Covariant loop quantum gravity, low-energy perturbation theory, and Einstein gravity with high-curvature UV corrections

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A low-energy perturbation theory is developed from the nonperturbative framework of covariant loop quantum gravity (LQG) by employing the background-field method. The resulting perturbation theory is a two-parameter expansion in the semiclassical and low-energy regime. The two expansion parameters are the large spin and small curvature. The leading-order effective action coincides with the Regge action, which well approximates the Einstein-Hilbert action in the regime. The subleading corrections organized by the two expansion parameters give the modifications of the Regge action in the quantum and high-energy regime from LQG. The perturbation theory developed here shows for the first time that covariant LQG produces the high-curvature corrections to Einstein-Regge gravity. This result means that LQG is not a naive quantization of Einstein gravity; rather, it provides the UV modification. The result of the paper may be viewed as the first step toward understanding the UV completeness of LQG.

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I. COVARIANT LQG AND THE SEMICLASSICAL LOW-ENERGY REGIME

The nonperturbative covariant formulation of loop quantum gravity (LQG) adapts the idea of path integral quantization to the framework of LQG [1]. In the formulation, a *spinfoam amplitude* $A(\mathcal{K})$ is defined on a given simplicial manifold $\mathcal{K}$ for the transition of boundary quantum 3-geometries (spin-network states in LQG).[1] The spinfoam amplitude sums over the history of spin networks, and suggests a foam-like quantum spacetime structure.

In this paper, a low-energy perturbation theory is developed from the nonperturbative framework of LQG. The perturbation theory explains how classical gravity emerges from the group-theoretic spinfoam formulation, and provides the high-energy (high-curvature) and quantum corrections. Here we show that an effective action can be derived after perturbatively integrating/summing over all types of spinfoam degrees of freedom $\{J_f \in \mathrm{Irrep}(\mathrm{SU}(2)), g_{ve} \in \mathrm{SL}(2,\mathbb{C}), z_{vf} \in \mathbb{CP}^1\}$ around a geometrical background configuration. The leading-order effective action coincides with the Regge action, which well approximates the Einstein-Hilbert action in the low-energy regime.

Importantly, in the subleading contributions, the perturbation theory developed here shows for the first time that covariant LQG produces the high-curvature corrections to the Einstein-Regge action, which modifies the UV behavior of Einstein gravity. It is the first time that a systematic method is developed to compute the high-curvature corrections from a full LQG framework.

[1]The spinfoam amplitude A is a $\mathcal{H}$-valued function on the space of triangulations, where $\mathcal{H}$ is the boundary Hilbert space and $\mathcal{H} = \mathbb{C}$ if the manifold has no boundary.

The discussion here focuses on the Lorentzian spinfoam amplitude proposed by Engle, Pereira, Rovelli, and Livine (EPRL) [2]. The nonperturbative construction of the EPRL spinfoam amplitude is purely (quantum-)group-theoretic. As one of the representations [3], the EPRL spinfoam amplitude reads

$$A(\mathcal{K}) = \sum_{J_f} d_{J_f} \mathrm{tr}\left[\prod_e P_e^{\mathrm{inv}}\right]. \tag{1}$$

P_e^{inv} is an invariant projector onto a certain subspace of the $\mathrm{SL}(2,\mathbb{C})$ intertwiners associated to each tetrahedron e in $\mathcal{K}$. Here f labels a triangle in $\mathcal{K}$, e labels a tetrahedron, and v labels a 4-simplex. J_f is the SU(2) spin assigned to each triangle. d_J is the dimension of the SU(2) irrep with spin J. The above nonperturbative spinfoam amplitude is *finite* in the quantum-group version [4], which includes the cosmological constant in LQG [5].

The EPRL spinfoam amplitude can be written in the following path-integral-like form (see Ref. [6] for a derivation):

$$A(\mathcal{K}) = \sum_{J_f} d_{J_f} \int_{\mathrm{SL}(2,\mathbb{C})} \mathrm{d}g_{ve} \int_{\mathbb{CP}^1} \mathrm{d}z_{vf} e^{S[J_f, g_{ve}, z_{vf}]}, \tag{2}$$

where g_{ve} is a $\mathrm{SL}(2,\mathbb{C})$ group variable associated with each dual half-edge. z_{vf} is a two-component spinor. The spinfoam action S given by

$$S[J_f, g_{ve}, z_{vf}] = \sum_{(e,f)} \{J_f \mathcal{V}_f[g_{ve}, z_{vf}] + i\gamma J_f \mathcal{K}_f[g_{ve}, z_{vf}]\}, \tag{3}$$

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where the shorthand notations $\mathcal{V}_f$ and $\mathcal{K}_f$ are defined by

$$\mathcal{V}_f \equiv \ln\left[\langle Z_{vef}, Z_{v'ef}\rangle^2 \langle Z_{v'ef}, Z_{v'ef}\rangle^{-1}\langle Z_{vef}, Z_{vef}\rangle^{-1}\right],$$
$$\mathcal{K}_f \equiv \ln\left[\langle Z_{vef}, Z_{vef}\rangle \langle Z_{v'ef}, Z_{v'ef}\rangle^{-1}\right], \tag{4}$$

with $Z_{vef} = g^\dagger_{ve} z_{vf}$. $\gamma \in \mathbb{R}$ is the Barbero-Immirzi paramter.

Practically, we apply the background-field method to Eq. (2) and consider the perturbations of the spinfoam variables around a given background configuration [spinfoam data (J_f, g_{ve}, z_{vf}) on $\mathcal{K}$].[2] The perturbative expansion is performed in the semiclassical and low-energy regime. Such a regime can be specified in the following way: the existing semiclassical results suggest that the semiclassical geometry emerging from the spinfoam is discrete with a (triangle-area) spacing $\alpha_f = \gamma J_f \ell_P^2$ [6,9–11]. Here we focus on the regime where the area scale α_f is much greater than the Planck scale ℓ_P^2, but much smaller than the mean curvature radius L^2 of the semiclassical geometry, i.e.,

$$\ell_P^2 \ll \alpha_f \ll L^2. \tag{5}$$

Equation (5) is a four-dimensional analog of the semiclassical regime in canonical LQG [12]. The relation $\ell_P^2 \ll \alpha_f$ comes from $\hbar \to 0$ and implies the semiclassicality. $\alpha_f \ll L^2$ implies the low-energy approximation, since it requires that the mean wavelength of the gravitational fluctuation is much larger than the lattice scale. Adapting Eq. (5) to the spinfoam formulation, $\ell_P^2 \ll \alpha_f$ can be implemented by $J_f \gg 1$ for all f, while $\alpha_f \ll L^2$ means that the deficit angle $|\Theta_f| \ll 1$ for all f, because $|\Theta_f| \sim \alpha_f/L^2[1 + o(\alpha_f/L^2)]$ [13]. In the following, the perturbative analysis of the spinfoam amplitude $A(\mathcal{K})$ is performed with respect to a certain background spinfoam configuration in the semiclassical low-energy regime (5). The analysis results in a low-energy effective action, whose leading contribution coincides with the Einstein-Hilbert action. The expansion parameters J_f^{-1} and Θ_f organize, respectively, the quantum and high-energy curvature corrections.

II. SEMICLASSICAL APPROXIMATION AND SIMPLICIAL GEOMETRY

Let us consider the spinfoam amplitude in the regime $\ell_P^2 \ll \alpha_f$. We write Eq. (2) as $A(\mathcal{K}) = \sum_{J_f} d_{J_f} A_{J_f}(\mathcal{K})$ and focus on the sum over fluctuations of J_f in the large-J regime. The partial amplitude $A_{J_f}(\mathcal{K})$ has been defined by collecting the (g, z) integrals in Eq. (2). The spins $J_f \equiv \lambda j_f$ are large for all f, where $\lambda \gg 1$ is the mean value of J_f. By the linearity of $S[J_f, g_{ve}, z_{vf}]$ in J_f, the stationary phase analysis is employed to study the asymptotic behavior of the partial amplitude $A_{J_f}(\mathcal{K})$ where J_f is uniformly large. Such an analysis has been developed in Refs. [6,9–11]. In the asymptotics, the leading contribution of $A_{J_f}(\mathcal{K})$ comes from the spinfoam critical configurations, i.e., the solutions of $\Re S = 0$ and $\partial_g S = \partial_z S = 0$. It turns out that each critical configuration is interpreted as a certain type of geometry on $\mathcal{K}$. Moreover, the critical configurations also know if the manifold is oriented and time-oriented [6]. As a result, the critical configurations are classified according to their geometrical interpretations and the information about orientations:

TABLE I.

	$\mathcal{V}_f$	$\mathcal{K}_f$
Lorentz time-oriented	0	$\varepsilon \mathrm{sgn}(V_4)\Theta_f$
Lorentz time-unoriented	$i\varepsilon\pi$	$\varepsilon \mathrm{sgn}(V_4)\Theta_f$
Euclidean	$i\varepsilon[\mathrm{sgn}(V_4^E)\Theta_f^E + \pi n_f]$	0
Vector	$i\Phi_f$	0

In Table I, The first two classes of critical configurations give the Lorentzian simplicial geometries on $\mathcal{K}$. Each critical configuration (j_f, g_{ve}, z_{vf}) in the first two classes is equivalent to a set of geometrical data $(\pm_v E_\ell(v), \varepsilon)$ [6] with $\varepsilon = \pm 1$. $E_\ell(v)$ is a cotetrad on $\mathcal{K}$ (where the edge vectors satisfy some conditions), up to an overall sign $\pm_v$ in each 4-simplex. $E_\ell(v)$ determines the oriented volume $V_4(v) = \det(E_\ell^I(v))$. The local spacetime orientation is defined by $\mathrm{sgn}(V_4)$. $E_\ell(v)$ also determines uniquely a spin connection $\Omega_e \in \mathrm{SO}(1,3)$ along each dual edge e. The critical configuration gives a locally time-oriented spacetime if the corresponding spin connection along a closed loop $\Omega_f = \prod_{e\subset\partial f} \Omega_e \in \mathrm{SO}^+(1,3)$. Additionally, the last two classes of critical configurations give the Euclidean simplicial geometry and degenerate vector geometry on $\mathcal{K}$. It turns out that $\mathcal{V}_f$ and $\mathcal{K}_f$ defined in Eq. (4) take different values in each class of critical configurations, as is shown in the above table. Here Θ_f (Θ_f^E) denotes the Lorentzian (Euclidean) deficit angle, Φ_f denotes the vector-geometry angle, and $n_f \in \{0, 1\}$.

III. LOW-ENERGY APPROXIMATION

Here we consider the perturbations of the spinfoam variables around a critical configuration $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ in the first class, which corresponds to the globally oriented and time-oriented Lorentzian simplicial geometry with $\mathrm{sgn}(\mathring{V}_4) = 1$, $\mathring{\varepsilon} = -1$ globally. It turns out that Einstein gravity is recovered from the perturbations around such a background. The background deficit angles $|\mathring{\Theta}_f| \ll 1$ since we are interested in the low-energy perturbations. The background spins $\mathring{J}_f = \lambda \mathring{j}_f$, with $\lambda \gg 1$, for the semiclassical approximation.

[2] See Refs. [7,8] for an early study of the spinfoam amplitude using the effective action.

The partial amplitude can be written as $A_{j_f}(\mathcal{K}) = \exp\lambda W[j_f]$, where $W[j_f]$ is an effective action obtained by integrating out the (g_{ve}, z_{vf}) variables in Eq. (2). $W[j_f]$ is computed in a neighborhood at the background $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ by generalizing the method of computing the effective action to the case of a complex action [14] (sometimes called the almost-analytic machinery). The general procedure is summarized in the following.

Let $S(j,x)$, $j \in \mathbb{R}^k$, $x \in \mathbb{R}^N$ be a smooth function in a neighborhood of $(\mathring{j}, \mathring{x})$. We suppose that $\Re[S(j,x)] \le 0$, $\Re[S(\mathring{j}, \mathring{x})] = 0$, $\delta_x S(\mathring{j}, \mathring{x}) = 0$, and $\delta^2_{x,x} S(\mathring{j}, \mathring{x})$ is nondegenerate. We denote by $\mathcal{S}(j,z)$, $j \in \mathbb{C}^k$, $z = x + iy \in \mathbb{C}^n$ an (nonunique) almost-analytic extension of $S(j,x)$ to a complex neighborhood of $(\mathring{j}, \mathring{x})$. The equations of motion $\delta_z \mathcal{S} = 0$ define an almost-analytic manifold M in a neighborhood of $(\mathring{j}, \mathring{x})$, which is of the form $z = Z(j)$. On M and inside the neighborhood, there is a positive constant C such that for $j \in \mathbb{R}^k$

$$-\Re[\mathcal{S}(j,z)] \ge C|\Im(z)|^2, \qquad z = Z(j). \tag{6}$$

We have the following asymptotic expansion for the integral:

$$\int e^{\lambda S(j,x)} u(x) \mathrm{d}x \sim e^{\lambda \mathcal{S}[j,Z(j)]} \left(\frac{1}{\lambda}\right)^{\frac{N}{2}} \sqrt{\det\left(\frac{2\pi i}{\mathcal{S}''[j,Z(j)]}\right)} \times \sum_{s=0}^{\infty} \left(\frac{1}{\lambda}\right)^s [L_s \tilde{u}](Z(j)),$$

where $u(x) \in C_0^\infty(K)$ is a compact support function on K inside the domain of integration. N is the number of independent x variables, which is the same as the number of holomorphic z variables. The differential operator L_s of order $2s$ operates on an almost-analytic extension $\tilde{u}(z)$ of $u(x)$ and evaluates the result at $z = Z(j)$. The branch of the square root is defined by requiring $\sqrt{\det(2\pi i/\mathcal{S}''[j,Z(j)])}$ to deform continuously to 1 under the homotopy,

$$(1-s)\frac{2\pi i}{\mathcal{S}''[j,Z(j)]} + s\mathbf{I} \in \mathrm{GL}(n,\mathbb{C}), \qquad s \in [0,1]. \tag{7}$$

Note that the asymptotic expansions from two different almost-analytic extensions of the pair $S(j,x)$, $u(x)$ are different only by a contribution bounded by $C_K \lambda^{-K}$ for all $K \in \mathbb{Z}_+$.

In our case the spinfoam action is an analytic function in a neighborhood at $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$. We write $\mathcal{S}[j_f; g_{ve}, \tilde{g}_{ve}; z_{vf}, \tilde{z}_{vf}]$ as the analytic continuation of the action $S[j_f, g_{ve}, z_{vf}]$ in a complex neighborhood at $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$. Then, by the above procedure, we obtain the following effective action after integrating out g_{ve}, z_{vf}:

$$W[j_f] = \mathcal{S}[j_f; g_{ve}(j), \tilde{g}_{ve}(j); z_{vf}(j), \tilde{z}_{vf}(j)] + \cdots, \tag{8}$$

where $\cdots$ stands for the subleading contributions of $o(1/\lambda)$. The leading contribution of $W[j]$ is given by evaluating $\mathcal{S}$ at the solution $\{g_{ve}(j), \tilde{g}_{ve}(j); z_{vf}(j), \tilde{z}_{vf}(j)\} \equiv Z(j)$ of $\partial_g \mathcal{S} = \partial_{\tilde{g}} \mathcal{S} = \partial_z \mathcal{S} = \partial_{\tilde{z}} \mathcal{S} = 0$. In a neighborhood of spins at $\mathring{j}_f$, the real part of $\mathcal{S}[j_f; Z(j)]$ is nonvanishing and negative unless $j_f = \mathring{j}_f$, where $Z(j)$ reduces to the real value $\mathring{g}_{ve}$, $\mathring{z}_{vf}$.

The leading contribution $\mathcal{S}[j_f; Z(j)]$ can be analyzed by a Taylor expansion in perturbations $\mathfrak{s}_f = j_f - \mathring{j}_f$,

$$\mathcal{S} = i\left[\sum_f \gamma \mathring{j}_f \mathring{\Theta}_f + \sum_f \gamma \mathring{\Theta}_f \mathfrak{s}_f + \sum_{f,f'} W_{f,f'} \mathfrak{s}_f \mathfrak{s}_{f'} + o(\mathfrak{s}^3)\right]. \tag{9}$$

The computations of the above coefficients at different orders are given in Ref. [15]. In particular, $W_{f,f'}$ is local in the sense that it vanishes unless f, f' belong to the same tetrahedron e.

The above result is for the partial amplitude $A_{j_f}(\mathcal{K})$. In order to compute $A(\mathcal{K})$, we implement the sum over perturbations $\mathfrak{s}_f$ inside a neighborhood at $\mathring{j}_f$. The spinfoam amplitude is written as $A(\mathcal{K}) \sim \sum_{\mathfrak{s}_f} d_{\lambda(\mathring{j}_f + \mathfrak{s}_f)} \exp\lambda W[\mathring{j}_f + \mathfrak{s}_f]$ and is studied perturbatively. The Poisson resummation formula can be applied to the sum over the perturbations $\mathfrak{s}_f$, which results in the following perturbative expression for $A(\mathcal{K})$:

$$e^{i\lambda\sum_f \gamma \mathring{j}_f \mathring{\Theta}_f} \sum_{k_f \in \mathbb{Z}} \int [\mathrm{d}\mathfrak{s}_f] e^{i\lambda[\sum_f (\gamma\mathring{\Theta}_f - 4\pi k_f)\mathfrak{s}_f + \sum_{f,f'} W_{f,f'}\mathfrak{s}_f \mathfrak{s}_{f'} + o(\mathfrak{s}^3)] + \cdots}, \tag{10}$$

where again $\cdots$ stands for the subleading contributions in $1/\lambda$.

The above discussion considers the large-J regime for the spinfoam amplitude for the semiclassical approximation. Now we implement the low-energy approximation. The low-energy regime is achieved when the background configuration $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ is such that $\Theta_f \ll 1$.

Firstly, let us consider the integrals with $k_f \neq 0$ in Eq. (10) and apply the stationary phase analysis as $\lambda \gg 1$. The equation of motion from $\mathcal{S}[j_f; Z(j_f)]$ is given by

$$0 = \partial_{j_f}\mathcal{S}|_{Z(j)} + \partial_{j_f}Z\partial_Z\mathcal{S}|_{Z(j)} = \partial_{j_f}\mathcal{S}|_{Z(j)}, \quad (11)$$

where $\partial_Z\mathcal{S}|_{Z(j)} = 0$ because $Z(j)$ is the solution of $\partial_g\mathcal{S}=\partial_{\bar{g}}\mathcal{S}=\partial_z\mathcal{S}=\partial_{\bar{z}}\mathcal{S}=0$. The condition $\Re\mathcal{S}[j_f; Z(j_f)] = 0$ implies the perturbation $\mathfrak{s}_f = 0$ where $Z(j)$ reduces to $\mathring{g}_{ve}$, $\mathring{z}_{vf}$. Taking into account both the equations of motion and $\Re\mathcal{S} = 0$ gives us that $\gamma\mathring{\Theta}_f - 4\pi k_f = 0$ for $k_f \neq 0$, which cannot be satisfied in the low-energy regime where $|\mathring{\Theta}_f| \ll 1$ [with $\gamma \sim o(1)$ or less]. As a result, all the integrals with $k \neq 0$ in Eq. (10) are exponentially decaying, according to the principle of stationary phase analysis [16].

We thus focus on the integral with $k_f = 0$ in Eq. (10),

$$\int [\mathrm{d}\mathfrak{s}_f] e^{i\lambda[\sum_{f,f'}\gamma\mathring{\Theta}_f\mathfrak{s}_f + \sum_f W_{f,f'}\mathfrak{s}_f\mathfrak{s}_{f'} + o(\mathfrak{s}^3)] + \cdots}. \quad (12)$$

We denote by $|\mathring{\Theta}| \ll 1$ the mean value of the background deficit angle and $\mathring{\Theta}_f = \mathring{\Theta}\Delta_f$. The two-dimensional space of $(\lambda, \mathring{\Theta})$ may be viewed as the parameter space for our perturbation theory, where the semiclassical and low-energy regime is located in $\lambda \gg 1$, $|\mathring{\Theta}| \ll 1$. Now a new parameter is defined by $\beta := \lambda\mathring{\Theta}$, or a coordinate transformation is defined from $(\lambda, \mathring{\Theta})$ to (λ, β), where β is treated as independent of λ. Then Eq. (12) reads

$$\int [\mathrm{d}\mathfrak{s}_f] e^{i\lambda[\sum_{f,f'} W_{f,f'}\mathfrak{s}_f\mathfrak{s}_{f'} + o(\mathfrak{s}^3)]} e^{i\beta\gamma\sum_f \Delta_f\mathfrak{s}_f + \cdots}. \quad (13)$$

Again the stationary phase analysis is applied as $\lambda \gg 1$. We find that $\mathfrak{s}_f = 0$ is a solution of both $\partial_{\mathfrak{s}_f}[\sum_{f,f'} W_{f,f'}\mathfrak{s}_f\mathfrak{s}_{f'} + o(\mathfrak{s}^3)] = 0$ and $\Re[\sum_{f,f'} W_{f,f'}\mathfrak{s}_f\mathfrak{s}_{f'} + o(\mathfrak{s}^3)] = 0$. Note that $\Re[\sum_{f,f'} W_{f,f'}\mathfrak{s}_f\mathfrak{s}_{f'} + o(\mathfrak{s}^3)] = \Re\mathcal{S}$ since $i\sum_f \gamma\mathring{\Theta}_f\mathfrak{s}_f$ is purely imaginary. The standard stationary phase formula [16] leads to the following result from Eq. (13) in the neighborhood of the background spins $\mathring{j}_f$ ($\mathfrak{s}_f = 0$):

$$\sum_{n=0}^{\infty} (1/\lambda)^n \mathfrak{L}_n[e^{i\beta\gamma\sum_f \Delta_f\mathfrak{s}_f + \cdots}]_{\mathfrak{s}_f = 0} = \sum_{n=0}^{\infty}\sum_{r=0}^{2n} (\gamma^r\beta^r/\lambda^n) f_{n,r}. \quad (14)$$

$\mathfrak{L}_n$ is a differential operator of order $2n$ (in $\partial_{\mathfrak{s}_f}$) where all the interactions from the Lagrangian are encoded (see Ref. [16] for a general expression). Applying the differential operator $\mathfrak{L}_n$ to $e^{i\beta\gamma\sum_f \Delta_f\mathfrak{s}_f}$ gives the power-counting result in Eq. (14). The coefficients $f_{n,r}$ are functions of λ and $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$, which are regular as $\lambda \to \infty$.[3] By inserting Eq. (14) into Eq. (10) and recalling $\beta = \lambda\mathring{\Theta}$, the following expansion for $A(\mathcal{K})$ is obtained:

$$A(\mathcal{K}) \sim e^{i\lambda\sum_f \gamma\mathring{j}_f\mathring{\Theta}_f} \sum_{n=0}^{\infty}\sum_{r=0}^{2n} (\gamma^r\mathring{\Theta}^r/\lambda^{n-r}) f_{n,r}, \quad (15)$$

where the exponentially decaying contributions have been neglected. We can read from the above result an effective action $I_{\mathrm{eff}}(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ by expressing $A(\mathcal{K}) \sim \exp iI_{\mathrm{eff}}$, where the effective action at the background $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ is an expansion with respect to $\mathring{\Theta}$ and λ^{-1},

$$iI_{\mathrm{eff}} = \lambda\left[i\sum_f \gamma\mathring{j}_f\mathring{\Theta}_f + \frac{\gamma^2}{4}\sum_{f,f'} W^{-1}_{f,f'}\mathring{\Theta}_f\mathring{\Theta}_{f'} + o(\gamma^3\mathring{\Theta}^3; \lambda^{-1})\right]. \quad (16)$$

The derivation of the above expression will be given in the next section. The coefficient $W^{-1}_{f,f'}$ is the inverse of $W_{f,f'}$ in Eq. (9). $W_{f,f'}$ has been computed in Ref. [15]. $W_{f,f'}$ is nonzero only when f, f' belong to the same tetrahedron e,

$$W_{f,f'} = \frac{2(1 + 2i\gamma - 4\gamma^2 - 2i\gamma^3)}{5 + 2i\gamma} \hat{n}^t_{ef}\mathbf{X}_e^{-1}\hat{n}_{ef'}, \quad (17)$$

where $\mathbf{X}_e^{ij} \equiv \sum_f j_f(-\delta^{ij} + \hat{n}^i_{ef}\hat{n}^j_{ef} + i\varepsilon^{ijk}\hat{n}^k_{ef})$. Here the unit 3-vector $\hat{n}_{ef}$ determined by $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ is the normal vector of the triangle f in the frame of the tetrahedron e [6,15]. Although $W_{f,f'}$ is local in f, f', the inverse $W^{-1}_{f,f'}$ is nonlocal in general; it may be nonzero for far away f, f'. So the $\gamma^2\mathring{\Theta}^2$ term is a nonlocal curvature correction in I_{eff}. Moreover, a systematic method was developed in Ref. [15] to compute in principle all the $\gamma^r\mathring{\Theta}^r$ corrections.

IV. SEMICLASSICAL LOW-ENERGY EFFECTIVE ACTION

Let us understand the expansion in I_{eff} in a more detailed way. We focus on the integral with $k_f = 0$ in Eq. (10). The integrals with nonzero k_f can be analyzed in the same way. We apply the technique of the almost-analytic machinery to the action

$$\mathcal{S}[\mathfrak{s}_f] = i\sum_f \gamma\mathring{j}_f\mathring{\Theta}_f + \sum_f i\gamma\mathring{\Theta}_f\mathfrak{s}_f + \sum_{f,f'} W_{f,f'}\mathfrak{s}_f\mathfrak{s}_{f'} + o(\mathfrak{s}^3),$$

with $\Re(\mathcal{S}) \leq 0$ by Eq. (6).

[3] If the λ^{-1}-corrections are neglected, the $\gamma\mathring{\Theta}$ expansion of I_{eff} is analytic in a neighborhood at $\gamma\mathring{\Theta} = 0$, by the analyticity of the spinfoam action [15].

If we consider the action $\mathcal{S}[\mathfrak{s}_f] \equiv \mathcal{S}[\gamma\mathring{\Theta}_f, \mathfrak{s}_f]$ where $\gamma\mathring{\Theta}_f$ is treated as a parameter here, we find

$$\gamma\mathring{\Theta}_f = 0 \quad \text{and} \quad \mathfrak{s}_f = 0 \Rightarrow \Re(\mathcal{S}) = 0 \quad \text{and} \quad \delta_{\mathfrak{s}_f}\mathcal{S} = 0. \tag{18}$$

The critical point $\gamma\mathring{\Theta}_f = 0$, $\mathfrak{s}_f = 0$ for the action $\mathcal{S}[\gamma\mathring{\Theta}_f, \mathfrak{s}_f]$ fulfills the assumption of the almost-analytic machinery. We apply the almost-analytic machinery to $\mathcal{S}[\gamma\mathring{\Theta}_f, \mathfrak{s}_f]$ by analytically continuing $\mathfrak{s}_f$ to the complex variables. $\mathcal{S}$ is an analytic function of $\mathfrak{s}_f$, and so the equation of motion $\delta_{\mathfrak{s}_f}\mathcal{S} = 0$ gives an analytic manifold $\mathfrak{s}_f = Z_f(\gamma\mathring{\Theta}_f)$ at least locally. The integral (12) is expressed as an asymptotic expansion,

$$e^{\lambda\mathcal{S}[\gamma\mathring{\Theta}_f, Z_f(\gamma\mathring{\Theta}_f)]}\left(\frac{1}{\lambda}\right)^{\frac{N_f}{2}}\sqrt{\det\left(\frac{2\pi i}{\delta^2_{\mathfrak{s}_f,\mathfrak{s}_{f'}}\mathcal{S}|_{\gamma\mathring{\Theta}_f, Z_f(\gamma\mathring{\Theta}_f)}}\right)}\left[1 + o\left(\frac{1}{\lambda}\right)\right]. \tag{19}$$

The leading effective action $\mathcal{S}[\gamma\mathring{\Theta}_f, Z_f(\gamma\mathring{\Theta}_f)]$ has the property [following from Eq. (6)] that

$$\Re(\mathcal{S}[\gamma\mathring{\Theta}_f, Z_f(\gamma\mathring{\Theta}_f)]) \leq -C|\Im(Z_f(\gamma\mathring{\Theta}_f))|^2. \tag{20}$$

We can compute more concretely the expression of the effective action $\mathcal{S}[\gamma\mathring{\Theta}_f, Z_f(\gamma\mathring{\Theta}_f)]$ as a power series of $\gamma\mathring{\Theta}_f$. We expand the action $\mathcal{S}[\gamma\mathring{\Theta}_f, \mathfrak{s}_f]$ at the first-order solution (in $\gamma\mathring{\Theta}_f$) from the equation of motion $\delta_{\mathfrak{s}_f}\mathcal{S} = 0$,

$$\mathfrak{s}_f = -\frac{i}{2}\sum_{f'} W^{-1}_{f,f'}\gamma\mathring{\Theta}_{f'}. \tag{21}$$

If we define $y_f \equiv \mathfrak{s}_f + \frac{i}{2}\sum_{f'} W^{-1}_{f,f'}\gamma\mathring{\Theta}_{f'}$, then the expansion of the action reads

$$\begin{aligned}\lambda\mathcal{S}[\gamma\mathring{\Theta}_f, y_f] &= i\lambda\sum_f \gamma\mathring{j}_f\mathring{\Theta}_f \\ &\quad + \lambda\left[\frac{1}{4}\sum_{f,f'} W^{-1}_{f,f'}\gamma\mathring{\Theta}_f\gamma\mathring{\Theta}_{f'} + o((\gamma\mathring{\Theta}_f)^3)\right] \\ &\quad + \lambda\left\{\sum_f [o((\gamma\mathring{\Theta}_f)^2)]y_f + \sum_{f,f'}[2W_{f,f'} \right. \\ &\quad \left. + o(\gamma\mathring{\Theta}_f)]y_f y_{f'} + o(y_f^3)\right\}\end{aligned}$$

The equation of motion $\delta_{y_f}\mathcal{S}[\gamma\mathring{\Theta}_f, y_f]$ with respect to y_f gives an $o((\gamma\mathring{\Theta}_f)^2)$ correction to the original approximating solution $y_f = 0$, i.e.,

$$y_f = o((\gamma\mathring{\Theta}_f)^2) \quad \text{or} \quad \mathfrak{s}_f = -\frac{i}{2}\sum_{f'} W^{-1}_{f,f'}\gamma\mathring{\Theta}_{f'} + o((\gamma\mathring{\Theta}_f)^2).$$

If we expand the action $\mathcal{S}[\gamma\mathring{\Theta}_f, \mathfrak{s}_f]$ at the new approximating solution and iterate the above procedure, we more accurately approximate the exact solution of $\delta_{\mathfrak{s}_f}\mathcal{S}[\gamma\mathring{\Theta}_f, \mathfrak{s}_f] = 0$ and obtain the exact solution $\mathfrak{s}_f$ as a power series of $\gamma\mathring{\Theta}_f$,

$$\mathfrak{s}_f = Z_f(\gamma\mathring{\Theta}_f) = \sum_{n=1}^{\infty} \alpha_{f,f_1,\ldots,f_n}\gamma\mathring{\Theta}_{f_1}\cdots\gamma\mathring{\Theta}_{f_n}, \tag{22}$$

where the series has a finite convergence radius since we know that $Z_f(\gamma\mathring{\Theta}_f)$ is analytic.

Evaluating the action $\mathcal{S}[\gamma\mathring{\Theta}_f, \mathfrak{s}_f]$ at this exact solution gives

$$\mathcal{S}[\gamma\mathring{\Theta}_f, Z_f(\gamma\mathring{\Theta}_f)] = i\sum_f \gamma\mathring{j}_f\mathring{\Theta}_f + \sum_{n=2}^{\infty}\beta^{(n)}_{f_1,\ldots,f_n}\gamma\mathring{\Theta}_{f_1}\cdots\gamma\mathring{\Theta}_{f_n},$$

where in particular the quadratic-order coefficient is given by

$$\beta^{(2)}_{f_1,f_2} = \frac{1}{4}W^{-1}_{f_1 f_2}. \tag{23}$$

The expression of the matrix W_{f_1,f_2} is given in Eq. (17).

As a result, when $\gamma\mathring{\Theta}_f$ is small,

$$\begin{aligned}\mathcal{S}[\gamma\mathring{\Theta}_f, Z_f(\gamma\mathring{\Theta}_f)] &= i\sum_f \gamma\mathring{j}_f\mathring{\Theta}_f + \frac{1}{4}\sum_{f,f'} W^{-1}_{f,f'}\gamma\mathring{\Theta}_f\gamma\mathring{\Theta}_{f'} \\ &\quad + o((\gamma\mathring{\Theta}_f)^3).\end{aligned} \tag{24}$$

Following the same procedure for the $k_f \neq 0$ branches in Eq. (10), we obtain that in general

$$\begin{aligned}&\mathcal{S}_k[\gamma\mathring{\Theta}_f - 4\pi k_f, Z_f(\gamma\mathring{\Theta}_f - 4\pi k_f)] \\ &\quad = i\sum_f \gamma\mathring{j}_f\mathring{\Theta}_f + \frac{1}{4}\sum_{f,f'} W^{-1}_{f,f'}(\gamma\mathring{\Theta}_f - 4\pi k_f)(\gamma\mathring{\Theta}_{f'} - 4\pi k_f) \\ &\quad + o((\gamma\mathring{\Theta}_f - 4\pi k_f)^3).\end{aligned} \tag{25}$$

In general $\mathcal{S}_k$ has a negative real part coming from the terms of quadratic and higher order in $(\gamma\mathring{\Theta}_f - 4\pi k_f)$. By Eq. (20), the exponential $e^{\lambda\mathcal{S}_k}$ in the asymptotic expansion Eq. (19)

for generic k decays exponentially unless $\gamma\mathring{\Theta}_f$ from the background data $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ is close to one of $\{4\pi k_f : k_f \in \mathbb{Z}\}$. The nondecaying $e^{\lambda \mathcal{S}_k}$ requires that the negative real part of $\lambda \mathcal{S}_k$ does not become large when $\lambda \gg 1$, which imposes a nontrivial restriction on $\gamma\mathring{\Theta}_f$, i.e., for a constant $C \sim o(1)$,

$$|\gamma\mathring{\Theta}_f| \le C\lambda^{-1/2} \quad \mathrm{mod} \quad 4\pi\mathbb{Z}. \tag{26}$$

If we assume $\gamma\mathring{\Theta}_f$ is small and the bound (29) is satisfied, the above analysis lets us obtain a perturbative effective action for the spinfoam state sum $A(\mathcal{K})$,

$$A(\mathcal{K}) \sim e^{\lambda I_{\mathrm{eff}}(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})}, \tag{27}$$

where the leading-order contribution to the effective action I_{eff} is given by a power series of $\gamma\mathring{\Theta}_f$,

$$I_{\mathrm{eff}} = i\sum_f \gamma \mathring{j}_f \mathring{\Theta}_f + \frac{1}{4}\sum_{f,f'} W^{-1}_{f,f'} \gamma\mathring{\Theta}_f \gamma\mathring{\Theta}_{f'} + o((\gamma\mathring{\Theta}_f)^3) + \\ - \frac{N_{g,z} - N_f}{2\lambda} \ln\lambda + o\left(\frac{1}{\lambda}\right). \tag{28}$$

$\mathring{\Theta}_f$ is the deficit angle of the background simplical geometry from $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$. Thus the above expression of the effective action is a curvature expansion whose leading order is the Regge action.

V. DISCUSSION

There are several remarks for the effective action (16) or (28):

(i) Low-energy effective action as curvature expansion: The terms $\propto \lambda(\gamma\mathring{\Theta})^{r\ge 2}$ are understood as the high-energy correction to the leading-order $i\lambda\sum_f \gamma \mathring{j}_f \mathring{\Theta}_f$, since the $|\mathring{\Theta}| \ll 1$ implements the low-energy approximation.[4] Therefore, as a power series of $\mathring{\Theta}$, I_{eff} is understood as a low-energy effective action from covariant LQG. The deficit angle $\mathring{\Theta} \sim \alpha\mathcal{R}$, where α is the mean (area) spacing of the lattice given by the background data $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$, $\mathcal{R}$ is the mean curvature of the background. Thus the effective action I_{eff} can be viewed as a curvature expansion, where the high-energy corrections are given by $\alpha^2\gamma^2\mathcal{R}^2 + \alpha^3\gamma^3\mathcal{R}^3 + \cdots$, with α being the (effective) coupling constant of the high-derivative interactions.

(ii) Two-parameter expansion: There are two parameters involved in the expression of the effective action I_{eff}, i.e., $\lambda \gg 1$ and $\mathring{\Theta} \ll 1$ (or α with dimension -2). $1/\lambda$ counts the quantum corrections, while $\mathring{\Theta}$ (or α) counts the high-energy corrections. The two expansion parameters implement the semiclassical low-energy regime $\ell_P^2 \ll \alpha \ll L^2$.

(iii) Restriction of Θ: The effective action iI_{eff} has a negative real part, which is contained in the terms of higher curvature [15], i.e., $\Re[iI_{\mathrm{eff}}] = \lambda\Re[\frac{1}{4}W^{-1}\gamma^2\mathring{\Theta}^2 + o(\gamma^3\mathring{\Theta}^3) + \cdots] \le 0$, where $\cdots$ stands for the terms suppressed by $1/\lambda$. This negative real part of the exponential would have given an exponentially decaying factor in $A(\mathcal{K})$ if $\gamma\mathring{\Theta}$ was of $o(1)$, which is *not* our case because $\mathring{\Theta} \ll 1$. The nondecaying $A(\mathcal{K})$ requires that $\Re[iI_{\mathrm{eff}}]$ does not become large when $\lambda \gg 1$, which results in a nontrivial bound for the deficit angle $\mathring{\Theta}$, i.e.,

$$|\mathring{\Theta}| \le \gamma^{-1}\lambda^{-\frac{1}{2}}. \tag{29}$$

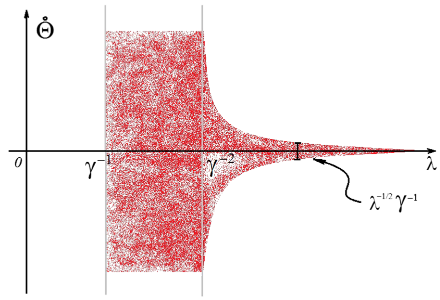

FIG. 1 (color online). The Einstein sector of spinfoam configurations.

The situation is illustrated in Fig. 1. The red region in Fig. 1 illustrates the space (in the coordinates λ and $\mathring{\Theta}$) of background configurations $(\mathring{J}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$, which validates the two-parameter expansion of the effective action I_{eff}. If $\mathring{\Theta}$ is beyond the bound (29), where the approximation (14) is invalid, the integral (12) is exponentially decaying as $\lambda \gg 1$ by the same argument for $k \ne 0$ integrals. Thus the red region in Fig. 1 illustrates the semiclassical low-energy effective degrees of freedom from the above approximation.

(iv) Einstein-Hilbert action: After the restriction (29), the leading contribution in I_{eff}, $i\lambda\sum_f \gamma \mathring{j}_f \mathring{\Theta}_f$, is the Regge action of general relativity (GR) as a functional of the edge lengths determined by

[4] The terms linear to $\gamma\mathring{\Theta}$ are suppressed by λ^{-1} except the leading Regge action.

$(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ (by identifying $\gamma\lambda\mathring{j}_f = \alpha_f/\ell_P^2$ to be the area of the triangle f in Planck units). Moreover, given that that $\mathring{\Theta}_f \sim \alpha_f/L^2 \ll 1$[5] and that $I_{\rm eff}$ is a power series in α_f, the leading-order contribution is essentially the Einstein-Hilbert action on a smooth manifold $\mathcal{M}$, i.e., as a functional (see, e.g., Ref. [13]),

$$i\lambda\sum_f \gamma\mathring{j}_f\mathring{\Theta}_f = \frac{i}{\ell_P^2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\mathring{g}}\mathcal{R}[\mathring{g}_{\alpha\beta}] \times [1 + o(\alpha_f/L^2)], \tag{30}$$

where $\mathring{g}_{\alpha\beta}$ is the Lorentzian metric approximated by the simplicial geometry from $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$. Therefore, $I_{\rm eff}$ can be written as

$$I_{\rm eff} = \frac{1}{\ell_P^2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-\mathring{g}}\mathcal{R}[\mathring{g}_{\alpha\beta}] \times [1 + o(\alpha\mathcal{R}) + o(1/\lambda)], \tag{31}$$

where the leading contribution is the Einstein-Hilbert action. Such a result immediately follows from the fact that the leading-order effective action is the Regge action from the semiclassical low-energy expansion. In this sense, the diffeomorphism invariance on $\mathcal{M}$ is then recovered as an approximated symmetry in the leading order of the semiclassical and low-energy approximation. However, it remains to be seen whether the theory can recover a continuum theory with diffeomeorphism invariance beyond the leading order or even nonperturbatively.

(v) Small Barbero-Immirzi parameter: Once $\gamma \ll 1$, the interesting regime $\gamma^{-1} \ll \lambda \le \gamma^{-2}$ appears in Fig. 1. $\gamma^{-1} \ll \lambda$ is required for $\ell_P^2 \ll \alpha_f$. As $\lambda \ll \gamma^{-2}$ and due to Eq. (29), $A(\mathcal{K})$ is not decaying even without the restriction of $\mathring{\Theta}$. Even a finite $\mathring{\Theta}$ is admitted in Eq. (16) without requiring $\mathring{\Theta} \ll 1$. Indeed, each $\mathring{\Theta}$ is accompanied by a γ in $I_{\rm eff}$ [and in Eq. (12) originally], where γ appears as an effective scaling of the deficit angle. One may choose $\beta = \lambda\gamma$ in Eq. (13) as $\gamma \ll 1$. Thus in the regime $\gamma^{-1} \ll \lambda \le \gamma^{-2}$, $I_{\rm eff}$ can be formulated as Eq. (16) with a finite deficit angle, where the leading order is the Regge action in general. Sending $\gamma \to 0$ effectively neglects the higher-curvature corrections. The analysis here may explain the spinfoam graviton propagator calculations [17–19] and the analysis in Ref. [20], which was the first to motivate $\gamma \ll 1$.

(vi) Flatness: $\lambda \to \infty$ asymptotically is another interesting regime in Fig. 1, where the deficit angle is so restricted that only $\mathring{\Theta} = 0$ (flat geometry) is allowed. It is related to the "flatness problem" in the spinfoam formulation discussed in Ref. [21]. However, the flatness problem disappears here for any finite $\lambda \gg 1$ via the low-energy perturbation theory. It is an open question as to how to interpret the regime where the spins are too large, as it seems to give a lattice spacing scale that is semiclassically too large, $\alpha_f \sim \lambda\ell_P^2$, and it contradicts the observation of smooth spacetime. In order to remove the regime, a spin cutoff may be introduced via q-deformation [4], which produces a relatively large bare cosmological constant [5].

The above discussion considers the fluctuations of spinfoam variables that touch a single critical configuration $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$. The fluctuations touching many critical configurations $(j_f, g_{ve}, z_{vf})_c$ result in a sum of the above perturbative expression of $A(\mathcal{K})$ over all the critical configurations. The red region in Fig. 1—which we call the *Einstein sector*, $\mathfrak{N}_E$—is a subspace of spinfoam configurations, in which all the critical configurations are interpreted as a globally oriented and time-oriented Lorentzian geometry [with $\mathrm{sgn}(V_4) = 1$, $\varepsilon = -1$ globally]. When the fluctuations of the spinfoam variables are considered within $\mathfrak{N}_E$, the perturbative expression of the spinfoam amplitude is then given by

$$A(\mathcal{K}) = \sum_{(j_f, g_{ve}, z_{vf})_c \in \mathfrak{N}_E} e^{\frac{i}{\ell_P^2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\mathcal{R}[g_{\alpha\beta}]\times[1+o(\alpha\mathcal{R})+o(1/\lambda)]}, \tag{32}$$

where $g_{\alpha\beta}$ is the Lorentzian metric approximated by $(j_f, g_{ve}, z_{vf})_c$. Equation (32) makes sense because the perturbations at a geometrical critical configuration (which is globally Lorentzian, oriented, and time-oriented) only touch the geometrical critical configuration of the same type. From Eq. (32) we see that the contributions to $A(\mathcal{K})$ from the perturbations within $\mathfrak{N}_E$ are given by the functional integration of the Einstein-Hilbert action (with a discrete measure) plus the high-energy and quantum corrections. The leading contributions to $A(\mathcal{K})$ in $\mathfrak{N}_E$ come from the critical configurations $(j_f, g_{ve}, z_{vf})_c$, which give $g_{\alpha\beta}$ satisfying the Einstein equation (with high-energy and quantum corrections).

The above discussion can be generalized straightforwardly to the analysis of correlation functions. In the Einstein sector $\mathfrak{N}_E$, the perturbative result of the spinfoam correlation function coincides with the corresponding

[5]Given $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ with nontrivial mean curvature radius, in order to obtain Eq. (31) a large triangulation is needed, e.g., if the size of $\mathcal{K}$ measured by $(\mathring{j}_f, \mathring{g}_{ve}, \mathring{z}_{vf})$ is of the same order as the curvature radius L, the number of simplices is at least of the order $N \sim L^4/\alpha^2 \gg 1$.

perturbative correlation function from Einstein gravity or Regge gravity, up to curvature and quantum corrections.

The corrections of higher order in curvature (in deficit angle) modify the Einstein-Regge gravity in the high-energy regime. It is interesting to further investigate these high-curvature terms predicted from covariant LQG, in order to see if LQG can provide a UV completion of perturbative Einstein gravity. The origin of high-curvature terms is the sum over *non-Regge-like* spins (the spins that cannot be viewed as Regge areas) in the spinfoam amplitude. The non-Regge-like spins are the extra UV degrees of freedom in addition to those from GR predicted by LQG. Their dynamics may be studied via the action (9) to see if they regulate Einstein gravity in the UV. Beside the UV corrections, the physical implication of quantum $1/\lambda$ corrections is also an open issue to be understood in the future. It may be important to understand the implications from both UV and quantum corrections in order to see if there exists a continuum limit of the theory beyond the leading-order effective action.

Finally, we remark that the work carried out in the present paper concerns the perturbation theory of covariant LQG when the background is chosen in the Einstein sector $\mathfrak{N}_E$. An analysis beyond the Einstein sector $\mathfrak{N}_E$ can also be carried out. There exist other different sectors—well separated from $\mathfrak{N}_E$—where a similar analysis results in leading-order effective actions that are different from Einstein gravity. As was shown in Refs. [6,9,11], there are other possible backgrounds (vacuum) corresponding to (i) nondegenerated Lorentzian geometry with a nonuniform orientation, or (ii) Lorentzian geometry, Euclidean geometry, or degenerate geometry, or a mixture of the three types. It is not yet understood if the contribution from the Einstein sector is dominating the spinfoam amplitude in the semiclassical low-energy regime, or if all types of background contribute in a democratic way. The same type of perturbative expansion can be carried out in principle for those types of geometries which do not correspond to the usual Einstein gravity at leading order. We refer the reader to Refs. [15,22] for some discussions. However, the physical implication of other types of backgrounds are thus far unclear, and this remains to be investigated in the future. It is possible that we may need a mechanism to stablize our theory in the semiclassical low-energy regime to the vacua in the Einstein sector. Such a mechanism is an open issue to be understood in the future.

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