PHYSICAL REVIEW D **89,** 125006 (2014)

Nonexistence of the self-accelerating dipole and related questions

Andrew M. Steane
Department of Atomic and Laser Physics, Clarendon Laboratory, Parks Road, Oxford OX1 3PU, United Kingdom
(Received 10 December 2013; published 6 June 2014)

We calculate the self-force of a constantly accelerating electric dipole, showing, in particular, that classical electromagnetism does not predict that an electric dipole could self-accelerate, nor could it levitate in a gravitational field. We also resolve a paradox concerning the inertial mass of a longitudinally accelerating dipole, showing that the combined system of dipole plus field can be assigned a well-defined energy-momentum four-vector, so that the principle of relativity is satisfied. We then present some general features of electromagnetic phenomena in a reference frame described by the Rindler metric, showing in particular that an observer fixed in a gravitational field described everywhere by the Rindler metric will find any charged object supported in the gravitational field to possess an electromagnetic self-force equal to that observed by an inertial observer relative to which the body undergoes rigid hyperbolic motion. It follows that the principle of equivalence is satisfied by these systems.

DOI: 10.1103/PhysRevD.89.125006 PACS numbers: 03.30.+p, 03.50.De, 04.20.-q, 04.40.Nr

I. INTRODUCTION

In 1984 Cornish proposed that according to classical electromagnetism, a sufficiently small or highly charged electric dipole (to be precise, a rigid body consisting of two point charges separated by a short rod) could undergo self-accelerated motion [1]. That is, after being placed in the right initial conditions, it would experience a self-force in the direction of its acceleration that was sufficient to maintain the acceleration, without the need for any applied external force. It follows that such a dipole could also self-levitate in a gravitational field.

This claim was accepted uncritically at the time [2], and the argument continues to be repeated [3]. We will show that the claim is wrong—but for interesting reasons. It turns out to be an example of a more general phenomenon that has long been misunderstood, and it continues to be widely misunderstood, namely the correct treatment of equations of motion when self-force is non-negligible.

It has been known for over a century that classical electromagnetism has difficulties in treating pointlike charges [4–8]. If a pointlike particle with a finite charge could exist, then it would produce around itself an electromagnetic field whose strength diverges near the particle and whose total energy is infinite. One might "live with" this problem by adopting the concept of "renormalization," arguing that only energy differences are physically relevant and, by the use of a suitable procedure to regularize the divergent integrals, sensible predictions could be obtained. However it turns out that this is not sufficient on its own, because it leads to equations of motion that have pathological runaway solutions. Cornish was well aware of this background and merely drew attention to a previously unnoticed but especially simple case.

In the case of the rigid spherical shell, the pathological cases can be ruled out by insisting that an entity of given charge and observed mass cannot have a radius below a certain minimum [7,9–11]. This is connected to the fact that no physical entity can have a negative mass—a simple enough fact, but one which can be hidden when electromagnetic energy and momentum has to be taken into account. We will show that the resolution in the case of the dipole is similar. The dipole case remains interesting, however, because the case for self-acceleration seems to be straightforward at first sight.

We also consider the fact that the electromagnetic self-force of a dipole depends on its orientation with respect to its acceleration. This appears to imply the inertial mass depends on the orientation, but that would contradict relativity and the principle of equivalence, because the field energy does not have such a dependence. Therefore it was considered paradoxical [12–15]. We resolve this paradox by appealing to the inertia of pressure.

An alternative resolution was offered by Ori and Rosenthal [14,15], based on a different, but well-motivated, definition of self-force also described by Pearle [16]. We reconcile the two approaches.

We also consider the case of a charged body at rest in an accelerating reference frame in flat spacetime. This is the frame described by the Rindler metric; it describes the simplest possible gravitational field (one which causes acceleration but not tidal effects). We present a general calculation of electromagnetic self-force in this case. Our approach to calculating the electromagnetic field agrees with several earlier treatments [17–19], but not, at first appearance, with a recent calculation by Pinto [13]. The difference is resolved by considering what is meant by observation in or relative to an accelerating frame; this influences the way forces acting at different positions should be summed or compared.

The paper is laid out as follows. Section II treats the self-force of an accelerating dipole. We first show that self-acceleration does not occur when the properties of the dipole are restricted to physically possible values, and then we address the mass paradox associated with the dependence of the self-force on orientation. The analysis is tractable when we model the dipole as two small spheres whose separation is large compared to their radius. In order to address the complete problem it is necessary also to consider the case where the spheres are close together; this is addressed by numerical calculations in Sec. II C. Section III presents the fact that there is more than one way to consider what is the rate of change of momentum of an extended object, owing to the relativity of simultaneity. Section IV presents the problem of electromagnetic self-force in the presence of a simple "gravitational" field (that is, a non-Minkowski metric but with zero spacetime curvature). An exact treatment turns out to be quite simple in the case of the Rindler metric. Section V summarizes the conclusions.

II. THE ACCELERATING DIPOLE

The "dipole" under consideration consists of a pair of charges $\pm q$ connected by a short rod of proper length d and undergoing rigid motion. By "rigid motion" we mean motion such that at each moment there is an inertial frame in which both charges are at rest, and their proper separation is constant; i.e., the separation is the same in all successive instantaneous rest frames. We need not assume that the rod is made of rigid material (which would be impossible)—instead we assume that d is the length it adopts, in equilibrium, under the influence of the compressive forces from the two charges as they attract one another, and any other external forces to which it may be subject, and we consider a case where the external force is such that d is constant.

In particular, we consider such a dipole undergoing motion at constant proper acceleration a_0 ("hyperbolic motion"), and in the first instance we treat the case where the rod is aligned perpendicular to the acceleration.

Figure 1 shows the lines of electric field from one of the charges, in the instantaneous rest frame. For illustration, the field $\mathbf{E}_+$ of the positive charge is shown. Note that, at the location of the other (negative) charge, $\mathbf{E}_+$ is directed outwards and somewhat in the direction opposed to the acceleration. Therefore, the force on the negative charge, owing to the field of the positive charge, is inward and somewhat in the direction along the acceleration. Similarly, the force on the positive charge, owing to the field of the negative one, is also inward and somewhat in the direction along the acceleration. By forming the sum of these two forces, one concludes that there is a net electromagnetic self-force along the direction of the acceleration. This seems to suggest that this self-force could provide the force required to make the dipole accelerate, and hence one would have a self-accelerating dipole.

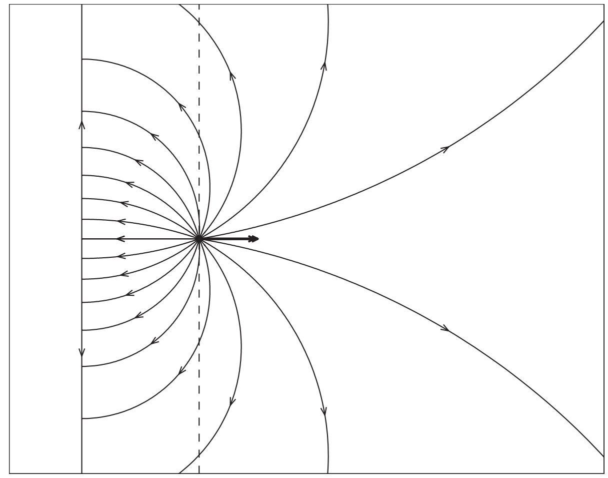

FIG. 1. Field lines of the electric field in the instantaneous rest frame of a positively charged small object undergoing constant proper acceleration in the $+x$ direction. A negative charge situated anywhere on the dashed line will experience a force with a component in the $+x$ direction and will itself produce an electric field similarly tending to accelerate the first charge.

We have presented this qualitative argument first, in order to show how natural the suggestion of self-acceleration is in this case. Next we back it up with some quantitative statements.

Let the motion be along x and let the charges by separated by a rod of fixed length d aligned along y. The field due to each charge at the position of the other has been obtained by Fulton and Rohrlich [20,21]. In the instantaneous rest frame it is given by

$$|E_x| = \frac{4qL^2}{4\pi\epsilon_0 d(4L^2 + d^2)^{3/2}}, \qquad |E_y| = 2|E_x|L/d, \tag{1}$$

where $L \equiv c^2/a_0$ and the signs are such that the charges attract in the y direction and each accelerates the other in the x direction.

Each charge also experiences a self-force which can be treated by using the Abraham-Lorentz-Dirac (ALD) equation. In order to do this, we first model each charge as a small spherical shell of radius R, and then take the limit $R \ll d$. At small but finite R, the electromagnetic self-four-force of such a shell is given by the ALD equation,

$$\mathsf{F}_{\text{shell}} = \frac{2}{3}e^2\left(-\frac{\dot{\mathsf{v}}}{R} + \ddot{\mathsf{v}} - \dot{\mathsf{v}}^2\mathsf{v} + O(R)\right), \tag{2}$$

where we introduced $e^2 \equiv q^2/4\pi\epsilon_0$ to reduce clutter, the dot signifies the derivative with respect to the proper time, the four-vectors are displayed in index-free notation, and we took $c = 1$. If the shell is not exploding under the influence of its own electromagnetic forces, then the material constituting it must be in tension. These internal stresses (Poincaré stresses) also give rise to a self-four-force, discussed in Appendix A, given by

$$\mathsf{F}_P = \frac{1}{6} e^2 \frac{\dot{\mathsf{v}}}{R} + O(R). \tag{3}$$

In the case of hyperbolic motion the second and third terms in Eq. (2) are equal and opposite. Hence one finds that, when $R \ll d$, the equation of motion for either charge of the dipole, when written in the instantaneous rest frame (and after reinstating c), is

$$f_{\text{ext}} + \frac{4e^2 L^2}{d(4L^2 + d^2)^{3/2}} - \frac{2e^2}{3Rc^2} a_0 + \frac{e^2}{6Rc^2} a_0 = m_{00} a_0, \tag{4}$$

where m_{00} is the bare rest mass of the spherical shell in the absence of internal stress.

Let us introduce

$$m_{\text{es}} \equiv \frac{e^2}{2Rc^2} \tag{5}$$

which is the total field energy of a spherical shell of charge that is permanently at rest (evaluated in the rest frame). Then we have

$$f_{\text{ext}} + \frac{4e^2 L^2}{d(4L^2 + d^2)^{3/2}} - \frac{4}{3} m_{\text{es}} a_0 + \frac{1}{3} m_{\text{es}} a_0 = m_{00} a_0.$$

It makes good physical sense to move the Poincaré stress term to the right hand side of the equation, writing

$$f_{\text{ext}} + \frac{4e^2 L^2}{d(4L^2 + d^2)^{3/2}} - \frac{4}{3} m_{\text{es}} a_0 = m_0 a_0, \tag{6}$$

where $m_0 \equiv m_{00} - m_{\text{es}}/3$. This is good practice because m_0 is the inertial mass of a well-defined physical entity (the material of the shell, including its internal stresses) that is being acted upon by forces external to it. This m_0 is often called the "bare mass." It is also customary to gather all the terms that are proportional to a_0 and introduce $m \equiv m_0 + (4/3) m_{\text{es}} = m_{00} + m_{\text{es}}$. m is the mass that will be "observed," i.e., deduced from measurements of the acceleration of a single shell under given applied forces, if one chooses to move all the inertial terms in the self-force to the right hand side of the equation of motion.

We want to know whether (6) has interesting solutions when $f_{\text{ext}} = 0$. After substituting $L = c^2/a_0$ and $f_{\text{ext}} = 0$, Eq. (6) gives a cubic equation for a_0^2. It has a single real solution for a_0^2, given by

$$a_0^2 = \left(\frac{2c^2}{d}\right)^2 \left[\left(\frac{e^2}{2mc^2 d}\right)^{2/3} - 1\right]. \tag{7}$$

This is the main result obtained by Cornish. One observes that for

$$d < \frac{e^2}{2mc^2} \tag{8}$$

one can have a solution of the equation of motion in which there is a constant acceleration with no applied force. This is the surprising result whose validity we will question.

It is instructive to consider the force exerted by each charge on the other, i.e., the term involving d in Eq. (6), also in terms of inertia. When $d \ll L$ this term is approximately $e^2/2Ld = \Delta m a_0$ where $\Delta m = e^2/2dc^2$. Therefore the pair of spheres has its total inertial mass reduced by

$$2\Delta m = e^2/dc^2, \tag{9}$$

which is precisely the "potential energy" of a pair of point charges at rest, separated by d. Of course this potential energy is really field energy: it is the amount by which the field energy is smaller, when the charges are brought to separation d, compared to when they are far apart, in the case $R \ll d$. One may then observe that if $\Delta m > m$, then one would have a negative total effective mass, and therefore self-acceleration. Thus one deduces the condition (8) again.

Since energy and momentum are exactly conserved in the interaction between particles and fields in classical electromagnetism, the existence of self-accelerated solutions has sometimes been interpreted, somewhat vaguely, as a way of drawing on the infinite reserves of energy to be found in the electromagnetic field near a pointlike particle. However, such an argument will not work, because we do not need R to be zero, only small, so the field energy is finite. It begins to look as if energy-momentum conservation is breaking down.

In fact, there is no such conclusion. The problem with the result is that the condition (8) cannot be satisfied if $m_0 \geq 0$. For then one has $m \geq (4/3) m_{\text{es}}$ so

$$\frac{e^2}{2mc^2} \leq \frac{3}{4} R. \tag{10}$$

Hence if we model the dipole as two small spherical shells, as above, then the condition for the self-accelerated solution is that the centers of the shells are separated by substantially less than twice their radius. But this implies that they overlap and therefore the calculation is invalid.

If one admits $m_0 \leq 0$, then it should not surprise us that self-acceleration could occur. As the matter with $m_0 < 0$ accelerates, its kinetic energy gets more and more negative, and a corresponding energy goes into the fields, since energy is conserved overall. But the whole situation remains unphysical.

A. Longitudinal dipole: Resolution of mass paradox

The self-force for the case of a longitudinally accelerating dipole is equally easy to extract using the equations for the field of an accelerating point charge. We consider a pair of point charges undergoing constant proper acceleration along the x axis. The condition for rigid motion (i.e., constant proper separation) is that the charges have proper accelerations given by [22,23] $a_i = c^2/x_i$ where x_i is the location of the ith charge in the instantaneous rest frame. It follows that if the particles are separated by a rod of proper length d, and the center of the rod has proper acceleration a_0, then the proper accelerations of the two particles are given by

$$\frac{a_0}{1 \pm a_0 d/2c^2},$$

the trailing particle having the higher acceleration.

The electric field at x_1 on the x axis due to an accelerating point charge q_2 located at x_2, in the instantaneous rest frame, is [24]

$$E^{(1,2)} = s\frac{4q_2 x_2^2}{4\pi\epsilon_0 (x_2^2 - x_1^2)^2}, \quad (11)$$

where s is the sign of $(x_1 - x_2)$ and the origin has been located such that $x_2 = c^2/a_2$. By interchanging the labels one finds that the total self-force (ignoring the self-force of each charge on itself) is

$$q_1 E^{(1,2)} + q_2 E^{(2,1)} = \frac{4q_1 q_2}{4\pi\epsilon_0 (x_1^2 - x_2^2)} = \frac{2e^2}{dc^2} a_0. \quad (12)$$

The calculation is exact in the limit $d \gg R$; it gives the self-force in the instantaneous rest frame. The final form on the right hand side of Eq. (12) suggests an interpretation in terms of mass, and it shows that in this case the inertial mass reduction is by twice what one might expect [for example, it is twice that observed in the transverse case, Eq. (9)]. This is the paradox noted by Griffiths and Owen [12] and taken up by Pinto [13] (see also [25]).

The paradox is not that the self-force depends on orientation, but with reconciling this fact with energy considerations. To lowest order in a_0 the field energy does not depend on orientation: it is given by $-e^2/d$ plus a contribution that is independent of the positions of the charges. Therefore it appears as if the energy and momentum of the complete system (dipole plus field) will not be able to form a four-vector at all orientations. This would violate basic principles of special relativity. It would also violate the principle of equivalence, since the passive gravitational mass of the complete system (matter plus field) is determined by the energy (divided by γc^2), whereas the inertial mass is determined by the momentum (divided by γv).

Since energy momentum is exactly conserved in classical electromagnetism, we can be sure that Eq. (12) gives the (negative of the) rate of change of the field momentum. To be precise, it matches the part of the field momentum associated with cross terms. This was checked to first order approximation by Griffiths and Owen, and we can rely on the consistency of the theory to be assured that it will be true exactly. The only mystery is that this momentum is not matching up with the field energy in the appropriate way: we have a "mysterious" factor 2.

The resolution is as follows.

This "2 problem" is just like the famous "4/3 problem" in the treatment of a charged sphere, and it can be understood in the same way. We have to take into account the pressure in the rod [26]. There is no choice about this: the physical system could be realized by placing two real, physical charged spheres at the end of a literal rod, and such a rod will certainly thus be placed in compression. The issue does not arise in the transverse case because in that case the pressure forces in the rod are transverse to the motion. In general, however, the pressure does influence the dynamics. One can think of this either in terms of "hidden momentum" [22] or in terms of the contribution of pressure to inertia (cf. Appendix A). When one calculates the contribution of the pressure to the inertia exhibited by the rod, one finds its inertia tensor is not isotropic.

Consider the two charged spheres separated by a rod lying along the direction of acceleration, which we continue to take as the x direction. Let us treat the material of the rod as an ideal fluid (imagine a fluid-filled tube with the charged spheres attached to pistons at each end). In the instantaneous rest frame, the pressure p in the fluid will obey the relativistic Navier-Stokes equation, which in the instantaneous rest frame takes the form

$$\left(\rho_0 + \frac{p}{c^2}\right)\frac{D\mathbf{u}}{Dt} = -\nabla p. \quad (13)$$

For positive pressure we can take the mass density of the fluid (ρ_0) to be negligible, and for the rigid hyperbolic motion under consideration, each part of the fluid has a proper acceleration given by $Du/Dt = c^2/x$. Hence the solution of the Navier-Stokes equation is

$$p \propto \frac{1}{x}. \quad (14)$$

If the cross section of each piston is A, then the equations of motion of the two charged spheres are

$$f_{\rm ext}^{(1)} + f_1 - p_1 A = m a_1,$$
$$f_{\rm ext}^{(2)} - f_2 + p_2 A = m a_2, \quad (15)$$

where m is the observed mass of each sphere, $f_{\rm ext}^{(i)}$ is the external force on sphere i, p_i are the pressures at the two ends of the tube, $a_i = c^2/x_i$ are the accelerations of the spheres, and f_i is the magnitude of the force on sphere i owing to the field of the other sphere, given by Eq. (11),

$$f_1 = \frac{e^2}{d^2L^2}x_2^2, \qquad f_2 = \frac{e^2}{d^2L^2}x_1^2, \tag{16}$$

where the two spheres are centered at $x_1 = L - d/2$, $x_2 = L + d/2$. By using (14) in (15) we find

$$\frac{f_1 + f_{\rm ext}^{(1)} - ma_1}{f_2 - f_{\rm ext}^{(2)} + ma_2} = \frac{x_2}{x_1}. \tag{17}$$

Hence

$$f_{\rm ext}^{(1)}x_1 + f_{\rm ext}^{(2)}x_2 = 2mc^2 - (f_1x_1 - f_2x_2). \tag{18}$$

In general there is no compelling reason why the external forces on the two spheres need be equal. However, if we suppose that they are, then we find that the total external force is related to the acceleration of the center of the rod, $a_0 = c^2/L$, by

$$2f_{\rm ext} = 2ma_0 - \frac{e^2}{dc^2}\left(1 - \frac{a_0^2d^2}{4c^4}\right)a_0. \tag{19}$$

Hence in the limit $a_0d \ll c^2$ we find that the self-force, after taking internal pressure into account, matches the result for transverse orientation of the dipole, Eq. (9). This confirms that the behavior of the momentum is consistent with the behavior of the energy, for small dipoles or small accelerations, for these two orientations of the dipole, and we will show it for all orientations in the next section.

For the case where the external force varies as $f_{\rm ext}^{(i)} \propto 1/x_i$, the left hand side of (18) evaluates to $2f_{\rm ext}L$, where $f_{\rm ext}$ is now the value of the force for $x = L$, and one obtains (19) again. In this case the total external force is not $2f_{\rm ext}$ but $2f_{\rm ext}(1 - d^2/4L^2)^{-1}$.

One may also interpret the physical picture in terms of "hidden momentum," as follows. Hidden momentum is momentum associated with energy transport through the body [22]. As the rod accelerates, the hidden momentum continually increases. This increases the inertia of the rod by the integral of the force along the length of the rod [28], which is e^2/d. This happens to be equal to the electrostatic field energy, but it is located in a completely different physical system, namely the material of the rod. Once this energy is added to the electrostatic field energy, we get a complete system (charges plus rod plus surrounding field) which can be treated as isolated and assigned a four-momentum. In particular, we find that the external force required to accelerate the dipole (or to keep it at a fixed location in a gravitational field) does not depend on the orientation of the dipole, for small a_0. In the longitudinal case, the dipole "pulls itself along" by its own electromagnetic forces more than in the transverse case; however, it has to do this in order to provide the hidden momentum associated with its internal pressure forces as well as its ordinary momentum, with the net result that its overall tendency to resist acceleration by outside forces is the same in the longitudinal as in the transverse case.

The condition for self-acceleration in the longitudinal case, after taking hidden momentum into account, is the same as for the transverse case, namely condition (8), but as before this is outside the range of validity of the calculation if we insist that the bare mass is non-negative. In either case, longitudinal or transverse, although we do not expect self-acceleration, we do expect that a dipole will be observably lighter than an object otherwise similar but with two charges of the same sign. The expected difference in the observed mass between the dipole and the dumbbell is twice Eq. (9), i.e., $2e^2/dc^2$.

All the above is valid for $a_0d \ll c^2$. More generally, the self-force given by Eq. (19) does not exactly match that given by (6). Both of these equations have been derived without restriction on the value of a_0, except for the restriction imposed by the horizon at $x = 0$, namely $d < 2L$ so $a_0d < 2c^2$, and we are still assuming $R \ll d$. However, comparing the dipole at one orientation with the dipole at another is nontrivial once the acceleration is substantial, because it is no longer clear what value should be considered "the acceleration of the dipole" when different parts have different accelerations (the longitudinal case), nor is it easy to locate the centroid of the field energy distribution. The following argument shows that a difference in self-force between the transverse and longitudinal cases is expected at $O(a_0^3)$. The situation is comparable to the case of an object fixed in a gravitational field whose strength varies as $g \propto 1/x$. Then for a prolate object of length d at height L, the total gravitational force when it is oriented vertically exceeds that when it is oriented horizontally by an amount of order $(d/L)^2f$, where f is the gravitational force in the horizontal case. We can apply this fact to the mass distribution associated with the field energy. The electromagnetic contribution to the mass is of order e^2/dc^2, and this mass is mostly concentrated in a prolate region of size approximately d. Therefore we expect an orientation-dependent contribution to the electromagnetic self-force of order

$$\left(\frac{d}{L}\right)^2\frac{e^2a_0}{dc^2} = \frac{e^2da_0^3}{c^6}. \tag{20}$$

B. Dipole at arbitrary orientation

We presented the transverse and longitudinal cases in detail in order to get clarity about the underlying physical mechanisms, and because it permits some simple exact results (in the limit $d \gg R$) to be exhibited, such as Eqs. (6), (7), (12) and (19). For a dipole at arbitrary orientation to its acceleration, we shall treat the problem to first order in the proper acceleration. The electric field produced by the first charge at the second is given by the standard expressions for the electric field of a charge in hyperbolic motion, and by expanding to first order in a_0 one finds

$$\mathbf{E}^{(2,1)} = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{d}}}{d^2} - \frac{\mathbf{a}_0 + (\mathbf{a}_0\cdot\hat{\mathbf{d}})\hat{\mathbf{d}}}{2c^2 d}\right] + O(a_0^2), \quad (21)$$

where $\hat{\mathbf{d}}$ is a unit vector in the direction from the first charge to the second. The total electromagnetic self-force of the dipole is therefore, to $O(a_0)$,

$$\mathbf{f}_{\rm self}^{(\rm e.m.)} \simeq \frac{e^2}{c^2 d}(\mathbf{a}_0 + (\mathbf{a}_0\cdot\hat{\mathbf{d}})\hat{\mathbf{d}}) + 2\mathbf{f}_{\rm sphere}^{(\rm e.m.)}, \quad (22)$$

where $\mathbf{f}_{\rm sphere}$ is the force of each charged sphere on itself (this is in the direction opposite to $\mathbf{a}_0$). The pressure force in the rod varies monotonically from one end to the other, but to $O(a_0)$ it is sufficient to use the average along the rod, which [again to $O(a_0)$] is equal to the average of the magnitudes of the two forces on the ends, i.e., $(e^2/d^3)\mathbf{d}$. Hence the rate of change of hidden momentum is

$$\frac{e^2}{c^2 d^3}\int_{-d/2}^{d/2}(\mathbf{d}\cdot\mathbf{a}_0)\hat{\mathbf{d}}\,\mathrm{d}s = \frac{e^2}{c^2 d^3}(\mathbf{d}\cdot\mathbf{a}_0)\mathbf{d}, \quad (23)$$

where, in the integral, s is the distance along the rod. The hidden momentum may be accounted for by bringing it to the other side of the equation of motion and regarding it as a contributer to the self-force. Hence, by subtracting (23) from (22), and also including the effect of internal stresses in the spheres, we find the total self-force of the system is

$$\mathbf{f}_{\rm self} \simeq \frac{e^2}{c^2 d}\mathbf{a}_0 + 2\mathbf{f}_{\rm sphere}. \quad (24)$$

Hence the resistance to acceleration by external forces is independent of the orientation of the rod, for all angles, to first order in a_0, when $d \gg R$. The whole situation is closely related to the Trouton-Noble experiment [29].

Historically the electromagnetic contribution to the mass of extended entities such as atoms and molecules has been considered to be of purely theoretical interest, being too small (of order 10^{-10} of the rest mass) to be observed experimentally. However, modern mass comparison techniques using ions trapped in Penning traps can achieve the required sensitivity [30]. It would be interesting, for example, to confirm that the inertial mass of a polar molecule such as lithium hydride is independent of its orientation. This would show that the quantum mechanical source of the internal pressure in the molecule, namely zero point energy when an electron is confined to a small region, gives rise to the requisite hidden momentum as special relativity says it must.

C. Dipole with large spheres

If we avoid the assumption of negative bare mass, then the remaining possibility, if we are searching for self-accelerating solutions, is to suggest that there might exist some charge distribution which gives a net electromagnetic self-force in the direction of the acceleration, even when the inertial terms are included. Given that the fields around an accelerating charge tend to retard any like charge moving alongside the first one, the most promising distribution would appear to be a dipolelike form, but made of a pair of larger spheres, so as to reduce $m_{\rm es}$ as much as possible without greatly changing the force exerted by each sphere on the other. Therefore let us consider two oppositely charged spherical shells, having total charges $\pm q$, radius R, with their centers separated by d (Fig. 2). The calculation in the previous section already applies to this dipole when $d \gg R$, but we would like to find out whether the case $d \simeq 2R$ (or indeed $d < 2R$, i.e., intersecting spheres) can yield self-acceleration.

The electromagnetic self-force of a single spherical shell of charge undergoing hyperbolic motion has been calculated exactly [23]. In the instantaneous rest frame, it is

$$f_{\rm shell} = \frac{2e^2}{Rc^2}a_0\sum_{n=0}^{\infty}\frac{(Ra_0/c^2)^{2n}}{(2n-1)(2n+1)^2(2n+3)}$$
$$\simeq \frac{e^2}{LR}\left[-\frac{2}{3} + \frac{2}{45}\left(\frac{R}{L}\right)^2 + \frac{2}{525}\left(\frac{R}{L}\right)^4 + \cdots\right], \quad (25)$$

and the condition that the sphere can maintain its proper size and shape is $R < L$. This shows that the further terms in the power series expansion do lower the absolute magnitude of the self-force of the shell, but one finds this reduction is not by enough to allow a self-accelerating dipole, as we now show. The force $f_{\rm dip}$ of each shell on the other is in the forward direction. It can be estimated for $d \gg R$ by using Eq. (1), which gives

$$f_{\rm dip} \simeq \frac{e^2}{Ld}\left[\frac{1}{2} - \frac{3}{16}\left(\frac{d}{L}\right)^2 + \frac{15}{256}\left(\frac{d}{L}\right)^4 + \cdots\right]. \quad (26)$$

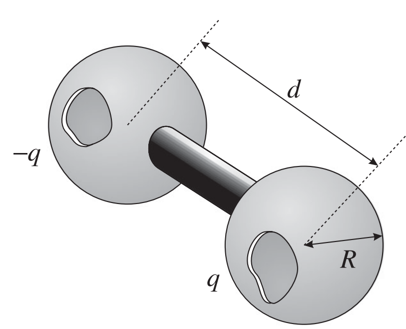

FIG. 2. A pair of charged spherical shells separated by a rigid rod.

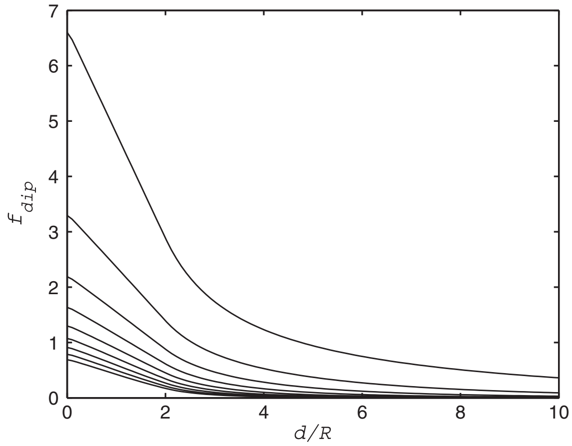

FIG. 3. The contribution f_{dip} to the self-force of a pair of oppositely charged spherical shells of radius R with centers separated by d and undergoing rigid hyperbolic motion in the transverse direction. The force is shown in units of e^2/L^2, for nine equispaced values of R between $0.1L$ and $0.9L$. The total self-force of the pair of spheres is $f = 2(f_{\mathrm{dip}} + f_{\mathrm{shell}})$ where f_{shell} is given by Eq. (25). As $d \to 0$ one finds that $f_{\mathrm{dip}} \to -f_{\mathrm{shell}}$ (see text) so $f \to 0$. The spheres are just touching when $d/R = 2$. At $d \gg R$, f_{dip} is independent of R and is given by the d-dependent term in Eq. (6).

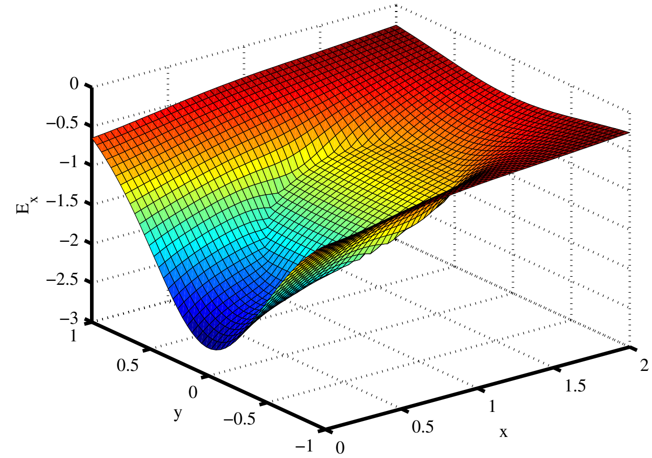

FIG. 4 (color online). The x component of the field $\bar{\mathbf{E}}_{\mathrm{shell}}$, which is the field of an accelerating sphere with a radially symmetric contribution removed, so as to leave a continuous function whose integral can be used to calculate the self-force for the case of a transversely oriented dipole. The figure shows the case $L = 1$, $R = 1/2$ for illustration. Note that $|\bar{\mathbf{E}}_{\mathrm{shell},x}|$ falls monotonically with y for points outside the shell.

For example, at $R = L$, $d = 2R$ (i.e., large, touching spheres) one finds $f_{\mathrm{shell}} = -(\pi^2/16)e^2/L^2$ and $f_{\mathrm{dip}} \simeq (8\sqrt{2})^{-1}e^2/L^2 \simeq 0.14|f_{\mathrm{shell}}|$.

In order to confirm this conclusion we need to replace the rough estimate for f_{dip} by a more accurate value. We did this by numerical integration, as described in Appendix B. Figure 3 shows the results. For all values of d/R and R/L we find that the total self-force opposes the acceleration.

One can prove that the self-force vanishes in the limit $d \to 0$ as follows. Consider the field $\mathbf{E}_{\mathrm{shell}}$ due to a single charged shell. It is discontinuous at the edge of the shell by $(\sigma/\epsilon_0)\hat{\mathbf{r}}$ where $\sigma = q/4\pi R^2$ is the surface charge density and $\hat{\mathbf{r}}$ is a unit vector in the direction radially outwards from the center of the shell. Therefore the field $\bar{\mathbf{E}}_{\mathrm{shell}} \equiv \mathbf{E}_{\mathrm{shell}} - \sigma H(r/R)\hat{\mathbf{r}}/\epsilon_0 r^2$ is continuous, where $H(x)$ is the Heaviside step function. The fields $\bar{\mathbf{E}}_{\mathrm{shell}}$ and $\mathbf{E}_{\mathrm{shell}}$ differ by a field that exerts no net force in the x direction on any charge distribution that is symmetric about $x = L$. Therefore we can use either of them for the purpose of calculating the self-force of the transversely oriented dipole. By using $\bar{\mathbf{E}}_{\mathrm{shell}}$ we eliminate the discontinuity; this allows the rest of the argument to proceed. Now consider the two contributions f_{shell} and f_{dip}. Both may be calculated by integrating $\bar{\mathbf{E}}_{\mathrm{shell}}$ over a spherical charge distribution. The two charge distributions in question have opposite signs and infinitesimally different locations in the limit $d \to 0$. Therefore in that limit one must find $f_{\mathrm{shell}} = -f_{\mathrm{dip}}$. This is expected since in this limit the fields produced by the two shells cancel. For $d > 0$ one can see from the overall form of the integrand (Fig. 4) that f_{dip} must fall monotonically as d increases. Therefore in this physical system the forward force arising from the presence of opposite charges can just approach the backward force arising from the presence of like charges, but cannot exceed it and thus produce self-acceleration.

The above considerations for $d \to 0$ will also apply if we model the pair of charged objects using some other shape or distribution of charge. This suggests that the overall conclusion, that the total self-force never points in the direction of acceleration, will hold true more generally. Further calculations would be needed to confirm this.

III. ALTERNATIVE DEFINITIONS OF SELF-FORCE

So far we have presented the self-force by adopting the policy of selecting a reference frame at the outset (the instantaneous rest frame) and summing the three-forces acting simultaneously in this frame. This is a valid method. However, owing to the relativity of simultaneity, it is not the only one that may be regarded as legitimate and useful.

Consider a composite object that can be decomposed into a set of discrete entities i. The total four-momentum of the composite object is

$$p^{\mu}_{\mathrm{tot}}(\tau_c, \chi) = \sum_i p^{\mu}_i(\tau_{i,\chi}), \tag{27}$$

where χ denotes a spacelike hypersurface, $\tau_{i,\chi}$ is the proper time on the ith worldline when that worldline intersects χ, and τ_c is the proper time on some reference worldline (e.g., the worldline of the centroid). In other words, χ is the hypersurface on which the individual four-momenta p^{μ}_i are evaluated in order to form the sum. Typically, one picks a spacelike hyperplane (so that the events $\{i\}_\chi$ are simultaneous in some frame). If the composite object is isolated, then the result does not depend on the choice of hyperplace [22]. If it is not isolated, which is the case for any calculation of self-force (since then the object in question is being pushed or pulled by its own electromagnetic field and by an external force), then $p^\mu_{\rm tot}$ does depend on χ. For an object whose motion is rigid—that is, its motion is such that at any given event on the world tube there is a reference frame in which all parts of the object are at rest, and at the same proper distances—a natural choice of χ is the hyperplace of simultaneity for the instantaneous rest frame at the given τ_c.

Suppose each discrete entity in the composite object experiences a four-force $\mathsf{F}^\mu_i = dp^\mu_i/d\tau_i$. Having established a definition of $p^\mu_{\rm tot}$ at one instant, one may take an interest in the rate of change of this quantity,

$$\frac{dp^\mu_{\rm tot}}{d\tau_c} = \lim_{\delta\tau_c\to 0} \frac{p^\mu_{\rm tot}(\tau_c+\delta\tau_c,\chi+\delta\chi) - p^\mu_{\rm tot}(\tau_c,\chi)}{\delta\tau_c}, \quad (28)$$

where we have assumed a one-to-one correspondence between χ and τ_c, such that $\delta\chi \to 0$ as $\delta\tau_c \to 0$.

The result (28) depends on what choice is made for the hyperplane $\chi + \delta\chi$. So far in this paper we have adopted the instantaneous rest frame in order to pick χ, and the method of summing three-forces acting simultaneously in that frame amounts to choosing for $\chi + \delta\chi$ a hyperplane parallel to χ and separated from it by a time δt in the given frame. The result for the spatial part of $dp^\mu_{\rm tot}/d\tau_c$ is

$$\frac{d\mathbf{p}_{\rm tot}}{dt} = \sum_i \frac{d\mathbf{p}_i}{dt}, \quad (29)$$

where the quantities $d\mathbf{p}_i/dt$ are evaluated on the hyperplane χ. For any given worldline we have $d\tau_i/dt = 1$ in the instantaneous rest frame; hence we may also write

$$\frac{dp^\mu_{\rm tot}}{d\tau_c} = \sum_i \frac{dp^\mu_i}{d\tau_i}. \quad (30)$$

Another interesting choice for $\chi + \delta\chi$ is a hyperplane of simultaneity for the new instantaneous rest frame at $\tau + \delta\tau$. For an accelerating object this is not parallel to χ, and one has

$$\frac{dp^\mu_{\rm tot}}{d\tau_c} = \lim_{\delta\tau_c\to 0} \sum_i \frac{p^\mu_i(\tau_i+\delta\tau_i) - p^\mu_i(\tau_i)}{\delta\tau_c} \quad (31)$$

$$= \sum_i \frac{dp^\mu_i}{d\tau_i}\frac{d\tau_i}{d\tau_c}, \quad (32)$$

where in the sum in (31), each $\delta\tau_i$ is the proper time elapsed on the ith worldline between the intersections of that worldline with χ and $\chi + \delta\chi$, and in (32) the quantities $dp^\mu_i/d\tau_c$ and $d\tau_i/d\tau_c$ are evaluated on the hyperplane χ. We now have two *different* definitions of the rate of change of the total momentum for the composite body: Eq. (30) is not the same as Eq. (32). Hence the phrase "the self-force" is ambiguous until one has specified which definition is adopted.

Ori and Rosenthal [14,15], following Pearle [16], have described the approach using (32). This approach has the advantage that for rigid motion, the internal forces cancel and the electromagnetic self-force one obtains is independent of the shape or orientation of the composite object. However, one should not ignore the internal forces altogether, and indeed in the approach using (30) they play an important role, as we have shown. [The statements in [14,15] suggesting the inadmissability of (30) are largely mistaken because they fail to take into account the fact that the internal stress tensor need not be spherically symmetric.]

IV. SELF-FORCE IN A SIMPLE GRAVITATIONAL FIELD

The general problem of self-force in a gravitational field is rich and subtle; for recent reviews see [31,32]. Here we consider only the case of a charged body held fixed in a spacetime described by the Rindler metric. This metric is appropriate to a uniformly accelerating reference frame in flat spacetime. Obviously, this case does not show the quintessential gravitational phenomena that are associated with curvature and tidal forces. However, the uniformly accelerating reference frame is an important basic case that can be used to explore phenomena that are associated purely with a spatial dependence of proper time, in the absence of spacetime curvature. It is also very useful for gaining physical insight.

We shall be concerned with the purely electromagnetic force which includes a divergent part (in the limit of pointlike objects) and a nondivergent part commonly called radiation reaction. The gravitationally induced self-force f_G discussed in [33] vanishes in flat spacetime, and therefore we shall not be concerned with it (even though it may dominate the radiation reaction in gravitational problems of practical interest).

Pinto [13] has presented a calculation of the field of a point charge in a reference frame described by the Rindler metric, by developing a formula for electric potential in the Rindler frame and evaluating its gradient. He thus finds that the electromagnetic self-force for a dipole is independent of orientation, to lowest order in the acceleration. Previously the electric field of a point charge in the constantly accelerating frame was obtained by several workers [17–19] using another method, namely to start with the field tensor in Minkowski space and then transform it; see also [34]. The field thus obtained differs from Pinto's and gives a self-force for a dipole that depends on orientation. We shall show that these differences arise from the difference between definitions (30) and (32).

Before considering the point charge, we examine the electromagnetic field in the Rindler frame in general. This will permit some observations more general than those given by Bradbury or Rohrlich, and we will bring out an interesting aspect not explicitly indicated by anyone. The derivation is quite simple.

We consider a region of flat spacetime. The region is mapped by a coordinate system (T, X, y, z) describing an inertial frame (one whose metric is Minkowskian), and also by another coordinate system (θ, h, y, z) related to the first by

$$\theta = \tanh^{-1}(T/X), \qquad h = \sqrt{X^2 - T^2} \tag{33}$$

in the region where h is real and positive (we will not need to consider the rest of spacetime). The metric for this second system is the Rindler metric, $g_{ab} = \mathrm{diag}(-h^2, 1, 1, 1)$. Any point fixed in the second system is undergoing hyperbolic motion relative to the first, with constant proper acceleration $1/h$ (we take $c = 1$ throughout this section). One can see immediately from the metric that the second system is static; i.e., the set of points at given (h, y, z) form a rigid lattice with fixed proper distances between them: it is the "constantly accelerating reference frame" in flat spacetime [19,22,35].

The coordinate transformation matrix is

$$\Lambda^a{}_{a'} \equiv \frac{\partial x^a}{\partial x^{a'}} = \begin{pmatrix} \frac{1}{h}\cosh\theta & -\frac{1}{h}\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{pmatrix} \tag{34}$$

in which the primed (unprimed) indices correspond to the Minkowski (Rindler) coordinate system. Note the similarity with the Lorentz transformation.

To calculate the electromagnetic effects we start in the first coordinate system and use Maxwell's equations in flat spacetime to find the fields in the standard way. This means that we neglect the effect of this electromagnetic field on the spacetime curvature; thus we neglect some nonlinear effects which are negligible in the limit of weak fields. We thus find the field tensor $F^{a'b'}$ in the Minkowski coordinate system. Since this transforms as an ordinary tensor, we may immediately find its form in the Rindler system, given by $F^{ab} = \Lambda^a{}_{a'}\Lambda^b{}_{b'}F^{a'b'}$. The result of this easy calculation is that the tensor transforms just like it would under a Lorentz transformation, except that the first row and column (i.e., F^{0b} and F^{a0}) pick up an additional factor $1/h$. Upon pre- and postmultiplying by g_{ab}, which introduces a factor h^2, we find that the covariant form F_{ab} has the first row and column multiplied by h, compared to a Lorentz-transformed version of $F_{a'b'}$.

To calculate the electromagnetic force on a charged particle in this field, we use

$$\frac{dp_a^{(\mathrm{EM})}}{d\tau} = qF_{a\lambda}\frac{dx^\lambda}{d\tau}, \tag{35}$$

where p_a is four-momentum and the superscript (EM) signifies that we are only writing down the contribution from electromagnetic effects [36]. [Note, however, that we shall introduce another definition of the "electromagnetic force" after Eq. (43).] For example, consider a particle fixed at height h in the Rindler frame. Its worldline in the Minkowski frame is $x^2 - t^2 = h^2$, and therefore its 4-velocity in the Minkowski frame is $u^{a'} = (\cosh\theta, \sinh\theta, 0, 0)$. Upon transforming we find [37] $u^a = \Lambda^a{}_a u^a = (1/h, 0, 0, 0)$. The factor $1/h$ in the 4-velocity exactly cancels the factor h in the first row of the field tensor, and we obtain

$$\frac{dp_a^{(\mathrm{EM})}}{d\tau} = \frac{dp_{\bar{a}}^{(\mathrm{EM})}}{d\tau}, \tag{36}$$

where the barred coordinates refer to the inertial frame obtained by the Lorentz boost from the original (t, x, y, z) frame. Equation (36) asserts that *for any given electromagnetic field in flat spacetime, the components of the electromagnetic four-force on a particle fixed in the Rindler frame, expressed in the coordinate system of that frame, are the same as those of the electromagnetic four-force on that same particle, expressed in the coordinate system of a Minkowski frame relative to which the Rindler frame is momentarily at rest*. Informally, one may say that, when calculating observations made by an observer at rest in the constantly accelerating reference frame, we do not need to worry about the general covariance of electromagnetism: just Lorentz boost to the instantaneous rest frame, and you will find the correct four-force.

In the case of the self-force of any arbitrary charge distribution undergoing rigid acceleration, the consequence of the above general statement is especially simple: the electromagnetic self-force is the same in the Rindler frame as in the Minkowski instantaneous rest frame. That is to say, the self-force as defined by (30) will agree in the two frames, and the self-force defined by (32) will also agree in the two frames.

It follows that all our previous statements about electromagnetic forces on an accelerating dipole also apply to electromagnetic forces on a dipole fixed in a gravitational field described by the Rindler metric. In particular, if we adopt the definition (30), then the electromagnetic self-force is larger when the dipole is oriented along the gravitational field than when it is oriented transverse to the gravitational field. Indeed, in view of the pressure forces in the rod, this must be the case if the principle of equivalence is to be satisfied. Having taken the internal pressure into account, the net result is that the weight of a dipole is independent of its orientation (assuming that it is in mechanical equilibrium). However, in the case of the uniformly accelerating reference frame, Eq. (30) is not the most natural definition of the total force on an extended object. Rather, Eq. (32) is more natural. If we adopt that definition, then we find the electromagnetic self-force is independent of orientation, to first order in the acceleration. We will now present this explicitly.

We begin by writing down the field of a charge undergoing hyperbolic motion with constant proper acceleration a_0, as observed in the inertial (Minkowski) frame in which the particle is momentarily at rest. If the particle is at (X_0, y_0, z_0), then the field at (X, y, z) is given by [20,22,38]

$$\bar{E}_\rho = \frac{q}{r^3}\frac{(\rho-\rho_0)(1+a_0(X-X_0))}{(1+a_0(X-X_0)+a_0^2r^2/4)^{3/2}},$$
$$\bar{E}_x = \frac{q}{r^3}\frac{(X-X_0)+\frac{a_0}{2}((X-X_0)^2-(\rho-\rho_0)^2)}{(1+a_0(X-X_0)+a_0^2r^2/4)^{3/2}}, \quad (37)$$

where $(\rho-\rho_0) = ((y-y_0)^2+(z-z_0)^2)^{1/2}$ and $r = ((X-X_0)^2+(\rho-\rho_0)^2)^{1/2}$, and we adopted Gaussian electromagnetic units.

In order to make the comparison with Pinto's calculation straightforward, introduce a change of coordinates to (t, x, y, z) where $t = \theta/g$, $x = h - 1/g$ and g is a constant. In these coordinates the metric is

$$\mathrm{d}s^2 = -(1+gx)^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \quad (38)$$

In a general (i.e., not Minkowski) frame, there is more than one way to define what may be called "electric field." One possible definition is the spatial part of the local four-force per unit charge on a charged particle that is not moving relative to the frame. This is given by

$$\mathcal{E}^i = F^i{}_\lambda u^\lambda \quad (39)$$

[cf. (35)] where u^λ is the 4-velocity of the local observer fixed in the frame. By the argument before Eq. (36), we have

$$\mathcal{E}^i = \bar{E}^i. \quad (40)$$

Also, since at $T = 0$ we have $x = X - 1/g$, it follows that $(X - X_0) = (x - x_0)$, so the electric field in the Rindler frame, as given by (39), is

$$\mathcal{E}_\rho = \frac{q}{r^3}\frac{(\rho-\rho_0)(1+a_0(x-x_0))}{(1+a_0(x-x_0)+a_0^2r^2/4)^{3/2}},$$
$$\mathcal{E}_x = \frac{q}{r^3}\frac{(x-x_0)+\frac{a_0}{2}((x-x_0)^2-(\rho-\rho_0)^2)}{(1+a_0(x-x_0)+a_0^2r^2/4)^{3/2}}. \quad (41)$$

In order that the charge at x_0 is fixed in the Rindler frame, its proper acceleration must match that of the local observer in the frame, so

$$a_0 = \frac{g}{1+gx_0}. \quad (42)$$

The field given by (41) is the field described in [17–19]. If we use it to calculate self-force, the results will agree with those found in the Minkowski frame, as already noted.

In the accelerating frame, the most natural way to form a sum of forces acting at different positions is the one given by (32). That is, one chooses for the hyperplane $\chi + d\chi$ the next hyperplane of simultaneity as defined by the accelerating frame. There remains a choice to be made about which worldline is the reference worldline, whose proper time is τ_c. Previously we suggested that one might use the centroid of the accelerating composite object, but in order to study dynamics more generally, one requires a reference worldline that is independent of the objects under consideration. Therefore one picks the worldline of a point fixed in the frame. The most natural such point is one at which g_{00} has the value -1, since then $d\tau_c = dt$. For the metric (38) this is the case for $x_c = 0$, and we have

$$\frac{d\tau_x}{d\tau_c} = \sqrt{-g_{00}} = 1 + gx. \quad (43)$$

We use this in (32). Thus we find that the force per unit charge that must be summed in order to calculate the self-force is given by

$$E^i = \sqrt{-g_{00}}F^i{}_\lambda u^\lambda = F^i{}_0, \quad (44)$$

which for our case is

$$E^i = (1+gx)\mathcal{E}^i. \quad (45)$$

For the case of a single point charge, to first order in g this is

$$E_\rho \simeq \frac{q\rho}{r^3}\left(1+\frac{g}{2}x\right),$$
$$E_x \simeq \frac{q}{r^3}\left(x+\frac{g}{2}[2x_0(x-x_0)-(\rho-\rho_0)^2]\right). \quad (46)$$

These equations agree with Eqs. (23)–(25) of Pinto [13].

Let $\mathbf{E}(\mathbf{r}_0, \mathbf{r})$ be the field at $\mathbf{r}$ due to a point charge at $\mathbf{r}_0$, as given by substituting (41) into (45). Then the electromagnetic self-force of a dipole formed by a pair of small charged spheres centered at (x_A, y_A, z_A) and (x_B, y_B, z_B) is (ignoring the force of each sphere on itself)

$$\mathbf{f}_{\text{self}} = -q^2(\mathbf{E}(\mathbf{r}_A, \mathbf{r}_B) + \mathbf{E}(\mathbf{r}_B, \mathbf{r}_A)). \quad (47)$$

This force is in the x direction. To fourth order in g we thus find

$$f_{\rm self} = \frac{q^2}{d}\left[g + \frac{3d^2 - \Delta x^2}{8}(-g^3 + (x_A + x_B)g^4)\right], \qquad (48)$$

where $d = |\mathbf{r}_B - \mathbf{r}_A|$ and $\Delta x = x_B - x_A$. This agrees with Eq. (26) of [13]. Note that the force is independent of orientation of the dipole at first order in g, and the lowest order orientation-dependent term is $O(g^3)$, in agreement with (20).

This completes the calculation of the electromagnetic self-force, but not the calculation of the total self-force, which must also include the effects of internal stress. However, the pressure in the rod varies as $1/(1 + gx)$ when calculated in the inertial instantaneous rest frame, and therefore it is uniform when calculated in the accelerating frame, so in the latter frame it does not contribute a net force when integrated over the whole surface of the rod.

In order to compare (48) with Eq. (24), one should divide (48) by $\sqrt{-g_{00}(x_0)} = 1 + gx_0$, where $x_0 = (x_A + x_B)/2$, and use (42). Thus one finds the lowest order term agrees exactly with the result for an accelerating dipole observed by an inertial observer. Therefore the system satisfies the equivalence principle.

A. Defining the electric field

In the above we discussed two definitions of what may be called electric field. One natural definition is to take the spatial part of the four-force per unit charge,

$$\mathcal{E}^i = u^\lambda F^i{}_\lambda; \qquad \mathcal{B}^i = \frac{1}{2}\epsilon^i{}_{\lambda\mu\nu}u^\lambda F^{\mu\nu}. \qquad (49)$$

This is recommended by Padmanabhan and Padmanabhan [34], but, as we have discussed, the notion of what is observed by observers fixed in the frame is better captured by including the metric in the definition, so that one obtains (44). This is the definition recommended by Landau and Lifshitz [6]. In the case of the Rindler frame (but not in general), the definition (49) has the following desirable feature: the field thus defined in the Rindler frame is equal to that observed in the instantaneous inertial rest frame of the local observer fixed in the Rindler frame. Since all observers fixed anywhere in the Rindler frame share the same instantaneous inertial rest frame, up to rotations (a special feature of certain frames, such as the Rindler frame), it follows that the electric field $\mathcal{E}$ in the Rindler frame will be independent of a translation of the coordinate system (a property not shared by F^i_0). This was noted in [34]; we have merely made an observation that allows it to be seen easily. Nevertheless, the field F^i_0 is the one best suited to examining what is observed by observers fixed in the frame, especially when comparing or summing forces observed at different locations.

V. CONCLUSION

To sum up, in this paper we have considered the electromagnetic self-force of the electric dipole. It is a mistake to treat a dipole as a pair of pointlike charged particles of finite charge, because this amounts to assuming that the object under discussion is equivalent to an unphysical one, namely one with a negative bare mass. Therefore one must consider something more realistic. An object that is physically possible, and that approximates to an electric dipole, is a pair of charged spherical shells of small radius R whose centers are a small distance d apart, moving rigidly (i.e., with fixed proper size and shape).

We first examined the supposed self-accelerating dipole. We concluded that the self-accelerating solution to the equation of motion is unphysical, because it is based on the assumption that a physical object could have negative bare mass, but that is not allowed in classical physics. In order to calculate this correctly, one must pay attention to all the terms, including the inertial term in the self-force of the charged spheres.

The above conclusion was obtained analytically for the case $d \gg R$, and then extended to all values of d by performing a numerical integration. We find that the total electromagnetic self-force is never along the direction of acceleration, for rigid hyperbolic motion of this system, and it vanishes in the limit $d \to 0$ (for any fixed value of R).

We also resolved a problem in relating the self-force to the expected inertia, when one compares the cases of transverse and longitudinal acceleration. We argued that it turns on the inertia of pressure (or equivalently, on the presence of hidden momentum), much like the famous "4/3 problem" for the charged sphere. In other words, one must include the effects of internal stresses in the physical object under discussion. The new feature is that the charge distribution is not spherically symmetric so neither is the stress-energy tensor. Hence the contribution of the internal stresses depends on the orientation of the system relative to its acceleration. In general, the electromagnetic self-force of a physical object *can* depend on the orientation of the object relative to its acceleration (and it does so depend for an accelerating dipole), but the rate of change of "hidden momentum" in the object also has such a dependence, with the net result that, for an isolated system in internal mechanical equilibrium, the ratio of momentum to velocity is independent of orientation, to first order in the acceleration, and is consistent with the mass-energy equivalence, as required by special relativity.

We then noted that self-force is open to more than one definition [Eqs. (30) and (32)]. This means that work based on the second definition [14–16] does not necessarily invalidate work based on the first, but in both cases one must pay attention to all the relevant forces.

We next considered the effects of gravity. We expect, of course, that the principle of equivalence will be upheld in any correct general relativistic treatment, but it is well known that that principle needs careful handling where self-force is concerned. It is useful to get precise algebraic statements of what can and cannot be said about self-force in a set of scenarios. We showed that an observer at rest in a gravitational field described everywhere by the Rindler metric will find any charged object supported in the field to possess an electromagnetic self-force equal to that observed in an inertial frame when the same object moves with constant acceleration and fixed proper size and shape. This is an exact statement about any charge distribution (not just a dipole or a sphere).

We showed how a recent calculation [13] of the fields of a point charge in the Rindler metric may be reconciled with several earlier calculations [17–19], and [34]. We then used this to obtain the self-force of a constantly accelerating dipole, as observed in the constantly accelerating reference frame in which the dipole is at rest.

All the effects described in this paper can be explored, in principle, through sensitive mass measurements of polar molecules.

ACKNOWLEDGMENTS

I thank an anonymous referee who drew my attention to Refs. [14,15]; this facilitated an improved discussion of the gravitational effects.

APPENDIX A: POINCARÉ STRESSES

The term given by Eq. (3) has been considered by many authors, starting with Poincaré in 1906 [9,10,39,40]; for a brief review see Rohrlich [7]. A simple way to calculate it is as follows. First consider a charged sphere in inertial motion. We assume the charge is all situated in a thin shell on the surface of the sphere. In the rest frame, the tension in the field at any point on the outer surface of this shell is

$$t = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0 R^2}\right)^2 = \frac{m_{\rm es}c^2}{4\pi R^3}.$$

Since the field inside the shell is zero, this is also the electromagnetic force per unit area on the shell of charge, in a radially outward direction (one can also obtain it by arguing that each element of charge experiences an average field equal to half that just outside the shell). For mechanical equilibrium, the material of the sphere must provide a compensating inward force. We can most simply model this by treating the sphere as an "ideal fluid," that is, a continuous system which has a rest frame in which there is no sheer stress, only pressure (or tension which is negative pressure). Then, for mechanical equilibrium, the pressure inside the sphere must be equal to $-t$. In the relativistic equations of motion for an ideal fluid, the energy density $\rho_0 c^2$ always enters in company with the pressure p, forming the combination $(\rho_0 c^2 + p)$ [41]. Consequently the inertia of a fluid is modified by its pressure. For inertial motion, mechanical equilibrium is attained if the pressure is uniform throughout the volume of the sphere. By integrating p/c^2 over this volume one finds that the inertial mass of the sphere is modified by

$$\frac{4}{3}\pi R^3 p/c^2 = -\frac{4}{3}\pi R^2 t/c^2 = -\frac{1}{3}m_{\rm es}.$$

When the sphere accelerates, the tension in the field changes somewhat, and the tension in the sphere is no longer uniform. However, such modifications are of higher than zeroth order in R. Therefore the above mass modification, multiplied by the acceleration, gives the leading order contribution to the self-force owing to Poincaré stresses, as given in Eq. (3).

APPENDIX B: NUMERICAL CALCULATIONS

We wish to calculate the self-force for a dipole consisting of two rigid spherical charged shells of radius R with centers separated by d and moving with constant proper acceleration in the transverse direction. Such an entity has an instantaneous rest frame. In this frame, let $\mathbf{f}^{(i,j)}$ be the net force on sphere i owing to the electric field sourced by sphere j. Then, owing to the linearity of electromagnetism, the total self-force is

$$\mathbf{f}^{(1,1)} + \mathbf{f}^{(1,2)} + \mathbf{f}^{(2,1)} + \mathbf{f}^{(2,2)} = 2(f_{\rm shell} + f_{\rm dip})\hat{\mathbf{x}},$$

where $f_{\rm shell}$ is given by Eq. ((25) and $f_{\rm dip}$ is equal to the x component of $\mathbf{f}^{(2,1)}$.

Choose the origin of coordinates so that the first sphere is centered at $(x,y,z)=(L_0,0,0)$ and the second at $(L_0,d,0)$, in the instantaneous rest frame, and both are accelerating in the positive x direction. Then

$$f_{\rm dip} = \int E_x^{(1)}(\mathbf{r}_2)dq_2, \tag{B1}$$

where $E_x^{(1)}(\mathbf{r}_2)$ is the x component of the electric field due to the first sphere at the location $\mathbf{r}_2$ of a point on the second sphere, and $\mathrm{d}q_2 = (-q/4\pi R)\mathrm{d}y_2\mathrm{d}\phi_2$ is an element of charge on the second sphere, in which ϕ_2 is an azimuthal angle about an axis through the centers of the spheres. ϕ_2 is in the range 0 to 2π, and y_2 ranges from $d-R$ to $d+R$. In the overall rectangular coordinate system, such an element is located at (x_2, y_2, z_2) given by

$$x_2 = L_0 + \sqrt{R^2-(y_2-d)^2}\cos(\phi_2),$$
$$y_2 = y_2,$$
$$z_2 = \sqrt{R^2-(y_2-d)^2}\sin(\phi_2),$$

where the positive square root should be taken. As explained in [23], to treat motion where the charge distribution undergoes acceleration at fixed proper dimensions, the electric field in the integrand is given by

$$E_x^{(1)}(\mathbf{r}_2) = \frac{q}{(4\pi)^2\epsilon_0 R}\int_{L_0-R}^{L_0+R}\mathrm{d}x_1 \times \int_0^{2\pi}\mathrm{d}\phi_1\{\tilde{E}_x(x_1; x_2, y_2-y_1, z_2-z_1)\}, \tag{B2}$$

where

$$\tilde{E}_x(L;x,y,z) \equiv \frac{-4L^2(L^2+y^2+z^2-x^2)}{((L^2+x^2+y^2+z^2)^2-4L^2x^2)^{3/2}},$$
$$y_1 = \sqrt{R^2-(x_1-L_0)^2}\cos(\phi_1),$$
$$z_1 = \sqrt{R^2-(x_1-L_0)^2}\sin(\phi_1). \tag{B3}$$

In order to handle the discontinuity in E_x, we used the "trick" described in Sec. II C. That is, before carrying out the integral to obtain f_{dip} we subtracted from E_x a field with the same discontinuity and whose contribution to the integral was zero.

[1] F. H. J. Cornish, Am. J. Phys. **54**, 166 (1986).
[2] D. J. Griffiths, Am. J. Phys. **54**, 744 (1986).
[3] V. Petkov, arXiv:physics/9906059.
[4] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1990), 3rd ed.
[5] J. D. Jackson, *Classical Electrodynamics* (John Wiley, New York, 1998), 3rd ed.
[6] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1971) [1st edition (Russian) 1941].
[7] F. Rohrlich, Am. J. Phys. **65**, 1051 (1997).
[8] H. Spohn, *Dynamics of Charged Particles and Their Radiation Field* (Cambridge University Press, Cambridge, UK, 2004).
[9] J. Schwinger, Found. Phys. **13**, 373 (1983).
[10] R. Medina, J. Phys. A **39**, 3801 (2006).
[11] D. J. Griffiths, T. C. Proctor, and D. F. Schroeter, Am. J. Phys. **78**, 391 (2010).
[12] D. J. Griffiths and R. E. Owen, Am. J. Phys. **51**, 1120 (1983).
[13] F. Pinto, Phys. Rev. D **73**, 104020 (2006).
[14] A. Ori and E. Rosenthal, Phys. Rev. D **68**, 041701 (2003).
[15] A. Ori and E. Rosenthal, J. Math. Phys. (N.Y.) **45**, 2347 (2004).
[16] P. Pearle, *in Electromagnetism: Paths to Research*, edited by D. Teplitz (Plenum Press, New York, 1982), p. 211.
[17] T. C. Bradbury, Ann. Phys. (N.Y.) **19**, 323 (1962).
[18] F. Rohrlich, Ann. Phys. (N.Y.) **22**, 169 (1963).
[19] E. Eriksen and O. Gron, Ann. Phys. (Amsterdam) **313**, 147 (2004).
[20] T. Fulton and F. Rohrlich, Ann. Phys. (N.Y.) **9**, 499 (1960).
[21] See [22], Sec. 8.2.5.
[22] A. M. Steane, *Relativity Made Relatively Easy* (Oxford University Press, Oxford, UK, 2012).
[23] A. M. Steane, Proc. R. Soc. A **470**, 20130480 (2013).
[24] See [20] or Eq. (8.74) of [22].
[25] J. A. E. Roa-Neri and J. L. Jimenez, Found. Phys. Lett. **7**, 403 (1994).
[26] Griffiths and Szeto [27] make this same observation without providing details; however, in [12] it was overlooked.
[27] D. J. Griffiths and E. W. Szeto, Am. J. Phys. **46**, 244 (1978).
[28] … for uniform acceleration. More generally, one must integrate Eq. (13).
[29] L. Page and N. I. Adams, Am. J. Phys. **13**, 141 (1945).
[30] S. Rainville, J. K. Thompson, and D. E. Pritchard, Science **303**, 334 (2004).
[31] E. Poisson, A. Pound, and I. Vega, Living Rev. Relativity **14**, 7 (2011).
[32] E. P. Soichiro Isoyama, Classical Quantum Gravity **29**, 155012 (2012).
[33] A. Vilenkin, Phys. Rev. D **20**, 373 (1979).
[34] H. Padmanabhan and T. Padmanabhan, Gen. Relativ. Gravit. **42**, 1153 (2010).
[35] W. Rindler, *Relativity: Special, General, and Cosmological* (Oxford University Press, Oxford, UK, 2006), 2nd ed.
[36] The full equation of motion has another term $\Gamma^{\mu}_{\alpha\nu} p_{\mu} dx^{\nu}/d\tau$ giving the gravitational contribution.
[37] This statement could also be derived directly from the metric, using $u_{\lambda}u^{\lambda} = -1$.
[38] E. Eriksen and O. Gron, Ann. Phys. (N.Y.) **286**, 320 (2000).
[39] H. Poincaré, Rend. Circ. Mat. Palermo **21**, 129 (1906).
[40] A. D. Yaghjian, *Relativistic Dynamics of a Charged Sphere* (Springer-Verlag, Berlin, 2006), 2nd ed.
[41] Equation (16.24) of [22].