parity, which we parametrize as

$$\tilde{\Psi}_E(p) = \delta(p - \sqrt{E^2 - m^2}) + \delta(p + \sqrt{E^2 - m^2}) + \tilde{\Phi}_E(p). \quad (4.1)$$

Later we will combine scattering states of even and odd parity in order to extract the reflection and transmission amplitudes. Inserting the ansatz from above in Eq. (2.3), we obtain

$$\left(\sqrt{p^2+m^2} - E\right)\tilde{\Phi}_E(p) + \frac{\lambda}{\pi} + \frac{\lambda}{2\pi}\int dp'\tilde{\Phi}_E(p') = 0 \Rightarrow \tilde{\Phi}_E(p) = \frac{\lambda}{\pi}\frac{1 + \pi\Phi_E(0)}{E - \sqrt{p^2+m^2}}. \quad (4.2)$$

Integrating Eq. (4.2) over all momenta, one finds

$$\Phi_E(0) = \frac{1}{2\pi}\int dp \tilde{\Phi}_E(p) = \frac{\lambda}{\pi}[1 + \pi\Phi_E(0)]\frac{1}{2\pi}\int dp \frac{1}{E - \sqrt{p^2+m^2}}. \quad (4.3)$$

Again, by replacing λ with $\lambda(\varepsilon)m^{-\varepsilon}$, and by using dimensional regularization, we then obtain

$$\Phi_E(0) = \frac{1}{\pi}\frac{\lambda(\varepsilon)m^{-\varepsilon}I(E)}{1 - \lambda(\varepsilon)m^{-\varepsilon}I(E)}. \quad (4.4)$$

For positive energy E the integral takes the form

$$I(E) = m^\varepsilon\left[\frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} + \frac{E}{\pi\sqrt{E^2-m^2}}\operatorname{arctanh}\frac{\sqrt{E^2-m^2}}{E}\right]. \quad (4.5)$$

Using Eq. (2.6), the function $\tilde{\Phi}_E(p)$ then results as

$$\tilde{\Phi}_E(p) = \frac{\lambda(E,E_B)}{\pi}\frac{1}{E - \sqrt{p^2+m^2}}, \quad (4.6)$$

with the energy-dependent running coupling constant (again renormalized at the scale E_B) given by

$$\lambda(E,E_B) = \frac{1}{I(E_B) - I(E)} = -\left[\frac{E}{\pi\sqrt{E^2-m^2}}\operatorname{arctanh}\frac{\sqrt{E^2-m^2}}{E} + \frac{E_B}{2\pi\sqrt{m^2-E_B^2}}\left(\pi + 2\arcsin\frac{E_B}{m}\right)\right]^{-1}. \quad (4.7)$$

Remarkably, using Eqs. (2.8) and (4.5), the ultraviolet divergences of $I(E)$ and $I(E_B)$ cancel, such that the running coupling constant is finite when we take the limit $\varepsilon \to 0$.

In order to investigate whether the resulting system is self-adjoint, let us now check the orthogonality of the various states. First, we calculate the scalar product of the bound state and the scattering states

$$\langle\Psi_B|\Psi_E\rangle = \frac{1}{2\pi}\int dp \frac{1}{E_B-\sqrt{p^2+m^2}}[\delta(p-\sqrt{E^2-m^2})+\delta(p+\sqrt{E^2-m^2})+\tilde{\Phi}_E(p)]$$
$$= \frac{1}{\pi(E_B-E)} + \frac{1}{\pi(I(E_B)-I(E))}\frac{1}{2\pi}\int dp \frac{1}{E_B-\sqrt{p^2+m^2}}\frac{1}{E-\sqrt{p^2+m^2}}. \quad (4.8)$$

The integral results in

$$\frac{1}{2\pi}\int dp \frac{1}{E_B-\sqrt{p^2+m^2}}\frac{1}{E-\sqrt{p^2+m^2}} = \frac{1}{E-E_B}\frac{1}{2\pi}\int dp\left(\frac{1}{E_B-\sqrt{p^2+m^2}}-\frac{1}{E-\sqrt{p^2+m^2}}\right)$$
$$= \frac{I(E_B)-I(E)}{E-E_B}, \quad (4.9)$$

such that indeed $\langle\Psi_B|\Psi_E\rangle = 0$. This is also the case for a strong bound state with $E_B < 0$, and even for an ultra-strong bound state with $E_B < -m$. Next we investigate the orthogonality of the scattering states

$$\langle\Psi_{E'}|\Psi_E\rangle = \frac{1}{\pi}\delta(\sqrt{E^2-m^2}-\sqrt{E'^2-m^2}) + \frac{\lambda(E,E_B)}{\pi^2(E-E')} + \frac{\lambda(E',E_B)}{\pi^2(E'-E)}$$
$$+ \frac{\lambda(E,E_B)\lambda(E',E_B)}{\pi^2}\frac{1}{2\pi}\int dp \frac{1}{E-\sqrt{p^2+m^2}}\frac{1}{E'-\sqrt{p^2+m^2}}$$
$$= \frac{1}{\pi}\delta(k-k') + \frac{1}{\pi^2(E-E')}\left[\frac{1}{I(E_B)-I(E)} - \frac{1}{I(E_B)-I(E')}\right]$$
$$+ \frac{1}{\pi^2}\frac{1}{I(E_B)-I(E)}\frac{1}{I(E_B)-I(E')}\frac{I(E')-I(E)}{E-E'}$$
$$= \frac{1}{\pi}\delta(k-k'). \quad (4.10)$$

Here we have introduced $k = \sqrt{E^2-m^2}$ and $k' = \sqrt{E'^2-m^2}$. The orthogonality of the various states shows explicitly that, after regularization and renormalization, the resulting Hamiltonian is indeed self-adjoint.

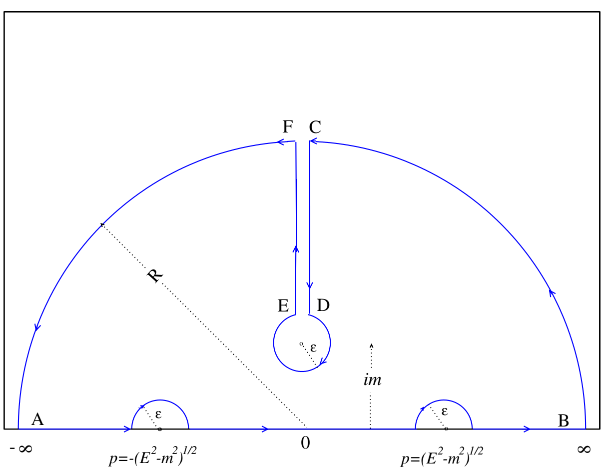

FIG. 3 (color online). Integration contour for the determination of the wave function of the scattering states with $E > m$. There is a branch cut along the positive imaginary axis, starting at $p = im$. In addition, there are two poles on the real axis at $p = \pm\sqrt{E^2-m^2}$.

The contour for the determination of the scattering wave function in coordinate space is illustrated in Fig. 3. In addition to the branch cut, there are two poles on the real axis at $p = \pm k = \pm\sqrt{E^2-m^2}$, which give rise to ingoing and outgoing plane waves. When these poles are avoided by the contour, one obtains the contribution $\tilde{\Phi}_E(x)$ to the total scattering wave function. The even-parity stationary scattering wave function in coordinate space

$$\Psi_E(x) = A(k)\left[\cos(kx) + \lambda(E,E_B)\frac{\sqrt{k^2+m^2}}{k}\sin(k|x|) - \frac{\lambda(E,E_B)}{\pi}\int_m^\infty d\mu \frac{\sqrt{\mu^2-m^2}}{\mu^2+k^2}\exp(-\mu|x|)\right],$$
$$E = \sqrt{k^2+m^2}, \quad (4.11)$$

is illustrated in Fig. 4. Like the bound state wave function, it is logarithmically divergent at the origin.

Let us again consider the nonrelativistic limit by considering small scattering energies, such that $\Delta E = E - m \ll m$, while also maintaining a small bound state energy $|\Delta E_B| = |E_B - m| \ll m$. In this case, the running coupling constant reduces to

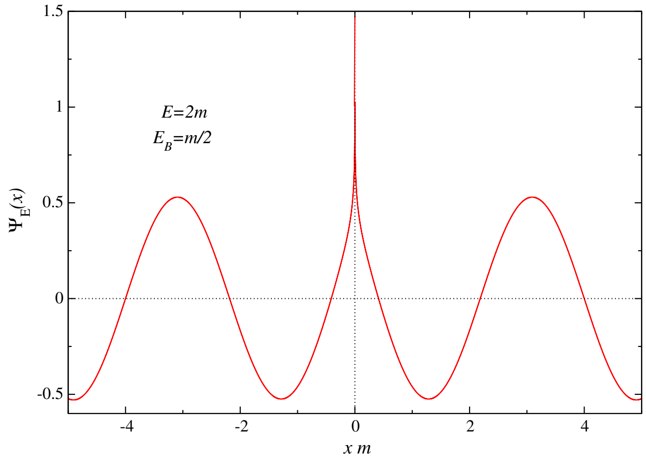

FIG. 4 (color online). Even-parity stationary scattering wave function in coordinate space for $E = 2m$ and $E_B = m/2$. Like the wave function of the bound state, the scattering wave function also diverges logarithmically at the origin.

$$\lambda(E, E_B) \to -\sqrt{-\frac{2\Delta E_B}{m}} = \lambda, \qquad (4.12)$$

where λ is indeed the energy-independent coupling constant of the nonrelativistic theory. As for the bound state wave function, the branch-cut contribution vanishes in the nonrelativistic limit, such that one recovers the nonrelativistic even-parity scattering wave function

$$\Psi_E(x) = A(k)\left[\cos(kx) + \frac{\lambda m}{k}\sin(k|x|)\right]. \qquad (4.13)$$

Here $k = \sqrt{2m\Delta E}$. It should be noted that the logarithmic divergence at the origin still persists for all nonzero values of $\Delta E = k^2/2m$.

V. REFLECTION AND TRANSMISSION AMPLITUDES

Let us now construct reflection and transmission amplitudes by superimposing the nontrivial even-parity scattering states $\Psi_E(x)$ with the trivial odd-parity scattering states $B\sin(kx)$. We will now adjust the amplitudes $A(k)$ and B of the even and odd scattering states such that the wave function takes the form

$$\Psi_I(x) = \exp(ikx) + R(k)\exp(-ikx) + C(k)\lambda(E, E_B)\chi_E(x), \qquad (5.1)$$

in region I to the left of the contact point, i.e. for $x < 0$. In region II, for $x > 0$, on the other hand, we demand

$$\Psi_{II}(x) = T(k)\exp(ikx) + C(k)\lambda(E, E_B)\chi_E(x). \qquad (5.2)$$

Here

$$\chi_E(x) = \frac{1}{\pi}\int_m^\infty d\mu \frac{\sqrt{\mu^2 - m^2}}{\mu^2 + E^2 - m^2}\exp(-\mu|x|), \qquad (5.3)$$

is the branch-cut contribution, which arises only in the relativistic case. Away from the contact point $x = 0$, this contribution decays exponentially and thus has no effect on the scattering wave function at asymptotic distances. After a straightforward calculation one obtains

$$A(k) = -C(k) = \frac{k}{k + i\sqrt{k^2 + m^2}\lambda(E, E_B)}, \qquad B = i, \qquad (5.4)$$

which leads to the reflection and transmission amplitudes

$$R(k) = -\frac{i\sqrt{k^2 + m^2}\lambda(E, E_B)}{k + i\sqrt{k^2 + m^2}\lambda(E, E_B)},$$
$$T(k) = \frac{k}{k + i\sqrt{k^2 + m^2}\lambda(E, E_B)}, \qquad (5.5)$$

which obey $1 + R(k) = T(k)$. Using Eq. (4.7), it is straightforward to convince oneself that $R(k)$ and $T(k)$ have a pole at $k = i\sqrt{m^2 - E_B^2}$, which corresponds to the bound state with energy $\sqrt{k^2 + m^2} = E_B$. The S matrix is given by

$$S(k) = R(k) + T(k) = \frac{k - i\sqrt{k^2 + m^2}\lambda(E, E_B)}{k + i\sqrt{k^2 + m^2}\lambda(E, E_B)} = \exp(2i\delta(k)), \qquad (5.6)$$

which determines the scattering phase shift

$$\tan\delta(k) = -\frac{\sqrt{k^2 + m^2}\lambda(E, E_B)}{k}, \qquad E = \sqrt{k^2 + m^2}. \qquad (5.7)$$

In Ref. [42], the problem has been investigated using the self-adjoint extension theory of the pseudodifferential operator $\sqrt{p^2 + m^2}$, which led to the same expression for the S matrix. This shows that dimensional regularization yields results that are consistent with the more abstract mathematical approach. We go significantly beyond the results of Ref. [42] by addressing numerous additional physics questions.

In three dimensions, it is common to consider the low-energy effective-range expansion, which corresponds to $k\cot\tilde{\delta}(k) = -1/a_0 + \frac{1}{2}r_0 k^2$, where a_0 is the scattering length and r_0 is the effective range. The one-dimensional scattering phase shift $\tilde{\delta}(k)$ measures the phase of the outgoing scattering wave relative to a sine wave that vanishes at the origin. In our one-dimensional problem, there is no scattering in the odd-parity sine-wave channel. The nontrivial one-dimensional scattering phase $\delta(k)$

measures the phase of the outgoing scattering wave relative to a cosine wave that has a maximum at the origin. Hence, compared to the one-dimensional case, $\delta(k)$ corresponds to $\tilde{\delta}(k) + \frac{\pi}{2}$, such that $\cot \tilde{\delta}(k)$ corresponds to $-\tan \delta(k)$. Hence, in our one-dimensional case, the effective-range expansion takes the form

$$-k \tan \delta(k) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + \cdots . \tag{5.8}$$

This yields the scattering length a_0 and the effective range r_0 as

$$a_0 = \frac{1}{m}\left(\frac{1}{\pi} - \frac{1}{\lambda(E_B)}\right), \qquad r_0 = -\frac{1}{a_0 m^2} + \frac{2}{3\pi a_0^2 m^3}. \tag{5.9}$$

Here $\lambda(E_B) < 0$ is the renormalized coupling constant defined in Eq. (2.11). When there is a bound state, the scattering length is positive, and it diverges when the bound state approaches zero energy. In the absence of a bound state, the scattering length would become negative. The scale of r_0 is set by the Compton wavelength $1/m$, while its particular value is also influenced by the scattering length through the dimensionless combination am. The effective range vanishes in the nonrelativistic limit $am \to \infty$, as one might naively expect for a contact interaction, but is nonzero in the relativistic case. This is due to the nonlocality of the Hamiltonian $\sqrt{p^2 + m^2}$, which senses the contact interaction already from some distance r_0. The phase shift $\delta(k)$ is illustrated in Fig. 5. It varies between $\delta(0) = \frac{\pi}{2}$ and $\delta(\infty) = 0$. This is consistent with the one-dimensional version of Levinson's theorem, which identifies the number of bound states as $n = 2[\delta(0) - \delta(\infty)]/\pi$ [50,51].

In the nonrelativistic limit, $\lambda(E, E_B)$ again reduces to the energy-independent coupling λ of the nonrelativistic theory, such that we indeed recover the nonrelativistic textbook results

$$R(k) = -\frac{im\lambda}{k + im\lambda}, \qquad T(k) = \frac{k}{k + im\lambda},$$
$$S(k) = \frac{k - im\lambda}{k + im\lambda}. \tag{5.10}$$

These quantities have a pole at $k = -im\lambda$, which determines the nonrelativistic bound state energy $\Delta E_B = k^2/2m = -m\lambda^2/2$. The scattering phase shift $\delta(k)$ is then given by

$$\tan \delta(k) = -\frac{m\lambda}{k}, \tag{5.11}$$

which yields the scattering length $a_0 = -1/(m\lambda)$ and the effective range $r_0 = 0$.

VI. RUNNING COUPLING CONSTANT, β FUNCTION, AND ASYMPTOTIC FREEDOM

Until now, we have introduced the coupling $\lambda(E_B)$ of Eq. (2.11), which is renormalized at the bound state energy, as well as the energy-dependent running coupling $\lambda(E, E_B)$ of Eq. (4.7), which again uses E_B as the renormalization condition, and enters the reflection and transmission amplitudes in the same way as the energy-independent coupling λ in the nonrelativistic case. Let us now investigate the dependence of the running coupling $\lambda(E, E_B)$ on the scattering energy E, which is illustrated in Fig. 6.

At high energies, $\lambda(E, E_B)$ vanishes logarithmically, thus indicating that the scattered particle becomes free in the infinite-energy limit. In particle physics, e.g. in QCD, this behavior is known as asymptotic freedom. The exact nonperturbative expression for the β function takes the form

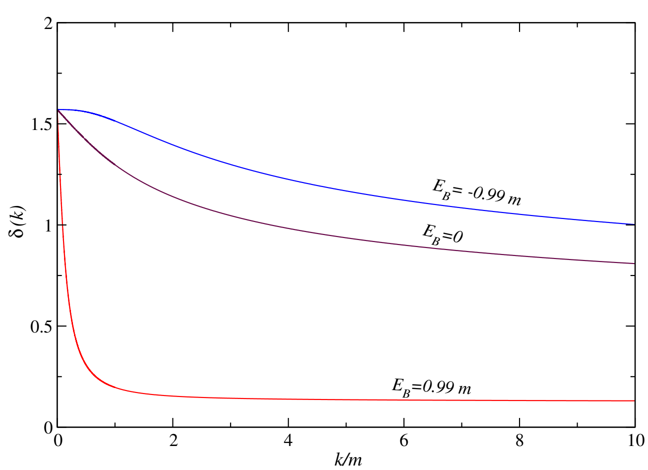

FIG. 5 (color online). Phase shift $\delta(k)$ as a function of the wave number k, for three different values of $E_B/m = 0.99, 0,$ and -0.99.

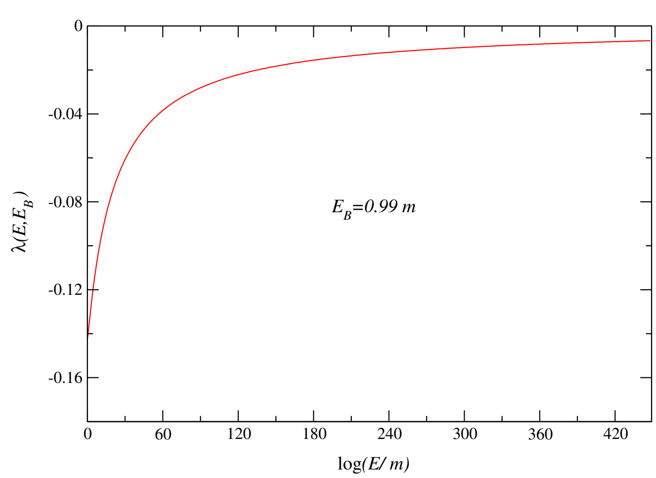

FIG. 6 (color online). Running coupling $\lambda(E, E_B)$ as a function of the scattering energy E, for $E_B = 0.99m$.

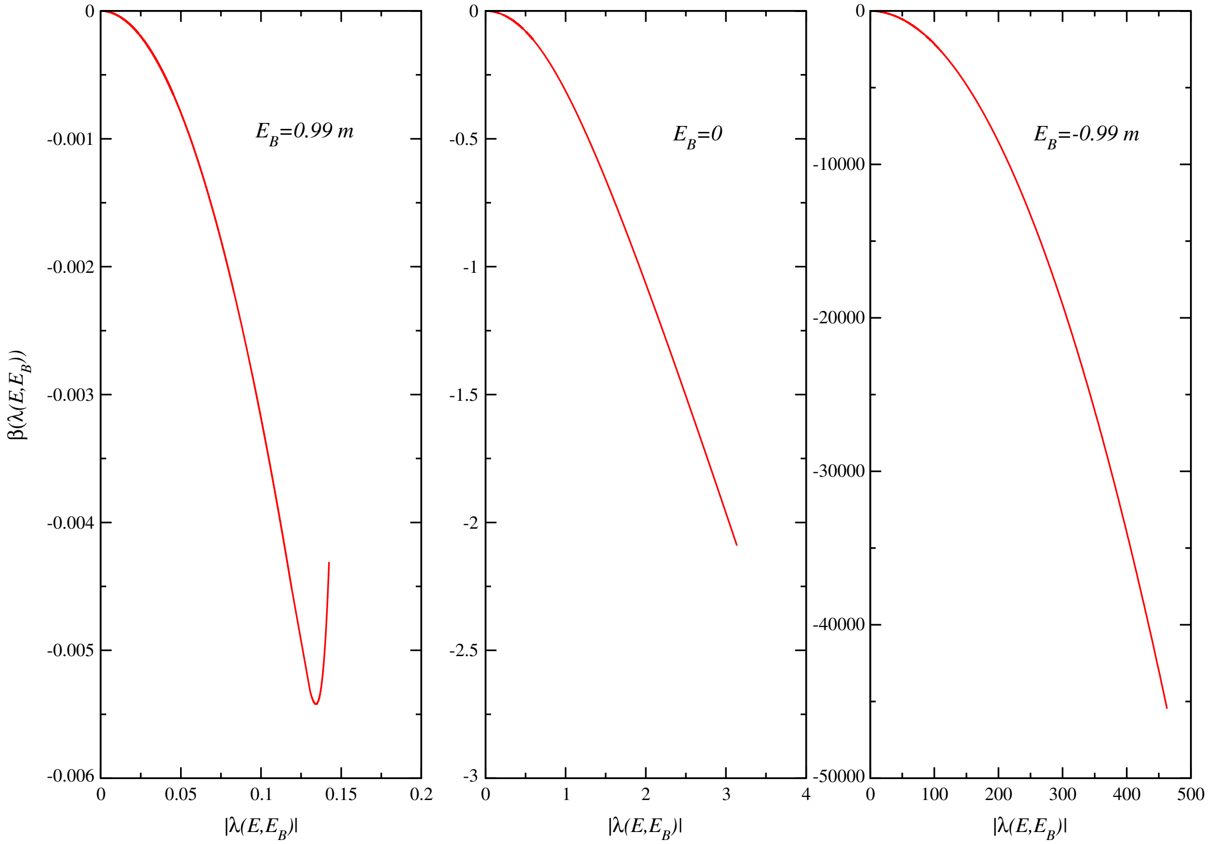

FIG. 7 (color online). The β function $\beta(\lambda(E,E_B))$ as a function of the running coupling $|\lambda(E,E_B)|$ for three values of $E_B/m = 0.99$, 0, and -0.99. The end points of the curves correspond to the maximal value of $|\lambda(m,E_B)|$, which is assumed in the low-energy limit $E \to m$. Note the different scales on the axes, which result from the very different ranges over which $\lambda(E,E_B)$ is varying.

$$\beta(\lambda(E,E_B)) = E\frac{\partial|\lambda(E,E_B)|}{\partial E} = -\frac{\lambda(E,E_B)^2}{\pi} + \frac{\lambda(E,E_B)^2\epsilon^2}{1-\epsilon^2} \times \left(\frac{1}{\lambda(E_B)} - \frac{1}{\lambda(E,E_B)} - \frac{1}{\pi}\right). \quad (6.1)$$

Since $\lambda(E,E_B)$ itself is negative, it is natural to use $|\lambda(E,E_B)|$ to define the β function. In the above expression, $\epsilon = m/E \sim 2\exp(-\pi/|\lambda(E,E_B)|)$ is nonperturbative and exponentially suppressed for small $\lambda(E,E_B)$. This implies that, to all orders in perturbation theory, the β function is given by its one-loop expression $-\lambda(E,E_B)^2/\pi$. The factor $1/\pi$ plays the role of the one-loop coefficient β_0. Nonperturbative corrections enter through ϵ, and become noticeable only at low energies. For asymptotically large energies, the β function behaves as

$$\beta(\lambda(E,E_B)) \to -\frac{\pi}{(\log(E/m))^2} \to -\frac{\lambda(E,E_B)^2}{\pi} < 0. \quad (6.2)$$

It vanishes at $\lambda(E,E_B) \to 0$, which corresponds to an ultraviolet fixed point. The negative sign of the β function again signals asymptotic freedom. In Fig. 7, the β function is illustrated for different values of E_B/m, which influences the behavior only far away from the ultraviolet fixed point at $\lambda(E,E_B) = 0$.

Another zero of the β function would require

$$\epsilon^2\left(\frac{1}{\lambda(E_B)} - \frac{1}{\lambda(E,E_B)}\right) = \frac{1}{\pi} \Rightarrow \frac{E}{m} = \frac{m}{\sqrt{E^2-m^2}}\operatorname{arctanh}\frac{\sqrt{E^2-m^2}}{E}, \quad (6.3)$$

provided that $\epsilon = m/E \neq 1$. However, the above condition is satisfied only for $E = m$, and hence, in this case, no other fixed point exists. As we will see in Sec. VII, in the massless case, $m = 0$, there is an additional infrared conformal fixed point.

VII. ULTRA-STRONG BOUND STATES AND REPULSIVE SCATTERING STATES

Until now, we have used the expression of Eq. (2.8) for $I(E_B)$, and thus we have implicitly assumed that $|E_B| < m$.

This includes the case of strong bound states with $-m < E_B < 0$, but it excludes ultra-strong bound states with energies $E_B < -m$. We again point out that the strong and ultra-strong bound states are not necessarily tachyonic, because the δ-function potential can be attributed to a hypothetical infinitely heavy particle. Hence, the total rest energy of the system always remains positive. For $E_B < -m$ one must use the expression of Eq. (2.9) for $I(E_B)$, with interesting consequences for the bound- and scattering-state wave functions. First of all, it should be pointed out that the various states are still mutually orthogonal, such that the Hamiltonian remains self-adjoint, even in the presence of an ultra-strong bound state. This is easy to see, because the orthogonality relations (4.8) and (4.10) do not depend on the explicit form of $I(E_B)$.

Let us first consider the extreme limit $E_B \to -\infty$. In this case, the running coupling constant takes the form

$$\lambda(E, E_B) \to -\left[\frac{E}{\pi\sqrt{E^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E^2 - m^2}}{E} - \frac{1}{\pi}\log\left(\frac{-2E_B}{m}\right)\right]^{-1}. \quad (7.1)$$

For small nonrelativistic energies $\Delta E = E - m \ll m$, this reduces to

$$\lambda \to \frac{\pi}{\log(-2E_B/m)} > 0. \quad (7.2)$$

Remarkably, this λ actually plays the role of the strength of the repulsive contact interaction $\lambda\delta(x)$ in the nonrelativistic theory. In other words, despite the fact that there is an infinitely strongly bound state, the low-energy scattering states approach those of the nonrelativistic repulsive potential $\lambda\delta(x)$, for which there is no bound state at all. In fact, in the limit $E_B \to -\infty$, the probability density of the relativistic ultra-strong bound state degenerates to a δ function. Because the scattering states still are logarithmically divergent at the origin, this is not in contradiction with orthogonality in the nonrelativistic limit. Figure 8 compares the even-parity scattering wave functions at low energy in the relativistic and nonrelativistic cases, which indeed coincide, except in the ultimate vicinity of the contact point. This indeed makes sense, because the short-distance behavior of the relativistic and the nonrelativistic theories are fundamentally different. We conclude that the relativistic contact interaction always produces a bound state. Remarkably, when this bound state becomes ultra-strong (with $E_B \to -\infty$), it decouples from the scattering states, which behave as if the contact interaction was repulsive.

Let us now discuss the case $-\infty < E_B < -m$. In this case, the running coupling constant is given by

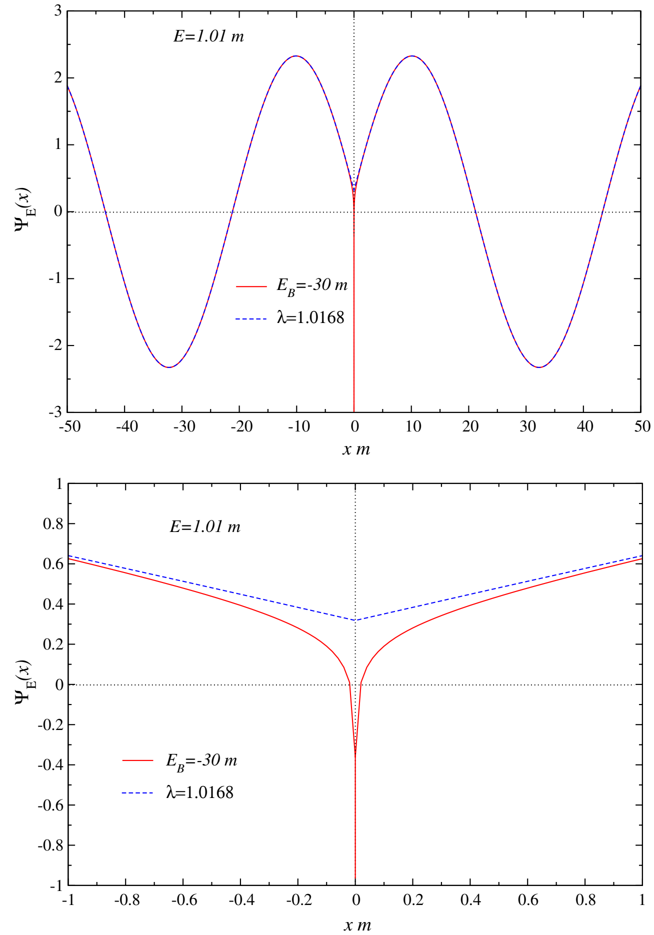

FIG. 8 (color online). Even-parity low-energy scattering wave function (with energy $E = 1.01m$) in the presence of an ultra-strong bound state with $E_B = -30m$, compared to the corresponding nonrelativistic wave function (with $\lambda = 1.0168$). The panel on the bottom zooms into the region around the contact point $x = 0$, in which the relativistic wave function is logarithmically divergent, while the nonrelativistic wave function remains finite.

$$\lambda(E, E_B) = -\left[\frac{E}{\pi\sqrt{E^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E^2 - m^2}}{E} - \frac{E_B}{\pi\sqrt{E_B^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E_B^2 - m^2}}{E_B}\right]^{-1}. \quad (7.3)$$

At low energies $m < E < -E_B$, the running coupling $\lambda(E, E_B) > 0$ is repulsive, it diverges at $E = -E_B$, and becomes attractive [i.e. $\lambda(E, E_B) < 0$] at high energies $E > -E_B$. The phase shift $\delta(k)$, illustrated in Fig. 9, goes through a resonance at $E = -E_B$ with $\delta(\sqrt{E_B^2 - m^2}) = \frac{\pi}{2}$. Since we still have $\delta(0) = \frac{\pi}{2}$, this behavior is still consistent with the one-dimensional version of Levinson's theorem.

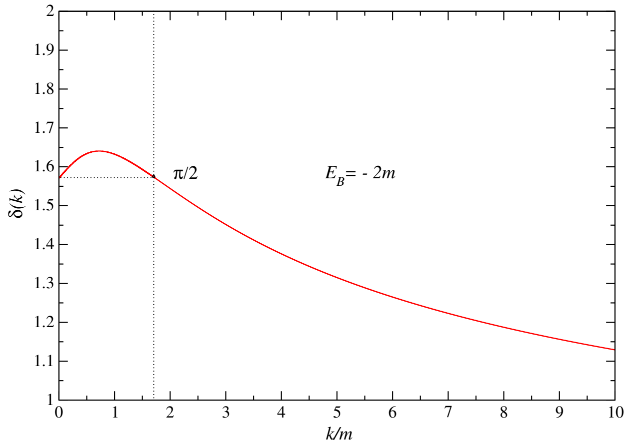

FIG. 9 (color online). Phase shift $\delta(k)$ as a function of the wave number k for $E_B = -2m$. The phase shift goes through a resonance at $E = -E_B$ with $\delta(k) = \delta(\sqrt{E_B^2 - m^2}) = \frac{\pi}{2}$.

VIII. THE MASSLESS CASE

Let us also consider the massless case $m = 0$. Since λ is dimensionless, the system is then scale invariant, at least at the classical level. For $m = 0$, we are automatically limited to ultra-strong bound states (with $E_B < -m$). The bound-state energy E_B is a scale generated nonperturbatively at the quantum level, in a similar way as the proton mass is generated in massless QCD. Scale invariance is then anomalously broken and a scale, in this case E_B, emerges by dimensional transmutation. All physical quantities can then be expressed in units of this scale.

First, let us consider the bound state wave function in momentum space

$$\tilde{\Psi}_B(p) = \sqrt{\frac{\pi}{-E_B}} \frac{1}{E_B - |p|}. \tag{8.1}$$

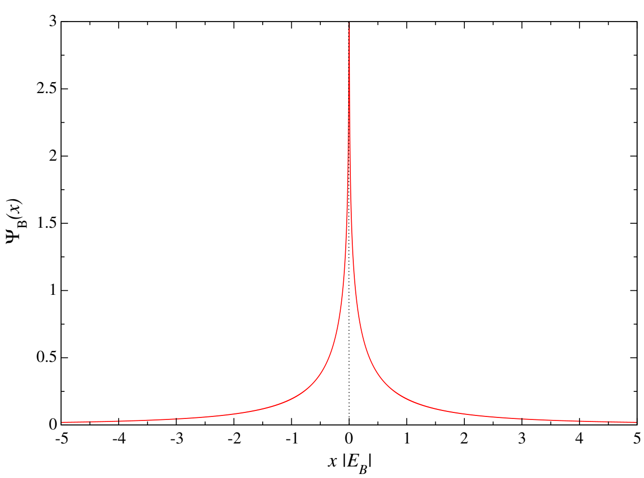

FIG. 10 (color online). Bound state wave function in the massless case.

In coordinate space, it takes the form

$$\Psi_B(x) = \sqrt{\frac{1}{-\pi E_B}} \int_0^\infty d\mu \frac{\mu}{\mu^2 + E_B^2} \exp(-\mu|x|), \tag{8.2}$$

which is illustrated in Fig. 10. As usual, the wave function diverges logarithmically at $x = 0$, but is still square integrable. Next, we consider the even-parity scattering state (with $E = k$)

$$\Psi_E(x) = A(k) \Big[\cos(kx) + \lambda(E, E_B) \sin(k|x|) - \frac{\lambda(E, E_B)}{\pi} \int_0^\infty d\mu \frac{\mu}{\mu^2 + k^2} \exp(-\mu|x|) \Big]. \tag{8.3}$$

Two scattering wave functions, one for $m < E < -E_B$ and one for $E > -E_B$, are shown in Fig. 11.

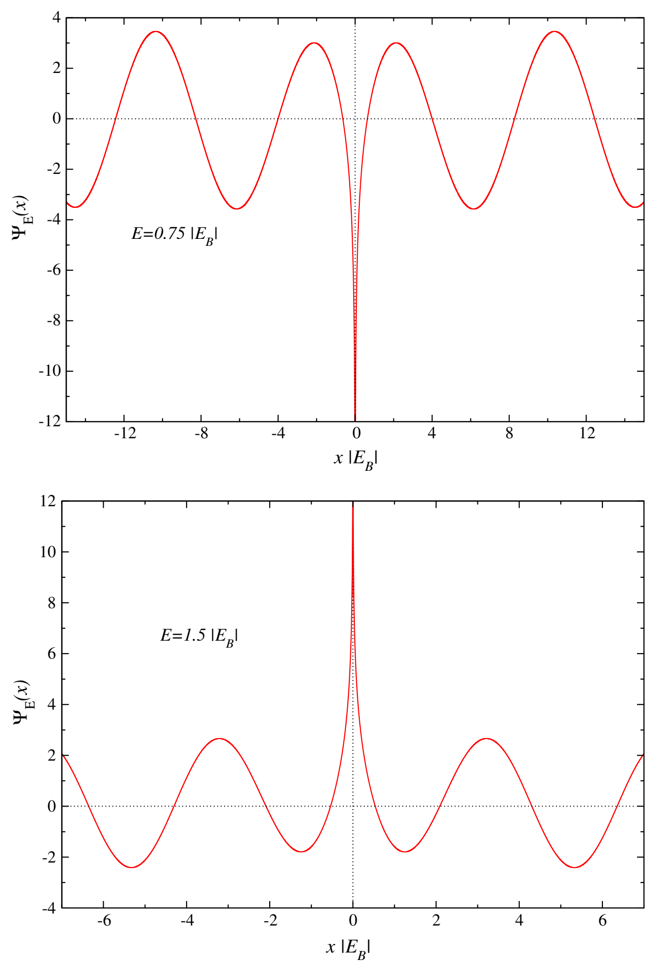

FIG. 11 (color online). Scattering wave functions in the massless case, for $E = 0.75|E_B| < -E_B$ (top panel), and for $E = 1.5|E_B| > -E_B$ (bottom panel).

The resulting reflection and transmission amplitudes as well as the S matrix are then given by

$$R(k) = -\frac{i\lambda(E,E_B)}{1+i\lambda(E,E_B)}, \qquad T(k) = \frac{1}{1+i\lambda(E,E_B)},$$
$$S(k) = \frac{1-i\lambda(E,E_B)}{1+i\lambda(E,E_B)}. \tag{8.4}$$

In the massless case, one obtains

$$\tan\delta(k) = -\lambda(E,E_B) = \frac{\pi}{\log(-E/E_B)} = \frac{\pi}{\log(k/|E_B|)}. \tag{8.5}$$

The β function then reduces to

$$\beta(\lambda) = E\frac{\partial|\lambda(E,E_B)|}{\partial E} = -\frac{\pi}{(\log(-E/E_B))^2} = -\frac{\lambda(E,E_B)^2}{\pi}, \tag{8.6}$$

which is now valid even at low energies. The running coupling and the β function are shown in Fig. 12. Remarkably, the running coupling vanishes not only at high, but also at low energies. In fact, the theory has both an ultraviolet and an infrared fixed point. At the ultraviolet fixed point, $\lambda(E,E_B)$ approaches 0 from below, as $E\to\infty$, while at the infrared fixed point, $\lambda(E,E_B)$ approaches 0 from above, as $E\to 0$. Both fixed points are described by the same zero of the β function of Eq. (8.6).

This situation resembles the one of an asymptotically free non-Abelian gauge theory near the so-called conformal window, which is relevant in the context of walking

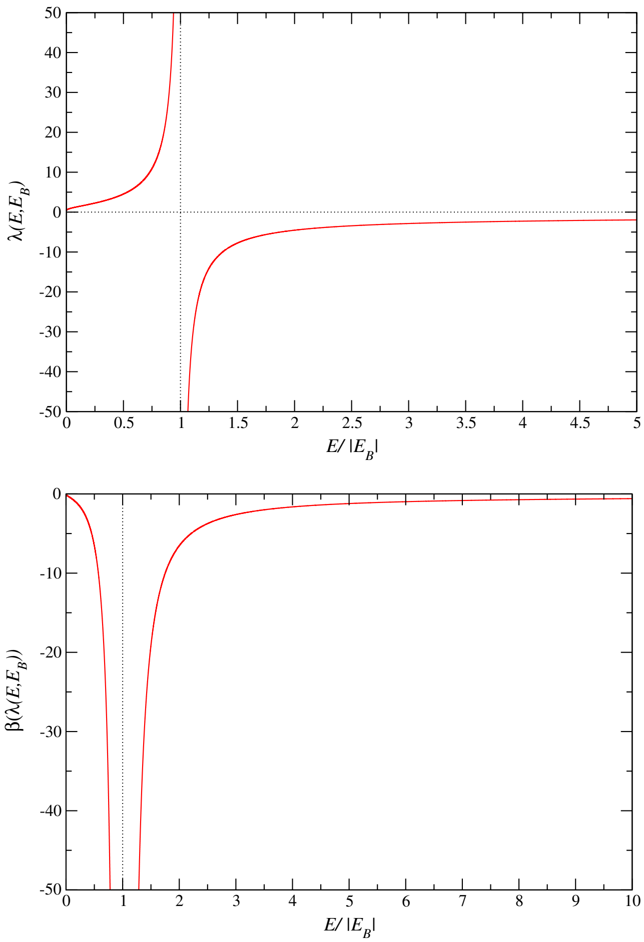

FIG. 12 (color online). Top panel: The running coupling $\lambda(E,E_B)$ as a function of the energy E in the massless case. The coupling goes to zero both at high and at low energies. Bottom panel: The β function $\beta(\lambda(E,E_B))$ as a function of the scattering energy E (in units of $|E_B|$) in the massless limit.

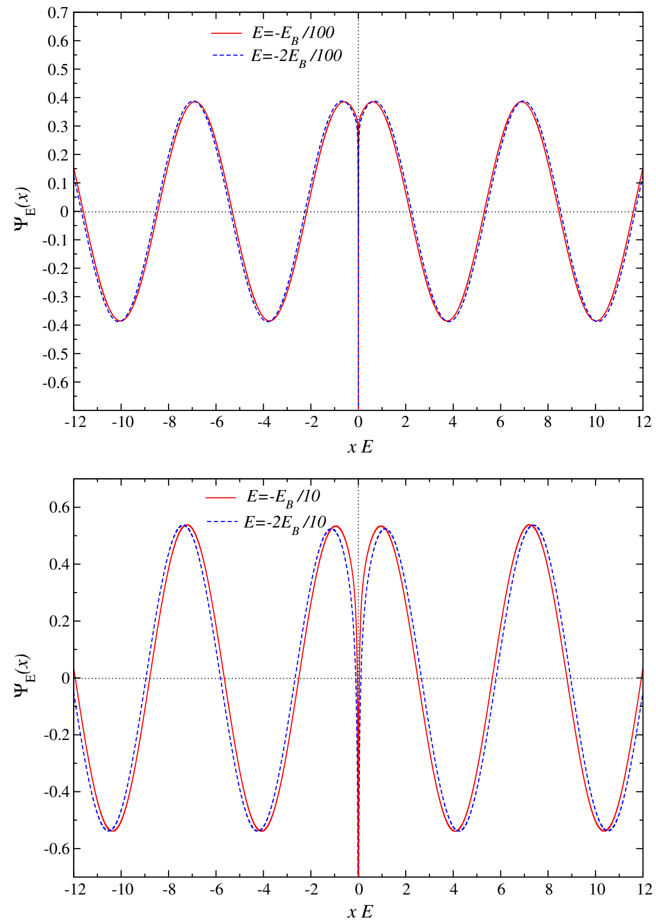

FIG. 13 (color online). Top panel: Even-parity scattering wave functions at low energies $E=|E_B|/100$ and $E=2|E_B|/100$, very close to the infrared conformal fixed point, as a function of the rescaled position xE. The two wave functions are related by a factor 2 scale transformation. Bottom panel: The scattering wave functions at somewhat higher energies $E=|E_B|/10$ and $E=2|E_B|/10$, further away from the conformal fixed point, show a visible deviation from scale invariance.

technicolor theories [52–57]. Another system of this kind - is the two-dimensional $O(3)$ model at vacuum angle $\theta = \pi$ [58,59], whose low-energy effective theory is the conformal $k = 1$ Wess-Zumino-Novikov-Witten model [60–62]. Such theories also have both an ultraviolet and an infrared fixed point. While the theory is scale invariant at very low energies, scale invariance is still explicitly violated, via dimensional transmutation, at a nonperturbatively generated higher energy scale. Thanks to asymptotic freedom, this scale is exponentially small compared to the ultimate ultraviolet cutoff (which can thus be sent to infinity). In our model, the energy $E_B < 0$ of the bound state sets the nonperturbatively generated energy scale, which still affects the scattering states at high energies $E > -E_B$. Low-energy scattering states (with $0 < E \ll -E_B$) are governed by the infrared fixed point and can thus be mapped into each other by scale transformations, as illustrated in Fig. 13.

IX. CONCLUSIONS

We have investigated contact interactions in one-dimensional relativistic quantum mechanics. In contrast to the nonrelativistic case, there is only a one-parameter family of self-adjoint extensions of the pseudodifferential operator $H = \sqrt{p^2 + m^2}$, which is characterized by the contact potential $\lambda\delta(x)$. Remarkably, this simple potential gives rise to rather rich physics. First of all, unlike in the nonrelativistic case, the δ-function potential requires regularization and subsequent renormalization, which we have performed using dimensional regularization. Indeed, using this physics approach, we obtained results that are consistent with the more abstract mathematical theory of self-adjoint extensions of pseudodifferential operators. That theory also implies that there are no nontrivial relativistic contact interactions in more than one spatial dimension. This is again in contrast to the nonrelativistic case, in which there is a one-parameter family of nontrivial contact interactions both in two and three spatial dimensions. In four and more spatial dimensions, on the other hand, there are no nontrivial self-adjoint extensions of the nonrelativistic free-particle Hamiltonian. It is interesting to investigate contact interactions in higher dimensions also using dimensional regularization. This has already been done in the nonrelativistic case. While dimensional regularization provides results that are consistent with the self-adjoint extension theory in two and three spatial dimensions, in contrast to the theory of self-adjoint extensions, it seems to lead to nontrivial contact interactions in higher dimensions [34]. However, it turns out that the resulting Hamiltonian is not self-adjoint and thus not physically meaningful. In this sense, dimensional regularization actually fails to produce the correct result. We suspect that the same may happen in the relativistic case, already in two and three spatial dimensions, which might be worth investigating.

As we discussed before, the external δ-function potential can be attributed to an infinitely heavy particle. It is interesting to ask whether this second particle can be treated fully dynamically, by giving it a finite mass. Only then may the system become Poincaré invariant, because translation invariance is no longer explicitly broken by the position of the external contact interaction. Leutwyler's noninteraction theorem suggests that Poincaré invariance is incompatible with interacting point particles. However, since the theorem operates at the classical level, and does not apply to quantum mechanical point interactions, there may be a quantum loophole that would be worth exploring. For the fully dynamical two-particle problem, the question arises whether both a self-adjoint Hamiltonian and a self-adjoint boost operator can be constructed, which obey the commutation relations of the Poincaré algebra together with the operator of the total momentum P. If so, the two-particle system will have a total energy $E = \sqrt{P^2 + M^2}$, where M is the rest energy of the system. In such a system, one could also investigate the Lorentz contraction of a moving wave packet, which, until now, has been investigated for free particles only [14]. Although we know that nature makes relativistic "particles" as nonlocal quantized field excitations, at least for pedagogical reasons, it is interesting to explore the alternative possibilities of local relativistic point particles. Based on the noninteraction theorem, such alternatives are expected to be very limited, which, in turn, underscores the strengths of relativistic quantum field theories.

As we have shown, asymptotic freedom, dimensional transmutation, and an infrared conformal fixed point in the massless limit, already arise in one-dimensional relativistic point particle quantum mechanics with a δ-function potential. This allowed us to illustrate nontrivial quantum field-theoretical phenomena as well as techniques including dimensional regularization and renormalization, avoiding the technical complications of quantum field theory. We conclude this paper by expressing our hope that the relatively simple system that we have investigated here will help to bridge the large gap that separates nonrelativistic quantum mechanics from relativistic quantum field theory in the teaching of fundamental physics.

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