Thermal property in Brownian motion of a particle coupled to vacuum fluctuations

Naritaka Oshita,[1] Kazuhiro Yamamoto,[1,2] and Sen Zhang[3]

[1]*Department of Physical Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan*
[2]*Hiroshima Astrophysical Science Center, Hiroshima University, Higashi-Hiroshima 739-8526, Japan*
[3]*Okayama Institute for Quantum Physics, Kyoyama 1-9-1, Kita-ku, Okayama 700-0015, Japan*

We investigate Brownian motions of a particle coupled to vacuum fluctuations of a quantum field. The Unruh effect predicts that an observer in an accelerated motion sees the Minkowski vacuum as thermally excited. This addresses the problem of whether or not a thermal property appears in a perturbative random motion of a particle in an accelerated motion due to the coupling. We revisit this problem by solving the equation of motion of a particle coupled to vacuum fluctuations including the radiation reaction force. We compute a Fourier integral for the variance of the random velocity in a rigorous manner. Similarly, we consider a particle coupled to vacuum fluctuations in de Sitter spacetime motivated by the argument that an observer in de Sitter spacetime sees the Bunch-Davies vacuum as a thermally excited state with the Gibbons-Hawking temperature. Our investigation clarifies the condition that the energy equipartition relation arises in the Brownian motions of a particle.

I. INTRODUCTION

Vacuum fluctuation is a key concept of quantum physics in curved spacetime. One of the famous phenomena is the Hawking radiation, which predicts thermal-like radiation from black hole spacetime [1]. From the equivalence principle, the Hawking radiation is closely related with the Unruh effect that predicts an observer in an accelerated motion sees the Minkowski vacuum as thermally excited [2]. It is an interesting question whether or not the Unruh effect can be tested in a laboratory. The radiation coming from the Unruh effect is called the Unruh radiation, though the existence of the Unruh radiation is under debate [3–5].

Motivated by the arguments [3–5], we investigate Brownian motion of a particle due to the coupling to quantum vacuum fluctuations. This issue was first formulated in Ref. [5], in which the authors considered the system that consists of a particle and a quantum field, which are coupled to each other. They derived a Langevin-like equation for a particle taking the random force from the quantum field fluctuations and the radiation reaction force into account. They applied the formulation to a particle in an uniformly accelerated motion, and found that the energy equipartition relation appears in the transverse fluctuations of a particle, which is perpendicular to the direction of the acceleration. However, the conclusion is based on a low-energy approximation, and the investigation is restricted to the transverse fluctuations in particle motions because of the limitation of the approximation. Therefore, the longitudinal fluctuations have not been investigated yet.

In the present paper, complementary to the previous work [5], we focus our investigation on the energy equipartition relation in random motions of a particle coupled to vacuum fluctuations of a quantum field. We consider the system that

consists of a particle and a quantum field, which are coupled to each other, in curved spacetime. We present basic formulas for a particle in random motions as a generalization of the previous work [5]. As applications, we consider a particle in de Sitter spacetime as well as a particle in an uniformly accelerated motion, which is well described by the Rindler spacetime coordinate. We compute the variance of the random velocity in a coincidence limit of the two point function of the random velocity, which we evaluate by counting all the poles in the Fourier integral, which is obtained by solving the equation of motion of a particle. Our results clarify the difference between the transverse and the longitudinal fluctuations of a particle in an accelerated motion. The energy equipartition relation appears for the transverse fluctuations, but it does not appear for the longitudinal fluctuations when a particle is uniformly accelerated.

A particle coupled to vacuum fluctuations in de Sitter spacetime is also considered, motivated by the prediction that an observer in de Sitter spacetime sees the Bunch-Davies vacuum as a thermally excited state with the Gibbons-Hawking temperature. For a particle coupled to the Bunch-Davies vacuum fluctuations in de Sitter spacetime, we find a similar structure of poles in the Fourier integral for the variance of the random velocity as that of the longitudinal fluctuations of a particle in an accelerated motion. Our rigorous method shows that the energy equipartition relation does not appear for $m \gg H$, where m is the mass of the particle and H is a Hubble constant specifying the de Sitter expansion rate. The energy equipartition relation only appears for $m \ll H$.

This paper is organized as follows: In Sec. II, we first present basis formulas, which describe the equation of

motion of a particle coupled to vacuum fluctuations of a quantum field, as a generalization of the previous work [5] to those in curved spacetime. We derive the Langevin-like equation for a particle taking random forces due to the coupling and the radiation reaction force into account. In Sec. III, we consider a particle coupled to the Bunch-Davies vacuum fluctuations in de Sitter spacetime. In Sec. IV, we consider a particle coupled to the Minkowski vacuum fluctuations in an accelerated motion. The transverse and the longitudinal random motions are investigated separately. Section V is devoted to summary and conclusions. We follow the metric convention $(-, +, +, +)$.

II. BASIC FORMULAS

We consider the system consisting of a particle and a scalar field in curved spacetime, whose action is given by

$$S = S_0(z) + S_0(\phi) + S_{\text{int}}(z, \phi), \qquad (2.1)$$

where $S_0(z)$ and $S_0(\phi)$ are the action for the fee particle and the field conformally coupled to the curvature,

$$S_0(z) = -m \int d\tau \sqrt{-g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu}, \qquad (2.2)$$

$$S_0(\phi) = \int d^4x \sqrt{-g} \frac{1}{2} \{-g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \xi R \phi^2\}, \quad (2.3)$$

and $S_{\text{int}}(z, \phi)$ describes the interaction,

$$S_{\text{int}}(z, \phi) = e \int d\tau d^4x \sqrt{-g_{\mu\nu}(x) \dot{z}^\mu \dot{z}^\nu} \phi(x) \delta^4(x - z(\tau))$$

$$= e \int d\tau \sqrt{-g_{\mu\nu}(z(\tau)) \dot{z}^\mu \dot{z}^\nu} \phi(z(\tau)). \qquad (2.4)$$

Note that $x^\mu = z^\mu(\tau)$ denotes the trajectory of the particle, which obeys,

$$m \frac{D\dot{z}^\mu}{D\tau} = e \left(\frac{D\dot{z}^\mu}{D\tau} \phi + \dot{z}^\mu \dot{z}^\alpha \frac{\partial\phi}{\partial x^\alpha} + g^{\mu\alpha} \frac{\partial\phi}{\partial x^\alpha} \right) \Big|_{x=z(\tau)}, \qquad (2.5)$$

where we used the notation $D/D\tau = \dot{z}^\mu \nabla_\mu$. The equation of motion for the scalar field is

$$(-\nabla^\mu \nabla_\mu + \xi R)\phi(x) = \frac{e}{\sqrt{-g}} \int d\tau \sqrt{-g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu} \delta^4(x - z(\tau)). \qquad (2.6)$$

The field equation has the solution, $\phi = \phi_{\text{h}} + \phi_{\text{inh}}$, where ϕ_{h} and ϕ_{inh} are the homogeneous and inhomogeneous solutions, respectively. The inhomogeneous solution is written as

$$\phi_{\text{inh}}(x) = e \int d^4x' G_R(x, x') \int d\tau' \sqrt{-g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu} \delta^4(x' - z(\tau'))$$

$$= e \int d\tau' G_R(x, z(\tau')), \qquad (2.7)$$

where $G_R(x, y)$ denotes the retarded Green function, which satisfies $\sqrt{-g}(-\nabla^\mu \nabla_\mu + \xi R)G_R(x, y) = \delta^4(x - y)$.

The terms from the inhomogeneous solution in Eq. (2.5) give rise to a radiation reaction force for the equation of motion of the particle [5–9]. We consider the conformally flat spacetime, in which it is known that the tail term in the radiation reaction force disappears [10]. The terms from the homogeneous solution ϕ_{h} in Eq. (2.5) give rise to random forces, and we have the Langevin-like equation of motion

$$m \frac{D\dot{z}^\mu}{D\tau} = \frac{e^2}{12\pi} \left(\frac{D^2 \dot{z}^\mu}{D\tau^2} - \dot{z}^\mu \left(\frac{D\dot{z}}{D\tau} \right)^2 \right) - \frac{e^2}{24\pi} (g^{\mu\nu} + \dot{z}^\mu \dot{z}^\nu) R_{\nu\alpha} \dot{z}^\alpha$$

$$+ e \left(\frac{D\dot{z}^\mu}{D\tau} \phi_{\text{h}} + \dot{z}^\mu \dot{z}^\alpha \frac{\partial\phi_{\text{h}}}{\partial x^\alpha} + g^{\mu\alpha} \frac{\partial\phi_{\text{h}}}{\partial x^\alpha} \right) \Big|_{x=z(\tau)}, \qquad (2.8)$$

where the mass of the particle is redefined. This equation is a generalization of that derived in [5], in which a particle in an uniformly accelerated motion in the Minkowski spacetime is considered.[1] Using this generalized equation, we first consider the particle in de Sitter spacetime in Sec. III.

III. A PARTICLE IN DE SITTER SPACETIME

We consider the particle in de Sitter spacetime, whose line element is written as

$$ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j, \qquad (3.1)$$

where $a(t) = e^{Ht}$ is the scale factor. The equation of motion is derived directly from Eq. (2.8). We write the trajectory of the particle $x^\mu = z^\mu(\tau) = (t(\tau), z^i(\tau))$ and derive the linearized equation of motion of $z^i(\tau)$, assuming that the particle moves around the origin of the spatial coordinate. Then we have

$$m(\dot{v}^i + Hv^i) = \frac{e^2}{12\pi}(\ddot{v}^i + H\dot{v}^i) + \frac{e}{a} \frac{\partial\phi_{\text{h}}}{\partial x^i} \Big|_{x=z(\tau)}, \qquad (3.2)$$

where we defined $v^i(\tau) = a(\tau)\dot{z}^i(\tau)$. This equation is solved in the Fourier space as

$$\tilde{V}^i(\omega) = e \frac{\tilde{\varphi}_i(\omega)}{(\omega - iH)(e^2\omega/12\pi + im)}, \qquad (3.3)$$

where we defined the Fourier expansion,

---

[1] A similar problem for a uniformly accelerated quark is considered in the context of the AdS/CFT correspondence in Ref. [11].

$$v^i(\tau) = \int \frac{d\omega}{2\pi} \tilde{V}^i(\omega) e^{-i\omega\tau}, \qquad (3.4)$$

$$\frac{1}{a(\tau)} \frac{\partial\phi}{\partial x^i}\bigg|_{x=z(\tau)} = \int \frac{d\omega}{2\pi} \tilde{\varphi}_i(\omega) e^{-i\omega\tau}. \qquad (3.5)$$

As we consider the scalar field conformally coupled to the curvature in the Bunch-Davies vacuum, we have [12,13]

$$\left\langle 0 \left| \frac{1}{a(t_x)} \frac{\partial\phi_{\rm h}(x)}{\partial x^i} \frac{1}{a(t_y)} \frac{\partial\phi_{\rm h}(y)}{\partial x^j} \right| 0 \right\rangle\bigg|_{x=z(\tau),y=z(\tau')}$$
$$= \frac{H^4}{32\pi^2} \frac{\delta_{ij}}{(\sinh H(\tau-\tau')/2 - i\epsilon)^4}, \qquad (3.6)$$

and

$$\langle v^i(\tau)v^j(\tau')\rangle_S = \delta_{ij} \frac{6}{e^2} \int_{-\infty}^{\infty} d\omega \frac{\omega}{\omega^2 + (12\pi m/e^2)^2}$$
$$\times \coth(\pi\omega/H) e^{i\omega(\tau'-\tau)}, \qquad (3.7)$$

where '$\langle\cdots\rangle_S$' means the symmetrization with respect to τ and τ'. The poles of the integrand are $\omega = \pm i12\pi m/e^2$ and $\pm inH$ with $n = 1, 2, \ldots$. The integration can be performed exactly. Then we have

$$\langle v^i(\tau)v^j(\tau')\rangle_S = \frac{6\delta_{ij}}{e^2} \left[e^{-H\sigma^{-1}\delta\tau} \pi \cot \pi\sigma^{-1} \right.$$
$$\left. + \sum_{n=1}^{\infty} \left(\frac{1}{n+\sigma^{-1}} + \frac{1}{n-\sigma^{-1}} \right) e^{-nH\delta\tau} \right]$$
$$= \delta_{ij} \frac{H}{2\pi m} \Xi[H, \sigma^{-1}; \delta\tau], \qquad (3.8)$$

where we defined $\sigma = e^2 H/12\pi m$, and $\delta\tau \equiv |\tau'-\tau|$. In the second line, we introduced the function defined by

$$\Xi[H, \zeta; \delta\tau] = \zeta \left\{ e^{-H\zeta\delta\tau} \pi \cot(\pi\zeta) \right.$$
$$\left. + \sum_{n=1}^{\infty} \left(\frac{1}{n+\zeta} + \frac{1}{n-\zeta} \right) e^{-nH\delta\tau} \right\}. \qquad (3.9)$$

With the use of the mathematical formulas,

$$\sum_{n=0}^{\infty} \frac{e^{-na\delta\tau}}{n+\zeta} = {}_2F_1(1, \zeta, 1+\zeta; e^{-a\delta\tau}) \frac{1}{\zeta} \qquad (3.10)$$

$${}_2F_1(a, b, a+b; z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2}$$
$$\times [2\psi(n+1) - \psi(a+n) - \psi(b+n)$$
$$- \ln(1-z)](1-z)^n, \qquad (3.11)$$

$$\psi(n+z) = \sum_{m=0}^{n-1} \frac{1}{m+z} + \psi(z), \qquad (3.12)$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric function, $\psi(z)$ is the polygamma function and $\Gamma(a)$ is the gamma function, the symbol $(a)_n$ is defined as $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, and $\Xi[H, \zeta; \delta\tau]$ can be rewritten as

$$\Xi[H, \zeta; \delta\tau] = \zeta \left\{ (e^{-H\zeta\delta\tau} + e^{H\zeta\delta\tau})(-\gamma - H\delta\tau - \psi(\zeta)) \right.$$
$$- \ln(1 - e^{-H\delta\tau})) - \frac{e^{-H\zeta\delta\tau}}{\zeta}$$
$$- \frac{\partial}{\partial c} {}_2F_1(1, \zeta, c; 1 - e^{-H\delta\tau})\bigg|_{c=1}$$
$$\left. - \frac{\partial}{\partial c} {}_2F_1(1, -\zeta, c; 1 - e^{-H\delta\tau})\bigg|_{c=1} \right\}, \qquad (3.13)$$

where γ is the Euler constant. In the limit of small $\delta\tau$, this function has the asymptotic formula,

$$\Xi[H, \zeta, \delta\tau] \simeq -1 + 2\zeta(-\ln(H\delta\tau) - \gamma - \psi(\zeta)) + \mathcal{O}(\delta\tau). \qquad (3.14)$$

Noting the definition $\sigma^{-1} = 12\pi m/He^2$ and using the asymptotic formula of the polygamma function,

$$\psi(\zeta) = \begin{cases} \ln\zeta - \frac{1}{2\zeta} - \frac{1}{12\zeta^2}, & \text{for } \zeta \gg 1, \\ -\frac{1}{\zeta} - \gamma, & \text{for } \zeta \ll 1, \end{cases} \qquad (3.15)$$

we finally have

$$\langle v^i(\tau)v^j(\tau')\rangle_S = \delta_{ij} \times \begin{cases} -\frac{12}{e^2}\left(\ln\frac{12\pi m\delta\tau}{e^2} + \gamma\right) + \mathcal{O}((H/m)^2) & \text{for } H \ll 12\pi m/e^2, \\ \frac{H}{2\pi m} - \frac{12}{e^2}\ln(H\delta\tau) + \mathcal{O}(m/H) & \text{for } H \gg 12\pi m/e^2. \end{cases} \qquad (3.16)$$

The results show that the variance of velocity has a logarithmic divergence in the limit that $\delta\tau$ approaches zero. However, since the coefficient of the divergence terms are independent of H, these terms don't contribute

to the thermal nature. Indeed the term $-\frac{12}{e^2}\left(\ln\frac{12\pi m\delta\tau}{e^2} + \gamma\right)$ remains in the limit of $H \to 0$, where the de Sitter spacetime reduces to the Minkowski spacetime. As will be described in the next section, we also find that the same

term remains for the case of particle in an uniformly accelerated motion in the limit that the acceleration constant approaches zero. So one may understand that this divergence comes from the short-distance motion of the particle, originated from our formulation based on point particle. The divergence coming from the short-distance motion of the particle will be removed by taking a finite size effect of the particle into account. Therefore, this suggests that $\delta\tau$ cannot be taken to be zero, and it is natural to introduce a finite value cutoff [14].

Equation (3.16) means that the energy equipartition relation does not appear for the case $H \ll 12\pi m/e^2$, where one cannot find a thermal property in the random motion. On the other hand, the energy equipartition relation looks to appear when $H \gg 12\pi m/e^2$ as the Gibbons-Hawking temperature is $H/2\pi$. It is well known that the scalar field in de Sitter spacetime has a stochastic property [15–17]. In particular, the stochastic inflation approach indicates that the massless scalar field coarse-grained over the horizon size H^{-1} follows an equation of motion with a white noise term. The results (3.16) in the case $H \gg 12\pi m/e^2$ would be closely related to this fact. However, we should also note that the amplitude of the velocity is larger than unity, even though the energy equipartition relation appears. This inconsistency would come from the fact that $H \gg m$ means that the particle's Compton wavelength is much longer than the horizon size. This would mean the inconsistency of our description of the point particle in this regime.

IV. A PARTICLE IN AN ACCELERATED MOTION

Now let us consider the particle in an accelerated motion with a uniform acceleration a. Equations of motion around a uniformly accelerated motion are derived in Ref. [5], which are summarized in the following. We consider the transverse and the longitudinal random motions separately.

A. Transverse fluctuations

For the transverse fluctuations perpendicular to the direction of the uniform acceleration, the equation of motion is

$$m\dot{v}^i = \frac{e^2}{12\pi}(\ddot{v}^i - a^2 v^i) + e\frac{\partial\phi_{\rm h}}{\partial x^i}\bigg|_{x=z(\tau)}, \qquad (4.1)$$

which leads to [5]

$$\langle v^i(\tau)v^j(\tau')\rangle_S = \delta_{ij}\frac{e^2}{6}\int_{-\infty}^{\infty}d\omega\,\frac{\omega(\omega^2+a^2)}{(e^2(\omega^2+a^2))^2+(12\pi m\omega)^2}$$
$$\times \coth(\pi\omega/a)e^{i\omega\delta\tau}. \qquad (4.2)$$

The integrand has the poles at $\omega = \pm ia\Omega_+, \pm ia\Omega_-$, and $\pm ina$ with $n = 2, 3, \ldots$, where $\Omega_\pm$ are defined as

$$\Omega_\pm = \sqrt{1+\frac{1}{2\sigma^2}\pm\sqrt{\left(1+\frac{1}{2\sigma^2}\right)^2-1}}$$
$$= \frac{1}{2\sigma}(\pm 1+\sqrt{1+4\sigma^2}), \qquad (4.3)$$

and here σ is defined as $\sigma = e^2 a/12\pi m$. This integration can be exactly performed, and we have

$$\langle v^i(\tau)v^j(\tau')\rangle_S = \delta_{ij}\frac{a}{2\pi m}\frac{1}{\sqrt{1+4\sigma^2}}\left[\Omega_+ e^{-a\Omega_+\delta\tau}\pi\cot\pi\Omega_+\right.$$
$$+\Omega_+\sum_{n=2}^{\infty}\left(\frac{1}{n+\Omega_+}+\frac{1}{n-\Omega_+}\right)e^{-na\delta\tau}$$
$$+\Omega_- e^{-a\Omega_-\delta\tau}\pi\cot\pi\Omega_-$$
$$\left.+\Omega_-\sum_{n=2}^{\infty}\left(\frac{1}{n+\Omega_-}+\frac{1}{n-\Omega_-}\right)e^{-na\delta\tau}\right], \qquad (4.4)$$

where we used

$$\frac{1-\Omega_-^2}{\Omega_+^2-\Omega_-^2}=\frac{1}{2}\left(1-\frac{1}{\sqrt{1+4\sigma^2}}\right),$$
$$\frac{1-\Omega_+^2}{\Omega_-^2-\Omega_+^2}=\frac{1}{2}\left(1+\frac{1}{\sqrt{1+4\sigma^2}}\right). \qquad (4.5)$$

Then, using the function $\Xi[a,\Omega_\pm;\delta\tau]$ defined by Eq. (3.9), we have

$$\langle v^i(\tau)v^j(\tau')\rangle_S = \delta_{ij}\frac{a}{2\pi m}\frac{1}{\sqrt{1+4\sigma^2}}$$
$$\times(\Xi[a,\Omega_+;\delta\tau]+\Xi[a,\Omega_-;\delta\tau]). \qquad (4.6)$$

Using the asymptotic formula (3.14) with replacing H with a, we have

$$\langle\delta v^i(\tau)\delta v^j(\tau')\rangle_S = \delta_{ij}\frac{a}{2\pi m}\frac{2}{\sqrt{1+4\sigma^2}}$$
$$\times\{-1-\Omega_+\psi(\Omega_+)-\Omega_-\psi(\Omega_-)$$
$$+(\Omega_++\Omega_-)(-\log(a\delta\tau)-\gamma)\}+\mathcal{O}(\delta\tau). \qquad (4.7)$$

In the limit of small $\delta\tau$, with the approximate formula (3.15), expression (4.7) reduces to

$$\langle v^i(\tau)v^j(\tau')\rangle_S = \delta_{ij} \times \begin{cases} \frac{a}{2\pi m} - \frac{12}{e^2}\left(\ln\frac{12\pi m\delta\tau}{e^2} + \gamma\right) + \mathcal{O}((a/m)^2) & \text{for } a \ll 12\pi m/e^2, \\ -\frac{12}{e^2}\left(\ln(a\delta\tau) + \frac{1}{2}\right) + \mathcal{O}(m/a) & \text{for } a \gg 12\pi m/e^2. \end{cases} \tag{4.8}$$

For the case $a \ll 12\pi m/e^2$, the equipartition relation appears as the Unruh temperature is $a/2\pi$, which was first discovered in Ref. [5]. On the other hand, it does not for the case $a \gg 12\pi m/e^2$, however, this case perhaps should not be considered because the Unruh temperature is much larger than the particle mass.

B. Longitudinal fluctuations

Now we consider the longitudinal fluctuations, whose equation of motion is given by (see Appendix in Ref. [5]),

$$m(\delta\ddot{\xi} - a^2\delta\xi) = \frac{e^2}{12\pi}(\delta\dddot{\xi} - a^2\delta\dot{\xi}) + e\vartheta\phi_h|_{x=z(\tau)}, \tag{4.9}$$

where $\vartheta \equiv (a + \partial/\partial\xi)$, $\delta\xi(\tau)$ specifies the position of the fluctuating particle in the longitudinal direction, and $\delta\xi = 0$ is the trajectory of the particle without fluctuations. This equation can be derived from (2.8) with an external force using the Rindler coordinate,

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) + dx_1^2 + dx_2^2. \tag{4.10}$$

The use of the Fourier expansion leads to the following formula for the variance of the random velocity:

$$\langle\delta\dot{\xi}(\tau)\delta\dot{\xi}(\tau')\rangle_S = \frac{6}{e^2}\int_{-\infty}^{\infty}d\omega\frac{\omega^3}{a^2+\omega^2}\frac{e^{i\omega\delta\tau}}{(12\pi m/e^2)^2+\omega^2}$$
$$\times \coth(\pi\omega/a). \tag{4.11}$$

The poles of the integrand are $\pm i\sigma^{-1}$, $\pm ina$ with $n = 1, 2, \ldots$. The poles at $\pm ia$ are the second order. The structure of the pole is similar to that of a particle in de Sitter spacetime. The integration can be exactly evaluated as

$$\langle\delta\dot{\xi}(\tau)\delta\dot{\xi}(\tau')\rangle_S = \frac{6}{e^2}\left[e^{-a\delta\tau/\sigma}\pi\cot\pi\sigma^{-1}\right.$$
$$+\frac{\sigma^4 - 5\sigma^2 - 2a\delta\tau\sigma^4 + 2a\delta\tau\sigma^2}{2(1-\sigma^2)^2}e^{-a\delta\tau}$$
$$\left.+\sum_{n=2}^{\infty}\frac{2n^3 e^{-na\delta\tau}}{(\sigma^{-2}-n^2)(1-n^2)}\right]. \tag{4.12}$$

By using the function $\Xi[a, \zeta; \delta\tau]$ defined by Eq. (3.9) with replacing variable H and ζ with a and σ^{-1}, respectively, we may rewrite

$$\langle\delta\dot{\xi}(\tau)\delta\dot{\xi}(\tau')\rangle_S = \frac{6}{e^2}\left[\frac{\sigma}{1-\sigma^2}\Xi[a, \sigma^{-1}; \delta\tau] + \frac{2\sigma^2 e^{-a\delta\tau}}{(1-\sigma^2)^2}\right.$$
$$+\frac{\sigma^4(1 - 5\sigma^{-2} - 2(1-\sigma^{-2})a\delta\tau)}{2(1-\sigma^2)^2}e^{-a\delta\tau}$$
$$+\frac{\sigma^2}{1-\sigma^2}\left((e^{a\delta\tau} + e^{-a\delta\tau})\ln(1 - e^{-a\delta\tau})\right.$$
$$\left.\left.+1 + \frac{1}{2}e^{-a\delta\tau}\right)\right], \tag{4.13}$$

which reduces to

$$\langle\delta\dot{\xi}(\tau)\delta\dot{\xi}(\tau')\rangle_S = \frac{6}{e^2}\left[-2\log(a\delta\tau) - \frac{2\gamma + 2\psi(\sigma^{-1}) + \sigma - \sigma^2}{1-\sigma^2}\right]$$
$$+\mathcal{O}(\delta\tau), \tag{4.14}$$

in the limit of small $\delta\tau$ with the use of Eq. (3.14). Furthermore, the use of the approximate formula (3.15) leads to

$$\langle\delta\dot{\xi}(\tau)\delta\dot{\xi}(\tau')\rangle_S = \begin{cases} -\frac{12}{e^2}\left(\ln\frac{12\pi m\delta\tau}{e^2} + \gamma\right) + \mathcal{O}((a/m)^2) & \text{for } a \ll 12\pi m/e^2, \\ -\frac{12}{e^2}\left(\log(a\delta\tau) + \frac{1}{2}\right) + \mathcal{O}(m/a) & \text{for } a \gg 12\pi m/e^2. \end{cases} \tag{4.15}$$

Thus, the results indicate that the energy equipartition relation does not appear in the longitudinal fluctuations, which is the contrast to the transverse fluctuations. This difference comes from the second term in the left hand side of equation of motion (4.9), $-ma^2\delta\xi$. Such a term in a classical dynamics makes its motion unstable. This prevents the energy equipartition relation in the longitudinal direction. There is the blocking of the energy equipartition relation in the direction of the accelerated motion.

V. CONCLUSIONS

In this paper, we investigated Brownian random motions of a particle coupled to vacuum fluctuations. We consider a

particle coupled to vacuum fluctuations in de Sitter spacetime and a particle in a uniformly accelerated motion in the Minkowski spacetime. A detector coupled to vacuum fluctuations is excited as if it is exposed to the thermal bath with the Gibbons-Hawking temperature in de Sitter spacetime or with the Unruh temperature of an acceleration motion. Then, we may expect that a thermal property may appear in the random motions of a particle in such situations. We investigate this problem by solving the equation of motion of a particle coupled to vacuum fluctuations including the radiation reaction force. We found that the energy equipartition relation may appear in the random motion of a particle. We also found that it does not always appear. For the particle in de Sitter spacetime, the energy equipartition relation apparently appears when $H/m \gg 1$, but it does not appear when $H/m \ll 1$. For the particle in an accelerated motion, the equipartition relation appears in the transverse motion when $a/m \ll 1$. We showed that the energy equipartition relation does not appear in the longitudinal fluctuations. In the previous work [5], the authors found that the energy equipartition relation appears in the transverse motion

when $a/m \ll 1$, but the conclusion was derived by evaluating only the pole of the lowest energy scale in the Fourier integration. In the present paper, our conclusions have been derived by evaluating all of the poles in the Fourier integration. The condition that the energy equipartition appears is that the lowest energy pole in the Fourier integral for the variance of velocity is smaller than the Unruh temperature for the particle in the accelerated motion or the Gibbons-Hawking temperature for the particle in de Sitter spacetime. It is interesting to investigate whether or not the random motions representing a thermal property can be detected as a signature of radiation, which will be investigated in a future work.

ACKNOWLEDGMENTS

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975); **46** 206(E) (1976).

[2] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).

[3] P. Chen and T. Tajima, Phys. Rev. Lett. **83**, 256 (1999).

[4] P. G. Thirolf, D. Habs, A. Henig, D. Jung, D. Kiefer, C. Lang, J. Schreiber1, C. Maia, G. Schalle, R. Schutzhold, and T. Tajima, Eur. Phys. J. D **55**, 379 (2009).

[5] S. Iso, Y. Yamamoto, and S. Zhang, Phys. Rev. D **84**, 025005 (2011).

[6] B. S. De Witt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).

[7] J. M. Hobbs, Ann. Phys. (N.Y.) **47**, 141 (1968).

[8] S-Y. Lin and B. L. Hu, Phys. Rev. D **73**, 124018 (2006).

[9] C. R. Galley and B. L. Hu, Phys. Rev. D **72**, 084023 (2005).

[10] J. M. Hobbs, Ann. Phys. (N.Y.) **47**, 166 (1968).

[11] E. Caceres, M. Chernicoff, A. Guijosa, and J. F. Pedraze, J. High Energy Phys. 06 (2010) 078.

[12] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Spacetime* (Cambridge University Press, Cambridge, England, 1982).

[13] T. Murata, K. Tsunoda, and K. Yamamoto, Int. J. Mod. Phys. A **16**, 2841 (2001).

[14] J.-T. Hsiang, T.-H. Wu, and D.-S. Lee, Found. Phys., **41**, 77 (2011).

[15] A. A. Starobinsky, *Current Topics in Field Theory, Quantum Gravity and Strings*, Lecture Notes in Physics Vol. 246, edited by H. J. de Vega and N. Sanchez (Springer, Berlin, 1986).

[16] K. Nakao, Y. Nambu, and M. Sasaki, Prog. Theor. Phys. **80**, 1041 (1988).

[17] G. Rigopoulos, arXiv:1305.0229.