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SLq(2) extension of the standard model

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We examine a quantum group extension of the standard model. The field operators of the extended theory are obtained by replacing the field operators Ψ of the standard model by $\hat{\Psi}\mathrm{D}^j_{mm'}$, where $\mathrm{D}^j_{mm'}$ are elements of a representation of the quantum algebra SLq(2), which is also the knot algebra. The $\mathrm{D}^j_{mm'}$ lie in this algebra and carry the new degrees of freedom of the field quanta. The $\mathrm{D}^j_{mm'}$ are restricted jointly by empirical constraints and by a postulated correspondence with classical knots. The elementary fermions are described by elements of the trefoil ($j=\frac{3}{2}$) representation and the weak vector bosons by elements of the ditrefoil ($j=3$) representation. The adjoint ($j=1$) and fundamental ($j=\frac{1}{2}$) representations define hypothetical bosonic and fermionic preons. All particles described by higher representations may be regarded as composed of the fermionic preons. This preon model unexpectedly agrees in important detail with the Harari-Shupe model. The new Lagrangian, which is invariant under gauge transformations of the SLq(2) algebra, fixes the relative masses of the elementary fermions within the same family. It also introduces form factors that modify the electroweak couplings and provide a parametrization of the Cabbibo-Kobayashi-Maskawa matrix. It is additionally postulated that the preons carry gluon charge and that the fermions, which are three preon systems, are in agreement with the color assignments of the standard model.

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I. INTRODUCTION

The possibility that the elementary particles are knots has been suggested by many authors, going back as far as Kelvin [1]. Among the different field theoretic attempts to construct classical knots, a model related to the Skyrme soliton has been described by Fadeev and Niemi [2]. There are also the familiar knots of magnetic field; and since these are macroscopic expressions of the electroweak field, it is natural to extrapolate from macroscopic to microscopic knots of this same field. One expects that the conjectured microscopic knots would be quantized, and that they would be observed as solitonic in virtue of both their topological and quantum stability. It is then natural to ask if the elementary particles might also be knotted. If they are, one expects that the most elementary particles, namely the elementary fermions, are also the most elementary knots, namely the trefoils. This possibility is suggested by the fact that there are four quantum trefoils and four classes of elementary fermions, and is supported by a unique one-to-one correspondence between the topological description of the four quantum trefoils and the quantum numbers of the four fermionic classes. We have attempted to determine the minimal restrictions on a model of the elementary particles in the context of weak interactions if the quantum knot is described only by its symmetry algebra SLq(2) independent of its field theoretic origin. The use of this symmetry algebra to define the quantum knot is similar to the use of the symmetry algebra of the rotation group to define the quantum spin.

II. KINEMATICS

A. The knot algebra and the quantum knot

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a two-dimensional representation of SLq(2), the knot algebra.

Then

$$\begin{array}{llll} ab = qba & bd = qdb & ad - qbc = 1 & bc = cb \\ ac = qca & cd = qdc & da - q_1 cb = 1 & q_1 \equiv q^{-1} \end{array} \tag{1}$$

where we take q real.

Let $\mathrm{D}^j_{mm'}$ be a $2j+1$ representation of SLq(2) [3,4].

Then

$$\mathrm{D}^j_{mm'}(a,b,c,d) = \sum_{\substack{0\le s\le n_+ \\ 0\le t\le n_-}} \mathrm{A}^j_{mm'}(q,s,t)\delta(s+t,n'_+)a^s b^{n_+-s} c^t d^{n_- - t} \tag{2}$$

where

$$\mathrm{A}^j_{mm'}(q,s,t) = \left[\frac{\langle n'_+\rangle_1!\langle n'_-\rangle_1!}{\langle n_+\rangle_1!\langle n_-\rangle_1!}\right]^{\frac{1}{2}} \frac{\langle n_+\rangle_1!}{\langle n_+ - s\rangle_1!\langle s\rangle_1!} \times \frac{\langle n_-\rangle_1!}{\langle n_- - t\rangle_1!\langle t\rangle_1!}$$

and where

$$\langle n\rangle_1 = \frac{q_1^n - 1}{q_1 - 1} \quad \text{and} \quad n_\pm = j \pm m \qquad n'_\pm = j \pm m'.$$

The algebra (1) is invariant under the gauge transformations:

$$\begin{aligned} \mathrm{U}_a(1)&: a' = e^{i\varphi_a}a \qquad d' = e^{-i\varphi_a}d \\ \mathrm{U}_b(1)&: b' = e^{i\varphi_b}b \qquad c' = e^{-i\varphi_b}c. \end{aligned} \tag{3}$$

Then $\mathrm{U}_a(1)\times\mathrm{U}_b(1)$ induces on $\mathrm{D}^j_{mp}(a,b,c,d)$ the gauge transformation [3]

$$\mathrm{D}^j_{mp}(a',b',c',d') = e^{i(\varphi_a+\varphi_b)m}e^{i(\varphi_a-\varphi_b)p}\mathrm{D}^j_{mp}(a,b,c,d) \tag{4}$$

or

$$\mathrm{D}^{j'}_{mp} = \mathrm{U}_m \times \mathrm{U}_p \mathrm{D}^j_{mp}. \tag{5}$$

If $|n\rangle$ is a ket lying in the state space of SLq(2), we define a quantum knot by the state function

$$\psi \mathrm{D}^j_{mm'}|n\rangle \tag{6}$$

where ψ is a standard quantum mechanical state function and

$$(j,m,m') = \frac{1}{2}(N,w,r+o). \tag{7}$$

Here (N,w,r) are the number of crossings, the writhe, and the rotation of the 2D projection of the corresponding oriented 3D-classical knot. The factor $\frac{1}{2}$ allows half integer representations of SLq(2). Since $2m$ and $2m'$ are of the same parity while w and r are topologically constrained to be of opposite parity, o is an odd integer which we set $=1$ for a trefoil knot [3].

The knot degrees of freedom are confined to the $\mathrm{D}^j_{mm'}$ factor and are introduced here similarly to the way spin degrees of freedom are introduced by adjoining a spin factor to a state without spin. This definition of the quantum knot allows only those selected states (j,m,m') of the full $2j+1$-dimensional representation that are permitted by the (N,w,r) spectrum of the 2D projection of the corresponding classical knot. Equation (7) is the "correspondence principle" of the model where (j,m,m') describes the quantum knot and (N,w,r) refers to the corresponding classical knot.

Note: One counts only two classical trefoils because classical trefoils of opposite r are not topologically different. Their 2D projections can be distinguished, however, and states of quantum trefoils, as here defined, with opposite r can be distinguished by different m'.

B. The knotted standard model

To go from the standard model to the knotted standard model we try to replace every field operator, Ψ, of the standard model by the "knotted field operator," $\Psi^j_{mn}(x|a,b,c,d)$, where

$$\Psi^j_{mn}(x|a,b,c,d) = \hat{\Psi}^j_{mn}(x)\mathrm{D}^j_{mn}(a,b,c,d) \tag{8}$$

Since D^j_{mn} lies in the SLq(2) algebra, (8) adds new degrees of freedom to the field quanta. Then after Ψ is replaced with Ψ^j_{mn} by (8) in the standard model Lagrangian, the spacetime factor $\hat{\Psi}^j_{mn}(x)$ will be determined by a new Lagrangian containing form factors generated by D^j_{mn}. Then $\hat{\Psi}^j_{mn}(x)$ will have an induced but clear dependence on (j,m,n).

Under $\mathrm{U}_a\times\mathrm{U}_b$ transformations of the algebra (1) the new field operators transform, by (4), as follows:

$$\Psi^{j'}_{mn} = \hat{\Psi}^j_{mn}(x)\mathrm{D}^j_{mn}(a',b',c',d') \tag{9}$$

$$= \mathrm{U}_m \times \mathrm{U}_n \Psi^j_{mn}. \tag{10}$$

For physical consistency the new field action must be invariant under (10), since (10) can be induced by $\mathrm{U}_a\times\mathrm{U}_b$ transformations that leave the defining algebra unchanged. There are then Noether charges associated with U_m and U_n that may be described as writhe and rotation charges, Q_w and Q_r, since $m=\frac{w}{2}$ and $n=\frac{1}{2}(r+1)$ for trefoils.

We define

$$Q_w \equiv -k_w m\left(\equiv -k_w\frac{w}{2}\right) \tag{11}$$

$$Q_r \equiv -k_r n\left(\equiv -k_r\frac{1}{2}(r+1)\right) \tag{12}$$

where k_m and k_r are undetermined constants with dimensions of electric charge.

We now compare the so defined Q_w and Q_r charges of the four quantum trefoils with the charge and hypercharge of the four fermion families in Table I [3].

Note that with the particular row to row correspondence, $(f_1,f_2,f_3)\leftrightarrow(w,r)$, in this table, *and only for this correspondence*, is there proportionality between the (t_3,t_0,Q_e) and the (Q_w,Q_r,Q_w+Q_r) columns.

To construct and interpret this table we have postulated that k is a universal constant in the following sense:

TABLE I. Comparison with the standard model.

Standard model					Quantum trefoil model				
(f_1, f_2, f_3)	t	t_3	t_0	Q_e	(w, r)	$D^{\frac{N}{2}}_{\frac{w}{2}\frac{r+1}{2}}$	Q_w	Q_r	$Q_w + Q_r$
$(e, \mu, \tau)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	(3,2)	$D^{\frac{3}{2}}_{\frac{3}{2}\frac{3}{2}}$	$-k(\frac{3}{2})$	$-k(\frac{3}{2})$	$-3k$
$(\nu_e, \nu_\mu, \nu_\tau)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$(-3, 2)$	$D^{\frac{3}{2}}_{-\frac{3}{2}\frac{3}{2}}$	$-k(-\frac{3}{2})$	$-k(\frac{3}{2})$	0
$(d, s, b)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}e$	$(3, -2)$	$D^{\frac{3}{2}}_{\frac{3}{2}-\frac{1}{2}}$	$-k(\frac{3}{2})$	$-k(-\frac{1}{2})$	$-k$
$(u, c, t)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}e$	$(-3, -2)$	$D^{\frac{3}{2}}_{-\frac{3}{2}-\frac{1}{2}}$	$-k(-\frac{3}{2})$	$-k(-\frac{1}{2})$	$2k$
							$Q_w = -k_w \frac{w}{2}$	$Q_r = -k_r \frac{r+1}{2}$	

$$k_w = k_r = k \tag{13}$$

and

$$k = \frac{e}{3} \tag{14}$$

with the same value for all trefoils.

Then it follows from the table that the total electric charge $(Q_w + Q_r)$ of each trefoil is the same as the electric charge Q_e of the corresponding family of fermions [3,5]:

$$Q_w + Q_r = Q_e \tag{15}$$

and that

$$Q_w = et_3 \tag{16}$$

and

$$Q_r = et_0. \tag{17}$$

Then (15)–(17) are in agreement with the standard model for which

$$Q_e = e(t_3 + t_0). \tag{18}$$

Since these relations hold for the special $(f_1, f_2, f_3) \leftrightarrow (N, w, r)$ row to row correspondence in the Table I, and only for this particular match between the trefoils and the fermionic families, the correspondence itself, in addition to the value of k as $\frac{e}{3}$, is empirically fixed and unique.

We accordingly identify the writhe charge, Q_w, of the trefoil with the isotopic charge of the standard fermion, measured by t_3, and the rotation charge, Q_r, of the trefoil with the hypercharge of the standard fermion, measured by t_0. We shall then assume that the elementary fermions are quantum trefoils and that their total electric charge may be written as either

$$Q_e = Q_w + Q_r \tag{19}$$

$$\text{or} \quad Q_e = e(t_3 + t_0) \tag{20}$$

similar to the way that their total angular momentum and magnetic moment may be written as the sum of the spin and orbital contributions. (The correspondence goes further since the spin and writhe are both localized: the spin is localized on the particles, and the writhe is localized at the crossings, while the orbital angular momentum describes the entire orbital motion, and the knot rotation is computed for the entire knot.)

From Table I one may also read the following relation between the quantum trefoils of the knot model, as measured by (N, w, r), and the fermions of the standard model, as described by the isotopic charge and hypercharge [3,5]:

$$(N, w, r + 1) = 6(t, -t_3, -t_0) \tag{21}$$

or by (7)

$$(j, m, n) = 3(t, -t_3, -t_0). \tag{22}$$

The empirical correspondence between the topological description of the four quantum trefoils and the quantum numbers of the four families of fermions is encapsulated in (21). Otherwise stated, there is a unique way of satisfying (21) with the four quantum trefoils and the four classes of fermions.

C. The Fermion-Boson interaction in the knot model

According to the rule (8) the Fermion-Boson interaction terms of the standard action are multiplied by the form factors

$$\bar{D}^{j''}_{m''p''} D^{j}_{mp} D^{j'}_{m'p'} \tag{23}$$

in passing from the standard model to the knot model. Here $D^{j'}_{m'p'}$ and $\bar{D}^{j''}_{m''p''}$ multiply the initial and final Fermi operators, respectively, while D^j_{mp} multiplies the mediating boson operator of the standard model.

By $U_m(1) \times U_p(1)$ invariance of (23) we have

$$(m,p) = (m'', p'') - (m', p'). \tag{24}$$

By (22) the empirical relations

$$\begin{aligned}(m', p') &= -3(t_3, t_0)' \\ (m'', p'') &= -3(t_3, t_0)''\end{aligned} \tag{25}$$

hold for the fermion operators.

Then by (24) and (25)

$$(m,p) = -3[(t_3,t_0)'' - (t_3,t_0)']. \tag{26}$$

In passing from the standard model to the knotted model we retain $SU(2) \times U(1)$ invariance and therefore the conservation of t_3 and t_0 separately: $t_3'' = t_3' + t_3$ and $t_0'' = t_0' + t_0$. The conservation of t_3 and t_0 is also a consequence of the required $U_a(1) \times U_b(1)$ invariance of the action, and is expressed by the separate conservation of the writhe and rotation charges.

Then by (26)

$$(m,p) = -3(t_3, t_0) \tag{27}$$

for the intermediate boson as well as for the initial and final fermions. Also since

$$j' + j'' \geq j \geq |j' - j''| \quad \text{and} \quad j' = j'' = \frac{3}{2} \tag{28}$$

j is fixed by

$$3 \geq j \geq |m|. \tag{29}$$

Then one has by (27)

$$D^j_{mp} = D^3_{\pm 30} \tag{30}$$

for the charged vector bosons, where $(t, t_3, t_0) = (1, \pm 1, 0)$ and we set $j = 3$. We assume a similar relation for the neutral vector boson where $(t, t_3, t_0) = (1, 0, 0)$.

Hence there is an empirical basis, dependent also on the postulated symmetries, for

$$(j, m, m') = 3(t, -t_3, -t_0) \tag{31}$$

for both the fermions and vector bosons of the knotted model.

In both cases one may write for the field operator of the knot model

$$\begin{aligned}\hat{\Psi}(t, t_3, t_0) &= \Psi(t, t_3, t_0) D^j_{mm'} \\ &= \Psi(t, t_3, t_0) D^{3t}_{-3t_3 -3t_0}\end{aligned} \tag{32}$$

where $\Psi(t, t_3, t_0)$ is the field operator of the standard model and in both cases we have (31).

Here Ψ means left chiral when it refers to the elementary fermion. As in the standard model, we assume that the right chiral field is an isotopic singlet, but in the knot extension we assume it has the same knot factor as its left chiral partner. The right chiral state does not satisfy (32).

D. The preon representations [4,5]

In the model that we are describing, the elementary fermions, with $t = \frac{1}{2}$, are the simplest quantum knots, the trefoils, with $N = 3$ and by (7)

$$j = \frac{1}{2} N = \frac{3}{2} (= 3t). \tag{33}$$

In the same model the electroweak bosons, with $t = 1$, are quantum ditrefoils, with $N = 6$ and

$$j = \frac{1}{2} N = \frac{6}{2} (= 3t). \tag{34}$$

Then the elementary fermions lie in the $j = \frac{3}{2}$ representation while the electroweak bosons lie in the $j = 3$ representation of SLq(2).

We now consider the adjoint ($j = 1$) representation and the fundamental ($j = \frac{1}{2}$) representation of SLq(2) as defined by (2). After dropping the $A^j_{mm'}$ these are shown in Tables II and III.

We shall refer to the members of the $D^{\frac{1}{2}}$ and D^1 representations as fermionic and bosonic preons respectively.

To determine (t_3, t_0, Q) for the fermionic and bosonic preons we shall extend the relations empirically established for the elementary fermions, then extended to the electroweak bosons, and generally expressed in $D^{3t}_{-3t_3 -3t_0}$.

The results for preons are shown in Tables IV and V where equations (18) and (22)

TABLE II. $D^{\frac{1}{2}}_{mm'}$

	m'	
m	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	a	b
$-\frac{1}{2}$	c	d

TABLE III. $D^1_{mm'}$

	m'		
m	1	0	−1
1	a^2	ab	b^2
0	ac	$ad + bc$	bd
−1	c^2	cd	d^2

TABLE IV. Fermionic preons $t = \frac{1}{6}$.

	t	t_3	t_0	Q
a	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{e}{3}$
b	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0
c	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0
d	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{e}{3}$

$$(t, t_3, t_0) = \tfrac{1}{3}(j, -m, -m') \tag{22}$$

$$Q = e(t_3 + t_0) = -\frac{e}{3}(m + m') \tag{18}$$

are now read from right to left.

The values of (t, t_3, t_0) in these tables have meaning for the knot model but not for isotopic spin. In this respect the knot model provides an extension of the isotopic spin. The fractional values of t_3 and t_0 follow from (22) and measure the writhe and rotation charges, respectively.

According to Table IV *there are two preons, a and b, charged and neutral respectively and their respective antiparticles, d and c, with opposite charge and hypercharge*. These preons agree with the preons proposed by Harari and by Shupe [6,7]. We may also regard a and c and d and b as belonging to t_3 doublets.

We now show that all particles belonging to higher representations may be regarded as built up out of preons (a, b, c, d) insofar as the values of (t_3, t_0, Q) for all the composite particles may be obtained by adding the (t_3, t_0, Q) of each of the constituent preons.

We have in general by (2)

$$\mathrm{D}^j_{mm'}(a,b,c,d) = \sum \mathrm{A}^j_{mm'}(q,s,t)\delta(s+t, n'_+) a^s b^{n_+ - s} c^t d^{n_- - t} \quad 0 \le s \le n_+ \quad 0 \le t \le n_-. \tag{35}$$

Denote the exponents of (a, b, c, d) by (n_a, n_b, n_c, n_d). These will vary from term to term but there are the following structural constraints on the sum (35):

$$n_a + n_b + n_c + n_d = 2j \tag{36}$$

$$n_a + n_b - n_c - n_d = 2m \tag{37}$$

$$n_a - n_b + n_c - n_d = 2m'. \tag{38}$$

But by (21) and (22)

$$(j, m, m') = 3(t, -t_3, -t_0) \tag{39}$$

and

$$(j, m, m') = \frac{1}{2}(N, w, r + o). \tag{40}$$

Equations (39) and (40) are the basic empirical and topological constraints defining the knot model. We have shown how they hold for the $j = 3/2$ and $j = 3$ representations. We now assume that they hold for all representations allowed by the model.

We may now rewrite the structural equations (36)–(38) in terms of (t, t_3, t_0) or alternatively in terms of (N, w, r).

We shall also retain (18)

$$Q = e(t_3 + t_0) = -\frac{e}{3}(m + m')$$

for all representations.

In terms of (t, t_3, t_0, Q) Eqs. (36), (37), and (38) become by (39)

$$t = \frac{1}{6}(n_a + n_b + n_c + n_d) \tag{41}$$

$$t_3 = -\frac{1}{6}(n_a + n_b - n_c - n_d) \tag{42}$$

$$t_0 = -\frac{1}{6}(n_a - n_b + n_c - n_d). \tag{43}$$

Then

$$Q = e(t_3 + t_0) = -\frac{e}{3}(n_a - n_d). \tag{44}$$

By Table IV Eqs. (41), (42), (43), and (44) may be written as follows:

$$t = n_a t_a + n_b t_b + n_c t_c + n_d t_d \tag{45}$$

$$t_3 = n_a(t_3)_a + n_b(t_3)_b + n_c(t_3)_c + n_d(t_3)_d \tag{46}$$

$$t_0 = n_a(t_0)_a + n_b(t_0)_b + n_c(t_0)_c + n_d(t_0)_d \tag{47}$$

$$Q = n_a Q_a + n_b Q_b + n_c Q_c + n_d Q_d \tag{48}$$

or

TABLE V. Bosonic preons ($t = \frac{1}{3}$).

	t_3	t_0	$\frac{Q}{e}$	$\mathrm{D}^1_{mm'}$		t_3	t_0	$\frac{Q}{e}$	$\mathrm{D}^1_{mm'}$		t_3	t_0	$\frac{Q}{e}$	$\mathrm{D}^1_{mm'}$
$\mathrm{D}^1_{1,1}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	a^2	$\mathrm{D}^1_{0,1}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	ac	$\mathrm{D}^1_{-1,1}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	c^2
$\mathrm{D}^1_{1,0}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	ab	$\mathrm{D}^1_{0,0}$	0	0	0	$ad + bc$	$\mathrm{D}^1_{-1,0}$	$\frac{1}{3}$	0	$\frac{1}{3}$	cd
$\mathrm{D}^1_{1,-1}$	$-\frac{1}{3}$	$\frac{1}{3}$	0	b^2	$\mathrm{D}^1_{0,-1}$	0	$\frac{1}{3}$	$\frac{1}{3}$	bd	$\mathrm{D}^1_{-1,-1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	d^2

TABLE VI. Preons ($j = \frac{1}{2}$).

	Q	t_3	t_0	$D^{3t}_{-3t_3-3t_0}$
a	$-\frac{e}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$D^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} \sim a$
b	0	$-\frac{1}{6}$	$\frac{1}{6}$	$D^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} \sim b$
c	0	$\frac{1}{6}$	$-\frac{1}{6}$	$D^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} \sim c$
d	$\frac{e}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$D^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} \sim d$

$$(t, t_3, t_0, Q) = \sum_{p=(a,b,c,d)} n_p(t_p, t_{3p}, t_{0p}, Q_p). \tag{49}$$

If we now interpret (a, b, c, d) as the creation operators for the (a, b, c, d) preons, then the (n_a, n_b, n_c, n_d) represent the number of (a, b, c, d) preons respectively in each term. Then (49) states that the composite particle on the left with quantum numbers (t, t_3, t_0, Q) may be regarded as a superposition of separate states, all of which have the same (t, t_3, t_0, Q) but contain different numbers of preons (n_a, n_b, n_c, n_d) with quantum numbers $(t_p, t_{3p}, t_{0p}, Q_p)$ where $p = (a, b, c, d)$.

We illustrate (45), (46), (47), and (48) in Tables VI, VII, and VIII.

These tables may be read in two ways:

(a) as describing creation operators representing the internal state of a composite particle, or

(b) as describing a product of creation operators for the component preons.

Tables VI, VII, and VIII are computed by (35) after dropping the $A^j_{mm'}$.

The field operators are now expanded in the complete polynomials $D^j_{mm'}$ expressed in terms of the preon operators (a, b, c, d). All terms in these polynomials have the same charge and hypercharge as the composite particle on the left side of (2). If $D^j_{mm'}$ is a monomial (like the elementary fermions) the field operator creates a single state, but otherwise it creates a superposition of several states.

According to Tables VI and VII *the leptons are composed of three a preons while the neutrinos are composed of three c preons. The down quarks contain one a and two b preons while the up quarks contain one c and two d preons.*

TABLE VII. Fermions ($j = \frac{3}{2}$).

	Q	t_3	t_0	$D^{3t}_{-3t_3-3t_0}$
l	$-e$	$-\frac{1}{2}$	$-\frac{1}{2}$	$D^{\frac{3}{2}}_{\frac{3}{2}\frac{3}{2}} \sim a^3$
ν	0	$\frac{1}{2}$	$-\frac{1}{2}$	$D^{\frac{3}{2}}_{-\frac{3}{2}\frac{3}{2}} \sim c^3$
d	$-\frac{1}{3}e$	$-\frac{1}{2}$	$\frac{1}{6}$	$D^{\frac{3}{2}}_{\frac{3}{2}-\frac{1}{2}} \sim ab^2$
u	$\frac{2}{3}e$	$\frac{1}{2}$	$\frac{1}{6}$	$D^{\frac{3}{2}}_{-\frac{3}{2}-\frac{1}{2}} \sim cd^2$

TABLE VIII. Electroweak vectors ($j = 3$).

	Q	t	t_3	t_0	$D^{3t}_{-3t_3-3t_0}$
W^+	e	1	1	0	$D^3_{-3,0} \sim c^3d^3$
W^-	$-e$	1	-1	0	$D^3_{3,0} \sim a^3b^3$
W^3	0	1	0	0	$D^3_{0,0} \sim f_3(bc)$
W^0	0	0	0	0	$D^0_{0,0} \sim f_0(bc)$

These descriptions of the elementary fermions as three preon structures are in agreement with the Harari-Shupe model [6,7]. In Table VIII the charged vectors are also in agreement with the same model, but the neutral vector W^3_μ is the superposition of four states of six preons given by

$$D^3_{00} = A(0,3)b^3c^3 + A(1,2)ab^2c^2d + A(2,1)a^2bcd^2 + A(3,0)a^3d^3$$

according to (2) and expressible by the algebra of (1) as a function of the neutral operator bc.

E. The complementary models

Since $(N, w, r + o) = 2(j, m, m')$, Eqs. (36)–(38) giving (j, m, m') may also be read as knot relations as follows:

$$N = n_a + n_b + n_c + n_d \tag{50}$$

$$w = n_a + n_b - n_c - n_d \tag{51}$$

$$r + o = n_a + n_c - n_b - n_d. \tag{52}$$

There are only three equations to determine the four (n_a, n_b, n_c, n_d). Therefore the composite particle, either (t, t_3, t_0) or (N, w, r), is in general a superposition of several components with different sets of (n_a, n_b, n_c, n_d).

Equation (50) states that the total number of preons equals the number of crossings (N).

Since we assume that the preons are fermions, the knot describes a fermion or a boson depending on whether the number of crossings is odd or even.

The meaning of Eqs. (51) and (52) becomes clear if we note that a and d are antiparticles with opposite charge and hypercharge, while b and c are neutral antiparticles with opposite values of the hypercharge.

We may therefore introduce the preon numbers

$$\nu_a = n_a - n_d \tag{53}$$

$$\nu_b = n_b - n_c. \tag{54}$$

Then (51) and (52) may be rewritten as

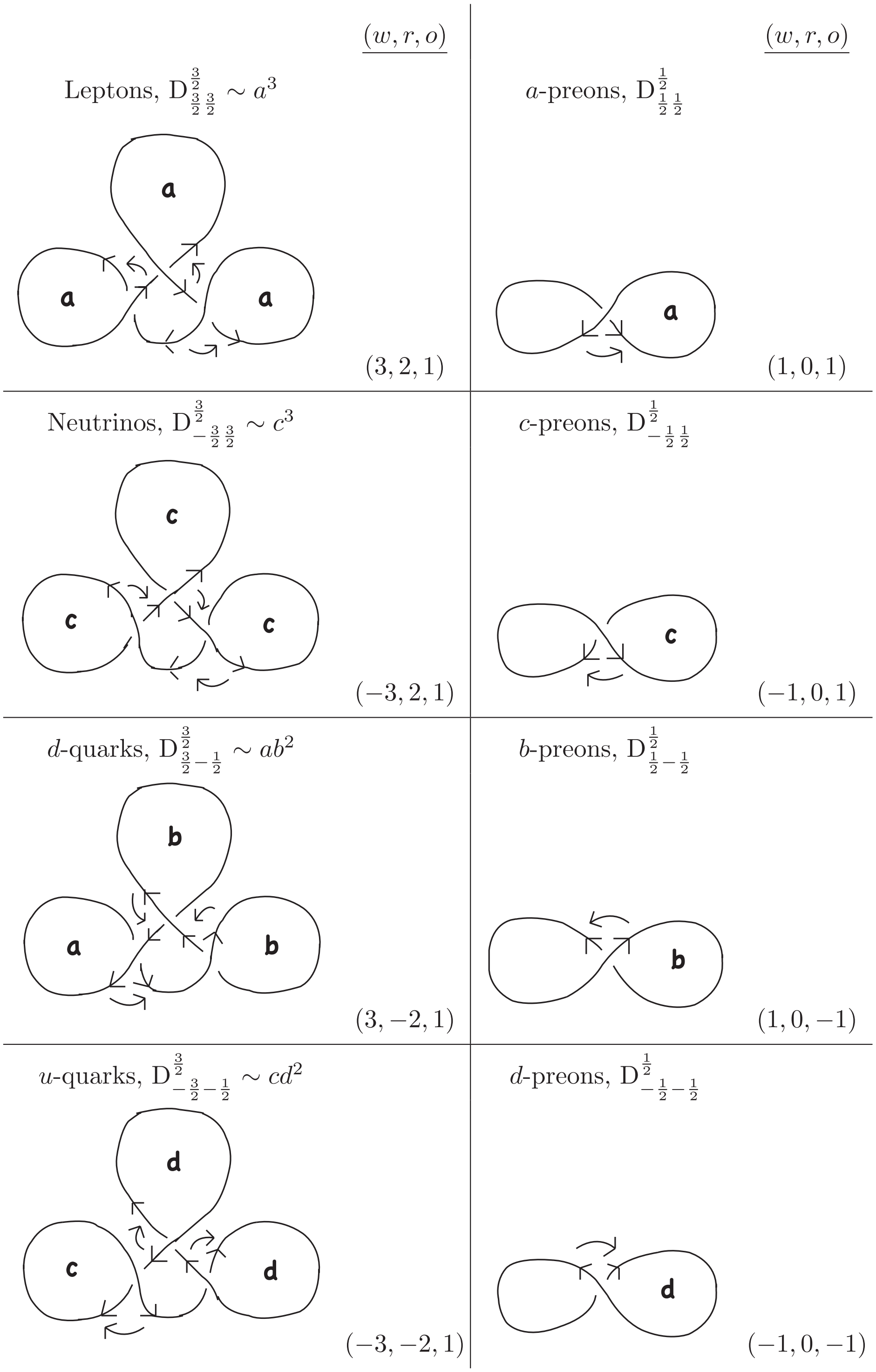

FIG. 1. $Q = -\frac{e}{6}(w + r + o)$, $(j, m, m') = \frac{1}{2}(N, w, r + o)$.

$$\nu_a + \nu_b = w = -6t_3 \tag{55}$$

$$\nu_a - \nu_b = r + o = -6t_0. \tag{56}$$

By (55) and (56) the conservation of the preon numbers and of charge and hypercharge is equivalent to the conservation of the writhe and rotation which are topologically conserved at the classical level. In this respect, these conservation laws may be regarded as topological.

The SLq(2) equations (50), (51), (52) hold for all representations and therefore for preons as well as for knots, although the preons are twisted loops rather than knots. If the indices (N, w, r) for the twisted loops interpreted as fermionic preons are determined in the same way as for knots, one finds $N = 1$, $w = \pm 1$, and $r = 0$. Then by (52) the odd integer, o, is for fermionic preons

$$o = n_a + n_b - n_c - n_d. \tag{57}$$

It follows that $o = 1$ for a and b, and that $o = -1$ for the antiparticles, d and c.

Viewed as a knot, a fermion becomes a boson when the number of crossings is changed by attaching or removing a curl. This picture is consistent with the view of a curl as an opened preon loop.

Corresponding to the representations of the four elementary fermions as three preon states, there is the complementary representation of the four trefoils as composed of three overlapping preon loops as shown in Fig. 1.

In interpreting Fig. 1 note that the two lobes of all the preons make opposite contributions to the rotation, r, so that the total rotation of each preon vanishes. When the three a preons and c preons are combined to form leptons and neutrinos, respectively, each of the three labeled circuits is counterclockwise and contributes $+1$ to the rotation while the single unlabeled shared circuit is clockwise and contributes -1 to the rotation so that the total r for both leptons and neutrinos is $+2$. For the quarks the three labeled loops contribute -1 and the shared loop $+1$ so that $r = -2$.

Written in terms of (N, w, r) and $(N, w, r)_p$ Eqs. (50)–(52) describing the composite particles are

$$N = \sum_p n_p N_p \tag{58a}$$

$$w = \sum_p n_p w_p \tag{58b}$$

$$\tilde{r} = \sum_p n_p \tilde{r}_p \tag{58c}$$

where $p = (a, b, c, d)$ and

$$\tilde{r} = r + o. \tag{59}$$

For preons

$$\tilde{r}_p = o_p. \tag{60}$$

For the elementary fermions of the standard model

$$\tilde{r} = r + 1. \tag{61}$$

These considerations lead one to view the symmetry of an elementary particle, defined by representations of the SLq(2) algebra, in any of the following ways:

$$\mathrm{D}^j_{mm'} = \mathrm{D}^{3t}_{-3t_3 - 3t_0} = \mathrm{D}^{\frac{N}{2}}_{\frac{w}{2}\frac{\tilde{r}}{2}} = \tilde{\mathrm{D}}^{N'}_{\nu_a \nu_b} \tag{62}$$

where N' is the total number of preons. The point particle-flux loop complementary representations are related by

$$\tilde{\mathrm{D}}^{N'}_{\nu_a \nu_b} = \sum_{Nwr} \delta(N', N)\delta(\nu_a + \nu_b, w)\delta(\nu_a - \nu_b, \tilde{r})\mathrm{D}^{\frac{N}{2}}_{\frac{w}{2}\frac{\tilde{r}}{2}}. \tag{63}$$

Since one may interpret the elements (a, b, c, d) of the SLq(2) algebra as creation operators for either preonic particles or flux loops, the D^j_{mp} may be interpreted as a creation operator for a composite particle composed of either preonic particles or flux loops. These two complementary views of the same structure may be reconciled as describing N-body systems bound by a knotted field having N crossings with the particles at the crossings, as illustrated in Fig. 2 for $N = 3$. In the limit where the three outside lobes become infinitesimal compared to the central circuit, the resultant structure will resemble a three particle system tied together by a Nambu-like string. Since the topological diagram of Fig. 2 describes loops that have no size or shape, one needs to introduce an explicit Lagrangian to go further.

F. Gluon charge

The previous considerations are based on electroweak physics. To describe the strong interactions it is necessary according to the standard model to introduce SU (3). We may assume that only the a and c preons carry gluon charge and that the b and d preons are color singlets [8]. The a and c preon operators then appear in triplicate as (a_i, c_i) where $i = (R, Y, G)$ without changing the (a, b, c, d) algebra. These colored preon operators provide a basis for the fundamental representation of SU(3) just as the colored quark operators do in standard theory. To adapt the electroweak operators to the requirements of gluon fields we make the following replacements:

$$\text{leptons}: a^3 \to \varepsilon^{ijk} a_i a_j a_k \tag{64}$$

$$\text{neutrinos}: c^3 \to \varepsilon^{ijk} c_i c_j c_k \tag{65}$$

$$\text{down quarks}: ab^2 \to a_i b^2 \tag{66}$$

$$\text{up quarks}: cd^2 \to c_i d^2. \tag{67}$$

Here (a_i, c_i) are creation operators for colored preons. Then the leptons and neutrinos are color singlets while the quark states correspond to the fundamental representations of SU(3), as required by the standard model.

The need for additional SU(3) symmetry to satisfy the Pauli exclusion principle thus appears already at the level of leptons (a^3) and neutrinos (c^3) in the SLq(2) *preon model*. Then there is gluon charge on the a and c preons, but not on the b and d preons. This distribution of gluon

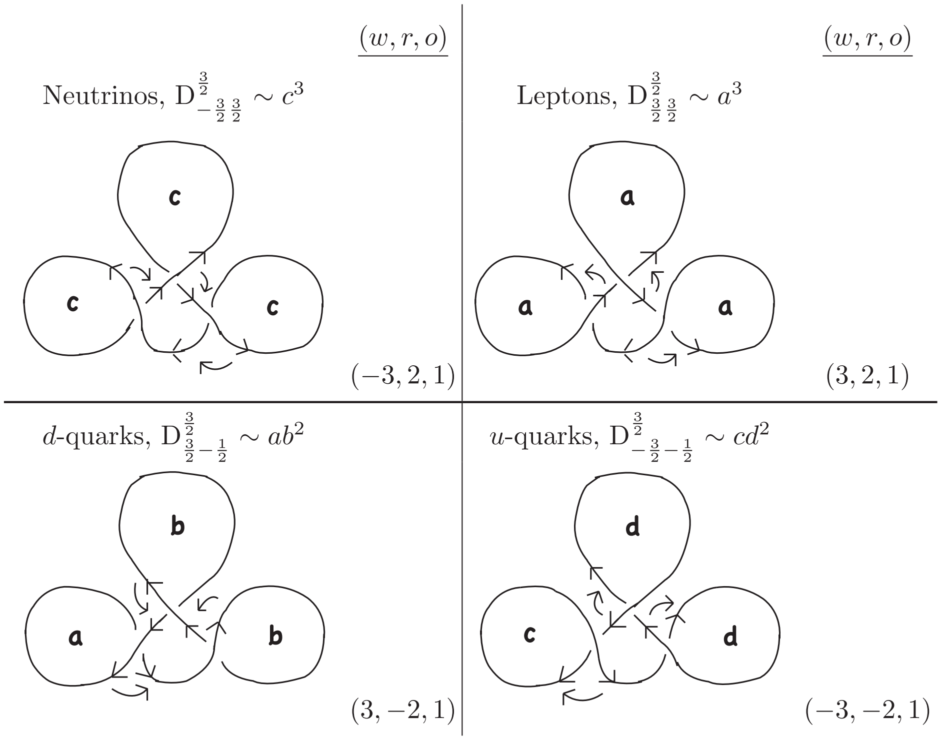

FIG. 2. The preons conjectured to be present at the crossings are not shown in these figures.

charge among the preons agrees with the distribution of gluon charge among the hadrons in the standard model [8].

III. DYNAMICS

A. The knot Lagrangian

The Lagrangian of the standard model at the electroweak level may be written as follows:

$$\mathcal{L}_{st} = -\frac{1}{4}\mathrm{Tr}\mathrm{W}^{\mu\lambda}\mathrm{W}_{\mu\lambda} - \frac{1}{4}\mathrm{H}^{\mu\lambda}\mathrm{H}_{\mu\lambda} + i[\bar{L}\nabla L + \bar{R}\nabla R] + \frac{1}{2}\nabla\bar{\varphi}\nabla\varphi - V(\bar{\varphi}\varphi) - \frac{m}{\rho}[\bar{L}\varphi R + \bar{R}\,\bar{\varphi}\,L]. \tag{68}$$

To obtain the knot Lagrangian we attempt to replace every field operator of the standard model by

$$\Psi \to \hat{\Psi}\mathrm{D}^{j}_{mm'} \tag{69}$$

where the $\mathrm{D}^{j}_{mm'}$ are determined empirically, as discussed in part I, Sec. 2 B subject to the requirement that every term of the modified Lagrangian be $\mathrm{SU}(2)\times\mathrm{U}(1)$ and $\mathrm{U}_a(1)\times\mathrm{U}_b(1)$ invariant.

To implement (69) we begin with

$$\Psi(t, t_3, t_0) \to \Psi(t, t_3, t_0)\mathrm{D}^{3t}_{-3t_3-3t_0} \tag{70}$$

and

$$Q = e(t_3 + t_0) \tag{71}$$

or

$$Q = -\frac{e}{3}(m + m') = -\frac{e}{6}(w + r + 1) \tag{72}$$

for the left chiral field, L, and for the vector bosons as discussed in part I, Sec. 2 B.

We assume that every right chiral field, R, has the same knot factor, $\mathrm{D}^{j}_{mm'}$, as the corresponding L field. We shall also assume that R is an isotopic singlet, with $t = 0$, here as in the standard model. Then R does not and is not required to satisfy (70). Since we shall, however, assume that

$$Q = -\frac{e}{3}(m + m') \tag{73}$$

holds for both L and R, it follows that L and R carry the same electric charge.

If the modification of the standard model is made according to the preceding substitutions, it will be shown that the new Lagrangian will be $\mathrm{U}_a(1)\times\mathrm{U}_b(1)$ invariant as required, and all new factors and terms appearing in the new Lagrangian will be functions of bc.

The new operator Lagrangian is then numerically valued on eigenstates of bc and is therefore a function of $\beta\gamma$, the eigenvalues of bc. In the simplest use of this model, the physical knot Lagrangian is defined on a single eigenstate, $|n\rangle$, which is identified by the empirical value of $\beta\gamma$, that is in turn determined by measurement of the physical observables from the new Lagrangian.

To obtain the knot Lagrangian the standard Lagrangian will be replaced term by term beginning with the mass terms.

B. The mass terms

In the standard model L and φ are isotopic doublets while $(\bar{L}\varphi)$ and R are isotopic singlets. We retain this isotopic structure and continue to follow the standard model by going to the unitary gauge where φ has a single component which is neutral. In passing to the SLq(2) algebra we assume that φ is a SLq(2) singlet.

To obtain the mass term for the leptons, we write for the left chiral fields

$$L(l) = \begin{pmatrix} \hat{\nu}_L & D^{\frac{3}{2}}_{-\frac{3}{2}\frac{3}{2}} \\ \hat{l}_L & D^{\frac{3}{2}}_{\frac{3}{2}\frac{3}{2}} \end{pmatrix} \to \begin{pmatrix} \hat{\nu}_L c^3 \\ \hat{l}_L a^3 \end{pmatrix} \tag{74}$$

where $\begin{pmatrix} \nu \\ l \end{pmatrix}_L$ is the doublet of the standard model and $\begin{pmatrix} \hat{\nu}_L \\ \hat{l}_L \end{pmatrix}$ is the corresponding doublet of the knot model. In the following equations the knot symbol, $\hat{\square}$, will be dropped in some terms but should be understood. The numerical coefficients in $D^j_{mm'}$ have been temporarily dropped and the monomials $D^{\frac{3}{2}}_{-\frac{3}{2}\frac{3}{2}}$ and $D^{\frac{3}{2}}_{\frac{3}{2}\frac{3}{2}}$ have been replaced by c^3 and a^3. Now, having assumed that φ is a SLq(2) singlet and that the knot factors for R and L are the same, one has

$$\bar{L}(l)\varphi_l R(l) = \left[\begin{pmatrix} \bar{\nu}_L\bar{c}^3 & \bar{l}_L\bar{a}^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \rho_l \end{pmatrix}\right](l_R a^3). \tag{75}$$

Here the Higgs doublet is in the unitary gauge and ρ_l is its neutral component.

Then

$$\bar{L}(l)\varphi_l R(l) = \rho_l(\bar{a}^3 a^3)\cdot \bar{l}_L l_R. \tag{76}$$

Similarly

$$\bar{R}(l)\bar{\varphi}_l L(l) = (\bar{a}^3\bar{l}_R)\cdot\left[\begin{pmatrix} 0 & \rho_l \end{pmatrix}\cdot\begin{pmatrix} \nu_L c^3 \\ l_L a^3 \end{pmatrix}\right] \tag{77}$$

$$= \rho_l(\bar{a}^3 a^3)\bar{l}_R l_L. \tag{78}$$

Hence

$$\bar{L}(l)\varphi_l R(l) + \bar{R}(l)\bar{\varphi}_l L(l) = \rho_l(\bar{a}^3 a^3)(\bar{l}_L l_R + \bar{l}_R l_L) \tag{79}$$

and

$$\langle n|\bar{L}(l)\varphi_l R(l) + \bar{R}(l)\bar{\varphi}_l L(l)|n\rangle = \rho_l\langle n|\bar{a}^3 a^3|n\rangle(\bar{l}l) = m_l\bar{l}l \tag{80}$$

where

$$m_l = \rho_l\langle n|\bar{a}^3 a^3|n\rangle \tag{81}$$

and ρ_l is the vacuum expectation value of the Higgs that fixes the lepton masses [multiplied by numerical factors dropped in (74)].

In (81) $\bar{a}^3 a^3$ is an operator holding for any member of the lepton family and by the algebra (1) is expressible as a simple polynomial in bc. We shall distinguish the three mass states by $|n\rangle$, $n = 0, 1, 2$, where $|n\rangle$ is an eigenstate of b and c and of the mass operator, expressed as a function of b and c. We therefore replace the lepton contribution to the mass term of the standard model by

$$\sum_n \langle n|\bar{L}_l\varphi_l R_l + \bar{R}_l\bar{\varphi}_l L_l|n\rangle \tag{82}$$

where n is summed over the three generations of leptons. Since $\langle n|\bar{a}^3 a^3|n\rangle$ depends on n while ρ_l does not depend on n, one may compute the mass ratios $\frac{m_\tau}{m_\mu}$ and $\frac{m_\mu}{m_e}$ from (81) in terms of the eigenvalues β and γ of b and c on the ground state $|0\rangle$[9,10].

To obtain the neutrino masses one needs a conjugate Higgs doublet $\begin{pmatrix} \rho_\nu \\ 0 \end{pmatrix}$. Then

$$\left[\begin{pmatrix} \bar{\nu}_L\bar{c}^3 & \bar{l}_L\bar{a}^3 \end{pmatrix}\begin{pmatrix} \rho_\nu \\ 0 \end{pmatrix}\right](\nu_R c^3) = \rho_\nu(\bar{c}^3 c^3)\bar{\nu}_L\nu_R \tag{83}$$

and

$$\bar{L}_\nu\varphi_\nu R_\nu + \bar{R}_\nu\bar{\varphi}_\nu L_\nu = m_\nu\bar{\nu}\nu \tag{84}$$

where

$$m_\nu = \rho_\nu\langle n|\bar{c}^3 c^3|n\rangle. \tag{85}$$

The same discussion may be repeated for the up and down members of the quark doublet, and summarized by replacing the mass term of the standard model by [9,10]

$$\sum_i\sum_n \langle n|\bar{L}(i)\varphi(i)R(i) + \bar{R}(i)\bar{\varphi}(i)L(i)|n\rangle \tag{86}$$

where n is summed over the three generations of each family and i is summed over the four families: $i = (l, \nu, u, d)$. The quark masses obtained in same way as (81) and (85) are

$$m_d = \rho_d\langle n|\bar{b}^2\bar{a}\cdot ab^2|n\rangle \tag{87}$$

and

$$m_u = \rho_u \langle n|\bar{d}^2\bar{c}\cdot cd^2|n\rangle. \quad (88)$$

C. The fermion-boson interaction

In the standard model this interaction is expressed by

$$i(\bar{L}\nabla L + \bar{R}\nabla R) \quad (89)$$

where ∇ is the covariant derivative

$$\nabla = \not{\partial} + \not{\mathcal{W}} \quad (90)$$

and $\mathcal{W}$ is the vector connection

$$\not{\mathcal{W}} = ig(\not{W}^+ t_+ + \not{W}^- t_- + \not{W}^3 t_3) + ig_0 \not{W}^0 t_0. \quad (91)$$

We shall describe in detail only the non-Abelian contribution to $\mathcal{W}$. [The Abelian (g_o) term may be described in a simpler way.]

To go over to the SLq(2) model, replace (W^+, W^-, W^3) according to (70) by

$$(W^+D^3_{-30}, W^-D^3_{30}, W^3D^3_{00}) \quad (92)$$

and replace

$$(W^+t_+, W^-t_-, W^3t_3) \quad (93)$$

in (91) by

$$(W^+D^3_{-30}\cdot t_+, W^-D^3_{30}\cdot t_-, W^3D^3_{00}\cdot t_3) \quad (94)$$

or by

$$(W^+\tau_+, W^-\tau_-, W^3\tau_3) \quad (95)$$

where

$$\tau_\pm = c_\pm t_\pm \mathcal{D}_\pm \quad (96)$$

$$\tau_3 = c_3 t_3 \mathcal{D}_3. \quad (97)$$

Here the $(c_\pm, c_3)$ are undetermined constants and

$$\mathcal{D}_+ = c^3 d^3 (\equiv D^3_{-30}/A^3_{-30}) \quad (98)$$

$$\mathcal{D}_- = a^3 b^3 (\equiv D^3_{30}/A^3_{30}) \quad (99)$$

$$\mathcal{D}_3 = f_3(bc)(\equiv D^3_{00}). \quad (100)$$

In defining $\mathcal{D}_+$ and $\mathcal{D}_-$ we may set

$$d = \bar{a} \quad c = \bar{b}$$

i.e., we may identify the creation operators for the d and c preons with the creation operators for the antiparticles of the a and b preons, respectively, in agreement with Table V.

Then

$$\mathcal{D}_- = \bar{\mathcal{D}}_+.$$

The $A^3_{\pm 30}$ numerical factors are absorbed in the $c_\pm$. The $(c_\pm, c_0)$ will be empirically determined in the next section 10.

The non-Abelian contribution to the covariant derivative in the knot model is now

$$\nabla = \not{\partial} + ig(\not{W}^+\tau_+ + \not{W}^-\tau_- + \not{W}^3\tau_3) \quad (101)$$

and the non-Abelian part of the fermion-boson interaction in the knot Lagrangian is

$$\sum_i \sum_n \langle n|\bar{L}(i)\nabla L(i)|n\rangle \quad (102)$$

where L, R, and ∇ are now all lying in the SLq(2) algebra, and where the sum over n is over the three generations, while the sum over i is over the two doublets. The only modification of ∇ in going over to the knot model is the replacement of $\vec{t}$ by $\vec{\tau}$.

We next consider the detailed dependence of (102) on knot form factors. For the lepton-neutrino doublet we have, dropping the Feynman slash,

$$\bar{L}\nabla L = \begin{pmatrix} \bar{\nu} & \bar{l} \end{pmatrix}_L (\partial + ig\mathcal{W}) \begin{pmatrix} \nu \\ l \end{pmatrix}_L. \quad (103)$$

The first term in (103) is

$$\begin{aligned} \begin{pmatrix} \bar{\nu} & \bar{l} \end{pmatrix}_L \partial \begin{pmatrix} \nu \\ l \end{pmatrix}_L &= (\bar{c}^3\bar{\nu}_L)\partial(c^3\nu_L) + (\bar{a}^3\bar{l}_L)\partial(a^3 l_L) \\ &= (\bar{c}^3 c^3)\bar{\nu}_L\partial\nu_L + (\bar{a}^3 a^3)\bar{l}_L\partial l_L. \end{aligned} \quad (104a)$$

Here

$$\begin{pmatrix} \nu \\ l \end{pmatrix}_L \equiv \begin{pmatrix} c^3\nu_L \\ a^3 l_L \end{pmatrix}$$

is the knot doublet and

$$\begin{pmatrix} \nu_L \\ l_L \end{pmatrix}$$

is the doublet of the standard model.

Equation (104a) may be rewritten as

$$\langle n|\begin{pmatrix} \bar{\nu} & \bar{l} \end{pmatrix}_L \partial \begin{pmatrix} \nu \\ l \end{pmatrix}_L |n\rangle = \bar{\nu}_L\Delta_\nu\nu_L + \bar{l}_L\Delta_l l_L \quad (104b)$$

where

$$\Delta_\nu = \langle n|\bar{c}^3 c^3|n\rangle\partial \qquad \Delta_l = \langle n|\bar{a}^3 a^3|n\rangle\partial. \tag{104c}$$

Then Δ_ν and Δ_l are modified momentum operators rescaled with the same factors that rescale the neutrino and lepton rest masses found in the previous section.

The second term of (103) is by (96), (97), and (101)

$$\begin{pmatrix}\bar\nu & \bar l\end{pmatrix}_L \mathcal{W}\begin{pmatrix}\nu \\ l\end{pmatrix}_L = \begin{bmatrix}\bar{c}^3\bar\nu_L & \bar{a}^3\bar{l}_L\end{bmatrix}\begin{bmatrix}c_3\mathcal{D}_3\mathrm{W}^3 & c_+\mathcal{D}_+\mathrm{W}^+ \\ c_-\mathcal{D}_-\mathrm{W}^- & -c_3\mathcal{D}_3\mathrm{W}^3\end{bmatrix}\begin{bmatrix}c^3\nu_L \\ a^3 l_L\end{bmatrix} \tag{105}$$

$$\begin{aligned} &= \begin{bmatrix}\bar{c}^3\bar\nu_L & \bar{a}^3\bar{l}_L\end{bmatrix}\begin{bmatrix}c_3\mathcal{D}_3\mathrm{W}^3\cdot c^3\nu_L + c_+\mathcal{D}_+\mathrm{W}^+\cdot a^3 l_L \\ c_-\mathcal{D}_-\mathrm{W}^-\cdot c^3\nu_L - c_3\mathcal{D}_3\mathrm{W}^3\cdot a^3 l_L\end{bmatrix} \\ &= c_3(\bar{c}^3\mathcal{D}_3 c^3)(\bar\nu_L\mathrm{W}^3\nu_L) + c_+(\bar{c}^3\mathcal{D}_+a^3)(\bar\nu_L\mathrm{W}^+l_L) \\ &\quad + c_-(\bar{a}^3\mathcal{D}_-c^3)(\bar{l}_L\mathrm{W}^-\nu_L) - c_3(\bar{a}^3\mathcal{D}_3a^3)(\bar{l}_L\mathrm{W}^3 l_L). \end{aligned} \tag{106}$$

There are four form factors stemming from the knot degrees of freedom, namely:

$$\mathrm{F}_{\bar\nu\nu} = c_3\langle n|\bar{c}^3\mathcal{D}_3c^3|n\rangle = c_3\langle n|\bar{c}^3 f_3(bc)c^3|n\rangle \tag{107}$$

$$\mathrm{F}_{\bar{l}l} = c_3\langle n|\bar{a}^3\mathcal{D}_3a^3|n\rangle = c_3\langle n|\bar{a}^3 f_3(bc)a^3|n\rangle \tag{108}$$

$$\mathrm{F}_{\bar\nu l} = c_+\langle n|\bar{c}^3\mathcal{D}_+a^3|n\rangle = c_+\langle\bar{c}^3(c^3d^3)a^3|n\rangle \tag{109}$$

$$\mathrm{F}_{\bar{l}\nu} = c_-\langle n|\bar{a}^3\mathcal{D}_-c^3|n\rangle = c_-\langle n|\bar{a}^3(a^3b^3)c^3|n\rangle. \tag{110}$$

Here $f_3(bc) = \mathrm{D}^3_{00}$ as in (100).

Then the interaction is

$$\mathrm{F}_{\bar\nu\nu}(\bar\nu_L\mathrm{W}^3\nu_L) - \mathrm{F}_{\bar{l}l}(\bar{l}_L\mathrm{W}^3l_L) + \mathrm{F}_{\bar\nu l}(\bar\nu_L\mathrm{W}^+l_L) + \mathrm{F}_{\bar{l}\nu}(\bar{l}_L\mathrm{W}^-\nu_L). \tag{111}$$

All of these form factors are invariant under $\mathrm{U}_a(1)\times\mathrm{U}_b(1)$ since a and d, as well as b and c, transform oppositely and each operator transforms oppositely to its adjoint.

For the up-down quark doublet we have

$$\bar{L}\nabla L = \begin{pmatrix}\bar u & \bar d\end{pmatrix}_L(\partial + ig\mathcal{W})\begin{pmatrix}u \\ d\end{pmatrix}_L \tag{112}$$

where

$$\begin{pmatrix}u \\ d\end{pmatrix}_L = \begin{pmatrix}cd^2\cdot u_L \\ ab^2\cdot d_L\end{pmatrix}. \tag{113}$$

Here again $\begin{pmatrix}u \\ d\end{pmatrix}_L$ is the knot doublet while $\begin{pmatrix}u_L \\ d_L\end{pmatrix}$ is the doublet in the standard model.

The first term of (112) is

$$\begin{aligned}\begin{pmatrix}\bar u & \bar d\end{pmatrix}_L\partial\begin{pmatrix}u \\ d\end{pmatrix}_L &= \overline{cd^2}\bar{u}_L\partial(cd^2)u_L + \overline{ab^2}\bar{d}_L\partial(ab^2)d_L \\ &= (\overline{cd^2}cd^2)\bar{u}_L\partial u_L + (\overline{ab^2}ab^2)\bar{d}_L\partial d_L.\end{aligned} \tag{114a}$$

Equation (114b) may be rewritten as

$$\langle n|\begin{pmatrix}\bar u & \bar d\end{pmatrix}_L\partial\begin{pmatrix}u \\ d\end{pmatrix}_L|n\rangle = \bar{u}_L\Delta_u u_L + \bar{d}_L\Delta_d d_L \tag{114b}$$

where

$$\Delta_u = \langle n|\overline{cd^2}cd^2|n\rangle\partial \qquad \Delta_d = \langle n|\overline{ab^2}ab^2|n\rangle\partial. \tag{114c}$$

Here Δ_u and Δ_d are modified momentum operators again rescaled with the same factors that rescale the rest masses of the u and d quarks in the previous section.

The second term is

$$\begin{aligned} &c_3[(\overline{cd^2})\mathcal{D}_3(cd^2)](\bar{u}_L\mathrm{W}^3u_L) \\ &+ c_+[(\overline{cd^2})\mathcal{D}_+(ab^2)](\bar{u}_L\mathrm{W}^+d_L) \\ &+ c_-[(\overline{ab^2})\mathcal{D}_-(cd^2)](\bar{d}_L\mathrm{W}^-u_L) \\ &- c_3[(\overline{ab^2})\mathcal{D}_3(ab^2)](\bar{d}_L\mathrm{W}^3d_L). \end{aligned} \tag{115}$$

The interaction term in Eq. (112) is then the sum of four parts.

$$\mathrm{F}_{\bar{u}u}(\bar{u}_L\mathrm{W}^3u_L) - \mathrm{F}_{\bar{d}d}(\bar{d}_L\mathrm{W}^3d_L) + \mathrm{F}_{\bar{u}d}(\bar{u}_L\mathrm{W}^+d_L) + \mathrm{F}_{\bar{d}u}(\bar{d}_L\mathrm{W}^-u_L) \tag{116}$$

where the four form factors are

$$\mathrm{F}_{\bar{u}u} = c_3\langle n|\overline{cd^2}f_3(bc)cd^2|n\rangle \tag{117}$$

$$\mathrm{F}_{\bar{d}d} = c_3\langle n|\overline{ab^2}f_3(bc)ab^2|n\rangle \tag{118}$$

$$\mathrm{F}_{\bar{u}d} = c_+\langle n|(\overline{cd^2})\mathcal{D}_+(ab^2)|n\rangle = c_+\langle n|\overline{cd^2}(c^3d^3)ab^2|n\rangle \tag{119}$$

$$\mathrm{F}_{\bar{d}u} = c_-\langle n|(\overline{ab^2})\mathcal{D}_-(cd^2)|n\rangle = c_-\langle n|\overline{ab^2}(a^3b^3)cd^2|n\rangle. \tag{120}$$

All of these form factors are invariant under $\mathrm{U}_a(1)\times\mathrm{U}_b(1)$ since a and d transform oppositely as do b and c.

After passing to SUq(2) all of the four form factors may be evaluated in terms of q and β where β is the eigenvalue of b on the ground state.

Since the R fields are SU(2) singlets, they are invariant under SU(2) transformations and are not subject to SU(2) interactions. They do transform according to hypercharge (t_0), or rotation charge. These are U(1) gauge transformations, and $\bar{R}\nabla R$ is the sum of the following four parts:

$$\langle n|\bar{c}^3c^3|n\rangle(\bar{\nu}_R(\partial + \mathrm{W}^0)\nu_R) \tag{121}$$

$$\langle n|\bar{a}^3a^3|n\rangle(\bar{l}_R(\partial + \mathrm{W}^0)l_R) \tag{122}$$

$$\langle n|\overline{cd^2}cd^2|n > (\bar{u}_R(\partial + \mathrm{W}^0)u_R) \tag{123}$$

$$\langle n|\overline{ab^2}ab^2|n > (\bar{d}_R(\partial + \mathrm{W}^0)d_R). \tag{124}$$

All these terms are again invariant under $\mathrm{U}_a(1)\times\mathrm{U}_b(1)$ gauge transformations on the SLq(2) algebra.

D. The Higgs kinetic energy term

The weak neutral couplings are

$$i(g\mathrm{W}_3\tau_3 + g_0\mathrm{W}_0\tau_0) = i(\mathcal{A}\mathrm{A} + \mathcal{Z}\mathrm{Z}) \tag{125}$$

where the t of the standard model has been replaced by τ as in (101) and

$$\mathrm{W}_0 = \mathrm{A}\cos\theta - \mathrm{Z}\sin\theta \tag{126}$$

$$\mathrm{W}_3 = \mathrm{A}\sin\theta + \mathrm{Z}\cos\theta. \tag{127}$$

Here θ is the Weinberg angle:

$$\tan\theta = \frac{g_0}{g}. \tag{128}$$

Then

$$\mathcal{A} = g_0(\tau_3 + \tau_0)\cos\theta \tag{129}$$

$$\mathcal{Z} = g(\tau_3 - \tau_0\tan^2\theta)\cos\theta. \tag{130}$$

If $|0\rangle$ is a neutral state

$$\mathcal{A}|0 >= 0. \tag{131}$$

By (129)

$$(\tau_3 + \tau_0)|0\rangle = 0. \tag{132}$$

Then by (130)

$$\mathcal{Z}|0\rangle = \left(\frac{\tau_3}{\cos\theta}\right)|0\rangle. \tag{133}$$

Then by (91), (125), (131), and (133)

$$\nabla(\varphi|0\rangle) = \partial(\varphi|0\rangle + ig\left[\mathrm{W}_+\tau_+ + \mathrm{W}_-\tau_- + \frac{\mathrm{Z}}{\cos\theta}\tau_3\right](\varphi|0\rangle) \tag{134}$$

and

$$\frac{1}{2}\langle 0|\mathrm{Tr}\overline{\nabla_\mu\varphi}\nabla^\mu\varphi|0\rangle = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + g^2\rho^2\left[\mathrm{I}_{++}\bar{\mathrm{W}}_+^\mu\mathrm{W}_{+\mu} + \mathrm{I}_{--}\bar{\mathrm{W}}_-^\mu\mathrm{W}_{-\mu} + \frac{\mathrm{I}_{33}}{\cos^2\theta}\bar{\mathrm{Z}}^\mu\mathrm{Z}_\mu\right] \tag{135}$$

where

$$\varphi = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

and

$$\mathrm{I}_{kk} = \frac{1}{2}\,\mathrm{Tr}\langle 0|\bar{\tau}_k\tau_k|0\rangle \qquad k = +, -, 3 \tag{136}$$

and Tr is taken over $\bar{t}_kt_k$.

To agree with the vector masses that are satisfactorily given by the standard model we have set $\langle 0|0\rangle = 1$ and shall also set

$$\mathrm{I}_{kk} = \frac{1}{2}\,\mathrm{Tr}\langle 0|\bar{\tau}_k\tau_k|0\rangle = 1. \tag{137}$$

Since

$$\tau_k = c_kt_k\mathcal{D}_k \qquad k = +, -, 3 \tag{138}$$

the previously introduced and undetermined constants in (96) and (97) are now fixed by

$$|c_k|^{-2} = \frac{1}{2}\langle 0|\bar{\mathcal{D}}_k\mathcal{D}_k|0\rangle \tag{139}$$

where the $\mathcal{D}_k$ are given by (98)–(100)

The c_k are properly invariant and may be evaluated as functions of q and $|\beta|^2$ in the same way that the form factors are evaluated in Sec. 9.

We now replace the Higgs kinetic energy term of the standard model by

$$\frac{1}{2}\mathrm{Tr}\langle 0|\bar{\nabla}_\mu\varphi\nabla^\mu\varphi|0\rangle \tag{140}$$

where $\nabla_\mu\varphi = \partial_\mu\varphi + ig[\mathrm{W}_+\tau_+ + \mathrm{W}_-\tau_- + \frac{\mathrm{Z}}{\cos\theta}\tau_3]\varphi$ is the knot covariant derivative of the Higgs field.

E. Field invariants

We replace the field invariant of the standard model by

$$\langle 0|\mathrm{Tr}\mathcal{W}_{\mu\lambda}\mathcal{W}^{\mu\lambda}|0\rangle \tag{141}$$

where $\mathcal{W}_{\mu\lambda}$ are the field strengths of the knot model and where $|0\rangle$ is the ground state of the commuting b and c operators.

The covariant derivative is

$$\nabla_\mu = \partial_\mu + \mathcal{W}_\mu \tag{142}$$

where $\mathcal{W}_\mu$ is the vector connection

$$\mathcal{W}_\mu = ig(\mathrm{W}_\mu^+\tau_+ + \mathrm{W}_\mu^-\tau_- + \mathrm{W}_\mu^3\tau_3) \tag{143}$$

where $\tau_\pm$ and τ_3 are given by (96)–(100).

The field strengths are

$$\mathcal{W}_{\mu\lambda} = [\nabla_\mu, \nabla_\lambda] = ig[\partial_\mu \mathrm{W}_\lambda^p - \partial_\lambda \mathrm{W}_\mu^p]\tau_p - g^2\mathrm{W}_\mu^k\mathrm{W}_\lambda^l[\tau_k, \tau_l] \tag{144}$$

and differ from the W_μ and $\mathrm{W}_{\mu\lambda}$ of the standard model by the substitution of τ_k for t_k.

The τ commutators lead to structure coefficients invariant under the gauge transformations $\mathrm{U}_a(1)\times\mathrm{U}_b(1)$ that leave the SLq(2) algebra invariant and hence are functions of bc only. The structure coefficients in (144) will therefore be functions of $\beta\gamma$, the value of bc on the ground state in (141).

F. Fermion and preon dynamics

Interactions and masses of the fermions and preons are in principle determined by the Lagrangian described in the preceding section. The fermions and preons, are described by the $\mathrm{D}^{\frac{3}{2}}$ and $\mathrm{D}^{\frac{1}{2}}$ representations, and interact by the D^3 vector bosons and by the D^1 vector bosons respectively.

Since the number of preons equals the number of crossings by (50), one may speculate that the crossings and preons are pointlike, that there is one preon at each crossing, and that the elementary fermions are composed of three preons bound by a trefoil of knot-electroweak and gluon fields. If this is a realistic picture, there should be three bound states, corresponding to the three members of each family, with their observed masses; and assuming that the preon dynamics is entirely determined by the knotted action, the calculation of these bound states could be formulated as a well-defined mathematical problem. On the other hand, to reach a physically credible picture, one needs some experimental guidance at relevant and presumably very high energies. For example, one should expect the electroproduction of a and d particles according to

$$e^+ + e^- \rightarrow a + d + \ldots$$

since they are charged $(\pm\frac{e}{3})$.

The following decay modes are also kinematically possible:

$$\text{Down quarks}: \mathrm{D}^{\frac{3}{2}}_{\frac{3}{2}-\frac{1}{2}} \rightarrow \mathrm{D}^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} + \mathrm{D}^1_{1-1} \quad (ab^2 \rightarrow a + b^2)$$

$$\text{Up quarks}: \mathrm{D}^{\frac{3}{2}}_{-\frac{3}{2}-\frac{1}{2}} \rightarrow \mathrm{D}^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} + \mathrm{D}^1_{-1-1} \quad (cd^2 \rightarrow c + d^2).$$

These decays could limit to three the number of generations by permitting the quark to decay if given a critical dissociation energy. In that case one would expect the formation of a preon-quark plasma at a sufficiently high temperature.

Currently there is data at hadronic energies on electroweak reaction rates and on the masses of the three generations. This data at present constrains and in principle is predicted by the knot model. To discuss this data we now introduce some simplifications based on the same physical picture and on SUq(2), the unitary version of SLq(2). Let us first consider the masses of the three generations of fermions.

G. The masses of the fermions [9,10]

The mass terms (81), (85), (87) and (88) of the knot Lagrangian contain the mass spectra of the four families that are listed in Table IX [9,10].

These masses are all of the form

$$\rho(m, m')\langle n|\bar{\mathrm{D}}^{\frac{3}{2}}_{mm'}\mathrm{D}^{\frac{3}{2}}_{mm'}|n\rangle$$

where (m, m') labels the family and n labels the generation. The $|n\rangle$ are eigenstates of $\bar{\mathrm{D}}^{\frac{3}{2}}_{mm'}\mathrm{D}^{\frac{3}{2}}_{mm'}$. Since the electric charge is $-\frac{e}{3}(m+m')$, the pair (m, m') determines both the mass and the charge.

In this table, as before, only the operator factor of the monomial $\mathrm{D}^{\frac{3}{2}}_{mm'}$ is recorded. The four prefactors $(\rho(l), \rho(\nu), \rho(d), \rho(u))$ represent the products of the numerical factors in $\mathrm{D}^{\frac{3}{2}}_{mm'}$ with the Higgs factor. The magnitude of ρ sets the energy scale and differs for each family.

The $M(i, n)$ in Table IX are invariant under $\mathrm{U}_a(1)\times\mathrm{U}_b(1)$ transformations since the preon operators (a, b, c, d) transform oppositely to their adjoints.

TABLE IX. Masses of Elementary Fermions.

i	$\mathrm{D}^{\frac{3}{2}}_{mm'}(i)$	$M(i,n)$
l	a^3	$\rho(l)\langle n\|\bar{a}^3a^3\|n\rangle$
ν	c^3	$\rho(\nu)\langle n\|\bar{c}^3c^3\|n\rangle$
d	ab^2	$\rho(d)\langle n\|\bar{b}^2\bar{a}\cdot ab^2\|n\rangle$
u	cd^2	$\rho(u)\langle n\|\bar{d}^2\bar{c}\cdot cd^2\|n\rangle$

To numerically evaluate the expectation values of these operator products we may go to the unitary version of SLq (2) by setting

$$d = \bar{a} \tag{145}$$

$$c = -q_1 \bar{b}. \tag{146}$$

Then

$$\begin{aligned} &ab = qba \qquad a\bar{a} + b\bar{b} = 1 \qquad a\bar{b} = q\bar{b}a \\ &\bar{a}a + q_1^2 \bar{b}b = 1 \qquad b\bar{b} = \bar{b}b. \end{aligned} \tag{147}$$

The identification of d with $\bar{a}$ and c with $\bar{b}$ is in agreement with the physical identification of the creation operators for the d and c preons with the creation operators, $\bar{a}$ and $\bar{b}$, for the antiparticles of the a and b preons respectively. Then the operators $\bar{a}^n a^n$ and $a^n \bar{a}^n$ are charge neutral and are expressible in terms of $b\bar{b}$ which is also charge neutral.

The reduction of $a^n d^n$ to a polynomial in bc may be shown as follows:

$$a^n d^n = a^{n-1} \cdot ad \cdot d^{n-1} \tag{148}$$

$$\begin{aligned} &= a^{n-1}(1 + qbc) d^{n-1} \\ &= a^{n-1} d^{n-1} (1 + q^{2n-1} bc). \end{aligned} \tag{149}$$

By iteration one finds

$$a^n d^n = \prod_{s=1}^{2n-1} (1 + q^s bc) \tag{150}$$

and in SUq(2)

$$a^n \bar{a}^n = \prod_{s=1}^{2n-1} (1 - q^{s-1} b\bar{b}) \tag{151}$$

and

$$\bar{a}^n a^n = \prod_{s=1}^{n} (1 - q_1^{2s} b\bar{b}). \tag{152}$$

We take the states $|n\rangle$ in Table IX to be eigenstates of a mass operator expressed as a function of $b\bar{b}$. Then the expectation values for these states are functions of $\beta\bar{\beta}$, the eigenvalue of $b\bar{b}$ on the ground state. The $M(i,n)$ are then functions of (q, β, n) and ρ, but the ratios of the masses in a single family depend only on (q, β, n) and not on ρ.

The three generations, corresponding to the ground and two excited states, may be labeled by any three choices of n. The three expressions for the mass $M(i,n)$ correspond to the three choices of n within a single family and are functions of the four parameters of the model (q, β, n, ρ) according to

$$M(i,n) = \rho(i) F(q^2, |\beta|^2, n) \tag{153}$$

where $F(q^2, |\beta|^2, n)$ is a polynomial in $|\beta|^2$ of the third degree, and a polynomial in q^2 of the degree determined by Table IX, Eqs. (151) and (152) and the algebra (1) [9,10].

Depending on the assignment of n to the three generations, one may determine q and β by Eq. (153) from the two ratios of the three observed masses [10].

H. Electroweak reaction rates [11]

The matrix elements of the standard model acquire the following form factors in the corresponding knot model:

$$\langle n'' | \bar{D}^{\frac{3}{2}}_{-3t_3'', -3t_0''} D^3_{-3t_3', -3t_0'} D^{\frac{3}{2}}_{-3t_3, -3t_0} | n \rangle \tag{154}$$

where n and n'' run over the three generations.

As an example consider

$$l^- + W^+ \to \nu_l \tag{155}$$

with the following form factor:

$$\langle \nu_l | \bar{D}^{\frac{3}{2}}_{-\frac{3}{2}\frac{3}{2}} D^3_{-30} D^{\frac{3}{2}}_{\frac{3}{2}\frac{3}{2}} | l \rangle. \tag{156}$$

If this form factor is reduced in the SUq(2) algebra, it becomes a function of q and β, where β is the eigenvalue of b on the ground state $|0\rangle$. Comparison of (156) with experimental data on lepton-neutrino interactions like (155) indicates that

$$(q, \beta) \cong \left(1, \frac{\sqrt{2}}{2}\right) \tag{157}$$

in approximate agreement with the universal Fermi interaction [10].

More demanding tests of the model are provided by the Cabbibo-Kobayashi-Maskawa (CKM) and the Pontecorvo-Maki-Nakagawa -Sakata (PMNS) matrices that relate to (154), depending on whether n and n'' label quarks or leptons and neutrinos. In making these tests we introduce the further assumption that the flavor states are the "coherent states," i.e., the eigenstates of the operators $\bar{a}$ and a, that are raising and lowering operators and thereby transmute particles of one generation into particles of the adjoining generation [11].

Starting from the mass states, one may obtain the flavor states as follows. The orthonormal mass states $|n>$ are defined to satisfy

$$|n\rangle = \bar{a}^n |0\rangle \tag{158}$$

$$\langle n | n' \rangle = \delta(n, n'). \tag{159}$$

Then $\bar{a}$ is a raising operator:

$$\bar{a}|n\rangle = \lambda_n|n+1\rangle \tag{160}$$

and

$$\langle n|a = \lambda_n^*\langle n+1|. \tag{161}$$

By (159) and (147)

$$|\lambda_n|^2 = 1 - q^{2n}|\beta|^2. \tag{162}$$

Similarly, a, working to the right on $|n\rangle$, is a lowering operator.

Let $|\alpha>$ be an eigenstate of a with eigenvalue α:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{163}$$

$$\langle\alpha|\bar{a} = \langle\alpha|\alpha^*. \tag{164}$$

We now compute the matrix element $\langle n|a|\alpha\rangle$ connecting mass and coherent states.

If a operates to the right, one has by (163)

$$\langle n|a|\alpha\rangle = \alpha\langle n|\alpha\rangle \tag{165}$$

and if it operates to the left, one has by (161)

$$\langle n|a|\alpha\rangle = \lambda_n^*\langle n+1|\alpha\rangle. \tag{166}$$

Then

$$\langle n+1|\alpha\rangle = \frac{\alpha}{\lambda_n^*}\langle n|\alpha\rangle. \tag{167}$$

By iteration,

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\prod_0^{n-1}\lambda_s^*}\langle 0|\alpha\rangle \tag{168}$$

and $\langle 0|\alpha\rangle$ may be fixed by normalizing $|\alpha\rangle$ [11].

Let the flavor states $|i\rangle$ be expressed as superpositions of the mass states $|n\rangle$:

$$|i\rangle = \sum |n\rangle\langle n|i\rangle \tag{169}$$

Then the matrix elements between flavor states are related to the matrix elements between mass states as follows:

$$\langle i|\mathrm{M}|i'\rangle = \sum\langle i|n\rangle\langle n|\mathrm{M}|n'\rangle\langle n'|i'\rangle. \tag{170}$$

The mass states are orthonormal but the flavor (coherent) states are not orthogonal and their normalizations are also left free to be fixed by the data.

Let

$$\mathrm{N}_i = \langle i|i\rangle. \tag{171}$$

We are now interested in the generalization of (156) to the cases for which

$$\mathrm{M} = \bar{\mathrm{D}}^{\frac{3}{2}}_{m''p''}\mathrm{D}^{3}_{m'p'}\mathrm{D}^{\frac{3}{2}}_{mp}, \tag{172}$$

when taken between flavor states, describes the weak vector interactions of all the elementary fermions. In particular

$$\langle u(i)|\mathrm{W}^+|d(i')\rangle = \sum_{nn'}\langle u(i)|u(n)\rangle\langle u(n)|\mathrm{W}^+|d(n')\rangle\langle d(n')|d(i')\rangle \tag{173}$$

$$\langle d(i)|\mathrm{W}^-|u(i')\rangle = \sum_{nn'}\langle d(i)|d(n)\rangle\langle d(n)|\mathrm{W}^-|u(n')\rangle\langle u(n')|u(i')\rangle \tag{174}$$

holding for the flavor states of the up and down quarks.

With the same model for the PMNS matrix the form factor is

$$\langle i|\bar{\mathrm{D}}^{\frac{3}{2}}_{-\frac{3}{2}\frac{3}{2}}\mathrm{D}^{3}_{00}\mathrm{D}^{\frac{3}{2}}_{-\frac{3}{2}\frac{3}{2}}|i'\rangle \tag{175}$$

where $i = 0, 1, 2$ label the three generations of neutrino flavor states.

Because of the $\mathrm{U}_m(1)\times\mathrm{U}_{m'}(1)$ symmetry, the matrix element M in (172) is neutral, i.e., $n_a - n_d = n_b - n_c = 0$. It is therefore a function of b and c only and has no off-diagonal elements:

$$\langle n|\mathrm{M}|n'\rangle\rangle = \mathrm{M}_n\delta(n, n') \tag{176}$$

and by (170)

$$\langle i|\mathrm{M}|i'\rangle = \sum_n\langle i|n\rangle\mathrm{M}_n\langle n|i'\rangle. \tag{177}$$

We propose that the quantities $|\langle n|i\rangle|$ correspond to the matrix elements of the CKM matrix and may be expressed as a function of

(a) the eigenvalues of a: α
(b) the norms of the eigenstates of a: N_i
(c) the matrix elements of a between neighboring mass states: $\lambda_n^* = \langle n|a|n+1\rangle$.

I. Comments on the present stage of the model

(1) This model has been used to parametrize flavor states, the CKM matrix, and mass ratios between generations.

(2) The states of composite particles described by $D^j_{mm'}$ are superpositions of preon states described by $D^{\frac{1}{2}}_{mm'}$ according to Eq. (49) where the (t, t_3, t_0, Q) describing $D^j_{mm'}$, are sums of the $(t, t_3, t_0, Q)_p$ describing the $D^{\frac{1}{2}}_{mm'}$ of the component preons. For $j = \frac{3}{2}$ the composite particles are the elementary fermions: leptons, neutrinos, up and down quarks, which are composed of 3 preons in exact agreement with the charge and hypercharge assignments of the Harari-Shupe model and with the experimental data on which their model is based.

(3) Since the knot dynamics and the preon dynamics are obtained by adjoining $D^{\frac{3}{2}}_{mm'}$, and $D^{\frac{1}{2}}_{mm'}$ factors, respectively, to the field operators of the standard model as described in Eq. (8), the knot and preon Lagrangians may be described as two SLq(2) representations of the Lagrangian of the standard model that differ by form factors induced by their different $D^j_{mm'}$. The so constructed knot and preon actions have to be either compatible or have to be supplemented or adjusted. The problem of relating the microscopic $(j = \frac{1}{2})$ to the macroscopic $(j = \frac{3}{2})$ descriptions has not been directly addressed but one may try to approach it with the aid of an effective Hamiltonian. Basic compatibility between the knot and preon actions would require that the effective preon Hamiltonian have three bound states with the masses and angular momentum of the observed $j = \frac{3}{2}$ leptons, neutrinos, and up and down quarks.

(4) It is possible to construct effective Hamiltonians describing three preon systems bound by a trefoil field as suggested in Fig. 2. If the external loops are infinitesimal, this structure would resemble a three particle nucleus like H^3. The interaction fields would be Coulomb- or Yukawa-like, depending on the masses of the vector preons by which the fermionic preons interact.
Current experimental knowledge requires very heavy preons and very small knots and therefore very strong binding. The binding problem remains a serious challenge for the model if the preons are observable as free particles. On the other hand, if the preons are not observable as free particles, it is natural to associate them with the crossing points of the trefoils since the number of preons equals the number of crossings, or $N' = N$ as shown in (50) and (63). Then one possibility is that the preons are solitonic field concentrations of mass, momentum, and charge, which occur at the crossing points.

J. The physical interpretation of "q" [12]

In the present context q is a parameter that measures the deformation of the standard model caused by the "knotting" of the elementary fermion. The empirical value of q obtained from electroweak reaction rates is in the neighborhood of unity. In particular, if the knot identification of the flavor states is accepted, then the observed CKM matrix indicates that the parameter, q, may be very close to unity. On the other hand, if there is any SLq(2) substructure at all, the possibility that q is precisely unity is excluded.

The primary substructure of quantum fields is determined by the Heisenberg algebra holding for the conjugate fields and realized by field quanta. Here there is another substructure determined by the SLq(2) algebra and implemented by preons.

The Heisenberg and SLq(2) algebras may be related by the following quadratic form:

$$K = A^t \varepsilon_q A \tag{178}$$

where

$$\varepsilon_q = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix} \qquad \varepsilon_q^2 = -1. \tag{179}$$

This form is invariant under SLq(2) transformations of A.

Choosing

$$A = \begin{pmatrix} D_x \\ x \end{pmatrix} \quad \text{and} \quad K = q^{-\frac{1}{2}} \tag{180}$$

one has by (178)–(180) the following SLq(2) invariant relation

$$D_x x - q x D_x = 1. \tag{181}$$

Equation (181) is identically satisfied if D_x is chosen as the q-difference operator, namely

$$D_x \Psi(x) = \frac{\Psi(qx) - \Psi(x)}{qx - x}. \tag{182}$$

If we introduce

$$P_x = \frac{\hbar}{i} D_x \tag{183}$$

then we have the SLq(2) invariant relation

$$(P_x x - q x P_x)\Psi(x) = \frac{\hbar}{i}\Psi(x). \tag{184}$$

If $q \to 1$ then (184) becomes the Heisenberg commutator applied to a quantum state. Otherwise D_x resembles, by (182), the differentiation operator on a lattice space and q may play the role of a dimensionless regulator.

In view of the physical evidence suggestive of substructure, which has been described here, as well as the natural appearance of the nonstandard q derivative, it may be possible to utilize SLq(2) to describe a finer level of structure than is currently considered.

We have ignored the gravitational field in this paper since it is not immediately relevant. As we have, however, discussed the knot symmetries of the fundamental particles, we have thereby also discussed the knot symmetries of these sources of the gravitational field. Since one expects that the symmetries of its source would in some measure be inherited by the gravitational field itself, it is interesting that knot states have emerged in a natural way in attempts to quantize general relativity [13].

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