

Angular-averaged consistency relations of large-scale structures

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The cosmological dynamics of gravitational clustering satisfies an approximate invariance with respect to the cosmological parameters that is often used to simplify analytical computations. We describe how this approximate symmetry gives rise to angular-averaged consistency relations for the matter density correlations. This allows one to write the $(\ell + n)$ density correlation, with ℓ large-scale linear wave numbers that are integrated over angles, and n fixed small-scale nonlinear wave numbers, in terms of the small-scale n -point density correlation and ℓ prefactors that involve the linear power spectra at the large-scale wave numbers. These relations, which do not vanish for equal-time statistics, go beyond the already known kinematic consistency relations. They could be used to detect primordial non-Gaussianities, modifications of gravity, limitations of galaxy biasing schemes, or to help design analytical models of gravitational clustering.

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I. INTRODUCTION

After the results of the WMAP and Planck missions [1,2], which have already uncovered a lot of information from the cosmic microwave background data, surveys of the large-scale structure of the Universe promise to be an important and complementary probe of cosmological scenarios [3,4]. In particular, they should shed light on the properties of the dark matter and dark energy components. Unfortunately, even without considering the very complex processes of galaxy and star formation [5–8] and focusing on the large-scale properties where gravity is the dominant driver, exact or well-controlled predictions for the statistical properties of the density and velocity fields are difficult. Large scales can be described by standard perturbative approaches [9,10], which can be improved to some degree by using resummation schemes [11–19]. However, these methods cannot reach the truly nonlinear regime where shell-crossing effects become important [20–22]. Small scales are studied through numerical simulations or phenomenological models [23] that rely on information gained through these simulations. However, these scales are difficult to model with high accuracy, even with simulations, and it would be useful to have analytical results that go beyond low-order perturbation theory.

Some exact results have recently been obtained [24–30] in the form of “kinematic consistency relations.” They relate the $(\ell + n)$ -density correlation, with ℓ large-scale wave numbers and n small-scale wave numbers, to the n -point small-scale density correlation, with ℓ prefactors that involve the linear power spectrum at the large-scale wave numbers. These relations, obtained at the leading order over the large-scale wave numbers k'_j , arise from the equivalence principle (in standard scenarios). It ensures that small-scale structures respond to a large-scale perturbation

(which at leading order corresponds to a constant gravitational force over the extent of the small-size object) by a uniform displacement. Therefore, these relations express a kinematic effect, due to the displacement of small-scale structures between different times. This also means that (at this order) they vanish for equal-time statistics, as a uniform displacement has no impact on the statistical properties of the density field observed at a given time.

In practice, it is difficult to measure different-time density correlations, as correlations between different redshift planes along our light cone (hence over distances of order c/H_0) are very small. Therefore, it would be useful to obtain similar relations that apply to single-time density correlations. This means that we must go beyond the kinematic effect and investigate how small-scale density fluctuations respond to nonuniform gravitational forces. At leading order over the large-scale wave numbers, this is given by the response to a change of the background density, which also corresponds to a large-scale curvature of the gravitational potential.

In this paper, we show how this problem can be addressed by using an approximate symmetry of the cosmological gravitational dynamics.

In Sec. II, we recall how most of the dependence on cosmological parameters can be absorbed by a remapping of the time coordinate, $t \rightarrow D_+(t)$, where $D_+(t)$ is the linear growing mode. This is a well-known approximate symmetry of the cosmological gravitational dynamics that is often used in analytical methods (e.g., perturbative schemes) to simplify the computations. Then, in Sec. III we show how this invariance dictates the response of density fluctuations to a small change of the background density, which corresponds to a change of the cosmological parameters Ω_m and Ω_K . This allows us to derive consistency relations that go beyond the kinematic effect and

remain nontrivial for the equal-time density correlations. In Sec. IV, we explicitly check this relation for the matter density bispectrum, at leading order of perturbation theory. We also present a fully nonlinear check in one dimension (using the fact that the Zel'dovich approximation becomes an exact solution), which provides a check for all many-body density correlations or polyspectra up to all orders of perturbation theory (and beyond the shell-crossing regime if we consider the system as defined by the Zel'dovich solution). We discuss our results and conclude in Sec. V.

II. APPROXIMATE SYMMETRY OF THE COSMOLOGICAL GRAVITATIONAL DYNAMICS

On scales much smaller than the horizon, where the Newtonian approximation is valid, the equations of motion read as [31]

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot [(1 + \delta) \mathbf{v}] = 0, \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + H \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{a} \nabla \phi, \quad (2)$$

$$\nabla^2 \phi = 4\pi G \bar{\rho} a^2 \delta, \quad (3)$$

where $a(t)$ is the scale factor, $H = \dot{a}/a$ the Hubble expansion rate, $\delta = (\rho - \bar{\rho})/\bar{\rho}$ the density contrast, and $\mathbf{v} = a d\mathbf{x}/dt$ the peculiar velocity. Here, we use the single-stream approximation to simplify the presentation, but the results remain valid beyond shell crossing. Linearizing these equations over $\{\delta, \mathbf{v}\}$, one obtains the linear growth rates $D_{\pm}(t)$, which are the independent solutions of [10,31]

$$\ddot{D} + 2H\dot{D} - 4\pi G \bar{\rho} D = 0. \quad (4)$$

For an Einstein–de Sitter universe, where $a(t) \propto t^{2/3}$, the linear growing mode is $D_+(t) \propto a \propto t^{2/3}$ and the linear decaying mode is $D_-(t) \propto a^{-3/2} \propto t^{-1}$. For a generic cosmology, with a nonzero cosmological constant and curvature, one must numerically solve Eq. (4).

As usual [11,32,33], it is convenient to make the change of variables

$$\eta = \ln D_+, \quad \mathbf{v} = \dot{a} f \mathbf{u}, \quad \phi = (\dot{a} f)^2 \varphi, \quad (5)$$

where $f = d \ln D_+ / d \ln a$. Then, the equations of motion read as

$$\frac{\partial \delta}{\partial \eta} + \nabla \cdot [(1 + \delta) \mathbf{u}] = 0, \quad (6)$$

$$\frac{\partial \mathbf{u}}{\partial \eta} + \left(\frac{3\Omega_m}{2f^2} - 1 \right) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \varphi, \quad (7)$$

$$\nabla^2 \varphi = \frac{3\Omega_m}{2f^2} \delta, \quad (8)$$

where $\Omega_m(t)$ is the matter density cosmological parameter as a function of time, which obeys $4\pi G \bar{\rho} = (3/2)\Omega_m H^2$.

It happens that for standard cosmologies (i.e., within general relativity), Ω_m/f^2 is always very close to 1 (which is exact for the Einstein–de Sitter case) [31]. Then, making the approximation $\Omega_m/f^2 \simeq 1$ removes all explicit time dependence in the equations of motion (6)–(8) and simplifies the analytical computations. This also removes all explicit dependence on the cosmological parameters. In particular, within a perturbative framework, one can use the results obtained for the Einstein–de Sitter case by making the replacement $a(t) \rightarrow D_+(t)$ [34,35]. The accuracy of this approximation was investigated in Refs. [13,17], where it was found that it performs to better than 1% at redshift $z = 0$ for $k \leq 2h \text{ Mpc}^{-1}$, and 0.1% at $z = 1$ on these scales. The approximation performs increasingly well at high redshift in the matter era (where we recover the Einstein–de Sitter cosmology). Although it degrades on small scales at $z = 0$, this approximation is used by most perturbative approaches [11–19] to simplify computations (in particular, it allows one to use the explicit exponential form of the linear response function or propagator adapted from the Einstein–de Sitter case, with factors e^η and $e^{-3\eta/2}$ [11,12,36]). Thus, it provides a sufficient approximation on perturbative scales and in the highly nonlinear regime at low redshift it is not the main source of inaccuracy, as uncertainties on the halo mass function, for instance, lead to greater error bars [22].

This approximate symmetry does not rely on the single-stream approximation, and instead of the Euler equations (2) and (7), we can use the equation of motion of the trajectories $\mathbf{x}(\mathbf{q}, t)$ of the particles. It reads as

$$\frac{\partial^2 \mathbf{x}}{\partial t^2} + 2H \frac{\partial \mathbf{x}}{\partial t} = -\frac{1}{a^2} \nabla \phi, \quad (9)$$

which becomes with the time coordinate η

$$\frac{\partial^2 \mathbf{x}}{\partial \eta^2} + \left(\frac{3\Omega_m}{2f^2} - 1 \right) \frac{\partial \mathbf{x}}{\partial \eta} = -\nabla \varphi, \quad (10)$$

where φ is the rescaled gravitational potential (8). This explicitly shows that it satisfies the same approximate symmetry. Therefore, our results are not restricted to the perturbative regime and also apply to small nonlinear scales governed by shell-crossing effects, as long as the approximation $\Omega_m/f^2 \simeq 1$ is sufficiently accurate (but this also means that we are restricted to scales dominated by gravity).

This standard approximation means that all the dependence on cosmological parameters is encapsulated in the linear growing mode $D_+(t)$. In this paper, we investigate the consequences of this approximate symmetry of the equations of motion, in terms of the “squeezed” limit of density correlations. This corresponds to Fourier-space

$(\ell + n)$ density correlations $\langle \tilde{\delta}(\mathbf{k}'_1) \dots \tilde{\delta}(\mathbf{k}'_\ell) \tilde{\delta}(\mathbf{k}_1) \dots \tilde{\delta}(\mathbf{k}_n) \rangle$, where the wave numbers k'_j are much smaller than all other wave numbers k_i and within the linear regime. Our method relies on the fact that a large-scale spherically symmetric perturbation of the initial density contrast is similar to a change of the mean density $\bar{\rho}$, from whence the cosmological parameters arise, from the point of view of a much smaller region at the center of this initial perturbation.

III. ANGULAR-AVERAGED CONSISTENCY RELATIONS

We first consider the case of a single large-scale wave number k' and we generalize to several large-scale wave numbers k'_j in Sec. III C 3.

A. Correlation and response functions

Because the cosmological density and velocity fields are statistically homogeneous and isotropic, it is often convenient to work in Fourier space. In this paper, we denote with a tilde Fourier-space fields, defining the Fourier transform as

$$\delta(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} \tilde{\delta}(\mathbf{k}). \quad (11)$$

To compare theoretical predictions with observations, one often computes correlation functions, $\langle \delta(\mathbf{x}_1) \dots \delta(\mathbf{x}_n) \rangle$, or multispectra, $\langle \tilde{\delta}(\mathbf{k}_1) \dots \tilde{\delta}(\mathbf{k}_n) \rangle$. In particular, the power spectrum $P(k)$ is defined as

$$\langle \tilde{\delta}(\mathbf{k}_1) \tilde{\delta}(\mathbf{k}_2) \rangle = \delta_D(\mathbf{k}_1 + \mathbf{k}_2) P(k_1), \quad (12)$$

where the Dirac factor arises from statistical homogeneity. We also denote with the subscript “ L ” the linear fields obtained by linearizing the equations of motion (1)–(3), and with the subscript “ $L0$ ” the linear fields today, at $z = 0$. Throughout this paper, we assume as usual that the linear decaying modes have had time to become negligible by the times of interest. Then, the initial conditions are fully defined by the linear growing mode, which is also set by the linear density field today $\delta_{L0}(\mathbf{x})$, which we assume to be Gaussian.

In analytical approaches, especially in perturbative schemes that use field-theoretic tools [11, 12, 14, 36–40], it is convenient to introduce response functions (also called propagators or Green functions), which we define in real space as

$$R^{\ell, n}(\mathbf{x}'_1, \dots, \mathbf{x}'_\ell; \mathbf{x}_1, t_1, \dots, \mathbf{x}_n, t_n) = \left\langle \frac{\mathcal{D}^\ell [\delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n)]}{\mathcal{D}\delta_{L0}(\mathbf{x}'_1) \dots \mathcal{D}\delta_{L0}(\mathbf{x}'_\ell)} \right\rangle, \quad (13)$$

and similarly in Fourier space (throughout this paper, we denote by the letter $\mathcal{D}$ the functional derivative). The quantities (13) describe how the nonlinear density

field, at positions and times $\{\mathbf{x}_1, t_1; \dots; \mathbf{x}_n, t_n\}$, responds to changes of the initial conditions [defined by $\delta_{L0}(\mathbf{x})$] at positions $\{\mathbf{x}'_1, \dots, \mathbf{x}'_\ell\}$.

As described, for instance, in Ref. [30], for Gaussian initial conditions, correlations between the nonlinear density contrast δ and the linear density contrast δ_{L0} that defines the initial conditions can be written in terms of response functions. This gives in Fourier space [30]

$$\langle \tilde{\delta}_{L0}(\mathbf{k}') \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle = P_{L0}(k') \left\langle \frac{\mathcal{D}[\tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n)]}{\mathcal{D}\delta_{L0}(-\mathbf{k}')} \right\rangle, \quad (14)$$

where $P_{L0}(k')$ is the linear power spectrum of the initial conditions δ_{L0} . This provides a simple method to obtain consistency relations for the density correlations by computing the response function [i.e., the last term in Eq. (14)] associated with a large-scale perturbation $\Delta\delta_{L0}$ of the initial condition.

The leading-order effect that arises in the large-scale limit, $k' \rightarrow 0$, is a constant force, $-\nabla(\Delta\phi_{L0})$, and velocity, $\Delta\mathbf{v}_{L0}$, over the small-scale region of size R , with $(k'R) \ll 1$. This also corresponds to a zero local density perturbation, because in the linear regime we have $\delta_L \propto \nabla \cdot \mathbf{v}_L$ (up to time-dependent factors), as seen from the continuity equation (1). This leads to a uniform displacement of small-scale structures. Then, one obtains kinematic consistency relations [24–30] that express a correlation of the form $\langle \tilde{\delta}_{L0}(\mathbf{k}'_1) \dots \tilde{\delta}_{L0}(\mathbf{k}'_\ell) \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle$, with ℓ low wave numbers and n high wave numbers, as a product of ℓ linear power spectra $P_{L0}(k'_j)$ with the small-scale nonlinear correlation $\langle \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle$, at lowest order over k'_j . Because this corresponds to a uniform displacement, this leading-order result vanishes at equal times, $t_1 = \dots = t_n = t$, and the results obtained for different times $t_1 \neq \dots \neq t_n$ simply describe how small-scale patches have moved in between these times because of the force exerted by a large-scale perturbation.

In this paper, we go beyond the kinematic effect recalled above and we consider the effect of a nonzero large-scale density fluctuation, that is, a nonzero curvature of the gravitational potential. This higher-order effect does not vanish for equal-time statistics because the large-scale perturbation of the gravitational potential curvature leads to a deformation of the small-scale structure (mostly a space-time dilatation, as the overall collapse is accelerated or decelerated). This leads to consistency relations for density correlations that remain nontrivial for single-time correlations. To remove constant gradients, which are absorbed by the kinematic effect and do not contribute to equal-time statistics, and to mimic a constant large-scale density fluctuation (and isotropic curvature of the gravitational potential), we consider spherical averages that read in configuration space as

$$C_W^n = \int d\mathbf{x}' W(x') \langle \delta_{L0}(\mathbf{x}') \delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n) \rangle, \quad (15)$$

and in Fourier space as

$$\tilde{C}_W^n = (2\pi)^3 \int d\mathbf{k}' \tilde{W}(k') \langle \tilde{\delta}_{L0}(\mathbf{k}') \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle, \quad (16)$$

where $W(x')$ [and its Fourier transform $\tilde{W}(k')$] is a large-scale window function. Using Eq. (14) and its configuration-space counterpart, Eqs. (15) and (16) read as

$$C_W^n = \int d\mathbf{x} d\mathbf{x}' W(x) C_{L0}(\mathbf{x}, \mathbf{x}') \left\langle \frac{\mathcal{D}[\delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n)]}{\mathcal{D}\delta_{L0}(\mathbf{x}')} \right\rangle, \quad (17)$$

where C_{L0} is the linear density correlation of the initial conditions, and

$$\tilde{C}_W^n = (2\pi)^3 \int d\mathbf{k}' \tilde{W}(k') P_{L0}(k') \left\langle \frac{\mathcal{D}[\tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n)]}{\mathcal{D}\tilde{\delta}_{L0}(-\mathbf{k}')} \right\rangle. \quad (18)$$

By definition of the functional derivatives, these expressions also mean that we must consider the change of the small-scale density correlation at linear order over a perturbation $\Delta\delta_{L0}$, as

$$C_W^n = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \langle \delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n) \rangle_\epsilon \quad (19)$$

and

$$\tilde{C}_W^n = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \langle \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle_\epsilon, \quad (20)$$

where $\langle \dots \rangle_\epsilon$ is the statistical average with respect to the Gaussian initial conditions δ_{L0} , when the linear density field is modified as

$$\delta_L(\mathbf{x}) \rightarrow \delta_L(\mathbf{x}) + \epsilon D_+(t) \int d\mathbf{x}' W(x') C_{L0}(\mathbf{x}, \mathbf{x}') \quad (21)$$

or

$$\tilde{\delta}_L(\mathbf{k}) \rightarrow \tilde{\delta}_L(\mathbf{k}) + \epsilon D_+(t) (2\pi)^3 \tilde{W}(k) P_{L0}(k). \quad (22)$$

Here and in the following, we normalize the linear growth rate and the linear density field as $\delta_L(\mathbf{x}, t) = D_+(t) \delta_{L0}(\mathbf{x})$. The spherical average over a much larger scale than the region of interest of size R in Eq. (21) ensures that over this small patch the density perturbation $\Delta\delta_{L0} \approx \epsilon \int d\mathbf{x}' W(x') C_{L0}(x')$ is constant (at leading order over $k'R$).

A similar idea was investigated in Ref. [41] in the context of single-field inflation, where it was noticed that the effect of a large-scale fluctuation is similar to changing the curvature of the universe, from the point of view of a small-scale region. However, this leads to a consistency relation between a $(1+n)$ correlation such as Eq. (15) and a small-scale n -point correlation in a different universe. As such, it cannot be directly measured because we have access to only one universe (unless one compares different large-scale regions characterized by different large-scale mean densities). In this paper, focusing on the late-time universe during the matter and dark energy epochs, we show in the next section how the approximate symmetry recalled in Sec. II allows us to derive consistency relations between correlations measured in the same universe. This is because this symmetry provides a link between the cosmological gravitational dynamics in different Friedmann-Lemaître-Robertson-Walker cosmologies.

B. Effect of a large-scale density perturbation

From the point of view of a small region, a much larger-scale almost uniform perturbation to the initial density contrast is similar to a change of the background density $\bar{\rho}$. Then, following Ref. [31], we first recall that such a small change of the background also corresponds to a linear growing mode of the density contrast. Thus, we consider two universes with close cosmological parameters, defined at the background level by the functions $\{\bar{\rho}(t), a(t)\}$ and $\{\bar{\rho}'(t), a'(t)\}$. The dynamics of the reference universe (i.e., our Universe) is given by the Friedmann equations,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\mathcal{G}}{3} (\bar{\rho} + \bar{\rho}_\Lambda) - \frac{K}{a^2}, \quad (23)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi\mathcal{G}}{3} \bar{\rho} + \frac{8\pi\mathcal{G}}{3} \bar{\rho}_\Lambda, \quad (24)$$

where we included the contributions from a cosmological constant and a curvature term and the dot denotes a derivative with respect to the time t . The auxiliary universe $\{\bar{\rho}'(t), a'(t)\}$, denoted with a prime, obeys the same equations with the change $\{\bar{\rho}, a, K\} \rightarrow \{\bar{\rho}', a', K'\}$ (the constant dark energy density $\bar{\rho}_\Lambda$ is not changed). It only differs from the reference universe by a small amount, of order ϵ , with

$$\begin{aligned} \bar{\rho} a^3 &= \bar{\rho}' a'^3 = \bar{\rho}_0, & a' &= a[1 - \epsilon(t)], \\ \bar{\rho}' &= \bar{\rho}[1 + 3\epsilon(t)]. \end{aligned} \quad (25)$$

Here and in the following, we only keep terms up to linear order over ϵ . Substituting Eq. (25) into the second Friedmann equation (24) written for the auxiliary universe, we obtain

$$\ddot{\epsilon} + 2H\dot{\epsilon} - 4\pi\mathcal{G}\bar{\rho}\epsilon = 0. \quad (26)$$

As is well known [31], we recover the evolution equation (4) of the linear growth rates $D_{\pm}(t)$. This is because spherically symmetric shells evolve independently as separate universes (before shell crossing), thanks to Birkhoff's theorem, and their density difference behaves as the linear growth rate (in the linear regime). Thus, we write

$$\epsilon(t) = D_+(t)\epsilon_0. \quad (27)$$

We now turn to the density and velocity fluctuations. In the reference universe, they follow the equations of motion (1)–(3). In the auxiliary universe, we have the same equations of motion with primed variables. For our purpose, these two sets of variables actually describe the same physical system, with two different choices for the background density $\bar{\rho}$ around which we study fluctuations. For instance, in the case $\epsilon > 0$, a density contrast δ' with a zero mean in the primed frame appears as a density contrast δ with a nonzero positive mean. Thus, a large-scale uniform density fluctuation $\Delta\delta_{L0}$ in the reference frame is absorbed by going to the primed frame. This will allow us to study the effect of a large-scale density fluctuation as in Eqs. (21)–(22).

Because both frames refer to the same physical system, we have

$$\mathbf{r}' = \mathbf{r} = a'\mathbf{x}' = a\mathbf{x}, \quad \bar{\rho}'(1 + \delta') = \bar{\rho}(1 + \delta), \quad (28)$$

where $\mathbf{r} = \mathbf{r}'$ is the physical coordinate. Thus, we have the relations

$$\begin{aligned} \mathbf{x}' &= (1 + \epsilon)\mathbf{x}, & \delta' &= \delta - 3\epsilon(1 + \delta), \\ \mathbf{v}' &= \mathbf{v} + \dot{\epsilon}a\mathbf{x}, \end{aligned} \quad (29)$$

where we used Eq. (25) and only kept terms up to linear order over ϵ . Then, we can check that if the fields $\{\delta', \mathbf{v}', \phi'\}$ satisfy the equations of motion (1)–(3) in the primed frame, the fields $\{\delta, \mathbf{v}, \phi\}$ satisfy the equations of motion (1)–(3) in the unprimed frame, with the gravitational potential transforming as

$$\phi' = \phi - a^2(\ddot{\epsilon} + 2H\dot{\epsilon})x^2/2. \quad (30)$$

This remains valid beyond shell crossing: if the trajectories $\mathbf{x}'(\mathbf{q}, t)$ satisfy the equation of motion (9) in the primed frame, the trajectories $\mathbf{x}(\mathbf{q}, t) = (1 - \epsilon)\mathbf{x}'(\mathbf{q}, t)$ satisfy the equation of motion (9) in the unprimed frame, with the gravitational potentials transforming as in Eq. (30).

Linearizing over the density contrast, the peculiar velocity, and the perturbation ϵ , we have

$$\begin{aligned} \delta_L &= \delta_L' + 3\epsilon, & \mathbf{v}_L &= \mathbf{v}_L' - \dot{\epsilon}a\mathbf{x}, \\ \phi_L &= \phi_L' + a^2(\ddot{\epsilon} + 2H\dot{\epsilon})\frac{x^2}{2}. \end{aligned} \quad (31)$$

In agreement with the remark above, we can again check that if $\{\delta_L', \mathbf{v}_L', \phi_L'\}$ is a valid linear growing mode in the primed frame, $\{\delta_L, \mathbf{v}_L, \phi_L\}$ is a valid linear growing mode in the unprimed frame. Moreover, the density contrast δ is equal to δ' , up to the dilatation (29), to which is added the uniform contribution $3\epsilon(t)$ that corresponds to a homogeneous linear growing mode, as seen from Eq. (27).

We can now compute the dependence of small-scale density correlations on ϵ_0 , that is, on changes of the background density. Thus, we consider the response function

$$R_{\epsilon_0}^n = \left\langle \frac{\partial[\tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n)]}{\partial \epsilon_0} \right\rangle_{\epsilon_0=0}. \quad (32)$$

As described above, adding a nonzero background ϵ corresponds to changing the initial background density from the reference $\bar{\rho}$ to the primed density $\bar{\rho}'$. This modifies the growth of large-scale structures, as the latter evolve in a new cosmology, defined by a new set of cosmological parameters. In particular, starting from a concordance Λ CDM flat cosmology with $\Omega_m + \Omega_\Lambda = 1$, the change of the background density generates a curvature term K/a^2 . For a given set of initial conditions δ_{L0} , the new field δ_{ϵ_0} , measured in the reference frame with the added background ϵ , can be expressed in terms of the density contrast δ' in the primed frame, where ϵ has been absorbed by the change $\bar{\rho} \rightarrow \bar{\rho}'$, through the mapping (29). This gives

$$\delta_{\epsilon_0}(\mathbf{x}, t) = (1 + 3\epsilon)\delta'[(1 + \epsilon)\mathbf{x}, t] + 3\epsilon, \quad (33)$$

which reads in Fourier space as

$$\tilde{\delta}_{\epsilon_0}(\mathbf{k}, t) = \tilde{\delta}'[(1 - \epsilon)\mathbf{k}, t] + 3\epsilon\delta_D(\mathbf{k}). \quad (34)$$

Next, we use the approximate symmetry described in Sec. II to write that the density contrast only depends on the cosmological parameters through the linear growth rate $D_+(t)$, whence $\tilde{\delta}'(\mathbf{k}, t; \Omega_m', \Omega_\Lambda', \Omega_K') = \tilde{\delta}(\mathbf{k}, D_+'(t))$, where $\tilde{\delta}(\mathbf{k}, D_+)$ is the functional that gives the nonlinear density contrast for any set of cosmological parameters, for a given initial condition of the zero-mean linear density contrast. Thus, Eq. (34) reads as

$$\tilde{\delta}_{\epsilon_0}(\mathbf{k}, t) = \tilde{\delta}[(1 - \epsilon)\mathbf{k}, D_{+\epsilon_0}] + 3\epsilon\delta_D(\mathbf{k}), \quad (35)$$

where $D_{+\epsilon_0}$ is the linear growth rate that is modified with respect to the initial D_+ by the perturbation ϵ . Then, the derivative of the density contrast with respect to ϵ_0 reads as

$$\left. \frac{\partial \tilde{\delta}(\mathbf{k}, t)}{\partial \epsilon_0} \right|_{\epsilon_0=0} = \left. \frac{\partial D_{+\epsilon_0}}{\partial \epsilon_0} \right|_0 \frac{\partial \tilde{\delta}}{\partial D_+} - D_+(t)\mathbf{k} \cdot \frac{\partial \tilde{\delta}}{\partial \mathbf{k}}, \quad (36)$$

where we disregarded the Dirac factor that does not contribute for wave numbers $\mathbf{k} \neq 0$.

We need to compute the dependence of the linear growing mode $D_{+\epsilon_0}$ on ϵ_0 . The linear growth rates D_+ and D'_+ obey Eq. (4), with unprimed and primed Hubble and density factors. Writing $D'_+(t) = D_+(t) + y(t)$, where y is of order ϵ , we obtain at linear order

$$\ddot{y} + 2H\dot{y} - 4\pi\mathcal{G}\bar{\rho}y = 2\dot{D}_+\dot{\epsilon} + 12\pi\mathcal{G}\bar{\rho}D_+\epsilon. \quad (37)$$

By definition of the matter density cosmological parameter Ω_m , the mean density also obeys

$$4\pi\mathcal{G}\bar{\rho} = \frac{3}{2}\Omega_m H^2 \simeq \frac{3\dot{D}_+^2}{2D_+^2}, \quad (38)$$

where in the last expression we again used the approximation $\Omega_m \simeq f^2$ associated with the approximated symmetry discussed in Sec. II. Then, using $\eta = \ln D_+$ as the time coordinate, Eq. (37) becomes

$$\frac{d^2 y}{d\eta^2} + \frac{1}{2}\frac{dy}{d\eta} - \frac{3}{2}y = \frac{13}{2}\epsilon_0 e^{2\eta}, \quad (39)$$

which gives

$$y(t) = \frac{13}{7}\epsilon_0 D_+(t)^2, \quad \left. \frac{\partial D_{+\epsilon_0}}{\partial \epsilon_0} \right|_0 = \frac{13}{7}D_+(t)^2. \quad (40)$$

This result was also obtained in Appendix D of Ref. [42]. Then, Eq. (36) also reads as

$$\left. \frac{\partial \tilde{\delta}(\mathbf{k}, t)}{\partial \epsilon_0} \right|_{\epsilon_0=0} = D_+(t) \left[\frac{13}{7} \frac{\partial \tilde{\delta}}{\partial \ln D_+} - \mathbf{k} \cdot \frac{\partial \tilde{\delta}}{\partial \mathbf{k}} \right], \quad (41)$$

which corresponds in configuration space to

$$\left. \frac{\partial \delta(\mathbf{x}, t)}{\partial \epsilon_0} \right|_{\epsilon_0=0} = D_+(t) \left[3\delta + \frac{13}{7} \frac{\partial \delta}{\partial \ln D_+} + \mathbf{x} \cdot \frac{\partial \delta}{\partial \mathbf{x}} \right]. \quad (42)$$

Equation (42) also follows from Eq. (33), where we disregard the constant factor 3ϵ because we consider small-scale wave numbers $k \neq 0$. In configuration space, this means that these relations are valid up to a constant density, which is irrelevant because we consider small-scale structures and disregard zero-mode (infinitely large-scale) normalizations.

This gives the impact of a large-scale uniform density perturbation, or a change of the background density, on the small-scale nonlinear density field. Indeed, from Eq. (31), the variable ϵ_0 corresponds to a change of the linear density contrast of

$$\Delta\delta_{L0} = 3\epsilon_0. \quad (43)$$

C. Consistency relations

1. One large-scale wave number

The comparison of Eq. (43) with Eq. (21) gives

$$\epsilon_0 = \frac{\epsilon}{3} \int d\mathbf{x}' W(x') C_{L0}(x'), \quad (44)$$

where we used the fact that W is a large-scale window and the integral over $\mathbf{x}'$ is independent of the position $\mathbf{x}$ in the small-scale region, at leading order in the ratio of scales. This gives

$$C_W^n = \frac{1}{3} \int d\mathbf{x}' W(x') C_{L0}(x') \frac{\partial \langle \delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n) \rangle_{\epsilon_0}}{\partial \epsilon_0}, \quad (45)$$

and using Eq. (42), we obtain

$$\begin{aligned} C_W^n &\rightarrow \int d\mathbf{x}' W(x') C_{L0}(x') \\ &\times \sum_{i=1}^n \frac{D_{+i}}{3} \left(3 + \frac{13}{7} \frac{\partial}{\partial \ln D_{+i}} + \mathbf{x}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} \right) \\ &\times \langle \delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n) \rangle. \end{aligned} \quad (46)$$

The small-scale correlation $\langle \delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n) \rangle$ is invariant through translations, thanks to statistical homogeneity. However, the dilatation operators $\mathbf{x}_i \cdot \partial / \partial \mathbf{x}_i$ break this invariance when the times t_i are not identical. Indeed, as described in Sec. III B, the change of the background density $\bar{\rho}$ due to the uniform density fluctuation (43) leads to a modified Hubble flow. This breaks the translation invariance for different-time statistics, as defining a different Hubble flow selects the origin from which comoving particles separate along with the global expansion. This is due to the large-scale approximation for the filter W where we considered that the small-scale region has a zero width at the center of the larger-scale fluctuation. To explicitly enforce this configuration, we set the center of the modified Hubble flow at the center of the small-scale region by writing

$$\begin{aligned} C_W^n &= \int d\mathbf{x}' W(x') C_{L0}(x') \sum_{i=1}^n \frac{D_{+i}}{3} \left[3 + \frac{13}{7} \frac{\partial}{\partial \ln D_{+i}} \right. \\ &\quad \left. + \left(\mathbf{x}_i - \frac{1}{n} \sum_{j=1}^n \mathbf{x}_j \right) \cdot \frac{\partial}{\partial \mathbf{x}_i} \right] \langle \delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n) \rangle, \end{aligned} \quad (47)$$

which is explicitly invariant through uniform translations of $\{\mathbf{x}_i\}$, as all terms only depend on relative distances.

In agreement with the remark above, Eq. (47) is identical to Eq. (46) when all times are equal, $D_{+1} = \dots = D_{+n}$. Using the definition (15), this gives the configuration-space consistency relation

$$\begin{aligned} & \int d\mathbf{x}' W(\mathbf{x}') \langle \delta_{L0}(\mathbf{x}') \delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n) \rangle \\ &= \int d\mathbf{x}' W C_{L0}(\mathbf{x}') \sum_{i=1}^n \frac{D_{+i}}{3} \left[3 + \frac{13}{7} \frac{\partial}{\partial \ln D_{+i}} \right. \\ & \quad \left. + \left(\mathbf{x}_i - \frac{1}{n} \sum_{j=1}^n \mathbf{x}_j \right) \cdot \frac{\partial}{\partial \mathbf{x}_i} \right] \\ & \quad \times \langle \delta(\mathbf{x}_1, t_1) \dots \delta(\mathbf{x}_n, t_n) \rangle. \end{aligned} \quad (48)$$

As explained above, this relation holds in the large-scale limit for the filter W , and up to uniform offsets for the density contrasts δ_i [i.e., the equality is valid when one integrates both sides with arbitrary weights $W_i(\mathbf{x}_i)$ that have a zero mean, $\int d\mathbf{x}_i W_i(\mathbf{x}_i) = 0$].

It is often more convenient to work in Fourier space (because the linearized equations of motion become diagonal). Because of statistical homogeneity, multispectra contain a Dirac factor that we explicitly factor out by defining

$$\langle \tilde{\delta}(\mathbf{k}_1) \dots \tilde{\delta}(\mathbf{k}_n) \rangle = \langle \tilde{\delta}(\mathbf{k}_1) \dots \tilde{\delta}(\mathbf{k}_n) \rangle' \delta_D(\mathbf{k}_1 + \dots + \mathbf{k}_n). \quad (49)$$

To simplify the analysis, it is convenient to consider $\langle \tilde{\delta}_1 \dots \tilde{\delta}_n \rangle'$ in Eq. (49) as a function of $\{\mathbf{k}_1, \dots, \mathbf{k}_{n-1}\}$ only, by substituting for $\mathbf{k}_n = -(\mathbf{k}_1 + \dots + \mathbf{k}_{n-1})$. Using the invariance through translations of $\langle \tilde{\delta}_1 \dots \tilde{\delta}_n \rangle$, which gives $\sum_{i=1}^n \partial/\partial \mathbf{x}_i \cdot \langle \tilde{\delta}_1 \dots \tilde{\delta}_n \rangle = 0$, we can write the dilatation factors of Eq. (48) (denoted as the overall operator D , without the factor $1/3$) as

$$\begin{aligned} D \cdot \langle \delta_1 \dots \delta_n \rangle &= \left\{ \sum_{i=1}^{n-1} \left[D_{+i}(\mathbf{x}_i - \mathbf{x}_n) + \frac{D_{+n} - D_{+i}}{n} \right. \right. \\ & \quad \left. \left. \times \sum_{j=1}^{n-1} (\mathbf{x}_j - \mathbf{x}_n) \right] \cdot \frac{\partial}{\partial \mathbf{x}_i} \right\} \langle \delta_1 \dots \delta_n \rangle. \end{aligned} \quad (50)$$

Using the Fourier transform of the density correlation as in Eq. (49) and integrating over $\mathbf{k}_n$, this yields

$$\begin{aligned} D \cdot \langle \delta_1 \dots \delta_n \rangle &= \int d\mathbf{k}_1 \dots d\mathbf{k}_{n-1} \left\{ \left[\sum_{i=1}^{n-1} D_{+i} \mathbf{k}_i \cdot \frac{\partial}{\partial \mathbf{k}_i} \right. \right. \\ & \quad \left. \left. + \sum_{i,j=1}^{n-1} \frac{D_{+n} - D_{+i}}{n} \mathbf{k}_i \cdot \frac{\partial}{\partial \mathbf{k}_j} \right] e^{i \sum_{i=1}^{n-1} \mathbf{k}_i \cdot (\mathbf{x}_i - \mathbf{x}_n)} \right\} \\ & \quad \times \langle \tilde{\delta}_1 \dots \tilde{\delta}_n \rangle'. \end{aligned} \quad (51)$$

Integrating by parts, using $\partial/\partial \mathbf{k}_n \cdot \langle \tilde{\delta}_1 \dots \tilde{\delta}_n \rangle' = 0$ and $\mathbf{k}_1 + \dots + \mathbf{k}_n = 0$, this also reads as

$$\begin{aligned} D \cdot \langle \delta_1 \dots \delta_n \rangle &= - \int d\mathbf{k}_1 \dots d\mathbf{k}_{n-1} e^{i \sum_{i=1}^{n-1} \mathbf{k}_i \cdot (\mathbf{x}_i - \mathbf{x}_n)} \\ & \quad \times \left[3 \frac{n-1}{n} \sum_{i=1}^n D_{+i} + \sum_{i,j=1}^n \left(\delta_{i,j}^K - \frac{1}{n} \right) \right. \\ & \quad \left. \times D_{+i} \mathbf{k}_i \cdot \frac{\partial}{\partial \mathbf{k}_j} \right] \langle \tilde{\delta}_1 \dots \tilde{\delta}_n \rangle', \end{aligned} \quad (52)$$

where $\delta_{i,j}^K$ is the Kronecker symbol. Therefore, Eq. (48) reads in Fourier space as

$$\begin{aligned} & \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} \langle \tilde{\delta}_{L0}(\mathbf{k}') \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle'_{k' \rightarrow 0} \\ &= P_{L0}(k') \sum_{i=1}^n D_{+i} \left[\frac{1}{n} + \frac{13}{21} \frac{\partial}{\partial \ln D_{+i}} \right. \\ & \quad \left. - \sum_{j=1}^n \left(\delta_{i,j}^K - \frac{1}{n} \right) \frac{\mathbf{k}_i}{3} \cdot \frac{\partial}{\partial \mathbf{k}_j} \right] \\ & \quad \times \langle \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle', \end{aligned} \quad (53)$$

where $\Omega_{\mathbf{k}'}$ is the unit vector along the direction of $\mathbf{k}'$. On large scales we recover the linear theory, with $\tilde{\delta}(\mathbf{k}', t') \simeq D_+(t') \tilde{\delta}_{L0}(\mathbf{k}')$. Thus, Eq. (53) also reads as

$$\begin{aligned} & \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} \langle \tilde{\delta}(\mathbf{k}', t') \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle'_{k' \rightarrow 0} \\ &= P_L(k', t') \sum_{i=1}^n \frac{D_{+i}}{D_+(t')} \left[\frac{1}{n} + \frac{13}{21} \frac{\partial}{\partial \ln D_{+i}} \right. \\ & \quad \left. - \sum_{j=1}^n \left(\delta_{i,j}^K - \frac{1}{n} \right) \frac{\mathbf{k}_i}{3} \cdot \frac{\partial}{\partial \mathbf{k}_j} \right] \\ & \quad \times \langle \tilde{\delta}(\mathbf{k}_1, t_1) \dots \tilde{\delta}(\mathbf{k}_n, t_n) \rangle'. \end{aligned} \quad (54)$$

Because we wrote the operator that acts over $\langle \tilde{\delta}_1 \dots \tilde{\delta}_n \rangle'$ in a symmetric form, in Eqs. (53)–(54) we can use any appropriate form for $\langle \tilde{\delta}_1 \dots \tilde{\delta}_n \rangle'$ [i.e., we can write the n -point correlation as a function of $\{\mathbf{k}_1, \dots, \mathbf{k}_{i-1}, \mathbf{k}_{i+1}, \dots, \mathbf{k}_n\}$, as $\mathbf{k}_i$ can be replaced by $-(\mathbf{k}_1 + \dots + \mathbf{k}_{i-1} + \mathbf{k}_{i+1} + \dots + \mathbf{k}_n)$ for any index i , or keep it as a function of the n wave numbers $\{\mathbf{k}_1, \dots, \mathbf{k}_n\}$, because of the constraint $\mathbf{k}_1 + \dots + \mathbf{k}_n = 0$].

In contrast with the kinematic consistency relations that express the transport of small-scale structures by large-scale fluctuations [24–30], the angular-averaged relations (54) do not vanish when all times are equal. Indeed, they go beyond this kinematic effect and express the deformation of small-scale structures by a large-scale isotropic curvature of the gravitational potential. When all times are equal, $t' = t_1 = \dots = t_n = t$, Eq. (54) becomes

$$\begin{aligned}
& \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} \langle \tilde{\delta}(\mathbf{k}', t) \tilde{\delta}(\mathbf{k}_1, t) \dots \tilde{\delta}(\mathbf{k}_n, t) \rangle'_{k' \rightarrow 0} \\
& = P_L(k', t) \left[1 + \frac{13}{21} \frac{\partial}{\partial \ln D_+} - \frac{1}{3} \sum_{i=1}^n \frac{\partial}{\partial \ln k_i} \right] \\
& \quad \times \langle \tilde{\delta}(\mathbf{k}_1, t) \dots \tilde{\delta}(\mathbf{k}_n, t) \rangle', \quad (55)
\end{aligned}$$

where we used $\mathbf{k}_1 + \dots + \mathbf{k}_n = 0$.

A nice property of the single-time consistency relation (55) is that it only involves single-time correlations on both sides (as opposed, for instance, to a relation that would involve the partial time derivative with respect to only one time t_i on the right-hand side). Moreover, thanks to the approximate symmetry discussed in Sec. II, both sides involve density correlations in the same (our) universe. This is because, although the effect of a large-scale curvature is similar to a change of cosmological parameters, the approximate symmetry allows us to express the density correlations in the modified cosmology in terms of the correlations measured in the original universe, through a rescaling of space and time coordinates. Therefore, the angular-averaged consistency relations (55) can be measured and tested in our Universe. However, this requires measuring the evolution with time of the density correlations to estimate the time derivative on the right-hand side.

2. Bispectrum

In practice, one does not measure density correlations up to very high orders, which become increasingly noisy, and most observational constraints from density correlations come from the two-point and three-point correlations. This corresponds in Fourier space to the power spectrum $P(k_1; t_1, t_2) = \langle \tilde{\delta}(\mathbf{k}_1, t_1) \tilde{\delta}(\mathbf{k}_2, t_2) \rangle'$ and bispectrum $B(k_1, k_2, k_3; t_1, t_2, t_3) = \langle \tilde{\delta}(\mathbf{k}_1, t_1) \tilde{\delta}(\mathbf{k}_2, t_2) \tilde{\delta}(\mathbf{k}_3, t_3) \rangle'$. (In practice, one measures single-time statistics, but for completeness we also consider the different-time statistics.) Taking into account the constraint $\mathbf{k}' + \mathbf{k}_1 + \mathbf{k}_2 = 0$ by writing $\mathbf{k}_1 = \mathbf{k} - \mathbf{k}'/2$ and $\mathbf{k}_2 = -\mathbf{k} - \mathbf{k}'/2$, with some arbitrary wave number $\mathbf{k}$, Eq. (54) reads for $n = 2$ as

$$\begin{aligned}
& \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} B\left(\mathbf{k}', \mathbf{k} - \frac{\mathbf{k}'}{2}, -\mathbf{k} - \frac{\mathbf{k}'}{2}; t', t_1, t_2\right)_{k' \rightarrow 0} \\
& = \frac{P_L(k', t')}{D_+(t')} \left[\frac{D_{+1} + D_{+2}}{2} \left(1 - \frac{1}{3} \frac{\partial}{\partial \ln k} \right) \right. \\
& \quad \left. + \frac{13}{21} \left(D_{+1}^2 \frac{\partial}{\partial D_{+1}} + D_{+2}^2 \frac{\partial}{\partial D_{+2}} \right) \right] P(k; t_1, t_2). \quad (56)
\end{aligned}$$

When all times are equal to t , this becomes, in agreement with Eq. (55),

$$\begin{aligned}
& \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} B\left(\mathbf{k}', \mathbf{k} - \frac{\mathbf{k}'}{2}, -\mathbf{k} - \frac{\mathbf{k}'}{2}; t\right)_{k' \rightarrow 0} \\
& = P_L(k', t) \left[1 + \frac{13}{21} \frac{\partial}{\partial \ln D_+} - \frac{1}{3} \frac{\partial}{\partial \ln k} \right] P(k, t). \quad (57)
\end{aligned}$$

3. Several large-scale wave numbers

It is possible to generalize the single-time consistency relation (55) to ℓ large-scale wave numbers by an iterative procedure, as long as they follow a hierarchy $k_1' \ll k_2' \ll \dots \ll k_{\ell}'$, because the angular average and the derivative $\partial/\partial \ln k$ commute. This gives

$$\begin{aligned}
& k_j' \ll k_{j+1}' : \int \prod_{j=1}^{\ell} \frac{d\Omega_{\mathbf{k}_j'}}{4\pi} \left\langle \prod_{j=1}^{\ell} \tilde{\delta}(\mathbf{k}_j') \prod_{i=1}^n \tilde{\delta}(\mathbf{k}_i) \right\rangle'_{k_j \rightarrow 0} \\
& = \mathcal{L}_1' \dots \mathcal{L}_{\ell}' \cdot \langle \tilde{\delta}(\mathbf{k}_1) \dots \tilde{\delta}(\mathbf{k}_n) \rangle', \quad (58)
\end{aligned}$$

where the operators $\mathcal{L}_j'$ read as

$$\begin{aligned}
\mathcal{L}_j' = P_L(k_j', t) & \left[1 + \frac{13}{21} \frac{\partial}{\partial \ln D_+} - \frac{1}{3} \sum_{m=j+1}^{\ell} \frac{\partial}{\partial \ln k_m'} \right. \\
& \quad \left. - \frac{1}{3} \sum_{i=1}^n \frac{\partial}{\partial \ln k_i} \right]. \quad (59)
\end{aligned}$$

The operators $\mathcal{L}_j'$ do not commute. This comes from the fact that the relation (58) is obtained from the iterated use of Eq. (55), where the large-scale limits are taken in a specific order, starting with k_1' and finishing with k_{ℓ}' , in agreement with the hierarchy $k_1' \ll k_2' \ll \dots \ll k_{\ell}'$. Expanding the product $\mathcal{L}_1' \dots \mathcal{L}_{\ell}'$ gives increasingly long expressions as the number ℓ of large-scale modes grows and we do not pursue this matter here.

4. Multicomponent case

If there are several fluids in the system, for instance, when we separate dark matter and baryons, each component (α) follows the equations of motion (1)–(2) or (6)–(7), written in terms of each doublet $\{\delta^{(\alpha)}, \mathbf{v}^{(\alpha)}\}$ or $\{\delta^{(\alpha)}, \mathbf{u}^{(\alpha)}\}$. The gravitational potential φ now obeys the Poisson equation $\nabla^2 \varphi = (3\Omega_m)/(2f^2) \sum_{\alpha} (\Omega_{(\alpha)}/\Omega_m) \delta^{(\alpha)}$. Because all fluids are subject to the same gravitational potential, in the linear growing mode all density contrasts $\delta^{(\alpha)}$ and velocities $\mathbf{v}^{(\alpha)}$ are equal (and the mean density ratios $\Omega_{(\alpha)}/\Omega_m$ remain constant). Then, assuming as for the single-fluid case that decaying modes have had time to become negligible, the single-fluid consistency relations (54)–(58) extend at once to the multifluid case.

This is no longer true when some fluids are subject to nongravitational forces that introduce new scales. For instance, when pressure forces or astrophysical processes, such as outflows from supernovae, have an impact on the baryon dynamics, the invariance with respect to changes of

cosmological parameters (in the approximation $\Omega_m/f^2 \approx 1$) is broken. Indeed, these new processes generically introduce different explicit dependencies on density and time scales that cannot be reduced to functions of Ω_m/f^2 . Thus, our results only hold in the regime where gravity is the dominant force.

D. Comments on the derivation of the consistency relations

The consistency relations (54)–(55), and their derivation through the effect of large-scale perturbations in Sec. III B, may appear somewhat counterintuitive, and we comment in this section on some points that may seem puzzling.

As explained in Sec. III A, the consistency relations derive from the exact relation (14), which shows how the correlation between n small-scale nonlinear modes and a large-scale linear mode can be obtained from the mean response of the small-scale modes to the large-scale mode. Then, the procedure presented in Secs. III B and III C describes how to obtain this response by identifying the large-scale mode with a change of the background density.

This gives rise to Eqs. (33)–(34), which are the first nontrivial result (no approximation has been used at this stage). However, they may already raise several questions that we address in turn.

First, it might seem that Eqs. (33)–(34), which give the impact of a large-scale perturbation on the small-scale nonlinear density contrast, contradict linear theory where different Fourier modes are known to decouple. This is not the case, because in the linear regime these equations read as $\delta_{e_0} = \delta' + 3\epsilon$ for Eq. (33), as in the first expression of Eq. (31), and as $\tilde{\delta}_{e_0} = \tilde{\delta}' + 3\epsilon\delta_D(\mathbf{k})$. This merely means that in the linear regime different growing mode solutions superpose. Since e_0 corresponds to a uniform density shift, it gives rise to the Dirac factor in Eq. (34), and we recover the fact that in the linear regime different wave numbers are decoupled (hence a perturbation at $\mathbf{k}' \rightarrow 0$ only affects the same mode with $\mathbf{k} = \mathbf{k}' \rightarrow 0$). Therefore, there is no contradiction between Eqs. (33)–(34) and linear theory.

In the nonlinear regime, different wave numbers become coupled, in agreement with the rescaling and amplification found in the first terms on the right-hand sides of Eqs. (33)–(34). However, one may worry that a first-order expansion over ϵ may not be sufficient to evaluate the response of small-scale nonlinear density contrasts. For instance, let us consider a single one-dimensional plane wave for the small-scale initial perturbation. This fluctuation will collapse at some time t_* to build an infinite-density two-dimensional sheet, of zero thickness. If we add a small linear large-scale mode ϵ , the collapse time t_* will be changed to a slightly different value $t_* + \Delta t_*$. Then, for any nonzero ϵ and Δt_* , the impact on the densities at time t_* along the position of the pancake is not small and not proportional to ϵ (since it is the difference between infinity and a finite value). However, this is a pathological case that

appears with a zero probability: in this example it only occurs at the precise time t_* at the position of the pancake. At later times, there remains an infinite-density sheet in both the zero- ϵ and nonzero- ϵ cases, and the main change is to slightly modify its position and mass. This still gives a large deviation if we measure the density contrast at the precise position of the sheet, but this is no longer the case if we perform some smoothing over an arbitrarily small window or if we consider the Fourier-space density contrast $\tilde{\delta}(\mathbf{k})$ (which actually integrates over all space).

Here, it is more convenient to think in terms of the particle trajectories $\mathbf{x}(\mathbf{q}, t)$. Then, using the fact that the Fourier-space density contrast can be written as a function of the particle positions, in a form similar to the one-dimensional expression (74) used below, we can see that as long as particle trajectories display a first-order expansion over ϵ , a similar expansion holds for the Fourier-space density contrast, even though shell crossings and infinite-density sheets may have appeared. This point will become obvious in Sec. IV B, where we show that the consistency relations are actually exact in one dimension until shell crossing, and even remain exact after shell crossings if we consider the Zel'dovich dynamics itself instead of the one-dimensional gravitational dynamics (both systems only being identical before shell crossing).

In three dimensions, even more pathological examples can be found, where particle trajectories themselves are singular with respect to small perturbations. For instance, spherically symmetric solutions with purely radial trajectories are strongly unstable with respect to nonspherical perturbations and the linear growth rate actually diverges [43]. Then, infinitesimal perturbations are sufficient to initiate the virialization of the cloud and generate significant transverse motions. However, these cases again appear with a zero probability (initial conditions are not exactly spherically symmetric with purely radial motions) and they should not impair the derivation presented in Secs. III B and III C.

The identification of the response of the system to a large-scale mode with the response to a change of the background density might seem puzzling, as even a large-scale mode is constrained to have a zero mean. However, it is rather clear that from the point of view of a small-scale region located at the center of the larger-scale perturbation, the latter acts as a uniform change of the background density. This is explicitly shown by the fact that Eq. (26), which describes the evolution of the difference between two close backgrounds, is identical to the evolution equation (4) of the linear growth rates $D_{\pm}(t)$. This is because they follow from the same fundamental set of equations, that describe the gravitational force.

Explicit examples that also clarify these points are presented in the next section, where we check the consistency relations at lowest order of perturbation theory for the bispectrum in three dimensions, and at all orders for all polyspectra in one dimension.

IV. EXPLICIT CHECKS

The angular-averaged consistency relations (54)–(55) are valid at all orders of perturbation theory and also beyond the perturbative regime, including shell-crossing effects, within the accuracy of the approximation $\Omega_m/f^2 \simeq 1$ (and as long as gravity is the dominant process).

We now provide two explicit checks of the angular-averaged consistency relations (54)–(55). First, we check these relations for the lowest-order case $n = 2$, that is, for the bispectrum, at lowest order of perturbation theory. Second, we present a fully nonlinear and nonperturbative check, for arbitrary n -point polyspectra, in the one-dimensional case.

A. Perturbative check

Here we briefly check the consistency relations for the lowest order case, $n = 2$, given by Eqs. (56)–(57), at lowest order of perturbation theory. At this order, the density bispectrum reads as [10]

$$\begin{aligned} B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; t_1, t_2, t_3) &= D_{+1} D_{+2} D_{+3}^2 P_{L0}(k_1) P_{L0}(k_2) \\ &\times \left[\frac{10}{7} + \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} + \frac{4}{7} \left(\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2 \right] \\ &+ 2\text{perm.} \end{aligned} \quad (60)$$

where “2perm.” stands for two other terms that are obtained from permutations over the indices $\{1, 2, 3\}$. In the small- k' limit we obtain

$$\begin{aligned} B(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2; t', t_1, t_2)_{k' \rightarrow 0} &= D_+' D_{+1} D_{+2}^2 P_{L0}(k') P_{L0}(k_1) \\ &\times \left[\frac{10}{7} + \frac{\mathbf{k}_1 \cdot \mathbf{k}'}{k'^2} + \frac{4}{7} \left(\frac{\mathbf{k}_1 \cdot \mathbf{k}'}{k_1 k'} \right)^2 \right] + (1 \leftrightarrow 2). \end{aligned} \quad (61)$$

Here we used the fact that the term in the bracket in Eq. (60) vanishes as k_3^2 for $k_3 \rightarrow 0$, whereas $P_{L0}(k_3) \sim k_3^{n_s}$ with $n_s \lesssim 1$. [If this is not the case, that is, there is very little initial power on large scales, we must go back to the consistency relation in the form of Eq. (53) rather than Eq. (54). However, this is not necessary in realistic models.] Then, taking into account the constraint $\mathbf{k}' + \mathbf{k}_1 + \mathbf{k}_2 = 0$ by writing $\mathbf{k}_1 = \mathbf{k} - \mathbf{k}'/2$ and $\mathbf{k}_2 = -\mathbf{k} - \mathbf{k}'/2$ as in Eq. (56), and expanding Eq. (61) over k' , we obtain

$$\begin{aligned} B_{k' \rightarrow 0} &= D_+' D_{+1} D_{+2}^2 P_{L0}(k') \\ &\times \left[P_{L0}(k) \left(\frac{13 + 8\mu^2}{14} + \frac{k}{k'} \mu \right) \right. \\ &\left. - \frac{dP_{L0}(k)}{d \ln k} \frac{\mu^2}{2} \right] + (1 \leftrightarrow 2), \end{aligned} \quad (62)$$

where $\mu = (\mathbf{k} \cdot \mathbf{k}')/(kk')$. The integration over angles gives

$$\begin{aligned} \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} B_{k' \rightarrow 0} &= D_+' D_{+1} D_{+2} \frac{D_{+1} + D_{+2}}{2} P_{L0}(k') \\ &\times \left[\frac{47}{21} P_{L0}(k) - \frac{1}{3} \frac{dP_{L0}(k)}{d \ln k} \right]. \end{aligned} \quad (63)$$

On the other hand, the right-hand side of Eq. (56) reads as

$$\begin{aligned} \int \frac{d\Omega_{\mathbf{k}'}}{4\pi} B_{k' \rightarrow 0} &= D_+' P_{L0}(k') \left[\frac{D_{+1} + D_{+2}}{2} \left(1 - \frac{1}{3} \frac{\partial}{\partial \ln k} \right) \right. \\ &\left. + \frac{13}{21} \left(D_{+1}^2 \frac{\partial}{\partial D_{+1}} + D_{+2}^2 \frac{\partial}{\partial D_{+2}} \right) \right] \\ &\times D_{+1} D_{+2} P_{L0}(k) \end{aligned} \quad (64)$$

and we recover Eq. (63). This provides a check of Eq. (56) and also of the single-time relation (57), which is a particular case of Eq. (56). Alternatively, the same procedure applied to the single-time bispectrum provides a direct check of Eq. (57).

Therefore, we have checked the angular-averaged consistency relation (54) for the bispectrum, at leading order of perturbation theory, within the approximate symmetry $\Omega_m/f^2 \simeq 1$ discussed in Sec. II. In this explicit check, the use of this approximate symmetry appears at the level of the expression (60) of the bispectrum, which only involves the linear growing mode D_+ . An exact calculation would give prefactors for the terms in the bracket that show new but weak dependencies on time and cosmology (and that are unity for the Einstein–de Sitter case) [10]. These deviations from Eq. (60) are usually neglected [for instance, when the cosmological constant is zero, they were shown to be well approximated by factors like $(\Omega_m^{-2/63} - 1)$ that are very small over the range of interest [44]].

B. One-dimensional nonlinear check

The explicit check presented in Sec. IV A only applies up to the lowest order of perturbation theory. Because the goal of the consistency relations is precisely to go beyond low-order perturbation theory, it is useful to obtain a fully nonlinear check. This is possible in one dimension, where the Zel’dovich solution [45] becomes exact (before shell crossing) and all quantities can be explicitly computed. Because of the change of dimensionality, we also need to rederive the one-dimensional form of the consistency relations. We present the details of our computations in the Appendix, and only give the main steps in this section.

1. One-dimensional equations of motion

First, as described in Appendix A 1, by using the change of variables (A12) the one-dimensional equations of motion can be written as

$$\frac{\partial \delta}{\partial \eta} + \frac{\partial}{\partial x} [(1 + \delta)u] = 0, \quad (65)$$

$$\frac{\partial u}{\partial \eta} + [\kappa(t) - 1]u + u \frac{\partial u}{\partial x} = -\frac{\partial \varphi}{\partial x}, \quad (66)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \kappa(t)\delta, \quad (67)$$

where we introduced the factor $\kappa(t)$ defined by

$$\kappa(t) = 4\pi\mathcal{G}(t)\bar{\rho}(t)\frac{D_+(t)^2}{\dot{D}_+(t)^2}. \quad (68)$$

As explained in Appendix A 1, we generalized the system to the case of a time-dependent Newton's constant $\mathcal{G}(t)$. This allows us to obtain scale factors $a(t)$ that expand forever as power laws, as in Eq. (A6), in a fashion that mimics the Einstein–de Sitter three-dimensional cosmology.

Thus, $\kappa(t)$ plays the role of the ratio $3\Omega_m/(2f^2)$ encountered in the three-dimensional case in Eqs. (6)–(8). In the one-dimensional cosmology (A6) with $c_1 = 0$, it is a constant given by $\kappa_0 = -(\alpha + 2)/(\alpha + 1)$. Then, the three-dimensional approximation $\Omega_m/f^2 \simeq 1$ used in the main text corresponds in our one-dimensional toy model to the approximation $\kappa \simeq \kappa_0$. That is, we neglect the dependence of κ on the cosmological parameters (here the coefficient c_1) and the dependence on the background is fully contained in the change of variables (A12). This is the one-dimensional approximate symmetry that is the equivalent of the three-dimensional approximate symmetry used in the previous sections. The generalization to the case of a time-dependent Newton's constant is not important at a formal level, because it does not modify the form of the equations of motion. However, it is necessary for this approximate symmetry to make practical sense, so that we can find a regime where κ is approximately constant [here, around $c_1 = 0$ with the choice (A5)].

The fluid equations (65)–(67) only apply to the single-stream regime, but we can again go beyond shell crossings by using the equation of motion of trajectories, which reads as

$$\frac{\partial^2 x}{\partial \eta^2} + [\kappa(t) - 1]\frac{\partial x}{\partial \eta} = -\frac{\partial \varphi}{\partial x}, \quad (69)$$

where φ is the rescaled gravitational potential (67). This is the one-dimensional version of Eq. (10) and it explicitly shows that particle trajectories obey the same approximate symmetry, before and after shell crossings.

2. One-dimensional consistency relations

To derive the consistency relations, we can follow the method described in the previous sections for the three-dimensional case. In a fashion similar to the analysis of

Sec. III B, we first derive in Appendix A 2 the impact of a large-scale linear density perturbation on the small-scale nonlinear density field. This gives in Fourier space

$$\left. \frac{\partial \tilde{\delta}(k, t)}{\partial \epsilon_0} \right|_{\epsilon_0=0} = D_+(t) \left[\frac{\partial \tilde{\delta}}{\partial \ln D_+} - k \frac{\partial \tilde{\delta}}{\partial k} \right], \quad (70)$$

which corresponds in configuration space to

$$\left. \frac{\partial \delta(x, t)}{\partial \epsilon_0} \right|_{\epsilon_0=0} = D_+(t) \left[\delta + \frac{\partial \delta}{\partial \ln D_+} + x \frac{\partial \delta}{\partial x} \right]. \quad (71)$$

As expected, we recover the same forms as in Eqs. (41)–(42), but with different numerical coefficients because the dimension of space has changed.

To obtain the one-dimensional consistency relations, we follow the method described in Sec. III C. The factors of $1/3$ are replaced by unity and we use the result (70). Then, Eq. (54) becomes

$$\begin{aligned} & \frac{1}{2} \sum_{\pm k'} \langle \tilde{\delta}(k', t') \tilde{\delta}(k_1, t_1) \dots \tilde{\delta}(k_n, t_n) \rangle'_{k' \rightarrow 0} \\ &= P_L(k', t') \sum_{i=1}^n \frac{D_{+i}}{D_+(t')} \left[\frac{1}{n} + \frac{\partial}{\partial \ln D_{+i}} \right. \\ & \quad \left. - \sum_{j=1}^n \left(\delta_{i,j}^K - \frac{1}{n} \right) k_i \frac{\partial}{\partial k_j} \right] \\ & \quad \times \langle \tilde{\delta}(k_1, t_1) \dots \tilde{\delta}(k_n, t_n) \rangle'. \end{aligned} \quad (72)$$

The three-dimensional angular average $\int d\Omega_{\mathbf{k}'} / (4\pi)$ of Eq. (54) is replaced by the one-dimensional average $\frac{1}{2} \sum_{\pm k'}$ over the two directions of k' (i.e., the two signs of k'). When all times are equal, $t' = t_1 = \dots = t_n = t$, Eq. (55) becomes

$$\begin{aligned} & \frac{1}{2} \sum_{\pm k'} \langle \tilde{\delta}(k', t) \tilde{\delta}(k_1, t) \dots \tilde{\delta}(k_n, t) \rangle'_{k' \rightarrow 0} \\ &= P_L(k', t) \left[1 + \frac{\partial}{\partial \ln D_+} - \sum_{i=1}^n \frac{\partial}{\partial \ln k_i} \right] \\ & \quad \times \langle \tilde{\delta}(k_1, t) \dots \tilde{\delta}(k_n, t) \rangle'. \end{aligned} \quad (73)$$

3. One-dimensional explicit checks

As is well known, in one dimension the Zel'dovich approximation [45] is actually exact before shell crossing. We briefly check this property in Appendix A 3 on the generalized system (A7)–(A9), where Newton's constant can vary with time. Then, using the conservation of matter, $(1 + \delta)dx = dq$, the Fourier-space nonlinear density contrast can be written as [37,46]

$$\tilde{\delta}(k) = \int \frac{dx}{2\pi} e^{-ikx} \delta(x) = \int \frac{dq}{2\pi} e^{-ikx(q)} - \delta_D(k). \quad (74)$$

Using the solution Eq. (A26), and disregarding the Dirac term for $k \neq 0$, this gives the usual expression

$$\tilde{\delta}(k, t) = \int \frac{dq}{2\pi} e^{-ikq} + \int dk' e^{ik'q} \frac{k}{k'} \tilde{\delta}_L(k', t). \quad (75)$$

The explicit nonlinear expression (75) allows us to check the one-dimensional consistency relations (72)–(73). We present in Appendix A 4 two different checks.

First, in Appendix A 4 a, following the derivation of the consistency relations, we check that the impact of a large-scale perturbation on the nonlinear density contrast is given by Eq. (70). This is the key relation from which the consistency relations derive and it provides a first one-dimensional nonlinear check.

Second, in Appendix A 4 b, we directly check the consistency relations (72)–(73) from the explicit expressions of the density polyspectra, without going through the intermediate step that considers the impact of a large-scale linear perturbation on the small-scale nonlinear density contrast. Thus, this also provides a check of the reasoning that underlies the derivation of these relations.

4. One-dimensional nonlinear and nonperturbative validity

The explicit checks presented in Appendix A 4 show that the consistency relations (72)–(73) are actually exact in the one-dimensional case, before shell crossing. Here there is a simplification, as compared with the three-dimensional case, that makes the consistency relations exact beyond the approximation of constant κ . This is because the nonlinear density contrast (75) only depends on the linear growing mode $D_+(t)$ and the approximation of constant κ is not needed. This is also apparent in Eqs. (A23)–(A25) or Eq. (A27). For the solution (A26), the right-hand sides of Eqs. (A24) and (A27) cancel out and we obtain a dynamics that only involves the linear growing mode $D_+(t)$ as the rescaled time coordinate. Then, for this class of solutions the symmetry that is the basis of the consistency relations is actually exact (before shell crossing).

This one-dimensional toy model provides an explicit nonlinear check of the consistency relations. This also ensures that they are valid to all orders of perturbation theory. However, because the single-stream equations of motion (A7)–(A9) and the solution (75) only apply before shell crossing, which is a nonperturbative effect, this one-dimensional gravitational toy model does not provide an explicit check in terms of nonperturbative shell-crossing contributions.

On the other hand, we can also consider the Zel'dovich solution (A26), and the associated density contrast (75), as a second one-dimensional toy model (which is no longer

related to “gravitational” forces). This second system only coincides with the one-dimensional gravitational dynamics before shell crossing and departs from it afterwards, but it is also a well-defined system at all times. Then, we can apply the same reasoning that underlies the derivation of the consistency relations to this system, which satisfies the same symmetries. From this point of view, the solution (75) now provides an explicit nonlinear check of the consistency relations that also applies beyond shell crossing.

As noticed above, it happens that in this one-dimensional case the function $\kappa(t)$ does not appear in the solution (75), so that the approximation $\kappa \simeq \kappa_0$ is not needed and the consistency relations (and the underlying symmetry) are exact, up to shell crossing for the one-dimensional gravitational dynamics, and even beyond shell crossing when we consider the second toy model defined by the solution (A26). Therefore, the time dependence of the scale factor $a(t)$ and of $\kappa(t)$ is irrelevant and we could as well keep the usual case of a time-independent Newton’s constant $\mathcal{G}$, with a finite collapse time of the system. Nevertheless, it is nice to consider the generalized one-dimensional system (A5) that can mimic the usual three-dimensional cosmological expansion up to infinite time. Moreover, if we consider the true one-dimensional gravitational dynamics, which departs from the Zel’dovich solution (A26) after shell crossing, the function $\kappa(t)$ will appear in the nonperturbative shell-crossing terms, when the right-hand side of Eq. (A27) no longer cancels out. Then, the consistency relations are only approximate in the shell-crossing regime, up to the accuracy of the $\kappa \simeq \kappa_0$ approximation.

V. CONCLUSION

As explained in the previous sections, the angular-averaged consistency relations (54)–(55) only rely on the approximate symmetry discussed in Sec. II, which states that the dependence on cosmological parameters can be absorbed through the mapping $t \rightarrow D_+$ of the time coordinate (within the approximation $\Omega_m/f^2 \simeq 1$). Therefore, our results are not restricted to the perturbative regime and also apply to small nonlinear scales governed by shell-crossing effects, as long as the approximation $\Omega_m/f^2 \simeq 1$ is sufficiently accurate.

We have also pointed out that these relations are actually exact in one-dimensional gravitational systems until shell crossing, and even beyond shell crossing for the Zel’dovich dynamics itself (which departs from one-dimensional gravity after shell crossing).

It is difficult to extend these results to the galaxy number density field in a rigorous fashion, because galaxy formation is not expected to satisfy the approximate symmetry of Sec. II. For instance, cooling processes and star formation introduce new time and density scales, which means that the time-coordinate mapping $t \rightarrow D_+$ is not sufficient to absorb all cosmological dependence.

However, these effects are likely to be subdominant if we assume, as in halo models, that galaxies are closely related to the dark matter density field. Thus, in most analytical approaches, one writes the galaxy number density fluctuations as a functional of the matter density fluctuations at the same time, $\delta_g(\mathbf{x}, t) = \delta_g[\delta(\cdot, t)]$. In the popular local bias model, this is simplified as a function of the local density contrast, smoothed over some scale R , $\delta_g(\mathbf{x}, t) = \delta_g[\delta_R(\mathbf{x}, t)] = \sum_{n=1}^{\infty} \frac{b_n}{n!} \delta_R(\mathbf{x}, t)^n$. However, this introduces an explicit model dependence, especially as the transformation (29) modifies the amplitude of the density contrast, which leads to a change of the number of halos in the framework of halo models. Then, the simplest method to derive consistency relations for the galaxy distribution from the matter density relations (54)–(58) is to start from an explicit galaxy bias model, that allows one to express galaxy correlations in terms of matter correlations. Then, one can directly use Eqs. (54) and (58) and obtain constraints on the galaxy correlations. The simplest case is the constant linear bias model, where $\delta_g = b_1 \delta$, with a scale-independent bias b_1 that only depends on the galaxy type (mass, luminosity, etc.). Then, Eqs. (54) and (58) directly extend to the galaxy density field, up to a factor b_1 .

In more general models of galaxy clustering, one takes into account higher orders or keeps a functional dependence, $\delta_g(\mathbf{x}) = \delta_g[\delta(\cdot)]$. In particular, one can use some derivative expansion of this functional, so that the galaxy density field also involves the deformation tensor or higher-order derivatives of the smoothed density field (e.g., Ref. [47]). Then, the general relation (14), where the nonlinear matter density contrasts $\tilde{\delta}(\mathbf{k}_i, t_i)$ are replaced by the galaxy number density contrasts $\tilde{\delta}_g(\mathbf{k}, t_i)$, is still valid, because it only assumes that the field $\tilde{\delta}_g$ is a functional of the Gaussian field $\tilde{\delta}_{L0}$ [30]. This allows one to write again Eqs. (17) or (18) for the galaxy density field. Next, one can write the derivatives $\mathcal{D}\delta_g/\mathcal{D}\delta_{L0}$ in terms of derivatives of the matter density contrast δ , and use Eqs. (41) or (42). However, in the generic case this gives expressions that involve the matter density field and cannot be written back in terms of the galaxy density field in a simple manner.

A detection (beyond the range authorized by the finite accuracy of the approximation $\Omega_m/f^2 \simeq 1$) of a violation of Eqs. (54) and (58), written in terms of the galaxy number density if that is possible, would signal a breakdown of the underlying galaxy biasing scheme. To remove this degeneracy, one can also use weak gravitational lensing observations, which directly probe the matter density field.

On the other hand, these consistency relations rely on the Gaussian hypothesis for the initial conditions. Indeed, this assumption is used to derive Eq. (14) [30], which is the basis of subsequent relations. Then, a violation of the angular-averaged consistency relations could signal primordial non-Gaussianities. Other possible interpretations could be effects from nonzero decaying modes, or a

departure from the “standard” cosmological scenarios, for instance a modified-gravity model where the approximation $\Omega_m/f^2 \simeq 1$ is strongly violated or where new terms in the equations of motion show an explicit dependence on cosmology (on the background density) that does not have the form Ω_m/f^2 .

Apart from these observational aspects, these angular-averaged consistency relations (54)–(58) might be used as a check of numerical simulations or algorithms. From a more theoretical perspective, they could also help in designing models for the matter density correlations. In particular, perturbative approaches, which often explicitly use the approximation $\Omega_m/f^2 \simeq 1$ to simplify the analysis, attempt to go beyond the standard perturbation theory by including partial resummations of higher-order diagrams. They can also be seen as different closure schemes, where one implements different truncations of the infinite hierarchy between n - and $n+1$ -point density correlations. (The standard perturbation theory amounts to setting all correlations above some finite order N to zero, while resummation schemes can be seen as assuming a specific ansatz for the $N+1$ correlation, expressed in terms of the lower-order ones, to close the hierarchy at order N .) Then, Eqs. (54) and (58) might serve as a guideline to write this $N+1$ correlation in terms of the lower-order correlations. We leave such investigations to future works.

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APPENDIX: ONE-DIMENSIONAL EXAMPLE

It is interesting to check the consistency relations obtained in this paper by using a simple one-dimensional example that can be exactly solved. This is provided by the Zel’dovich dynamics [45], which is exact in one dimension (before shell crossing).

1. One-dimensional equations of motion

In physical coordinates, the one-dimensional continuity, Euler and Poisson equations, read as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho v) = 0, \quad (\text{A1})$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{\partial \Psi}{\partial r}, \quad (\text{A2})$$

$$\frac{\partial^2 \Psi}{\partial r^2} = 4\pi \mathcal{G}(t) \rho, \quad (\text{A3})$$

where $\rho(r, t)$ and $v(r, t)$ are the one-dimensional density and velocity fields. Here we generalized the one-dimensional

gravitational dynamics to the case of a time-dependent Newton's constant $\mathcal{G}(t)$. The background solution corresponds to $\nu = Hr$ and $\Psi = 2\pi\mathcal{G}\bar{\rho}r^2$, with $H = \dot{a}/a$. The one-dimensional continuity equation yields $\dot{\bar{\rho}} = \bar{\rho}_0/a$ and the Euler equation gives

$$\frac{\ddot{a}}{a} = -4\pi\mathcal{G}(t)\bar{\rho}, \quad (\text{A4})$$

which corresponds to the Friedman equation. In the three-dimensional case, the expansion of the universe gives $\bar{\rho} \propto a^{-3}$, which dilutes the gravitational attraction (as $1/r^2$), and there are ever-expanding solutions (without cosmological constant). In the one-dimensional case, we have $\bar{\rho} \propto a^{-1}$ and the gravitational force is not diluted by the expansion. (As is well known, in one dimension the gravitational force is constant and independent of the distance between particles before shell crossing.) Then, there are no ever-expanding solutions and the system collapses after a finite time. However, by generalizing to a time-dependent Newton's constant $\mathcal{G}(t)$, we can again obtain solutions that expand forever. Thus, we can consider the power-law models

$$-2 < \alpha < -1: \mathcal{G}(t) = \mathcal{G}_0 \left(\frac{t}{t_0} \right)^\alpha, \quad (\text{A5})$$

which lead to the expansion laws

$$a(t) = -\frac{4\pi\mathcal{G}_0\bar{\rho}_0 t_0^2}{(\alpha+1)(\alpha+2)} \left(\frac{t}{t_0} \right)^{\alpha+2} + c_1 t, \quad (\text{A6})$$

where c_1 is an arbitrary integration constant. The case $c_1 = 0$ is the one-dimensional version of the standard three-dimensional Einstein–de Sitter cosmology, while the term $c_1 t$ plays the role of the three-dimensional curvature term.

We can now switch to comoving coordinates, with $x = r/a$, $v = \nu - Hr$, $\rho = \bar{\rho}(1 + \delta)$, $\phi = \Psi + a\ddot{a}x^2/2$, and we obtain the one-dimensional version of Eqs. (1)–(3),

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x} [(1 + \delta)v] = 0, \quad (\text{A7})$$

$$\frac{\partial v}{\partial t} + Hv + \frac{1}{a} v \frac{\partial v}{\partial x} = -\frac{1}{a} \frac{\partial \phi}{\partial x}, \quad (\text{A8})$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi\mathcal{G}(t)\bar{\rho}a^2\delta. \quad (\text{A9})$$

Linearizing these equations, we obtain the evolution equation of the linear modes of the density contrast. It takes the same form as the usual three-dimensional equation (4),

$$\ddot{D} + 2H(t)\dot{D} - 4\pi\mathcal{G}(t)\bar{\rho}(t)D = 0, \quad (\text{A10})$$

but with a time-dependent Newton's constant and the one-dimensional scale factor (A6). In particular, in the case where $c_1 = 0$ in Eq. (A6), which corresponds to the three-dimensional Einstein–de Sitter cosmology, with a scale factor $a(t)$ that keeps expanding forever but at a decelerated rate, we have the power-law linear growing and decaying modes

$$c_1 = 0: D_+(t) \propto t^{-\alpha-1}, \quad D_-(t) \propto t^{-\alpha-2}. \quad (\text{A11})$$

In a fashion similar to the change of variables (5), we make the change of variables

$$\eta = \ln D_+, \quad v = \frac{a\dot{D}_+}{D_+} u, \quad \phi = \left(\frac{a\dot{D}_+}{D_+} \right)^2 \varphi, \quad (\text{A12})$$

and we obtain the equations of motion (65)–(68) given in the main text. We can again go beyond shell crossings by using the equation of motion of trajectories (69).

2. One-dimensional background density perturbation

To derive the one-dimensional consistency relations, we can follow the method described in the main text for the three-dimensional case and first consider the impact of small changes to the background density. As in Eq. (25), we consider two universes with close cosmological parameters,

$$a' = a[1 - \epsilon(t)], \quad \bar{\rho}' = \bar{\rho}[1 + \epsilon(t)], \quad (\text{A13})$$

and substituting into the “Friedmann equation” (A4) we obtain

$$\ddot{\epsilon} + 2H\dot{\epsilon} - 4\pi\mathcal{G}(t)\bar{\rho}\epsilon = 0. \quad (\text{A14})$$

Again, we recover the evolution equation (A10) of the linear density modes and we can write $\epsilon(t) = \epsilon_0 D_+(t)$. Next, the change of frame described in Eq. (29) becomes

$$x' = (1 + \epsilon)x, \quad \delta' = \delta - \epsilon(1 + \delta), \quad v' = v + \dot{\epsilon}ax, \quad (\text{A15})$$

and $\delta_L = \delta_L' + \epsilon$. This means that, as in Eq. (33), the background density perturbation ϵ is absorbed by the change of frame as

$$\delta_{\epsilon_0}(x, t) = (1 + \epsilon)\delta'[(1 + \epsilon)x, t] + \epsilon, \quad (\text{A16})$$

which reads in Fourier space as

$$\tilde{\delta}_{\epsilon_0}(k, t) = \tilde{\delta}'[(1 - \epsilon)k, t] + \epsilon\delta_D(k). \quad (\text{A17})$$

Using the approximate symmetry $\kappa \simeq \kappa_0$, we neglect the dependence of the dynamics on variations of κ , so that the impact of the background only comes through the mapping (A12). Therefore, as in Eq. (35), we write

$\tilde{\delta}_{\epsilon_0}(k, t) = \tilde{\delta}[(1 - \epsilon)k, D_{+\epsilon_0}] + \epsilon \delta_D(k)$, where $D_{+\epsilon_0}$ is the linear growth rate that is modified with respect to the initial D_+ by the perturbation ϵ . Then, the derivative of the density contrast with respect to ϵ_0 reads as

$$\left. \frac{\partial \tilde{\delta}(k, t)}{\partial \epsilon_0} \right|_{\epsilon_0=0} = \left. \frac{\partial D_{+\alpha_0}}{\partial \epsilon_0} \right|_0 \frac{\partial \tilde{\delta}}{\partial D_+} - D_+(t) k \frac{\partial \tilde{\delta}}{\partial k}, \quad (\text{A18})$$

where we disregarded the Dirac factor that does not contribute for wave numbers $k \neq 0$.

Next, writing again the linear growing mode in the primed frame as $D'_+(t) = D_+(t) + y(t)$, and substituting into Eq. (A10) with the primed background, we obtain the one-dimensional version of Eq. (37) as

$$\ddot{y} + 2H\dot{y} - 4\pi\mathcal{G}\bar{\rho}y = 2\dot{D}_+\dot{\epsilon} + 4\pi\mathcal{G}\bar{\rho}D_+\epsilon. \quad (\text{A19})$$

Using Eqs. (68) and (A10), this can be written in terms of the time coordinate $\eta = \ln D_+$ as

$$\frac{d^2 y}{d\eta^2} + (\kappa - 1) \frac{dy}{d\eta} - \kappa y = (\kappa + 2) \epsilon_0 e^{2\eta}, \quad (\text{A20})$$

which gives (using the approximation of constant κ)

$$y(t) = \epsilon_0 D_+(t)^2, \quad \left. \frac{\partial D_{+\epsilon_0}}{\partial \epsilon_0} \right|_0 = D_+(t)^2. \quad (\text{A21})$$

Then, Eq. (A18) also reads as Eq. (70) in Fourier space, and Eq. (71) in configuration space, where again we disregarded the constant factor ϵ because we consider small-scale wave numbers with $k \neq 0$.

Next, this gives the one-dimensional consistency relations (72)–(73) as described in Sec. IV B 2.

3. Zel'dovich solution

Making the change of variable [48]

$$v = a\dot{D}_+w, \quad \phi = 4\pi\mathcal{G}\bar{\rho}a^2D_+\psi, \quad (\text{A22})$$

and using $D_+(t)$ as the time coordinate, the equations of motion (A7)–(A9) can be written as

$$\frac{\partial \delta}{\partial D_+} + \frac{\partial}{\partial x} [(1 + \delta)w] = 0, \quad (\text{A23})$$

$$\frac{\partial w}{\partial D_+} + w \frac{\partial w}{\partial x} = -\frac{\kappa}{D_+} \left(\frac{\partial \psi}{\partial x} + w \right) \quad (\text{A24})$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\delta}{D_+}. \quad (\text{A25})$$

Then, we can check that $w = -\partial\psi/\partial x$ is a solution of the equations of motion, with the continuity and Euler equations reducing to $\partial w/\partial D_+ + w\partial w/\partial x = 0$. This gives the solution

$$x(q, t) = q + D_+(t)s_{L0}(q), \quad (\text{A26})$$

for the trajectories of the particles, where q is the Lagrangian coordinate and $s(q, t) = D_+s_{L0}$ the nonlinear displacement field, which is identical to the linear displacement field in this one-dimensional case. This solution breaks down after shell crossing, as the gravitational force on a particle changes when there is some exchange of matter between the left and right sides of this particle. Then, the fluid equations no longer apply and we must solve the equation of motion of the particles, which reads as

$$\frac{\partial^2 x}{\partial D_+^2} = -\frac{\kappa}{D_+} \left(\frac{\partial x}{\partial D_+} + \frac{\partial \psi}{\partial x} \right). \quad (\text{A27})$$

The displacement field is related to the linear density contrast by

$$\delta_L(q, t) = D_+(t)\delta_{L0}(q) \quad \text{with} \quad \delta_{L0} = -\frac{ds_{L0}}{dq}, \quad (\text{A28})$$

which also reads as

$$s(q, t) = \int_{-\infty}^{+\infty} dk e^{ikq} \frac{i}{k} \tilde{\delta}_L(k, t). \quad (\text{A29})$$

Then, using the conservation of matter, the Fourier-space nonlinear density contrast can be written as Eq. (75).

4. Check of the one-dimensional consistency relations

a. Impact of a large-scale perturbation on the nonlinear density contrast

To check the validity of the one-dimensional consistency relations from the exact solution (75), we simply need the change of the nonlinear density contrast $\tilde{\delta}(k)$ when we make a small perturbation $\Delta\delta_L$ to the initial conditions on much larger scales. Let us consider the impact of a small large-scale perturbation $\Delta\delta_L$ to the initial conditions. Here we also restrict to even perturbations, $\Delta\tilde{\delta}_L(-k') = \Delta\tilde{\delta}_L(k')$, as the consistency relations studied in this paper apply to spherically averaged statistics, which correspond to the $\pm k'$ averages in the one-dimensional relations (72)–(73). Then, expanding Eq. (75) up to first order over $\Delta\delta_L$, and over powers of k' , we obtain

$$k' \rightarrow 0: \Delta\tilde{\delta}(k) = \left[\int dk' \Delta\tilde{\delta}_L(k') \right] \times \int \frac{dq}{2\pi} e^{-ikq} \int dk'' e^{ik''q} \frac{k}{k''} \tilde{\delta}_L(k'') (ikq). \quad (\text{A30})$$

Here the limit $k' \rightarrow 0$ means that we consider a perturbation of the initial conditions $\Delta\tilde{\delta}_L(k')$ that is restricted to low wave numbers $k' < \Lambda$ with a cutoff Λ that goes to zero (i.e., that is much smaller than the wave numbers k and $2\pi/q$ of interest).

On the other hand, from the expression (75) we obtain at once the exact result

$$\frac{\partial \tilde{\delta}}{\partial \ln D_+} - k \frac{\partial \tilde{\delta}}{\partial k} = \int \frac{dq}{2\pi} e^{-ikq} \int dk'' e^{ik''q} \frac{1}{k''} \tilde{\delta}_L(k'') (ikq). \quad (\text{A31})$$

The comparison with Eq. (A30) gives

$$k' \rightarrow 0: \Delta \tilde{\delta}(k) = \left[\int dk' \Delta \tilde{\delta}_L(k') \right] \left(\frac{\partial \tilde{\delta}(k)}{\partial \ln D_+} - k \frac{\partial \tilde{\delta}(k)}{\partial k} \right) \quad (\text{A32})$$

The consistency relations (72)–(73) only rely on the expression (70), which also reads (at linear order over ϵ_0) as

$$\Delta \tilde{\delta}(k) = \epsilon_0 D_+(t) \left(\frac{\partial \tilde{\delta}(k)}{\partial \ln D_+} - k \frac{\partial \tilde{\delta}(k)}{\partial k} \right). \quad (\text{A33})$$

Since we have $\epsilon_0 = \Delta \tilde{\delta}_L / D_+$, we recover Eq. (A32).

b. Explicit check on the density polyspectra

Instead of looking for the impact of a large-scale linear perturbation on the nonlinear density contrast, as in the previous section, we can directly check the consistency relations in their forms (72) or (73). Considering for simplicity the equal-time polyspectra (73), we define

$$\begin{aligned} E_n(k'; k_1, \dots, k_n; t) &\equiv \langle \tilde{\delta}_L(k', t) \tilde{\delta}(k_1, t) \dots \tilde{\delta}(k_n, t) \rangle \\ &= D_+ \left\langle \tilde{\delta}_{L0}(k') \int \frac{dq_1 \dots dq_n}{(2\pi)^n} e^{-i \sum_{j=1}^n k_j q_j} \right. \\ &\quad \left. \times e^{D_+ \int dk/k \tilde{\delta}_{L0}(k) \sum_{j=1}^n k_j e^{ikq_j}} \right\rangle, \end{aligned} \quad (\text{A34})$$

where in the last expression we used Eq. (75). The Gaussian average over the initial conditions $\tilde{\delta}_{L0}$ gives

$$\begin{aligned} E_n &= -\frac{P_L(k')}{k'} \int \frac{dq_1 \dots dq_n}{(2\pi)^n} \sum_{j=1}^n k_j e^{-ik'q_j} \\ &\quad \times e^{-i \sum_{j=1}^n k_j q_j} e^{-D_+^2/2 \int dk/k^2 P_{L0}(k) \left| \sum_{j=1}^n k_j e^{ikq_j} - 1 \right|^2}. \end{aligned} \quad (\text{A35})$$

Making the changes of variable $q_1 = q_1' + q_n, \dots, q_{n-1} = q_{n-1}' + q_n$, the argument of the last exponential does not depend on q_n . Then, the integration over q_n yields a Dirac factor $\delta_D(k' + k_1 + \dots + k_n)$, that we factor out by defining $E_n = E_n' \delta_D(k' + k_1 + \dots + k_n)$, with a primed notation as in Eq. (49), and we replace k_n by $-(k' + k_1 + \dots + k_{n-1})$. Finally, in the limit $k' \rightarrow 0$ we expand the terms $e^{-ik'q_j}$ up to first order over k' , and we obtain

$$\begin{aligned} k' \rightarrow 0: E_n' &= P_L(k') \int \frac{dq_1 \dots dq_{n-1}}{(2\pi)^{n-1}} \\ &\quad \times \left[1 + i \sum_{j=1}^{n-1} k_j q_j \right] e^{-i \sum_{j=1}^{n-1} k_j q_j} \\ &\quad \times e^{-D_+^2/2 \int dk/k^2 P_{L0}(k) \left| \sum_{j=1}^{n-1} k_j (e^{ikq_j} - 1) \right|^2}. \end{aligned} \quad (\text{A36})$$

Proceeding in the same fashion, the n -point polyspectra read as

$$\begin{aligned} P_n &\equiv \langle \tilde{\delta}(k_1, t) \dots \tilde{\delta}(k_n, t) \rangle' \\ &= \int \frac{dq_1 \dots dq_{n-1}}{(2\pi)^{n-1}} e^{-i \sum_{j=1}^{n-1} k_j q_j} \\ &\quad \times e^{-D_+^2/2 \int dk/k^2 P_{L0}(k) \left| \sum_{j=1}^{n-1} k_j (e^{ikq_j} - 1) \right|^2}. \end{aligned} \quad (\text{A37})$$

Then, we can explicitly check from the comparison with Eq. (A36) that we have the relation

$$k' \rightarrow 0: E_n' = P_L(k') \left[1 + \frac{\partial}{\partial \ln D_+} - \sum_{j=1}^{n-1} \frac{\partial}{\partial \ln k_j} \right] P_n, \quad (\text{A38})$$

and we recover the consistency relation (73). [In Eq. (A38) the right-hand side does not involve k_n because it has been replaced by $-(k_1 + \dots + k_{n-1})$ in Eq. (A37), using the Dirac factor $\delta_D(k_1 + \dots + k_n)$.]

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