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True self-energy function and reducibility in effective scalar theories

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This is the eighth paper in the series devoted to the systematic study of effective theories. Below, I discuss the renormalization of the one-loop two-leg functions in multicomponent effective scalar theory. It is shown that only a part of numerous contributions that appear in the general expression for a two-leg graph can be considered as the true self-energy function. This part is completely fixed by the values of minimal coupling constants; it is the only one that should be taken into account in the conventional process of the summation of Dyson's chain that results in explicit expression for the full propagator. The other parts provide the well-defined finite corrections for the graphs with the number of legs $n > 2$. It is also shown that there is no need to attract the renormalization prescriptions for the higher derivatives of the two-leg function on the mass shell; the requirements of finiteness and diagonability turn out to be quite sufficient.

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I. INTRODUCTION

First of all, it is necessary to recall the definition of the term "effective theory" suggested in Ref. [1] and used throughout the paper. The theory is called effective if the corresponding Hamiltonian in the interaction picture contains all the local monomials consistent with a given linear symmetry.[1] In Ref. [2], the definition of the *effective scattering theory* was given: this is just an effective theory only designed for calculating the S matrix (not the Green functions). As pointed out in Ref. [3], the Green functions, as well as the effective Lagrangian, depend on the infinite set of redundant parameters[2] (see, e.g., Ref. [4]), while the S-matrix elements only depend on the essential parameters. What is important is that when the essential parameters are concentrated in a certain area it looks possible to construct the renormalizable S matrix (see Ref. [3] and also Ref. [5]). For this reason, I find it interesting to make an attempt to construct the iteration scheme suitable for the effective scattering theory. Such a scheme should result in finite expressions for the S-matrix elements at every step of the iteration procedure. The finiteness of the Green functions (off the mass shell) is not required; only the finiteness of their residues at $p_i^2 \to m_i^2$ is required.

The obvious problem emerging immediately with this way is that of the two-point Green function (self-energy). In contrast to the S-matrix elements, we need to know this function off the mass shell. One more problem manifests itself when one performs the conventional Dyson summation of the chain of two-point functions to obtain the full propagator. The point is that the result demonstrates the obvious contradiction with the limitation imposed by the famous Källen—Lehmann representation. Besides, when inserted in the external line of a Green function, the two-leg graph of the effective theory brings unwanted poles, which makes the physical interpretation contradictory. At last, the presence of many similar particles[3] in a theory makes the problem of diagonalization difficult.

The present paper is devoted to a discussion of the above-mentioned problems in the framework of multicomponent effective *scalar* theory. I use (and explain when necessary) the terminology from our previous publications (see Refs. [1,2] and references therein).

Three notes are in order. First, it is implied that in the theory considered below there are no massless particles. This eliminates infrared problems. Second, as usual, the diverging integrals are considered regularized by a one-parametric cutoff. At last, I only consider the case of space-time dimension $D = 4$.

Below, I often use the following commonly accepted abbreviations: one-particle-irreducible (1PI), one-particle-reducibile (1PR), Lehman—Simanzik—Zimmerman (LSZ), and renormalization prescription (RP).

II. PRELIMINARY NOTES

First of all, it is necessary to remind the reader of some results obtained in the previous papers (see Refs. [1,2,9,10]) and the terminology introduced therein. For simplicity, I consider here only the case of scalar theories. I refer the reader to the above-cited papers for more detailed discussion and the relevant figures.

References [1] and [2] considered the phenomenon of disappearance of the pole associated with the propagator line of a particle with mass m and momentum p in the S-matrix graph due to the presence of "killing" factors

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[1]This is just a slight modification of the definition suggested in Ref. [6].

[2]This is just because an infinite set of different Lagrangians may result in the same S matrix; see, e.g., Refs. [7,8].

[3]Particles with identical quantum numbers except mass.

[those proportional to $(p^2 - m^2)$] in adjacent vertices and the corresponding confluence of these latter ones. This phenomenon is called the *reduction* of a line. The vertex is called minimal with respect to its line p if it does not contain the corresponding killing' factor $(p^2 - m^2)$. The line p of a graph is called *minimal* if it cannot be reduced or, equivalently, if the adjacent vertex (or both adjacent vertices if the line is inner) is minimal with respect to it. The graph may be called minimal[4] if all its external lines are minimal. Clearly, the reduction of all internal lines of the minimal graph results in the new graph that is built entirely of *minimal vertices* each of which is minimal with respect to all its lines. Note that the analytic expressions that correspond to the graphs under consideration (the original and reduced) are identical.

It can be easily understood that an arbitrary graph that provides the nonzero contribution to the S matrix is minimal, and hence it can only depend on the minimal parameters (coupling constants at the minimal vertices). This means that the set of *essential* parameters only contains the minimal coupling constants. This set is much more narrow as compared to the total number of coupling constants (minimal plus nonminimal) of the effective theory. Nevertheless, it is still infinite. This follows from the fact that all the vertices of the form $g_n\phi^n(x)$ $(n = 5, 6, \ldots)$ are minimal. The theories that contain vertices of these types $(n \geq 5)$ are nonrenormalizable. This means that one needs to attract an infinite number of counterterms constructed from the field and its derivatives of arbitrary order to eliminate the occurrence of infinities in S-matrix elements, and hence it is necessary to formulate an infinite number of corresponding RPs including those fixing the finite parts of nonminimal parameters. It turns out that the renormalization of the S-matrix graph constructed from the minimal vertices, in principle, might introduce dependence on nonminimal parameters. This contradicts what is written above. Is there any way out of this contradiction? I think the answer is yes. It is necessary to reconstruct the renormalization procedure in such a way that the need to fix the nonminimal counterterms would not appear at all. Surely, this might be only possible if the nonminimal coupling constants are certain functions of the minimal ones. In other words, the renormalizability of the effective scattering theory requires the existence of certain complicated symmetry that establishes linkage between the values of different coupling constants. In this case, it looks like the number of independent essential constants in the effective theory with a single scalar particle should be three: two minimal coupling constants g_3 and g_4 and the physical mass m. To check/prove this guess, it is necessary to construct the explicit form of the corresponding symmetry relations:

$$F(m^2, g_3, g_4, \ldots) = 0.$$

Here, I imply that the set of arguments of the function F contains all parameters that appear in the basic Lagrangian (both minimal and nonminimal).

In this paper, I make the very first step toward constructing the relevant renormalization procedure. I follow the conventional logical scheme. First of all, one needs to perform the renormalization of 1PI one-loop n-leg graphs for $n = 1, 2, 3, \ldots$. Then, these renormalized (finite) graphs can be considered subgraphs in the structure of 1PI two-loop n-leg graphs, which, in turn, must be renormalized, and so on. The new feature that manifests itself in the case of the effective theory is the emergence of the possibility to introduce two different definitions of one-particle irreducibility—the graphical or the analytical 1PI. This problem is discussed in Sec. IV

III. MOST GENERAL FORM OF LOCAL VERTICES

Let us first consider the simplest effective theory, that containing only one real scalar field $\phi(x)$:

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p_0}[a^+(p)\exp(ipx) + \text{H.c.}].$$

The creation and annihilation operators fulfill the conventional commutation relation

$$[a^-, a^+]_- = (2\pi)^3 2p_0\delta(\mathbf{p} - \mathbf{q}).$$

Here, $p_0 = \sqrt{\mathbf{p}^2 + m^2}$, and m stands for the physical mass.

Note that I rely upon the renormalized perturbation scheme with on-mass-shell renormalization prescriptions. R operation is precisely that described in Ref. [11] (see also Ref. [12]).

The full interaction Hamiltonian density of the effective theory is the sum of an infinite number of local terms of the form

$$H(x) = \sum_{n=0}^{\infty}[H_n(x) + C_n], \tag{1}$$

where $H_n(x)$ is an infinite sum of *all* Lorentz-invariant n-leg local vertices constructed from the field and its derivatives of various orders. C_n stands for the full sum of n-leg counterterms.

To present H_n in explicit form, it is necessary to introduce a contracted notation for the field derivatives of various orders. Let us define

$$\partial^{[s]} \stackrel{\text{def}}{=} \partial^{\mu_1}\ldots\partial^{\mu_s}.$$

The most general triple interaction Hamiltonian density may be written as an infinite sum of local terms of the form

[4]This notion will be further refined when we consider the four-leg graphs.

$$H_3 = \frac{1}{3!}\sum_{s=0}\tilde{D}^{jk;s}:\phi(\partial^{[s]}\phi^j)(\partial_{[s]}\phi^k):, \qquad (2)$$

where $:\ldots:$ denotes the normal product,

$$\phi^i \stackrel{\text{def}}{=} K^i\phi, \quad K \stackrel{\text{def}}{=} -(\partial^\mu\partial_\mu + m^2), \quad K^i \stackrel{\text{def}}{=} \underbrace{K\ldots K}_{\text{i times}},$$

and $\tilde{D}^{jk;s}$ are real (dimensional) coupling constants. In Eq. (30), there are no derivatives acting on the field ϕ_a because one can make use of the integration by parts.

For the following, we do not need to know the form of vertices with $l > 3$ lines. Nevertheless, it may be useful to show how one can write down, say, the vertex with four lines:

$$H_4 = \frac{1}{4!}\sum_{ijk}^{\infty}\sum_{s_1s_2s_3}^{\infty}\tilde{D}^{ijk;s_1s_2s_3}:\phi(\partial^{[s_1]}\partial^{[s_2]}\phi^i)(\partial_{[s_2]}\partial^{[s_3]}\phi^j) \times(\partial_{[s_3]}\partial_{[s_1]}\phi^k):.$$

The generalization for the case of $l > 4$ lines is straightforward.

In momentum space, the Feynman rules needed to write down the two-leg graphs are constructed from the elements of bare propagator π,

$$\pi(p) = \frac{1}{p^2 - m^2} \stackrel{\text{def}}{=} \frac{1}{\kappa_p},$$

and the vertices of the form

$$V(\kappa_1,\kappa_2,\kappa_3) = i(2\pi)^4\delta(k_1 + k_2 + k_3)\sum_{i,j,k=0}^{\infty}D^{ijk}(\kappa_1)^i(\kappa_2)^j(\kappa_3)^k \qquad (3)$$

(recall that we only consider the one-loop two-leg graphs). Here and below,

$$\kappa_i \equiv (k_i^2 - m^2),$$

and D^{ijk} are just certain sums constructed from the above-introduced coupling constants $\tilde{D}^{jk;s}$ and masses.

IV. ONE-LOOP TWO-LEG FUNCTION, SELF-ENERGY, AND IRREDUCIBILITY

Using the above-given form (3), one can construct the most general expression for the one-loop two-point function that is conventionally called self-energy. It reads[5]

$$S(p^2) = \sum_{ijklmn=0}^{\infty}D^{ijk}D^{lmn}\kappa_p^i\kappa_q^l\int dr\,dt\,\delta(p + r - t) \times\delta(q + t - r)\kappa_t^{j+n-1}\kappa_r^{k+m-1} + C(p^2,\Lambda)\delta(p + q). \qquad (4)$$

Here, $C(p^2,\Lambda)$ stands for the counterterm series

$$C(p^2) = \left[C^{[\log]}(p^2)\cdot\log\Lambda + \sum_{n=0}^{\infty}C^{[n]}(p^2)\Lambda^{2n}\right], \qquad (5)$$

where Λ is the cutoff parameter and every $C^{[x]}(p^2)$ $(x = \log, 0, 1, \ldots)$ is a power series in p^2:

$$C^{[x]}(p^2) = \sum_{n=0}^{\infty}c_n^{[x]}p^{2n}.$$

Recall that in effective theory all the types of two-leg counterterms are presented in Eq. (5). The counterterms of the types $C^{[x]}(p^2)$ $(x = \log, 1, 2, \ldots)$ are needed to remove infinities, while $C^{[0]}(p^2)$ are used for the finite renormalization required by RPs.

It can be easily shown that the sum (4) contains only one nontrivial integral (it corresponds to $j + n = k + m = 0$):

$$I^{0,0} = \int\frac{dr\,dt}{\kappa_r\kappa_t}\delta(p + r - t)\delta(q + t - r) \equiv \delta(p + q)[J(p^2) + a_1\log\Lambda + a_2]. \qquad (6)$$

All the other integrals diverge like the powers of Λ. In Eq. (6), a_1 and a_2 are just arbitrary constants (depending on m^2), while the integral

$$J(p^2) \equiv -\int dr\frac{p^2 + 2rp}{(r^2 - m^2)^2\cdot[(r + p)^2 - m^2]} \qquad (7)$$

is finite.

Clearly, all the diverging terms (as well as a_1 and a_2) can be absorbed by the yet unfixed counterterm coefficients $c_n^{[x]}$ $(x = \log, 1, 2, \ldots)$, and the expression (4) can be rewritten as

$$S(p^2) = \sum_{il=0}^{\infty}D^{i00}D^{l00}\kappa_p^i\kappa_p^lJ(p^2) + \sum_{n=0}^{\infty}c_n(p^2)^n. \qquad (8)$$

Here, c_n are the new (finite) counterterm coefficients to be fixed with the help of renormalization prescriptions. Let us present Eq. (8) in the form most suitable for the following analysis. For this, it is convenient to reorder the terms in both sums as follows:

[5]For the following discussion, factors $\frac{i}{(2\pi)^4}$ are not essential and are therefore omitted. Both external momenta are considered incoming.

$$S(p^2) = \sum_{i=0}^{\infty} G^i J(p^2) \kappa_p{}^i + \sum_{i=0}^{\infty} \tilde{d}_i \, \kappa_p{}^i. \tag{9}$$

Here,

$$G^i = \sum_{k=0}^{i} D^{k00} D^{(i-k)00}, \tag{10}$$

and the coefficients $\tilde{d}_i$ (free parameters) are certain combinations of c_n and various degrees of m^2.

The problem is that the number of unknown parameters $\tilde{d}_i$ $(i = 0, 1, 2, \ldots)$ in our theory is actually infinite, while we have only two physically motivated restrictions that can be used to fix them. They are

$$S(p^2)|_{p^2=m^2} = 0 \tag{11}$$

(which fixes the pole position of the two-leg Green function) and

$$\frac{\partial}{\partial p^2} S(p^2)\bigg|_{p^2=m^2} = 0 \tag{12}$$

(which fixes the true normalization of the wave function). Let us try to fulfill formally these restrictions and analyze the results. Substituting Eq. (9) in Eq. (11), we obtain

$$\tilde{d}_0 = -G^0 J(m^2). \tag{13}$$

Then, from Eq. (12), it follows that

$$\tilde{d}_1 = G^0 J'(m^2) - G^1 J(m^2). \tag{14}$$

So, the counterterm coefficients $\tilde{d}_i$ with $i \geq 2$ remain unfixed (recall that they are certainly nonzero).

Here is a point to recall that both the requirements (11) and (12) are based on the result of the formal computation of the full propagator $P(p^2)$ by way of summing Dyson's chain constructed from an infinite number of links (two-leg insertions) connected with one another by the simple propagator.[6] Every link is considered as the 1PI full two-leg function S (conventionally called "self-energy"):

$$P(p^2) = \pi + \pi S \pi + \pi S \pi S \pi + \ldots \overset{\text{formally}}{=} \frac{1}{1 - \pi S} = \frac{1}{p^2 - m^2 - S}. \tag{15}$$

The result in the rhs of Eq. (15) is only valid under the condition that[7]

$$|\pi(p^2) S(p^2)| < 1. \tag{16}$$

In familiar renormalizable theories, this limitation is certainly fulfilled. That is why in such a case the conditions (11) and (12) can be used as legitimate RPs. However, this is not true in the case of the effective theory. To show this, let us make use of the requirement that follows[8] from Eq. (16):

$$S(p^2)|_{p^2 \to \infty} \leq O(p^2). \tag{17}$$

If this limitation is broken, the use of RPs (11) and (12) as the normalizing conditions for the two-leg function turns out groundless.

There is a different argumentation (not based on the full summing of Dyson's chain) in favor of using those RPs for the normalization of the two-leg function. It is based on the quite natural requirement: neither the pole location nor the residue should be changed by the higher orders of the loop expansion. This argumentation is no less correct than that discussed above. The problem is that in the effective theory the straightforward use of RPs (11) and (12) looks a bit naive since it certainly leads to an unsatisfactory result.

Note that the expression (14) requires attracting the RP for the nonminimal parameter

$$G^1 = 2 D^{000} D^{100}. \tag{18}$$

It can be shown that the renormalization of three-leg one-loop graphs would, in turn, require fixing the parameters G^i with $i = 2, 3, \ldots$. This contradicts what was written in Ref. [1] (and compactly recalled in Sec. II). Similarly, as has been shown above, the direct summing of Dyson's chain leads to contradiction with the Källen—Lehmann representation.

I think the reason for these contradictions lies in erroneous (naive) identification of the full expression for the one-loop two-leg function with the expression for the one-loop self-energy function that should appear in correct representation of the two-leg Green function in the area close to the pole position. Such an approach seems to me too forthright.

The physically motivated form for the expression of the two-leg Green function reads

[6] I would like to stress that at this point it is tacitly implied that every interim propagator is really presented it the chain. This is not always the case in the effective theory just because some of them might be "killed" by the corresponding factors stemming from the adjacent vertices. For this reason, it turns out to be possible to rely upon the alternative definition for the notion 1PI.

[7] The violation of this condition was a key point that allowed Veltman (see Ref. [13]) to obtain his famous conclusions concerning the description of unstable particles in the framework of quantum field theory.

[8] In fact, this is just a version of the wel-known consequence of Källen—Lehmann representation (see Chapter 10.7 in the monograph [4]).

$$G_2(p^2) = \frac{1}{p^2 - m^2 - \Sigma(p^2)} + Q(p^2), \qquad (19)$$

where $Q(m^2) = 0$ and $\Sigma(p^2)$ meets the conditions (11), (12), and (16). This form is no less grounded than Eq. (15); at the same time, it does not lead to the above-mentioned contradictions. One can rewrite Eq. (19) as follows:

$$G_2(p^2) = \pi[1 + \Sigma\pi + \Sigma\pi\Sigma\pi + \ldots] + Q(p^2). \qquad (20)$$

The form (20) gives us a hint as to what part of the full expression (9) for the two-leg function should be considered as the true self-energy $\Sigma(p^2)$. First, to avoid too rapid a growth, this part should not require the counterterms $\tilde{d}_i$ with $i > 1$ [see Eq. (9)]. Second, to avoid the necessity of attracting the nonminimal RPs, it should only contain the constants G_0; the appearance of G_i with $i \neq 0$ is inadmissible. At last, the true self-energy function $\Sigma(p^2)$ must be normalized as follows:

$$\begin{cases} \Sigma(p^2)|_{p^2=m^2} = O, \\ \frac{\partial}{\partial p^2}\Sigma(p^2)\Big|_{p^2=m^2} = O. \end{cases} \qquad (21)$$

Taking all this into account, one can present the one-loop two-leg function (9) as follows:

$$S(\kappa_p) = G^0\bar{J}(\kappa_p) + d_0 + \kappa_p d_1 + \sum_{j=1}^{\infty}\kappa_p^j G^j\bar{J}(\kappa_p) + \sum_{j=2}^{\infty}\kappa_p^j d_j. \qquad (22)$$

Here, $\bar{J}(\kappa_p)$ stands for the normalized integral (7)

$$\bar{J}(\kappa_p) \equiv J(p^2) - J(m^2), \qquad (23)$$

and

$$d_0 = 0; \qquad d_1 \equiv -G^0 J'(0); \qquad d_j \equiv \tilde{d}_j + J(0), \quad (j \geq 2). \qquad (24)$$

Note that the counterterm coefficients d_j $(j = 2, 3, \ldots)$ remain unfixed.

Now, we can rewrite the expression (22) as follows:

$$S(\kappa_p) = \Sigma(\kappa_p) + \sum_{j=0}^{\infty}\kappa_p^j R^j(\kappa_p) + \sum_{j=2}^{\infty}\kappa_p^j b_j. \qquad (25)$$

Here,

$$R^j(\kappa_p) \equiv G^j\bar{J}(\kappa_p), \qquad (26)$$

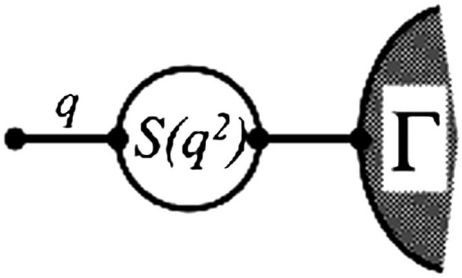

FIG. 1. One-loop insertion in the external line.

while

$$\Sigma(\kappa_p) \equiv G^0(\bar{J}(\kappa_p) - \kappa_p\bar{J}'(\kappa_p)) \qquad (27)$$

is exactly what I call the "true self-energy function" (TSE). This function meets the conditions (21) and (17); when used in Eq. (19), it does not lead to any contradiction with the Källen—Lehmann representation. Surely, those conditions only fix the first item in Eq. (25). They do not allow us to fix completely the form (8) because, first, the second item depends on the nonminimal parameters G^i $(i = 1, 2, \ldots)$ and, second, the last one contains the higher counterterms d_i with $i = 2, 3, \ldots$. Nevertheless, as will be shown below, the form (25) turns out quite sufficient for computing the one-loop dressed lines in graphs of the effective scattering theory.[9]

It is pertinent to note that the compliance of the TSE function with the conditions (21) and (17) automatically entails the conditions (11) and (12) for the full one-loop two-leg function $S(\kappa_p)$. From this, it follows that the insertion of $S(\kappa_q)$ in the external line of the arbitrary S-matrix graph (see Fig. 1) can be neglected; such insertions make no influence on the amplitudes of physical processes.

Now, we should consider the effect caused by the insertion of $S(\kappa_r)$ in the internal line (see Fig. 2).

From the fact that both the second and third items in Eq. (25) are nonminimal, it follows that this effect leads to the disappearance of the corresponding propagator (or both propagators) and the emerging of two kinds of new n-leg vertices with $n \geq 3$. In terms of equations, it looks as follows. Let $V_1(p_1, \ldots, p_m, r)$ and $V_2(q_1, \ldots, q_n, r)$ be the original (basic) vertices with $m + 1$ and $n + 1$ lines, respectively.[10] Further, let r be the line that connects them with one another. Inserting $S(q)$ in this line results in the graph Γ that can be written as follows (the common delta function is taken into account and then omitted, so $r = p_1 + \cdots + p_m = q_1 + \cdots + q_n$):

[9]Recall that we are only interested in the renormalization of S-matrix elements; the Green functions *may contain* infinite terms, which make no influence on the amplitudes of physical processes.

[10]Both l, $m \geq 2$ because the line in question is internal.

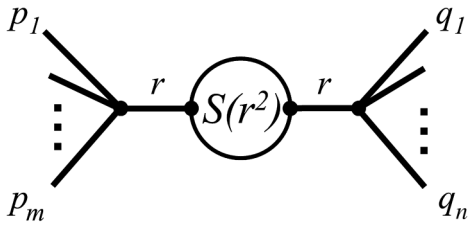

FIG. 2. One-loop insertion in the internal line.

$$\Gamma(p_1,\ldots,p_m;q_1,\ldots,q_n) = V_1(p_1,\ldots,p_m,r)\frac{1}{\kappa_r}S(r)\frac{1}{\kappa_r}V_2(q_1,\ldots,q_n,r). \quad (28)$$

According to the conventional definition,[11] this is the 1PR graph. However, as will become clear below, one can introduce the alternative ("analytical") definition of what is irreducibility [graphical 1PI (G1PI) and (A1PI)].

Let us substitute in Eq. (28) the explicit expression (25). This gives [with account taken of Eq. (18)]

$$\begin{aligned}\Gamma(p,q) &= V_1(p)\frac{1}{\kappa_r}\left[\Sigma(\kappa_r)+\sum_{j=1}^{\infty}\kappa_r^j R^j(\kappa_r)+\sum_{j=2}^{\infty}\kappa_r^j b_j\right]\frac{1}{\kappa_r}V_2(q)\\ &= V_1(p)\frac{1}{\kappa_r}\Sigma(\kappa_r)\frac{1}{\kappa_r}V_2(q)+\sum_{j=2}^{\infty}b_j[V_1(p)\kappa_r^j V_2(q)]\\ &\quad+\sum_{j=2}^{\infty}G^j\bar{J}(\kappa_p)[V_1(p)\kappa_r^{j-2}V_2(q)]\\ &\quad+D^{000}D^{100}[V_1(p)\bar{J}(\kappa_p)V_2(q)]\frac{1}{\kappa_r}\\ &\quad+\frac{1}{\kappa_r}D^{000}D^{100}[V_1(p)\bar{J}(\kappa_p)V_2(q)]. \quad (29)\end{aligned}$$

Let us analyze this result term by term.

In the left side of Eq. (29), we have the G1PR graph for the process $m \to n$ with the one-loop dressed internal line. In contrast, in the rhs, we have a sum of graphs that describe the same process, and only a part of these graphs look reducible, while the others look irreducible. It is this point where the difference between two notions (G1PI and A1PI) manifests itself clearly. In conventional renormalizable theories, there is no difference between G1PI and A1PI just because such theories do not contain the nonminimal vertices.

The first item in the rhs of Eq. (29) is just a familiar graph of the theory in which the one-loop line dressing is implemented by the vertices $g_3/3!\cdot\phi^3$. The second item presents the sum of local vertices of the counterterm type (it has the loop index $l=1$) with $(m+n)$ lines. These vertices can be absorbed by (or, equivalently, combined with) the corresponding counterterms initially presented in the full Hamiltonian (1). Those resultant conterterms should be fixed on the next steps of the renormalization procedure [one-loop renormalization of the graphs with $(m+n)$ external lines]. At the moment (and for the future), they should be simply discarded.

The third item in the rhs of Eq. (29) is a new element that has no analog in renormalizable theories. It can be treated as the *nonlocal vertex*[12] with $n \geq 4$ lines and the loop index $l=1$. When renormalizing the one-loop graphs with $n \geq 4$ legs, these elements should be taken into account on the same footing as the ordinary counterterms.

At last, the fourth and fifth items together. See the last two lines of Eq. (29) show that it emerges one more type of countervertex with $n \geq 3$ legs. The role of the countervertex with three legs will be demonstrated in the next paper devoted to the renormalization of the three-leg function.

I would like to note that $\Sigma(p^2)$ only depends on the minimal parameters. Surely, the countervertices depend on nonminimal ones. This fact, however, does not prevent one from computing the one-loop dressed lines with two-leg insertions because the nonminimal parameters only may appear in the one-loop graphs with $n \geq 3$ legs. This means that the problem of nonminimal RPs turns out to be shifted to the next stage of the renormalization procedure. As I will show in separate publication, the same phenomenon happens with three-leg graphs.

It might be useful to analyze the process of reduction of the simple (undressed) internal line.[13] This point has been discussed already in Refs. [1,2], to which I refer the reader.

What is the meaning of the above analysis? The point is that for the renormalization of a given graph it is necessary (and sufficient) to renormalize all its 1PI subgraphs. In the case in which we rely on the G1PI concept, we would need to fix the nonminimal counterterms (just because the subgraphs may be nonminimal). In contrast, when all the lines of the graph in question have been reduced (the graph is made minimal), all its subgraphs turn out minimal, and one only needs to fix the minimal counterterms. This confirms the general logical line described in Refs. [1,2].

One more note seems to be pertinent. The dressing of a line by the completely reduced two-leg graphs is nothing but the summing of Dyson's chain constructed from the minimal full two-leg insertions $\Sigma^{\infty}(p^2)$.

[11] In what follows, I call the conventional definition of one-particle irreducibility the "graphical 1PI."

[12] I call these elements "countervertices." It is important to note that the countervertices have the opposite sign as compared to the ordinary ones. This is due to the factors i presenting in both vertices V_1 and V_2.

[13] Recall that there is no need to consider the reduction of external lines because in nontrivial S-matrix graphs all the external lines are minimal.

V. DIAGONALIZATION PROBLEM

In this section, I consider the more general case, namely, the multicomponent effective scalar theory. This is because I need to point out the solution to another problem mentioned in Sec. I, namely, the problem of diagonalization (or, equivalently, the mixing problem).

Let us consider the multicomponent effective scalar theory that describes the interaction of a set (possibly infinite) of fields ϕ_k $(k = 1, 2, \ldots)$,

$$\phi_c(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p_0}[a_c^+(p)\exp(ipx) + \text{H.c.}],$$

with the conventional commutation relations

$$[a_r^-(p), a_k^+(q)]_- = (2\pi)^3 2p_0\delta_{rk}\delta(\mathbf{p}-\mathbf{q})$$

($p_0 = \sqrt{\mathbf{p}^2 + m_k^2}$). Here, m_a stands for the *mass parameter* of the particle a; this parameter is just the real part of the pole position of the full propagator. It is implied that $m_k \neq m_p$ when $k \neq p$.

The full interaction Hamiltonian density of this theory reads:

$$H(x) = \sum_{n=0}^{\infty}[H_n(x) + C_n],$$

where $H_n(x)$ is (just as above) an infinite sum of *all* Lorentz-invariant monomials constructed from the fields and their derivatives of arbitrary orders, and C_n stand for the counterterms.

The most general triple interaction Hamiltonian density may be written as

$$H_3 = \frac{1}{3!}\sum_{abc}\sum_{s=0}\tilde{D}_{abc}^{jk;s}:\phi_a(\partial^{[s]}\phi_b^j)(\partial_{[s]}\phi_c^k):, \tag{30}$$

where $:\ldots:$ denotes the normal product,

$$\phi_a^i \stackrel{\text{def}}{=} K_a^i\phi_a, \quad K_a \stackrel{\text{def}}{=} -(\partial^\mu\partial_\mu + m_a^2), \quad K_a^i \stackrel{\text{def}}{=} \underbrace{K_a\ldots K_a}_{i\text{ times}},$$

and $\tilde{D}_{abc}^{jk;s}$ are real (dimensional) coupling constants. The symbol $\sum_{abc}$ is used for the sum over the whole set of particles under consideration.

In momentum space, the Feynman rules needed to write down the two-leg graphs are constructed from the elements of bare propagator, which is the diagonal matrix π,

$$\pi_{ab}(k) = \delta_{ab}\frac{1}{\kappa_a} \equiv \delta_{ab}\frac{1}{k^2 - m_a^2},$$

and the vertices of the form

$$V_{abc}(\kappa_a, \kappa_b, \kappa_c) = i(2\pi)^4\delta(k_a + k_b + k_c) \times \sum_{i,j,k=0}^{\infty} D_{abc}^{ijk}(\kappa_a)^i(\kappa_b)^j(\kappa_c)^k. \tag{31}$$

Here,

$$\kappa_x = \kappa_x(k) \equiv k^2 - m_x^2,$$

and D_{abc}^{ijk} are certain sums constructed from the coupling constants $\tilde{D}_{abc}^{jk;s}$ and masses.

The most general expression for the one-loop two-point function reads (both lines a and b are considered incoming)

$$S_{ab}(k_a^2) = \sum_{ef}\Bigg[\int dk_e dk_f\delta(k_a + k_e - k_f)\delta(k_b + k_f - k_e) \times \sum_{ijk=\frac{0}{lmn}=0}^{\infty} D_{aef}^{ijk}D_{bfe}^{lmn}\frac{\kappa_a^i\kappa_e^j\kappa_f^k\kappa_b^l\kappa_f^m\kappa_e^n}{\kappa_e\kappa_f} + C_{abef}\delta(k_a + k_b)\Bigg]. \tag{32}$$

Here, the summation $\sum_{ef}$ is done over the whole set of particles that create the loop, and C_{abef} stands for the counterterm series

$$C_{abef}(q^2) = \left[C_{abef}^{[\log]}(q^2)\cdot\log\Lambda + \sum_{n=0}^{\infty}C_{abef}^{[n]}(q^2)\Lambda^{2n}\right], \tag{33}$$

where Λ is the cutoff parameter and every $C_{abef}^{[x]}(q^2)$ $(x = \log, 0, 1, \ldots)$ is just a power series in q^2.

Precisely as in Sec. III, one obtains the *finite* expression for the individual items of the sum over the particle set:

$$S_{abef}(q^2) = \sum_{i,l=0}^{\infty}\kappa_a^i\kappa_b^l D_{aef}^{i00}D_{bfe}^{l00}J_{ef}(q^2) + \sum_{p=0}^{\infty}\tilde{C}_{abef}^{p}(q^2)^p. \tag{34}$$

Here,

$$J_{ef}(k^2; m_e^2, m_f^2) \stackrel{\text{def}}{=} \frac{1}{2}[F_{ef} + F_{fe}], \tag{35}$$

and

$$F_{fe}(k^2; m_f^2, m_e^2) = -\int dq\frac{k^2 + 2qk}{(q^2 - m_f^2)(q^2 - m_e^2)[(q+k)^2 - m_e^2]}. \tag{36}$$

Let us present the series of finite counterterms in the equivalent (though more complex) form that is most suitable for subsequent calculations:

$$\sum_{p=0}^{\infty}\tilde{C}^{p}_{abef}(q^2)^p = q^2 C_{abef} + \sum_{i,l=0}^{\infty} S^{il}_{abef}\kappa^i_a\kappa^l_b. \quad (37)$$

Now, the expression (34) can be rewritten as follows:

$$S_{abef}(q^2) = \Sigma_{ab}(ef|q^2) + \sum_{i=1}^{\infty}\{[\kappa^i_a(G^{i0}_{abef}J_{ef}(q^2) + S^{i0}_{abef}] + [\kappa^i_b(G^{0i}_{abef}J_{ef}(q^2) + S^{0i}_{abef}]\} + \sum_{i,l\geq 1}^{\infty}\sum_{ef}\kappa^i_a\kappa^l_b\{G^{il}_{abef}J_{ef}(q^2) + S^{il}_{abef}\}. \quad (38)$$

Here (as in Sec. III), I have introduced the object $\Sigma_{ab}(ef|q^2)$, hereafter referred to as the self-energy matrix[14]:

$$\Sigma_{ab}(ef|q^2)\stackrel{\text{def}}{=}G^{00}_{abef}J_{ef}(q^2) + S^{00}_{abef} + q^2 C_{abef}. \quad (39)$$

To proceed further, one needs to fix the unknown coefficients in Eq. (38). The results of the previous section suggest that perhaps not all the coefficients are needed for the renormalization of two-leg insertion in the lines of S-matrix graphs. So, first, we need to understand what very coefficients should be fixed. For this, we should turn to the physical interpretation of the external lines of Green function graphs.

The interpretation of a given external line of the Green function is based on the LSZ formula. In short, the external line a with the momentum q corresponds to the particle with the mass parameter m_a if the relevant Green function develops the only (simple) pole at $q^2 = m_a^2$. Once there is another pole, say, at $q^2 = m_b^2$ (or the pole at $q^2 = m_a^2$ is not simple), the interpretation becomes ambiguous.

Let us discuss this point in the framework of the conventional perturbation scheme. Consider the one-loop level of a certain Green function graph with a given external line, which we would like to interpret as that corresponding to the particle with mass m_a. The interpretation problem appears when this graph is 1PR and the line in question (the a-line with momentum q) contains the one-loop self-energy insertion $S_{abef}(q^2)$ (see Fig. 1). The analytical expression for the Green function graph (in fact, this is a sum of individual graphs) under consideration reads

$$G_{a\ldots} = \frac{1}{q^2 - m_a^2}\sum_{bef}S_{abef}(q^2)\frac{1}{q^2 - m_b^2}\Gamma_{b\ldots}, \quad (40)$$

where $\Gamma_{b\ldots}$ stands for the remaining (loopless) part of the graph (the ellipsis stands for the indices corresponding to the set of external lines). The summation indices b, e, f run over the whole set of particles.

[14]Below, it will be shown that the diagonal elements of this matrix play the role of TSE functions for the corresponding particles.

The form (40) [with Eq. (34) taken into account] clearly demonstrates the presence of many poles in addition to that at $q^2 = m_a^2$. The extra poles arise from the terms with $l = 0$ and arbitrary i in the first item of Eq. (34). Moreover, when $i = l = 0$, the pole at $p^2 = m_a^2$ is of the second order. This means that in contrast to initially suggested identification we cannot uniquely associate the dressed external line [that with two-leg insertion $S(q^2)$] with any concrete particle.

In the framework of the renormalized perturbation scheme, the solution to this problem is simple. It consists of imposing the following set of limitations[15] on the nondiagonal elements $(a \neq b)$,

$$\begin{cases}\lim\limits_{q^2\to m_a^2} S_{abef}(q^2) = O\\ \lim\limits_{q^2\to m_b^2} S_{abef}(q^2) = O,\end{cases} \quad (41)$$

and

$$\begin{cases}\lim\limits_{q^2\to m_a^2} S_{aa,ef}(q^2) = O\\ \lim\limits_{q^2\to m_a^2} \frac{\partial}{\partial q^2} S_{aa,ef}(q^2) = O\end{cases} \quad (42)$$

on the diagonal ones. In what follows, I call these conditions the diagonalizability requirements.

The restrictions (41) ensure that the graph (40) does describe the interaction of the field ϕ_a associated with the particle m_a. The first of the restrictions (42) is nothing but the conventional RP that fixes the value m_a of the particle a mass parameter. The second provides a guarantee that the wave function is properly normalized. The prescriptions (42) are also suitable when there is only one particle in a theory. The RPs (41) and (42) are necessary to assign meaning to the effective scattering theory.

To proceed further, let us introduce the shortened notation:

$$G^{il}_{abef} = D^{i00}_{aef}D^{l00}_{bfe}.$$

Now, let us turn to the expression (38) and see what coefficients are fixed by the conditions (41) and (42). The conditions (41) give

$$\begin{aligned} S^{00}_{abef} &= -G^{00}_{abef}\frac{m_b^2 J_{ef}(m_a^2) - m_a^2 J_{ef}(m_b^2)}{m_b^2 - m_a^2},\\ C_{abef} &= -G^{00}_{abef}\frac{J_{ef}(m_a^2) - J_{ef}(m_b^2)}{m_a^2 - m_b^2},\\ S^{0i}_{abef} &= -G^{0i}_{abef}J_{ef}(m_a^2),\\ S^{i0}_{abef} &= -G^{i0}_{abef}J_{ef}(m_b^2). \end{aligned} \quad (43)$$

[15]In the case of unstable particles, these limitations should be imposed on the real parts; see, e.g., Ref. [14].

Similarly, the conditions (42) give

$$S^{00}_{aaef} = -G^{00}_{aaef}[J_{ef}(m_a^2) - m_a^2 J_{ef}(m_a^2)],$$
$$C_{aaef} = -G^{00}_{aaef} J_{ef}(m_a^2). \quad (44)$$

It can be easily shown that Eq. (44) follows from the two upper lines of Eq. (43) in the limit $m_a \to m_b$. This means that the diagonal elements of self-energy matrix, indeed, play the role of the self-energy functions corresponding to individual particles. The above-obtained results allow one to state that the insertion $S(q^2)$ in the external line of the S-matrix graph makes no influence on the amplitudes of physical processes. One can simply neglect them.

The influence of S^{il}_{abef} on the internal line of the S-matrix graph can be analyzed precisely in the same way as in Sec. IV.

It is obvious that the terms in the last line of Eq. (38) are insensitive to the constraints (41) and (42). Clearly, these terms play a role that is quite similar to that of corresponding terms in the last two lines of Eq. (29). The finite two-leg counterterms S^{il}_{abef} with $i, l \geq 1$ should be dropped because they are absorbed by the n-leg counterterms with $n \geq 4$ that will be fixed at the next steps of the renormalization procedure. In contrast, the terms

$$\sum_{i=1}^{\infty} \{[\kappa_a^i (G^{i0}_{abef} J_{ef}(q^2) + S^{i0}_{abef}] + [\kappa_b^i (G^{0i}_{abef} J_{ef}(q^2) + S^{0i}_{abef}]\}$$

must be taken into account because they define the nonlocal countervertices with three legs. Similarly, the terms

$$\kappa_a^i \kappa_b^l G^{il}_{abef} J_{ef}(q^2)$$

with $i, l \geq 1$ define the nonlocal countervertices with $n \geq 4$ legs.

Thus, the renormalization of two-leg insertions in the lines of the S-matrix graph of the single- and multicomponent effective scalar theories is completed. The result of the one-loop dressing of a line is finite and only depends on minimal parameters. This confirms the conclusions made in Refs. [1,2]. It is important to stress that the problem of dependence on nonminimal parameters turns out shifted to the next step of the renormalization procedure. This point will be discussed in a subsequent publication.

VI. CONCLUSION

The attractive features of the effective field theories have been demonstrated already in many papers.[16] Unfortunately, the phenomenological advantage of such theories turns out to be strongly limited by the "problem of couplings": the number of unknown phenomenological constants catastrophically increases with the number of loops taken into account.

Meanwhile, the relations obtained in Refs. [17,18,19] clearly demonstrate that the concept of the effective scattering theory together with the quasiparticle method (see Refs. [20–24] result in quite reasonable sum rules (bootstrap relations) connecting among themselves the values of hadron masses and on-shell coupling constants, which are nothing but the right sides of the renormalization prescriptions. This means that the solution to the problem of couplings requires developing the suitable renormalization procedure. The very first step on this way is done in the present paper.

The main result obtained above is that there is no need to attract the renormalization prescriptions for the higher derivatives of the two-leg graph; it turns out to be quite sufficient to rely upon the requirements of finiteness and diagonalizability. Another—no less interesting—result is the demonstration of the difference between the notions of graphical and analytical irreducibility. In fact, this result shows that until the complete reduction (as described in Refs. [1,2]) of a given graph is done there is no sense in singling out the 1PI subgraphs. This, in particular, allows one to avoid the contradiction with limitations imposed by the Källen—Lehmann representation. Of course, this preserves the correct loop counting.

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[16] The excellent review can be found in Ref. [15]; see also Ref. [16].

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