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Absence of gyratons in the Robinson-Trautman class

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We present the Riemann and Ricci tensors for a fully general nontwisting and shear-free geometry in arbitrary dimension D. This includes both the nonexpanding Kundt and expanding Robinson–Trautman family of spacetimes. As an interesting application of these explicit expressions, we then integrate the Einstein equations and prove a surprising fact that in any D the Robinson–Trautman class does not admit solutions representing gyratonic sources, i.e., a matter field in the form of a null fluid (or particles propagating with the speed of light) with an additional internal spin. Contrary to the closely related Kundt class and pp-waves, the corresponding off-diagonal metric components thus do not encode the angular momentum of some gyraton. Instead, we demonstrate that in standard $D = 4$ general relativity they directly determine two independent amplitudes of the Robinson–Trautman exact gravitational waves.

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I. INTRODUCTION

The Robinson–Trautman class of spacetimes, discovered more than fifty years ago [1,2], is one of the most fundamental families of exact solutions to Einstein's field equations. Geometrically, it is defined by admitting a geodesic, shear-free, twist-free but *expanding* null congruence. This group of spacetimes contains many important vacuum solutions, in particular, Schwarzschild-like static black holes, accelerating black holes (C metric), and radiative spacetimes of various algebraic types. It also admits a cosmological constant, electromagnetic field, or pure radiation, as in the case of the Vaidya metric or Kinnersley photon rockets. More details and a substantial list of references can be found in chapter 28 of Ref. [3] or chapter 19 of Ref. [4].

In Ref. [5], the Robinson–Trautman family of solutions was extended to higher dimensions D in the case of empty space (with any value of the cosmological constant) and for aligned pure radiation. Interestingly, there are great differences with respect to the usual $D = 4$ case (see also Ref. [6]). Aligned electromagnetic fields were subsequently also incorporated into the Robinson–Trautman higher-dimensional spacetimes within the Einstein–Maxwell theory [7], and an additional Chern–Simons term for $D \geq 5$ odd dimensions was also considered. The results were recently summarized in the review work [8] on algebraic properties of spacetimes of higher dimensions.

The complementary *nonexpanding* Kundt class of twist-free and shear-free geometries also admits explicit vacuum solutions with an arbitrary cosmological constant, electromagnetic fields, and pure radiation (null fluid); see chapter 31 of Ref. [3] or chapter 18 of Ref. [4] for summaries concerning the Einstein theory in $D = 4$. The corresponding extensions to higher dimensions were presented in the work [9]. Interestingly, the whole Kundt class also admits spacetimes representing null fields of *gyratonic matter* with internal spin/helicity. It turns out that the angular momentum of such rotating sources is encoded in the nondiagonal metric functions.

This observation was made by Bonnor already in 1970 [10,11]. He studied both the interior and the exterior field of a "spinning null fluid" in the class of axially symmetric pp-wave spacetimes, which are the simplest representatives of the Kundt family. In the natural coordinates of nontwisting geometries (see Sec. II), the energy-momentum tensor in the interior region is phenomenologically described by the radiation energy density T_{uu} and by the components T_{up} representing the spinning character of the source (its nonzero angular momentum density). Spacetimes with such localized spinning sources moving at the speed of light were independently rediscovered and investigated (in four and higher dimensions) in 2005 by Frolov and his collaborators, who called them gyratons [12,13]. These pp-wave-type gyratons were later studied in greater detail and generalized to include a negative cosmological constant [14], an electromagnetic field [15], and various other settings including nonflat backgrounds. An extensive summary can be found in Ref. [16]. This recent work presents and investigates gyratons in a fully general class of Kundt spacetimes in any dimension.

In fact, all the so far known spacetimes with gyratonic sources belong to the Kundt class. The following question thus arises: Is it possible to find gyratons in other geometries? The most natural candidate is clearly the Robinson–Trautman family because it shares the nontwisting and shear-free property and in $D = 4$ it admits a similar algebraic structure. It differs only in having a nonvanishing expansion of the geometrically privileged null congruence.

This is the purpose of the present paper: We systematically study the possible existence of Robinson–Trautman

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gyratonic solutions (in any dimension), which would be analogous to those known in the Kundt class. First, in Sec. II, we present the general form of the nontwisting shear-free line element and all its components of the Christoffel symbols and the Riemann and the Ricci tensors. In subsequent Sec. III, we derive the explicit solutions to Einstein's equations in such a setting by performing their step-by-step integration. We summarize the obtained spacetimes and discuss them in Sec. IV. Appendix A contains the proof of some useful identities.

II. GENERAL ROBINSON–TRAUTMAN AND KUNDT GEOMETRY

In the most natural coordinates, the line element of a general nontwisting D-dimensional spacetime is given by [5]

$$\begin{aligned} \mathrm{d}s^2 = {} & g_{pq}(r,u,x)\mathrm{d}x^p\mathrm{d}x^q + 2g_{up}(r,u,x)\mathrm{d}u\mathrm{d}x^p \\ & - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\mathrm{d}u^2, \end{aligned} \tag{1}$$

where x is a shorthand for $(D-2)$ spatial coordinates x^p.[1] The nonvanishing contravariant metric components are g^{pq} (an inverse matrix to g_{pq}), $g^{ru}=-1$, $g^{rp}=g^{pq}g_{uq}$, and $g^{rr}=-g_{uu}+g^{pq}g_{up}g_{uq}$, so that

$$g_{up}=g_{pq}g^{rq}, \qquad g_{uu}=-g^{rr}+g_{pq}g^{rp}g^{rq}. \tag{2}$$

The geometrically privileged null vector field $\mathbf{k}=\partial_r$ generates a geodesic and affinely parametrized congruence. A direct calculation for the metric (1) immediately shows that the covariant derivative of $\mathbf{k}$ is given by $k_{a;b}=\Gamma^u_{ab}=\frac{1}{2}g_{ab,r}$, so that $k_{r;b}=0=k_{a;r}$. The optical matrix [8] defined as $\rho_{ij}\equiv k_{a;b}m^a_i m^b_j$, where $\mathbf{m}_i\equiv m^p_i(g_{up}\partial_r+\partial_p)$ are $(D-2)$ unit vectors forming the orthonormal basis in the transverse Riemannian space, is thus simply given by

$$\rho_{ij}=k_{p;q}m^p_i m^q_j=\frac{1}{2}g_{pq,r}m^p_i m^q_j. \tag{3}$$

This can be decomposed into the antisymmetric twist matrix $A_{ij}\equiv\rho_{[ij]}$, symmetric traceless shear matrix σ_{ij}, and the trace Θ determining the expansion such that $\sigma_{ij}+\Theta\delta_{ij}=\rho_{(ij)}$ with $\delta^{ij}\sigma_{ij}=0$, i.e., $\rho_{ij}=A_{ij}+\sigma_{ij}+\Theta\delta_{ij}$. From Eq. (3), we immediately see that $A_{ij}=0$, which confirms that the metric (1) is *nontwisting*. If we impose the additional condition that the metric is *shear free*, $\sigma_{ij}=0$, we obtain the relation $\Theta\delta_{ij}=\frac{1}{2}g_{pq,r}m^p_i m^q_j$. Using $g_{pq}m^p_i m^q_j=\delta_{ij}$, we thus infer

$$g_{pq,r}=2\Theta g_{pq}, \quad \text{so that} \quad g_{pq,rr}=2(\Theta_{,r}+2\Theta^2)g_{pq}. \tag{4}$$

The first expression can be integrated as

$$g_{pq}=R^2(r,u,x)h_{pq}(u,x), \qquad \text{where} \quad R=\exp\left(\int\Theta(r,u,x)\mathrm{d}r\right). \tag{5}$$

When the expansion vanishes, $\Theta=0$, this effectively reduces to $R=1$ so that the spatial metric $g_{pq}(u,x)$ is independent of the affine parameter r. It yields exactly the *Kundt class* of nonexpanding, twist-free and shear-free geometries [3,4,8,9]. The other case $\Theta\neq0$ gives the expanding *Robinson–Trautman class*, which we will study in this contribution.

The Christoffel symbols for the general nontwisting spacetime (1) after applying the shear-free condition (4) are

$$\Gamma^r_{rr}=0, \tag{6}$$

$$\Gamma^r_{ru}=-\frac{1}{2}g_{uu,r}+\frac{1}{2}g^{rp}g_{up,r}, \tag{7}$$

$$\Gamma^r_{rp}=-\frac{1}{2}g_{up,r}+\Theta g_{up}, \tag{8}$$

$$\Gamma^r_{uu}=\frac{1}{2}[-g^{rr}g_{uu,r}-g_{uu,u}+g^{rp}(2g_{up,u}-g_{uu,p})], \tag{9}$$

$$\Gamma^r_{up}=\frac{1}{2}[-g^{rr}g_{up,r}-g_{uu,p}+g^{rq}(2g_{u[q,p]}+g_{qp,u})], \tag{10}$$

$$\Gamma^r_{pq}=-\Theta g^{rr}g_{pq}-g_{u(p||q)}+\frac{1}{2}g_{pq,u}, \tag{11}$$

$$\Gamma^u_{rr}=\Gamma^u_{ru}=\Gamma^u_{rp}=0, \tag{12}$$

$$\Gamma^u_{uu}=\frac{1}{2}g_{uu,r}, \tag{13}$$

$$\Gamma^u_{up}=\frac{1}{2}g_{up,r}, \tag{14}$$

$$\Gamma^u_{pq}=\Theta g_{pq}, \tag{15}$$

$$\Gamma^m_{rr}=0, \tag{16}$$

$$\Gamma^m_{ru}=\frac{1}{2}g^{mn}g_{un,r}, \tag{17}$$

$$\Gamma^m_{rp}=\Theta\delta^m_p, \tag{18}$$

$$\Gamma^m_{uu}=\frac{1}{2}[-g^{rm}g_{uu,r}+g^{mn}(2g_{un,u}-g_{uu,n})], \tag{19}$$

$$\Gamma^m_{up}=\frac{1}{2}[-g^{rm}g_{up,r}+g^{mn}(2g_{u[n,p]}+g_{np,u})], \tag{20}$$

$$\Gamma^m_{pq}=-\Theta g^{rm}g_{pq}+{}^S\Gamma^m_{pq}, \tag{21}$$

where ${}^S\Gamma^m_{pq}\equiv\frac{1}{2}g^{mn}(2g_{n(p,q)}-g_{pq,n})$ are the Christoffel symbols with respect to the spatial coordinates only, i.e.,

[1]Throughout this paper, the indices m, n, p, and q label the spatial coordinates and range from 2 to $D-1$.

the coefficients of the covariant derivative on the transverse $(D-2)$-dimensional Riemannian space.

The Riemann curvature tensor components are then obtained (after straightforward but lengthy calculation) in the form

$$R_{rprq} = -(\Theta_{,r} + \Theta^2) g_{pq}, \tag{22}$$

$$R_{rpru} = -\frac{1}{2} g_{up,rr} + \frac{1}{2}\Theta g_{up,r}, \tag{23}$$

$$R_{ruru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{4} g^{pq} g_{up,r} g_{uq,r}, \tag{24}$$

$$R_{rpmq} = 2 g_{p[m}\Theta_{,q]} - 2\Theta^2 g_{p[m} g_{q]u} + \Theta g_{p[m} g_{q]u,r}, \tag{25}$$

$$R_{rupq} = g_{u[p,q],r} + \Theta(g_{u[p} g_{q]u,r} - 2 g_{u[p,q]}), \tag{26}$$

$$R_{rpuq} = \frac{1}{2} g_{up,r||q} + \frac{1}{4} g_{up,r} g_{uq,r} - g_{pq}\Theta_{,u} - \frac{1}{2}\Theta\left(g_{pq,u} + g_{pq} g_{uu,r} + g_{uq} g_{up,r} - g_{pq} g^{rn} g_{un,r} + 2 g_{u[p,q]} \right), \tag{27}$$

$$R_{ruup} = g_{u[u,p],r} + \frac{1}{4} g^{rn} g_{un,r} g_{up,r} - \frac{1}{2} g^{mn} g_{um,r} E_{np} + \Theta\left(g_{up,u} - \frac{1}{2} g_{uu,p} - \frac{1}{2} g_{up} g_{uu,r} \right), \tag{28}$$

$$R_{mpnq} = {}^{S}R_{mpnq} - \Theta^2 g^{rr}(g_{mn} g_{pq} - g_{mq} g_{pn}) - \Theta(g_{mn} e_{pq} + g_{pq} e_{mn} - g_{mq} e_{pn} - g_{pn} e_{mq}), \tag{29}$$

$$R_{upmq} = g_{p[m,u||q]} + g_{u[q,m]||p} + e_{p[m} g_{q]u,r} + \Theta(g^{rr} g_{p[m} g_{q]u,r} + g_{uu,[q} g_{m]p} - 2 g^{rn} E_{n[q} g_{m]p}), \tag{30}$$

$$\begin{aligned} R_{upuq} = {} & -\frac{1}{2}(g_{uu})_{||p||q} + g_{u(p,u||q)} - \frac{1}{2} g_{pq,uu} + \frac{1}{4} g^{rr} g_{up,r} g_{uq,r} - \frac{1}{2} g_{uu,r} e_{pq} + \frac{1}{2} g_{uu,(p} g_{q)u,r} - g^{rn} E_{n(p} g_{q)u,r} + g^{mn} E_{mp} E_{nq} \\ & - \frac{1}{2}\Theta g_{pq}[g^{rr} g_{uu,r} + g_{uu,u} - g^{rn}(2 g_{un,u} - g_{uu,n})]. \end{aligned} \tag{31}$$

Finally, the components of the Ricci tensor are

$$R_{rr} = -(D-2)(\Theta_{,r} + \Theta^2), \tag{32}$$

$$R_{rp} = -\frac{1}{2} g_{up,rr} - \frac{1}{2}(D-4)\Theta g_{up,r} + g_{up}\Theta_{,r} - (D-3)\Theta_{,p} + (D-2)\Theta^2 g_{up}, \tag{33}$$

$$R_{ru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{2} g^{rp} g_{up,rr} + \frac{1}{2} g^{pq}(g_{up,r||q} + g_{up,r} g_{uq,r}) - (D-2)\Theta_{,u} - \frac{1}{2}\Theta[g^{pq} g_{pq,u} - (D-4) g^{rp} g_{up,r} + (D-2) g_{uu,r}], \tag{34}$$

$$\begin{aligned} R_{pq} = {} & {}^{S}R_{pq} - f_{pq} - g_{pq}(g^{rr}\Theta_{,r} - 2\Theta_{,u} + 2 g^{rn}\Theta_{,n}) + 2 g_{u(p}\Theta_{,q)} + \Theta^2[2 g_{pq} g^{rn} g_{un} - (D-2) g_{pq} g^{rr} - 2 g_{up} g_{uq}] \\ & + \Theta[2 g_{u(p||q)} + 2 g_{u(p} g_{q)u,r} - (D-2) e_{pq} + g_{pq}(g_{uu,r} - 2 g^{rn} g_{un,r} - g^{mn} e_{mn})], \end{aligned} \tag{35}$$

$$\begin{aligned} R_{up} = {} & -\frac{1}{2} g^{rr} g_{up,rr} - \frac{1}{2} g_{uu,rp} + \frac{1}{2} g_{up,ru} + g^{rn} g_{u[n,p],r} - \frac{1}{2} g^{rn}(g_{up,r||n} + g_{un,r} g_{up,r}) \\ & + g^{mn}\left(\frac{1}{2} g_{um,r} g_{un||p} + g_{m[p,u||n]} + g_{u[m,p]||n} - \frac{1}{2} e_{mn} g_{up,r} \right) \\ & + g_{up}\Theta_{,u} + \Theta\Big[g_{up} g_{uu,r} + \frac{1}{2}(D-4)(g_{uu} g_{up,r} - g_{uu,p}) - g_{up,u} \\ & - g^{rn} g_{un,r} g_{up} + (D-6) g^{rn}\left(g_{u[n,p]} - \frac{1}{2} g_{un} g_{up,r} \right) + \frac{1}{2}(D-2) g^{rn} g_{np,u} \Big], \end{aligned} \tag{36}$$

$$
\begin{aligned}
R_{uu} = {} & -\frac{1}{2}g^{rr}g_{uu,rr} - g^{rn}g_{uu,rn} - \frac{1}{2}g^{mn}e_{mn}g_{uu,r} + g^{rn}g_{un,ru} - \frac{1}{2}g^{mn}g_{mn,uu} \\
& + g^{mn}\left(g_{um,u||n} - \frac{1}{2}g_{uu||m||n}\right) + \frac{1}{2}(g^{rr}g^{mn} - g^{rm}g^{rn})g_{um,r}g_{un,r} \\
& + 2g^{mn}g^{rp}g_{um,r}g_{u[n,p]} + \frac{1}{2}g^{mn}g_{um,r}g_{uu,n} + g^{mn}g^{pq}E_{pm}E_{qn} \\
& + \frac{1}{2}\Theta[(D-4)g^{rn}(2g_{un,u} - g_{uu,n} - g_{un}g_{uu,r}) + (D-2)(g_{uu}g_{uu,r} - g_{uu,u})], \qquad (37)
\end{aligned}
$$

and the Ricci scalar is

$$
\begin{aligned}
R = {} & {}^{S}R + g_{uu,rr} - 2g^{rn}g_{un,rr} - 2g^{pq}g_{up,r||q} - \frac{3}{2}g^{pq}g_{up,r}g_{uq,r} \\
& + 2\Theta_{,r}[(D-2)g_{uu} - (D-3)g^{rn}g_{un}] + 4(D-2)\Theta_{,u} - 4(D-3)g^{rn}\Theta_{,n} \\
& - \Theta^2[(D-1)(D-2)g^{rr} - 2(2D-5)g^{rn}g_{un}] \\
& + \Theta[2(D-2)g_{uu,r} - 2(2D-7)g^{rn}g_{un,r} + (D-1)g^{pq}g_{pq,u} - 2(D-3)g^{pq}g_{up||q}]. \qquad (38)
\end{aligned}
$$

In the above expressions, ${}^{S}R_{mpnq}$, ${}^{S}R_{pq}$, and ${}^{S}R$ are the Riemann tensor, Ricci tensor, and Ricci scalar for the transverse-space metric g_{pq}, respectively. The symbol $||$ denotes the covariant derivative with respect to g_{pq},

$$g_{up||q} = g_{up,q} - g_{um}{}^{S}\Gamma^m_{pq}, \qquad (39)$$

$$g_{up,r||q} = g_{up,rq} - g_{um,r}{}^{S}\Gamma^m_{pq}, \qquad (40)$$

$$g_{p[m,u||q]} = g_{p[m,q],u} + \frac{1}{2}({}^{S}\Gamma^n_{pm}g_{nq,u} - \Gamma^n_{pq}g_{nm,u}), \qquad (41)$$

$$g_{u[q||m]||p} = g_{u[q,m],p} - {}^{S}\Gamma^n_{pq}g_{u[n,m]} - {}^{S}\Gamma^n_{pm}g_{u[q,n]}, \qquad (42)$$

$$(g_{uu})_{||p||q} = g_{uu,pq} - g_{uu,n}{}^{S}\Gamma^n_{pq}, \qquad (43)$$

$$g_{up,u||q} = g_{up,uq} - g_{um,u}{}^{S}\Gamma^m_{pq}, \qquad (44)$$

and e_{pq}, E_{pq}, and f_{pq} are convenient shorthands defined as

$$e_{pq} = g_{u(p||q)} - \frac{1}{2}g_{pq,u}, \qquad (45)$$

$$E_{pq} = g_{u[p,q]} + \frac{1}{2}g_{pq,u}, \qquad (46)$$

$$f_{pq} = g_{u(p,r||q)} + \frac{1}{2}g_{up,r}g_{uq,r}, \qquad (47)$$

where, of course, $g_{u[p,q]} = g_{u[p||q]}$. It will also be useful to rewrite the following r derivatives of the metric functions in terms of the contravariant components [see Eq. (2)]:

$$g_{up,r} = g_{pq}(g^{rq}{}_{,r} + 2\Theta g^{rq}), \qquad (48)$$

$$g_{up,rr} = g_{pq}(g^{rq}{}_{,rr} + 2\Theta_{,r}g^{rq} + 4\Theta^2 g^{rq} + 4\Theta g^{rq}{}_{,r}), \qquad (49)$$

$$g_{uu,r} = -g^{rr}{}_{,r} + 2g_{pq}(g^{rp}g^{rq}{}_{,r} + \Theta g^{rp}g^{rq}), \qquad (50)$$

$$
\begin{aligned}
g_{uu,rr} = {} & -g^{rr}{}_{,rr} + 2g_{pq}(g^{rp}g^{rq}{}_{,rr} + g^{rp}{}_{,r}g^{rq}{}_{,r} + \Theta_{,r}g^{rp}g^{rq} \\
& + 2\Theta^2 g^{rp}g^{rq} + 4\Theta g^{rp}g^{rq}{}_{,r}). \qquad (51)
\end{aligned}
$$

The expressions (32)–(37) of the Ricci tensor enable us to write explicitly the gravitational field equations for any nontwisting and shear-free geometry of an arbitrary dimension D, that is, for any Kundt or Robinson–Trautman spacetime.

III. EINSTEIN'S FIELD EQUATIONS WITH GYRATONS AND THEIR COMPLETE INTEGRATION

General Einstein's equations for the metric g_{ab} have the form $R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$, where we admit a nonvanishing cosmological constant Λ and an arbitrary matter field given by its energy momentum-tensor T_{ab} with the trace $T = g^{ab}T_{ab}$. By substituting their trace $R = \frac{2}{D-2}(\Lambda D - 8\pi T)$, we obtain

$$R_{ab} = \frac{2}{D-2}\Lambda g_{ab} + 8\pi\left(T_{ab} - \frac{1}{D-2}Tg_{ab}\right). \qquad (52)$$

Our main aim here is to solve the Einstein field equations (52) in the case of expanding Robinson–Trautman geometry with a *gyratonic matter*, which is a natural generalization of a pure radiation field to admit a spin of the null source [10,12,16]. We thus assume that the only nonvanishing components of the energy-momentum tensor are $T_{uu}(r,u,x)$ corresponding to the classical pure radiation component and $T_{up}(r,u,x)$, which encodes inner gyratonic angular momentum. We immediately observe from Eqs. (1) and (2) that the trace of such an energy-momentum tensor vanishes, $T=0$.

Moreover, the condition $T^{ab}{}_{;b}=0$, which follows from Bianchi identities, after a straightforward manipulation, gives the constraints

$$T_{up;r}=0, \qquad T_{uu;r}=g^{pq}T_{up;q}. \tag{53}$$

These can be explicitly rewritten using Eqs. (6)–(21) as

$$T_{up,r}-\Theta T_{up}=0, \tag{54}$$

$$T_{uu,r}+(D-2)\Theta T_{uu} = g^{pq}T_{up||q}+\left[\frac{1}{2}g^{rp}{}_{,r}+(D-1)\Theta g^{rp}\right]T_{up}. \tag{55}$$

We can now perform a step-by-step integration of the Einstein field equations (52).

A. Equation $R_{rr}=0$

From Eq. (32), we get the explicit form of this equation,

$$\Theta_{,r}+\Theta^2=0, \tag{56}$$

which obviously determines the r dependence of the expansion scalar Θ. Its general solution can be written as $\Theta^{-1}=r+r_0(u,x)$. However, the metric (1) is invariant under the gauge transformation $r\to r-r_0(u,x)$, and we can thus, without loss of generality, set the integration function $r_0(u,x)$ to zero. The expansion simply becomes

$$\Theta=\frac{1}{r}. \tag{57}$$

The integral form (5) of the shear-free condition (4) with the expansion given by Eq. (57) completely determines the r dependence of the $(D-2)$-dimensional spatial metric $g_{pq}(r,u,x)$, namely,

$$g_{pq}=r^2h_{pq}(u,x), \tag{58}$$

so that $g^{pq}=r^{-2}h^{pq}$, where h^{pq} is the inverse matrix of h_{pq}. The r-independent metric part h_{pq} will be constrained by the next Einstein's equations.

B. Equation $R_{rp}=0$

Using Eqs. (33), (48), and (49), we rewrite the Ricci tensor component R_{rp} in a more compact way:

$$R_{rp}=-\frac{1}{2}g_{pq}(g^{rq}{}_{,rr}+D\Theta g^{rq}{}_{,r})-(D-3)\Theta_{,p}. \tag{59}$$

Employing now the restriction given by $R_{rr}=0$, i.e., the explicit form of expansion (57), the equation $R_{rp}=0$ becomes

$$g^{rq}{}_{,rr}+\frac{D}{r}g^{rq}{}_{,r}=0. \tag{60}$$

We easily find its general solution $g^{rq}(r,u,x)$ in the form

$$g^{rq}=e^q(u,x)+r^{1-D}f^q(u,x), \tag{61}$$

where e^q and f^q are arbitrary integration functions of u and x. In view of Eqs. (2) and (58), the corresponding covariant components of the Robinson–Trautman metric are

$$g_{up}=r^2e_p(u,x)+r^{3-D}f_p(u,x), \tag{62}$$

where $e_p\equiv h_{pq}e^q$ and $f_p\equiv h_{pq}f^q$.

At this stage, we can also fully integrate the energy-momentum conservation equations (54) and (55), which determine the r dependence of the gyratonic energy-momentum tensor:

$$T_{up}=\mathcal{J}_p r, \tag{63}$$

$$T_{uu}=\frac{1}{D-2}h^{pq}\mathcal{J}_{p||q}+\mathcal{J}_p\left[e^pr+\frac{1}{2}(D-1)f^pr^{2-D}\ln r\right]+\mathcal{N}r^{2-D}, \tag{64}$$

where $\mathcal{J}_p(u,x)$ and $\mathcal{N}(u,x)$ are integration functions of u and x.

C. Equation $R_{ru}=-\frac{2}{D-2}\Lambda$

It is convenient to rewrite the general Ricci tensor component (34) using the contravariant metric components,

$$2R_{ru}=g^{rr}{}_{,rr}+(D-2)\Theta g^{rr}{}_{,r}-g_{pq}(g^{rp}g^{rq}{}_{,rr}+g^{rp}{}_{,r}g^{rq}{}_{,r}+D\Theta g^{rp}g^{rq}{}_{,r})+(g^{rp}{}_{,r}+2\Theta g^{rp})_{||p}-\Theta g^{pq}g_{pq,u}-2(D-2)\Theta_{,u}. \tag{65}$$

Employing the previous results of (57), (58), and (61), the corresponding Einstein equation becomes

$$g^{rr}{}_{,rr} + (D-2)r^{-1}g^{rr}{}_{,r} = -2\left(e^p{}_{||p} - \frac{1}{2}h^{pq}h_{pq,u}\right)r^{-1} - \frac{4}{D-2}\Lambda + (D-3)f^p{}_{||p}r^{-D} + (D-1)^2 f^p f_p r^{2(1-D)}. \tag{66}$$

Its homogeneous solution is $g_0^{rr} = a + br^{3-D}$, where $a(u,x)$ and $b(u,x)$ are integration functions. The particular solution can be obtained as a superposition of terms $g_{(k)}^{rr} = \frac{1}{(k+2)(D-1+k)}\gamma r^{k+2}$ corresponding to all terms of the form γr^k on the right-hand side of Eq. (66). The general solution with an explicit r dependence of the metric component g^{rr} thus becomes

$$g^{rr} = a + br^{3-D} - \frac{2}{D-2}\left(e^p{}_{||p} - \frac{1}{2}h^{pq}h_{pq,u}\right)r - \frac{2\Lambda}{(D-1)(D-2)}r^2 + \frac{D-3}{D-2}f^p{}_{||p}r^{2-D} + \frac{D-1}{2(D-2)}f^p f_p r^{2(2-D)}. \tag{67}$$

Notice that g_{uu} is then simply obtained using Eq. (2) as

$$g_{uu} = -g^{rr} + r^2 e^p e_p + 2r^{3-D}e^p f_p + r^{2(2-D)}f^p f_p. \tag{68}$$

D. Equation $R_{pq} = \frac{2}{D-2}\Lambda g_{pq}$

Using Eqs. (50), (57), (58), (61), and (62), the general Ricci tensor component (35) becomes

$$\begin{aligned} R_{pq} = {}^S R_{pq} - [(D-3)g^{rr} + rg^{rr}{}_{,r}]h_{pq} - \left[\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)h_{pq} + (D-2)\left(e_{(p||q)} - \frac{1}{2}h_{pq,u}\right)\right]r \\ + (f_{(p||q)} - f^n{}_{||n}h_{pq})r^{2-D} - \frac{1}{2}(D-1)^2 f_p f_q r^{2(2-D)}, \end{aligned} \tag{69}$$

in which, employing Eq. (67),

$$(D-3)g^{rr} + rg^{rr}{}_{,r} = (D-3)a - 2\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)r - \frac{2\Lambda}{D-2}r^2 - \frac{D-3}{D-2}f^n{}_{||n}r^{2-D} - \frac{(D-1)^2}{2(D-2)}f^n f_n r^{2(2-D)}. \tag{70}$$

The corresponding Einstein equations (52) thus take the form

$$\begin{aligned} {}^S R_{pq} - (D-3)ah_{pq} + \left[\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)h_{pq} - (D-2)\left(e_{(p||q)} - \frac{1}{2}h_{pq,u}\right)\right]r \\ + \left(f_{(p||q)} - \frac{h_{pq}}{D-2}f^n{}_{||n}\right)r^{2-D} - \frac{1}{2}(D-1)^2\left(f_p f_q - \frac{h_{pq}}{D-2}f^n f_n\right)r^{2(2-D)} = 0. \end{aligned} \tag{71}$$

The trace of this equation explicitly determines the function $a(u,x)$ introduced in Eq. (67), namely,

$$a = \frac{\mathcal{R}}{(D-2)(D-3)}, \tag{72}$$

where $\mathcal{R} \equiv h^{pq}\mathcal{R}_{pq}$ is the Ricci scalar curvature of the spatial metric h_{pq}, which is the r-independent part of g_{pq}. Notice that due to Eq. (58) the corresponding Ricci tensor is $\mathcal{R}_{pq} \equiv {}^S R_{pq}$, while $\mathcal{R} \equiv {}^S R r^2$. Decomposing Eq. (71) into the terms with different powers of r, we obtain the following constraints on the metric functions:

$$\mathcal{R}_{pq} = \frac{h_{pq}}{D-2}\mathcal{R}, \tag{73}$$

$$\frac{1}{2}h_{pq,u} = e_{(p||q)} - \frac{h_{pq}}{D-2}\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right), \tag{74}$$

$$f_{(p||q)} = \frac{h_{pq}}{D-2}f^n{}_{||n}, \tag{75}$$

$$f_p f_q = \frac{h_{pq}}{D-2}f^n f_n. \tag{76}$$

Now, if we multiply both sides of Eq. (76) by f^q, we obtain $f_p(f^q f_q) = \frac{1}{D-2}f_p(f^q f_q)$. This necessarily implies

$$f_p = 0 \tag{77}$$

whenever $f^q f_q \neq 0$. If $f^q f_q \equiv h^{pq} f_p f_q = 0$, then again $f_p = 0$ for all p because the Riemannian metric h^{pq} is a positive definite matrix. In such a case, the condition (75) is trivially satisfied.

At this stage, the most general Robinson–Trautman line element (possibly admitting the gyratonic matter) takes the form

$$\mathrm{d}s^2 = r^2 h_{pq}\mathrm{d}x^p\mathrm{d}x^q + 2r^2 e_p \mathrm{d}u\mathrm{d}x^p - 2\mathrm{d}u\mathrm{d}r + (r^2 e^p e_p - g^{rr})\mathrm{d}u^2, \tag{78}$$

where

$$g^{rr} = a + br^{3-D} + cr - \frac{2}{(D-1)(D-2)}\Lambda r^2, \tag{79}$$

with

$$c \equiv -\frac{2}{D-2}\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right). \tag{80}$$

The functions h_{pq} and e_p are constrained by Eqs. (73) and (74). Because of Eq. (73), the transverse $(D-2)$-dimensional Riemannian space must be an Einstein space.

E. Equation $R_{up} = \frac{2}{D-2}\Lambda g_{up} + 8\pi T_{up}$

Using Eqs. (57), (58), (61), (62), and (68) with Eq. (77), the Ricci tensor component R_{up} (36) becomes

$$R_{up} = -e_p[(D-3)g^{rr} + rg^{rr}{}_{,r}] + \frac{1}{2}[g^{rr}{}_{,rp} + (D-4)r^{-1}g^{rr}{}_{,p}] + h^{mn}(h_{m[p,u||n]} + e_{[m,p]||n}) + \left[(D-2)\left(e^n e_{[n,p]} - \frac{1}{2}(e^n e_n)_{,p} + \frac{1}{2}e^n h_{np,u}\right) - e_p\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)\right]r, \tag{81}$$

where

$$(D-3)g^{rr} + rg^{rr}{}_{,r} = \frac{1}{D-2}\mathcal{R} - 2\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)r - \frac{2}{D-2}\Lambda r^2,$$

$$g^{rr}{}_{,rp} + (D-4)r^{-1}g^{rr}{}_{,p} = -2\frac{D-3}{D-2}\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)_{,p} + \frac{(D-4)}{(D-2)(D-3)}\mathcal{R}_{,p}r^{-1} - b_{,p}r^{2-D}; \tag{82}$$

see Eqs. (67) and (70) with Eqs. (72) and (77). The corresponding Einstein equations (52) with Eq. (63) are thus

$$-\frac{1}{D-2}\mathcal{R}e_p - \frac{D-3}{D-2}\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)_{,p} + h^{mn}(h_{m[p,u||n]} + e_{[m,p]||n}) + \frac{(D-4)}{2(D-2)(D-3)}\mathcal{R}_{,p}r^{-1} - \frac{1}{2}b_{,p}r^{2-D} + \left[(D-2)\left(e^n e_{[n,p]} - \frac{1}{2}(e^n e_n)_{,p} + \frac{1}{2}e^n h_{np,u}\right) + e_p\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)\right]r = 8\pi\mathcal{J}_p r. \tag{83}$$

This gives the following conditions:

$$\mathcal{R}e_p + (D-3)\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)_{,p} - (D-2)h^{mn}(h_{m[p,u||n]} + e_{[m,p]||n}) = 0, \tag{84}$$

$$(D-4)\mathcal{R}_{,p} = 0, \tag{85}$$

$$b_{,p} = 0, \tag{86}$$

$$(D-2)\left(e^n e_{[n,p]} - \frac{1}{2}(e^n e_n)_{,p} + \frac{1}{2}e^n h_{np,u}\right) + e_p\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right) = 8\pi\mathcal{J}_p. \tag{87}$$

Using Eq. (74), the relation $e_{m||p||n} = e_{m||n||p} + e_q\mathcal{R}^q_{mpn}$, and Eq. (73), we find that Eq. (84) is satisfied identically. Equation (85) clearly restricts the dependence of the spatial Ricci scalar $\mathcal{R}$ on the spatial coordinates x^p, namely,

$$\mathcal{R} = \mathcal{R}(u) \quad \text{for } D > 4, \tag{88}$$

$$\mathcal{R} = \mathcal{R}(u, x) \quad \text{for } D = 4. \tag{89}$$

There is thus a significant difference between the $D = 4$ case of classical relativity and the extension of Robinson–Trautman spacetimes to higher dimensions. Similarly, Eq. (86) gives

$$b = b(u). \tag{90}$$

Finally, by substituting the expression (74) into Eq. (87), we get

$$(D-2)\left(e^n e_{n||p} - \frac{1}{2}(e^n e_n)_{,p}\right) = 8\pi \mathcal{J}_p. \tag{91}$$

Since $(e^n e_n)_{,p} = (e^n e_n)_{||p}$, its left-hand side *always vanishes*, and we obtain the condition for the energy-momentum tensor (63), (64)

$$\mathcal{J}_p = 0. \tag{92}$$

Necessarily, in *any* dimension D, we thus obtain

$$T_{up} = 0, \qquad T_{uu} = \mathcal{N} r^{2-D}, \tag{93}$$

which is just the well-known pure radiation field (null fluid) *without* the "rotational" components T_{up} of the energy-momentum tensor. We have thus proved that there are *no* solutions with gyratonic sources in the Robinson–Trautman class of spacetimes.

F. Equation $R_{uu} = \frac{2}{D-2}\Lambda g_{uu} + 8\pi T_{uu}$

This final equation determines the relation between the Robinson–Trautman geometry and the pure radiation matter field represented by the profile $\mathcal{N}(u,x)$. Using Eqs. (57), (58), (61), (62), and (68) with Eq. (77), the Ricci tensor component R_{uu} given by Eq. (37) becomes

$$\begin{aligned} R_{uu} = {} & \frac{1}{2}g^{rr}g^{rr}{}_{,rr} + \frac{1}{2}\left[e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u} + (D-2)g^{rr}r^{-1} - 2e^n e_n r\right]g^{rr}{}_{,r} \\ & + e^n\left[g^{rr}{}_{,r} + \frac{1}{2}(D-6)g^{rr}r^{-1}\right]_{,n} + \frac{1}{2}h^{mn}g^{rr}_{||m||n}r^{-2} + \frac{1}{2}(D-2)g^{rr}{}_{,u}r^{-1} \\ & - (D-3)e^n e_n g^{rr} + h^{mn}\left[e_{m,u||n} - \frac{1}{2}(e^p e_p)_{||m||n} - \frac{1}{2}h_{mn,uu}\right] \\ & + h^{mn}h^{pq}\left(e_{[p,m]} + \frac{1}{2}h_{pm,u}\right)\left(e_{[q,n]} + \frac{1}{2}h_{qn,u}\right) \\ & + \left[\frac{1}{2}(D-2)(e^m e^n h_{mn,u} - e^n(e^p e_p)_{,n}) - e^p e_p\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)\right]r. \end{aligned} \tag{94}$$

Moreover, employing the explicit form (79) of g^{rr} with the help of Eqs. (74) and (90), we get

$$\begin{aligned} R_{uu} = {} & \frac{2}{D-2}\Lambda g_{uu} + \frac{1}{2}(D-2)\left[b_{,u} + \frac{1}{2}(D-1)bc\right]r^{2-D} + \frac{1}{2}h^{mn}a_{||m||n}r^{-2} \\ & + \frac{1}{2}[(D-2)(a_{,u} + ac) + (D-6)e^n a_{,n} + h^{mn}c_{||m||n}]r^{-1} \\ & + \frac{1}{2}(D-2)(c_{,u} + c^2) + e^n{}_{||n}c + \frac{1}{2}(D-4)e^n c_{,n} - (D-3)e^p e_p a \\ & + h^{mn}\left[e_{m,u||n} - \frac{1}{2}h_{mn,uu} - \frac{1}{2}(e^p e_p)_{||m||n} + h^{pq}e_{p||m}e_{q||n}\right] \\ & + \frac{1}{2}(D-2)[e^m e^n h_{mn,u} - e^n(e^p e_p)_{,n} - e^n e_n c]r, \end{aligned} \tag{95}$$

where a is given by Eq. (72) and c by Eq. (80). Now, lengthy calculations using the previously derived constraints lead to the identities (proved in the Appendix)

$$e^m e^n h_{mn,u} - e^n(e^p e_p)_{,n} - e^n e_n c = 0, \tag{96}$$

$$\frac{1}{2}(D-2)(c_{,u} + c^2) + e^n{}_{||n}c + \frac{1}{2}(D-4)e^n c_{,n} - (D-3)e^p e_p a + h^{mn}\left[e_{m,u||n} - \frac{1}{2}h_{mn,uu} - \frac{1}{2}(e^p e_p)_{||m||n} + h^{pq}e_{p||m}e_{q||n}\right] = 0, \tag{97}$$

$$(D-2)(a_{,u}+ac)+(D-6)e^n a_{,n}+h^{mn}c_{||m||n}=(D-4)e^n a_{,n}, \tag{98}$$

which are exactly the terms in Eq. (95) proportional to r, r^0, and r^{-1}, respectively. The corresponding Einstein equation $R_{uu}=\frac{2}{D-2}\Lambda g_{uu}+8\pi T_{uu}$ with Eq. (93) thus takes a very simple form

$$\frac{1}{2}(D-2)\left[b_{,u}+\frac{1}{2}(D-1)bc\right]r^{2-D}+\frac{1}{2}h^{mn}a_{||m||n}r^{-2}+\frac{1}{2}(D-4)e^n a_{,n}r^{-1}=8\pi\mathcal{N}r^{2-D}. \tag{99}$$

It is interesting that also the last term on the left-hand side always vanishes since $(D-4)a_{,n}=0$ in any dimension D; see Eq. (85). Consequently, in any dimension, the last Einstein equation can be compactly[2] written as

$$\Delta a+\frac{1}{2}(D-1)(D-2)bc+(D-2)b_{,u}=16\pi\mathcal{N}, \tag{100}$$

where $\Delta a\equiv h^{mn}a_{||m||n}$ is the covariant Laplace operator on the $(D-2)$-dimensional transverse Riemannian space. In particular, this Einstein field equation reads

$$\frac{1}{2}(D-1)bc+b_{,u}=\frac{16\pi}{D-2}\mathcal{N}\quad \text{for } D>4, \tag{101}$$

$$\Delta\left(\frac{1}{2}\mathcal{R}\right)+3bc+2b_{,u}=16\pi\mathcal{N}\quad \text{for } D=4. \tag{102}$$

For the special choice $e^p=0$, Eq. (102) reduces exactly to the classical Robinson–Trautman equation [3,4] [with the identification $a=\frac{1}{2}\mathcal{R}=\Delta(\log P)=K$, $b=-2m(u)$, and $c=-2(\log P)_{,u}$, where K is the Gaussian curvature of the spatial metric $h_{pq}=P^{-2}\delta_{pq}$]. Equation (101) generalizes the field equation previously derived in Ref. [5] in the sense that now it also includes the contribution from the off-diagonal metric components e^p entering the function c as their covariant spatial divergence $e^n{}_{||n}$; see Eq. (80).

IV. SUMMARY AND CONCLUDING DISCUSSION

The most general D-dimensional Robinson–Trautman line element in vacuum, with a cosmological constant Λ and possibly the pure radiation field $T_{uu}=\mathcal{N}r^{2-D}$ can thus be written [in the natural gauge in which the expansion is $\Theta=r^{-1}$; see Eq. (57)] as

$$\mathrm{d}s^2=r^2h_{pq}(\mathrm{d}x^p+e^p\mathrm{d}u)(\mathrm{d}x^q+e^q\mathrm{d}u)-2\mathrm{d}u\mathrm{d}r-g^{rr}\mathrm{d}u^2, \tag{103}$$

where

$$g^{rr}=\frac{\mathcal{R}}{(D-2)(D-3)}+\frac{b(u)}{r^{D-3}}-\frac{2}{D-2}\left(e^p{}_{||p}-\frac{1}{2}h^{pq}h_{pq,u}\right)r-\frac{2\Lambda}{(D-1)(D-2)}r^2, \tag{104}$$

with the functions $h_{pq}(u,x)$ and $e^p(u,x)$ constrained by Eqs. (73), (74), and (100), namely,

$$\mathcal{R}_{pq}=\frac{h_{pq}}{D-2}\mathcal{R}, \tag{105}$$

$$e_{(p||q)}-\frac{1}{2}h_{pq,u}=\frac{h_{pq}}{D-2}\left(e^n{}_{||n}-\frac{1}{2}h^{mn}h_{mn,u}\right), \tag{106}$$

$$\frac{\Delta\mathcal{R}}{(D-2)(D-3)}-(D-1)\left(e^n{}_{||n}-\frac{1}{2}h^{mn}h_{mn,u}\right)b+(D-2)b_{,u}=16\pi\mathcal{N}, \tag{107}$$

in which b is a function of the null coordinate u only. The first equation (105) restricts just the Riemannian metric h_{pq} of the transverse $(D-2)$-dimensional space covered by the coordinates x^p, with $\mathcal{R}_{pq}$ and $\mathcal{R}$ being its Ricci tensor and Ricci scalar. Therefore, any Einstein space metric h_{pq} is admitted. The second constraint (106) imposes a *specific coupling* between the spatial metric h_{pq} and the off-diagonal metric components represented by $(D-2)$ functions e^p. In addition, there is the Einstein equation (107), which relates these metric functions and an arbitrary "mass" function $b(u)$ to the pure radiation profile $\mathcal{N}$. In the vacuum case, $\mathcal{N}=0$.

After the step-by-step integration of all Einstein's equations, we proved that there are no gyratons in the Robinson–Trautman class. In any dimension including $D=4$, we necessarily obtained $\mathcal{J}_p=0$ so that $T_{up}=\mathcal{J}_p r=0$; see Eq. (63) with Eq. (92) or (93), which means that the null matter field cannot have an "internal spin" (angular momentum).

This is in striking contrast to the closely related Kundt family of spacetimes, which in general (and in any D) admits such gyrating sources, as recently demonstrated in Ref. [16] (there is also a comprehensive list of previous works studying particular subclasses of the Kundt

[2]The term proportional to r^{-2} in Eq. (99) is always zero in the case $D>4$ since $a_{||m||n}=0$ due to Eq. (88). In the $D=4$ case, it is combined with the terms proportional to $r^{2-D}=r^{-2}$ into the expression (100).

gyratons). Such a conclusion is surprising because the Robinson–Trautman family of geometries is the closest to the Kundt family—both are nontwisting and shear free, and (at least in $D=4$) they admit similar algebraic structures and matter fields.

The questions thus arise about the nature of such a difference and also concerning the possible physical interpretation of the off-diagonal metric functions e^p. In the following two short sections, we will tackle these two problems.

A. Robinson–Trautman vs gyratonic Kundt spacetimes

Of course, the Robinson–Trautman class of spacetimes is expanding ($\Theta = r^{-1} \neq 0$), while the Kundt class is nonexpanding ($\Theta = 0$). The absence or presence of the gyratons thus must be traced to this geometric difference. In Sec. II, we presented the complete list of all curvature tensor components for any nontwisting and shear-free geometry that contains both the Robinson–Trautman and the Kundt family. We are thus able to trace the point at which the integration of Einstein's equations with gyratonic energy-momentum tensor starts to differ significantly.

To be specific, by setting $\Theta = 0$ for the Kundt class in Eq. (4), we immediately obtain $g_{pq} = h_{pq}$ independent of r, instead of Eq. (58), which reads $g_{pq} = r^2 h_{pq}$ in the Robinson–Trautman case. The second field equation (59) for $\Theta = 0$ yields $g_{up} = e_p + r f_p$ instead of Eq. (62), which is $g_{up} = r^2 e_p + r^{3-D} f_p$. The third field equation (34) gives, instead of Eqs. (67) and (68), $g_{uu} = a + br + [\frac{2}{D-2}\Lambda + \frac{1}{2}(f^p{}_{||p} + f^p f_p)]r^2$ in full agreement with Eqs. (64) and (65) of Ref. [9]. Apart from different powers of r, the metric coefficients for the Kundt and Robinson–Trautman spacetimes thus look very similar.

The main difference between these two types of geometries occurs after employing the next field equation for the spatial Ricci components R_{pq} given by Eq. (35). For the Kundt class, this equation is *independent* of r, namely, $\mathcal{R}_{pq} = \frac{2}{D-2}\Lambda h_{pq} + f_{pq}$, where $f_{pq} \equiv f_{(p||q)} + \frac{1}{2} f_p f_q$. Its trace $\mathcal{R} = 2\Lambda + h^{mn} f_{mn}$ enables us to rewrite it as

$$\begin{aligned}\mathcal{R}_{pq} - \frac{h_{pq}}{D-2}\mathcal{R} &= f_{pq} - \frac{h_{pq}}{D-2} h^{mn} f_{mn} \\ &= \left(f_{(p||q)} - \frac{h_{pq}}{D-2} f^n{}_{||n} \right) + \frac{1}{2}\left(f_p f_q - \frac{h_{pq}}{D-2} f^n f_n \right).\end{aligned} \tag{108}$$

This imposes a specific coupling between the traceless part of the Ricci curvature $\mathcal{R}_{pq}$ of the $(D-2)$-dimensional Riemannian space and the traceless part of the tensor f_{pq} constructed from the functions f_p determining (part of) the off-diagonal Kundt metric components g_{up}.

It can now be observed from Eq. (71) that exactly the same terms occur in the corresponding field equation for the Robinson–Trautman metric, but with *different powers* of r. This is the key point: In the nonexpanding Kundt class, we have obtained just one condition (108), whereas in the expanding Robinson–Trautman class, there are *four separate constraints*, namely, Eqs. (73)–(76). It is the severe constraint (76) that necessarily requires $f_p = 0$, see Eq. (77), which then fulfills Eq. (75) identically. In the Robinson–Trautman case, we are thus left with the condition (73), which for $f_p = 0$ is the *same as* Eq. (108) in the nonexpanding case. However, in the Robinson–Trautman case, there is the *additional constraint* (74), i.e., Eq. (106), that couples $e_{(p||q)} - \frac{1}{2} h_{pq,u}$ to its trace. It turns out that this specific restriction on possible functions e_p determining the other part of the off-diagonal metric components g_{up} forbids—after applying the following Einstein's equation for R_{up}—the presence of gyratonic matter fields in the Robinson–Trautman geometries; see Eq. (92). In contrast, there is *no such constraint* on e_p in the nonexpanding family, which enables gyratons to be included in the Kundt geometries.

B. Robinson–Trautman gravitational waves in $D = 4$

Finally, we will elucidate the physical meaning of the functions e_p. Instead of representing the angular momentum of a gyratonic matter they directly encode amplitudes of the Robinson–Trautman gravitational waves. In usual $D = 4$ dimensions, the transverse Riemannian space is two-dimensional. If such a 2-space were to have *constant curvature*, its metric h_{pq} can be written in the conformally flat form

$$\begin{aligned}&\mathrm{d}s_0^2 = h_{pq}\mathrm{d}x^p\mathrm{d}x^q = \psi^{-2}[(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2], \quad \text{where} \\ &\psi \equiv 1 + \frac{1}{2}\epsilon[(x^2)^2 + (x^3)^2],\end{aligned} \tag{109}$$

with $\epsilon = 0, +1$, or -1. For such a metric, the Christoffel symbols are

$$\begin{aligned}{}^S\Gamma^2_{22} &= {}^S\Gamma^3_{23} = -{}^S\Gamma^2_{33} = -\epsilon x^2 \psi^{-1}, \\ {}^S\Gamma^3_{33} &= {}^S\Gamma^2_{32} = -{}^S\Gamma^3_{22} = -\epsilon x^3 \psi^{-1}.\end{aligned} \tag{110}$$

Nontrivial Riemann and Ricci tensor components read $\mathcal{R}_{2323} = 2\epsilon\psi^{-4}$ and $\mathcal{R}_{22} = \mathcal{R}_{33} = 2\epsilon\psi^{-2}$, so that the Ricci scalar is $\mathcal{R} = 4\epsilon = 2K$, where K is the (constant) Gaussian curvature. This obviously satisfies the constraint (105). It remains to fulfill the constraint (106). Since h_{pq} given by Eq. (109) is independent of u, it reduces to $e_{(p||q)} = \frac{1}{2} h_{pq} e^n{}_{||n}$, which is, using Eq. (110),

$$
\begin{aligned}
e_{(2||2)} &= \psi^{-2} e^2{}_{,2} - \epsilon\psi^{-3}(x^2 e^2 + x^3 e^3),\\
e_{(3||3)} &= \psi^{-2} e^3{}_{,3} - \epsilon\psi^{-3}(x^2 e^2 + x^3 e^3),\\
e_{(2||3)} &= \frac{1}{2}\psi^{-2}(e^2{}_{,3} + e^3{}_{,2}),\\
e^n{}_{||n} &= (e^2{}_{,2} + e^3{}_{,3}) - 2\epsilon\psi^{-1}(x^2 e^2 + x^3 e^3). \qquad (111)
\end{aligned}
$$

The constraint is thus equivalent to very simple two conditions[3]:

$$e^2{}_{,2} = e^3{}_{,3} \quad \text{and} \quad e^2{}_{,3} = -e^3{}_{,2}. \qquad (112)$$

Clearly, these are just the Cauchy–Riemann conditions for the *complex* function f constructed from the real functions $e^2(u, x^2, x^3)$ and $e^3(u, x^2, x^3)$, which depend on an external parameter u and the complex variable ξ composed from the spatial coordinates x^2 and x^3. In particular, introducing the complex quantities

$$\xi \equiv \frac{1}{\sqrt{2}}(x^2 + \mathrm{i}x^3) \quad \text{and} \quad f \equiv -\frac{1}{\sqrt{2}}(e^2 + \mathrm{i}e^3), \qquad (113)$$

we obtain that f is a *holomorphic function* of the complex variable ξ since Eq. (112) is equivalent to $f_{,\bar{\xi}} = 0$, while $f_{,\xi} = -(e^2{}_{,2} + \mathrm{i}e^3{}_{,2})$. Any complex function $f(u, \xi)$ holomorphic in ξ thus automatically satisfies the constraint (106).

Therefore, in Einstein's $D = 4$ general relativity, it is convenient to adopt the complex representation ξ of the spatial coordinates in the transverse 2-space and the complex function $f(\xi)$ to represent the off-diagonal metric functions e^2 and e^3. Performing the transformation (113), the general vacuum Robinson–Trautman solution (103), (104) (possibly with a cosmological constant Λ and/or pure radiation field) for which the transverse 2-space has constant curvature takes the form

$$\mathrm{d}s^2 = 2\frac{r^2}{\psi^2}|\mathrm{d}\xi - f(u,\xi)\mathrm{d}u|^2 - 2\mathrm{d}u\mathrm{d}r - g^{rr}\mathrm{d}u^2, \qquad (114)$$

$\psi \equiv 1 + \epsilon\xi\bar{\xi}$ with $\epsilon = 0$, $+1$, or -1, and

$$g^{rr} = 2\epsilon + \frac{b}{r} + \left((f_{,\xi} + \bar{f}_{,\bar{\xi}}) - \frac{2\epsilon}{\psi}(\bar{\xi}f + \xi\bar{f})\right)r - \frac{\Lambda}{3}r^2. \qquad (115)$$

Here, $b = b(u)$ is an arbitrary function, and we used the fact that

$$e^p{}_{||p} = -(f_{,\xi} + \bar{f}_{,\bar{\xi}}) + \frac{2\epsilon}{\psi}(\bar{\xi}f + \xi\bar{f}). \qquad (116)$$

The field equation (107) is now reduced to a simple relation $2b_{,u} - 3be^p{}_{||p} = 16\pi\mathcal{N}$, which in the *vacuum case* is just

$$2b_{,u} = 3be^p{}_{||p}. \qquad (117)$$

For $b = 0$, this vacuum field equation is identically satisfied. In such a case, we obtain the complete family of Robinson–Trautman gravitational waves of algebraic type N. Indeed, the metric (114), (115) with $b = 0$ is exactly the line element written in Sec. 4 of Ref. [17] above Eq. (16). This is related to the García Díaz–Plebański 1981 form [18] of these exact radiative spacetimes,

$$
\begin{aligned}
\mathrm{d}s^2 = {}& 2v^2\mathrm{d}\xi\mathrm{d}\bar{\xi} + 2v\bar{A}\mathrm{d}\xi\mathrm{d}u + 2vA\mathrm{d}\bar{\xi}\mathrm{d}u + 2\psi\mathrm{d}u\mathrm{d}v\\
& + 2(A\bar{A} + \psi B)\mathrm{d}u^2, \qquad (118)
\end{aligned}
$$

where $A \equiv \epsilon\xi - vf$, $B \equiv -\epsilon + \frac{1}{2}v(f_{,\xi} + \bar{f}_{,\bar{\xi}}) + \frac{1}{6}\Lambda v^2\psi$, via a simple transformation $r = -v\psi$; see also Ref. [19]. The corresponding gravitational wave amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$ of the two independent polarizations are directly determined by the *second covariant derivatives of the function* $c = -e^p{}_{||p}$, cf. Eq. (80), namely, by the symmetric traceless 2×2 matrix

$$w_{pq} \equiv c_{||p||q} - \frac{1}{2}h_{pq}h^{mn}c_{||m||n}. \qquad (119)$$

Indeed, using Eq. (110), it follows that

$$
\begin{aligned}
w_{22} = -w_{33} &= \frac{1}{2}(c_{,2,2} - c_{,3,3}) + \epsilon\psi^{-1}(x^2 c_{,2} - x^3 c_{,3}),\\
w_{23} = w_{32} &= c_{,2,3} + \epsilon\psi^{-1}(x^3 c_{,2} + x^2 c_{,3}), \qquad (120)
\end{aligned}
$$

which can be rewritten in the complex notation (113) as $w_{33} = -2\mathcal{R}e\Psi$, $w_{23} = -2\mathcal{I}m\Psi$, where $\Psi \equiv \frac{1}{2}\psi^{-2}(\psi^2 c_{,\xi})_{,\xi}$. Substituting for $c = -e^p{}_{||p}$ from Eq. (116), we immediately obtain $\Psi = \frac{1}{2}f_{,\xi\xi\xi}$. This yields very simple explicit relations:

$$w_{33} = -\mathcal{R}e f_{,\xi\xi\xi}, \quad w_{23} = -\mathcal{I}m f_{,\xi\xi\xi}. \qquad (121)$$

By comparing with the expressions (34) of Ref. [20] determining the two amplitudes of the Robinson–Trautman gravitational waves (measured by a geodesic deviation in a suitable orthonormal frame), we observe that $\mathcal{A}_+ \propto w_{33}$ and $\mathcal{A}_\times \propto w_{23}$.

We thus conclude that although the off-diagonal metric components $g_{up} = r^2 e_p$ can be locally removed from the metric (103) by a gauge transformation $x'(x, u)$ such that $\mathrm{d}x'^p = \mathrm{d}x^p + e^p\mathrm{d}u$, it is in fact *convenient* to keep e^p (or, equivalently, the complex function f) nontrivial because these functions directly encode the amplitudes of the Robinson–Trautman gravitational waves. The physical meaning of these metric components in higher dimensions remains an open question since various independent results

[3]They imply that e^2 and e^3 are harmonic conjugate functions in *flat* 2-space, $\triangle e^2 = 0 = \triangle e^3$.

indicate that there are no Robinson–Trautman gravitational waves in $D > 4$ [5,8,21].

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APPENDIX: PROOF OF THE USEFUL IDENTITIES

Here, we present the steps that enable us to prove the nontrivial identities (96)–(98).

1. Identity (96)

This identity immediately follows from the constraint (74), which, in view of the definition (80), can be written as

$$h_{mn,u} = 2e_{(m||n)} + h_{mn}c. \tag{A1}$$

Multiplying this equation by $e^m e^n$, we obtain $e^m e^n h_{mn,u} = 2e^n e^m e_{m||n} + e^n e_n c$, which is equal to

$$e^m e^n h_{mn,u} = e^n (e^m e_m)_{,n} + e^n e_n c. \tag{A2}$$

2. Identity (97)

First, it can be shown using Eqs. (80) and (A1) and the relation $h^{mn}{}_{,u} = -h^{mp}h^{nq}h_{pq,u}$ that

$$\frac{1}{2}(D-2)(c_{,u}+c^2) + e^n{}_{||n}c + h^{mn}\left[e_{m,u||n} - \frac{1}{2}h_{mn,uu}\right] = h^{mn}[e_{m,u||n} - e_{m||n,u}]. \tag{A3}$$

Moreover, using the explicit expressions for the spatial covariant derivatives

$$e_{m,u||n} = e_{m,un} - e_{p,u}{}^S\Gamma^p_{mn}, \tag{A4}$$

$$e_{m||n,u} = (e_{m,n} - e_p{}^S\Gamma^p_{mn})_{,u}, \tag{A5}$$

$$e_{(p||m)||n} = e_{(p||m),n} - e_{(q||m)}{}^S\Gamma^q_{pn} - e_{(p||q)}{}^S\Gamma^q_{mn}, \tag{A6}$$

we obtain

$$h^{mn}[e_{m,u||n} - e_{m||n,u}] = -\frac{1}{2}(D-4)e^n c_{,n} + e^p h^{mn} e_{p||m||n} + e^p h^{mn}[e_{m||p||n} - e_{m||n||p}]. \tag{A7}$$

Simple calculation yields

$$h^{mn}\left[-\frac{1}{2}(e^p e_p)_{||m||n} + h^{pq}e_{p||m}e_{q||n}\right] = -e^p h^{mn} e_{p||m||n}. \tag{A8}$$

Putting Eqs. (A3), (A7), and (A8) together, applying the definition $e_{m||p||n} - e_{m||n||p} \equiv -\mathcal{R}^q{}_{mnp}e_q$, and using the constraint $\mathcal{R}_{pq} = (D-3)h_{pq}a$, which follow from Eqs. (72) and (73), we thus prove

$$\frac{1}{2}(D-2)(c_{,u}+c^2) + e^n{}_{||n}c + \frac{1}{2}(D-4)e^n c_{,n} - (D-3)e^p e_p a + h^{mn}\left[e_{m,u||n} - \frac{1}{2}h_{mn,uu} - \frac{1}{2}(e^p e_p)_{||m||n} + h^{pq}e_{p||m}e_{q||n}\right] = 0, \tag{A9}$$

which is the identity (97).

3. Identity (98)

Using the explicit form (72) of a, where $\mathcal{R} = h^{pq}\mathcal{R}_{pq}$, with (73) and (80) it can be shown that

$$(D-2)(a_{,u} + ac) = \frac{1}{D-3}h^{pq}\mathcal{R}_{pq,u} - 2ae^n{}_{||n}. \tag{A10}$$

It remains to evaluate the term $h^{pq}\mathcal{R}_{pq,u}$, which (from the definition of the Ricci tensor) is

$$h^{pq}\mathcal{R}_{pq,u} = h^{pq}[{}^S\Gamma^m_{pq,mu} - {}^S\Gamma^m_{pm,qu} + {}^S\Gamma^m_{pq,u}{}^S\Gamma^n_{mn} + {}^S\Gamma^n_{pq}{}^S\Gamma^m_{nm,u} - 2{}^S\Gamma^m_{pn,u}{}^S\Gamma^n_{mq}]. \tag{A11}$$

Direct calculation using the identity $h^{mn}{}_{,u} = -h^{mp}h^{nq}h_{pq,u}$ followed by relations (A1), (A6), and $e_{m||p||n} - e_{m||n||p} = -\mathcal{R}^q{}_{mnp}e_q$ reveals that

$$T^m_{pq} \equiv {}^S\Gamma^m_{pq,u} = h^{mn}[e_{n||(p||q)} - e^k\mathcal{R}_{k(pq)n}] + \delta^m_{(p}c_{,q)} - \frac{1}{2}h^{mn}h_{pq}c_{,n}. \tag{A12}$$

It is important to observe that the quantity T^m_{pq} is a *tensor in the transverse* $(D-2)$*-dimensional space*. Therefore, reexpressing Eq. (A11) using the covariant derivative $T^m_{pq||m}$, we get the tensor relation

$$h^{pq}\mathcal{R}_{pq,u} = h^{pq}[T^m_{pq||m} - T^m_{pm||q}]. \tag{A13}$$

Substituting Eq. (A12) into Eq. (A13), we obtain

$$h^{pq}\mathcal{R}_{pq,u} = h^{mn}h^{pq}[e_{n||p||q||m} - e_{n||p||m||q}] + 2h^{mn}(e^p\mathcal{R}_{pm})_{||n} - (D-3)h^{mn}c_{||m||n}. \tag{A14}$$

Now, the contraction of the identity (3.2.21) of Ref. [22] yields the identity

$$h^{mn}h^{pq}[e_{n||p||q||m} - e_{n||p||m||q}] = 0, \tag{A15}$$

while the direct evaluation using Eqs. (72) and (73) gives

$$2h^{mn}(e^p\mathcal{R}_{pm})_{||n} = 2(D-3)[e^n a_{,n} + ae^n{}_{||n}]. \tag{A16}$$

Putting Eq. (A14) with Eqs. (A15) and (A16) into Eq. (A10), we finally obtain the identity

$$(D-2)(a_{,u} + ac) + (D-6)e^n a_{,n} + h^{mn}c_{||m||n} = (D-4)e^n a_{,n}, \tag{A17}$$

which completes the proof.

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