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Coulomb corrections to the parameters of the Molière multiple scattering theory

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High-energy Coulomb corrections to the parameters of the Molière multiple scattering theory are obtained. Numerical calculations are presented in the range of the nuclear charge number of the target atom $6 \leqslant Z \leqslant 92$. It is shown that these corrections have a large value for sufficiently heavy elements of the target material and should be taken into account in describing high-energy experiments with nuclear targets.

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I. INTRODUCTION

The Coulomb correction (CC) is the difference between the exact Born parameter $\xi = zZ\alpha/\beta$[1] result and the Born approximation. At intermediate energies, formulas for the Coulomb corrections are not available in analytical form [1]. The analytic formulas for the high-energy CCs are known as the Bethe-Bloch formulas for the ionization losses [2] and those for the Bethe-Heitler cross section [3,4] and the spectrum of bremsstrahlung [3–5].

A similar expression was found for the total cross section of the Coulomb interaction of hadronic atoms with ordinary target atoms [6]. Also, Coulomb corrections were obtained to the cross sections of the quasielastic and elastic electron scattering [7,8], the coherent electroproduction of vector mesons [8,9], the pair production in nuclear collisions [3,4,10,11], and a two-dimensional screened potential in an impact-parameter space [12].

The specificity of the expressions presented in this work is that they determine the Coulomb corrections to the parameters of the Molière multiple scattering theory, i.e., the screening angular parameter and some other parameters of the Molière expansion method [13].

Molière's theory is of interest for numerous applications related to particle transport in matter, is widely used in most of the transport codes, and also presents the most used tool for taking into account the multiple scattering effects in experimental data processing (the DIRAC experiment [14] like many others [15–18]).

As the Molière theory is currently used roughly for 1 MeV–200 GeV proton beams [16,17] and extremely high-energy cosmic rays and can be applied to investigate the IceCubes neutrino-induced showers with energies above 1 PeV [19,20], the role of the high-energy CCs to the parameters of this theory becomes significant. Of special importance is the Coulomb correction to the screening angular parameter, as just this single parameter enters into other important quantities of the Molière theory and describes the scattering.

In his original paper, Molière obtained an approximate semianalytical expression for this parameter, valid to second order in ξ, where only the first term is determined quite accurately, while the coefficient in the second term is found numerically and approximately.

In this work, we obtained for the screening angle and other parameters of the Molière theory exact analytical results valid to all orders in ξ. We also evaluated numerically Coulomb corrections to the Born approximation of these parameters accounting for all orders in ξ over the range $6 \leqslant Z \leqslant 92$. Additionally, we estimated the accuracy of the Molière theory in determining the screening angle.

This paper is organized as follows: We start (in Sec. II) from the consideration of the standard approach to the multiple scattering theory proposed by Molière. Then, in Sec. III, we obtain the analytical and numerical results for the Coulomb corrections to the parameters of the Molière theory. In Sec. IV, we summarize our results and discuss some perspectives. The Appendix contains a derivation of the transport equation for the Bessel-transformed probability distribution function.

II. MOLIÈRE MULTIPLE SCATTERING THEORY

The small-angle multiple scattering of charged high-energy particles in the Coulomb field of nuclei is equivalent to a diffusion process in the angular plane of $\vec{\theta} = (\theta, \varphi)$ normal to the incident particle direction z, where θ and φ indicate the polar and azimuthal angles of the track of the scattered particle measured with respect to the initial direction [21,22].

The small-angle approximation assumes that θ is small; it consists in replacing $\sin\theta \sim \theta$, $\cos\theta \sim 1$, and the upper limit π for θ by infinity. Owing to the axial symmetry of the

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[1]Here z is the charge number of the scattered particle, Z is the nuclear charge number of the target atom, $\alpha = 1/137$ is the fine structure constant, and $\beta = v/c$ is the velocity of a projectile in units of the velocity of light. The dimensionless parameter ξ governing the validity of the Born approximation becomes exact for small ξ; it is commonly known as the Born parameter.

problem in most cases of interest, the distribution $W(\theta,\varphi,t)\theta d\theta d\varphi$ will be independent of the azimuthal angle, and we will use the quantity $W(\theta,t)\theta d\theta$ with the normalization condition $\int W(\theta,t)\theta d\theta = 1$ to represent the number of projectiles scattered in the angular interval $d\theta$ after traveling through an absorber of a thickness t. Also, we introduce the differential cross section $d\sigma(\vec{\chi})$ for the single elastic scattering at a single-scattering angle $\vec{\chi} = (\chi, \vec{\phi})$ determined by the polar χ and azimuthal ϕ angles and assume that $\sin\chi \sim \chi \ll 1$.

The basis for studies of multiple scattering effects within the semiclassical approach to particle transport in matter is the Boltzmann transport equation [21]. For a thin homogeneous absorber and fast charged particles, the standard transport (diffusion) equation holds

$$\frac{\partial W(\theta,t)}{\partial t} = -nW(\theta,t)\int \sigma(\chi)\chi d\chi + n\int W(\vec{\theta}-\vec{\chi},t)\sigma(\chi)d^2\chi, \qquad (1)$$

where n is the density of the scattering centers per unit volume and $d^2\chi = \chi d\chi d\phi/(2\pi)$. The first term in the right-hand side of (1) describes the decrease in the number of projectiles from the cone θ, and the second one, the increase in the cone from the outside of the cone [22].

Following Molière [13,22], we introduce the Bessel transformation of distribution

$$g(\eta,t) = \int_0^\infty \theta J_0(\eta\theta)W(\theta,t)d\theta,$$
$$W(\theta,t) = \int_0^\infty \eta J_0(\eta\theta)g(\eta,t)d\eta. \qquad (2)$$

Using the folding theorem we can also obtain the transport equation for $g(\eta,t)$ (see details in the Appendix):

$$\frac{\partial g(\eta,t)}{\partial t} = -ng(\eta,t)\int_0^\infty \sigma(\chi)\chi d\chi[1-J_0(\eta\chi)]. \qquad (3)$$

Its solution is

$$g(\eta,t) = \exp\{N(\eta,t) - N_0(0,t)\}, \qquad (4)$$

$$N(\eta,t) = nt\int \sigma(\chi)\chi d\chi J_0(\eta\chi), \qquad (5)$$

where N_0 denotes the value of (5) for $\eta = 0$, i.e., a total number of collisions. Molière's theory is valid for $N_0 \geqslant 20$. The quantity $N_0(0,t) - N(\eta,t)$ is much smaller than N_0 and can be regarded as an effective number of collisions.

Inserting (4) with (5) back into (2), we get

$$W(\theta,t) = \int_0^\infty \eta d\eta J_0(\eta\theta) \times \exp\left\{-nt\int_0^\infty \sigma(\chi)\chi d\chi[1-J_0(\eta\chi)]\right\}. \qquad (6)$$

This expression, written here with the small-angle approximation, is independent of the exact form of the single-scattering law.

If the target thickness satisfies the condition $t \ll l$, where $l = 1/(n\sigma)$ and σ is the total single-scattering cross section, the distribution function can be written as $W(\theta,t) = nt\sigma$; in this case it represents the single-scattering probability. In the case when $t \gg l$, the accounting of the multiple scattering is necessary, and the distribution function should be determined by (6).

In order to obtain a power series expansion for $W(\theta,t)$, we first write the small-angle version a modified Rutherford law

$$nt\sigma(\chi)\chi d\chi = 2\chi_c^2\chi d\chi q(\chi)/\chi^4, \qquad (7)$$

$$\chi_c^2 = 4\pi ntz^2 Z(Z+1)e^4/(pv)^2. \qquad (8)$$

The quantity $q(\chi)$ in (7) is the ratio of the actual differential scattering cross section, written in the form used in [22–24]

$$\frac{d\sigma(\chi)}{dO} = 2\pi\left(\frac{2zZe^2}{pv}\right)^2\left(\frac{1}{\chi^2+\chi_0^2}\right)^2, \qquad (9)$$

to the Rutherford one for the unscreened Coulomb potential

$$\frac{d\sigma_R(\chi)}{dO} = 2\pi\left(\frac{2zZe^2}{pv}\right)^2\frac{1}{\chi^4}. \qquad (10)$$

Here $dO = \chi d\chi d\phi$ represents the angular phase volume; e is the elementary charge, $e^2/\hbar c = 4\pi\alpha$; $p = mv$, m, and v are the mass of the charged scattered particle and its velocity at large distances from the scattering center which is assumed to be at rest; $\chi_0 = \hbar/pa$, $a = 0.885a_0Z^{-1/3}$, a_0 is the Bohr radius, and a denotes the Fermi radius of the target atom.

The screening factor $q(\chi) = \chi^4/(\chi^2+\chi_0^2)^2$ contains a deviation from the Rutherford formulas due to the effects of screening of atomic electrons and the Coulomb corrections arising from multiphoton exchanges between the scattered particle and the atomic nuclei. It is equal to unity for large values of $\chi \geqslant \chi_0$ and tends to zero at $\chi = 0$.

The physical meaning of χ_c (8) can be understood from the requirement that the probability of scattering on the angles exceeding χ_c is unity:

$$nt\int_{\chi_c}^{\infty} d\sigma(\chi) = \frac{4\pi ntz^2 Z(Z+1)e^4}{(pv)^2}\int_{\chi_c}^{\infty}\frac{d\chi}{\chi^3} = 1.$$

Typically, $\chi_c/\chi_0 \sim 100$. The given formula is based on the Rutherford cross section and is the definition of the angle χ_c. Here we replace $Z^2 \to Z(Z+1)$ keeping in mind the scattering on atomic electrons. Below, we assume that $z = 1$.

In terms of χ_c, the solution (4) of (3) reads [13,22]

$$-\ln g(\eta,t) = N_0(0,t) - N(\eta,t) = 2\chi_c^2\int_0^{\infty}\frac{d\chi}{\chi^3}q(\chi)[1-J_0(\chi\eta)]. \quad (11)$$

To estimate the value of this integral, we introduce (following [13,22]) some quantity ς from the region (χ_0, χ_c):

$$\chi_0 \ll \varsigma \ll \chi_0.$$

For the part of integral from 0 to ς, we can use a good approximation $1 - J_0(\chi\eta) = (\chi\eta)^2/4$, and the integral becomes

$$\int_0^{\varsigma}\frac{d\chi}{\chi^3}q(\chi)[1-J_0(\chi\eta)] = \frac{1}{4}\eta^2\int_0^{\varsigma}\frac{d\chi}{\chi}q(\chi). \quad (12)$$

For the part from ς to infinity, we can put $q(\chi) = 1$ and obtain

$$\int_{\varsigma}^{\infty}\frac{d\chi}{\chi^3}[1-J_0(\chi\eta)] = \frac{1}{4}\eta^2 I_1(\varsigma\eta), \quad (13)$$

$$I_1(x) = 4\int_x^{\infty}\frac{dt}{t^3}[1-J_0(t)] = \frac{2}{x^2}[1-J_0(x)] + 2\int_x^{\infty}\frac{dt}{t^2}J_1(t),$$

$$2\int_x^{\infty}\frac{dt}{t^2}J_1(t) = \frac{1}{x}J_1(x) + \int_x^{\infty}\frac{dt}{t}J_0(t) \quad (14)$$

with $x = \varsigma\eta$. For $x \ll 1$, using $\int_x^{\infty} J_0(t)dt/t = \ln(2/x) - C_E + O(x^2)$ we get

$$\int_{\varsigma}^{\infty}\frac{d\chi}{\chi^3}[1-J_0(\chi\eta)] = \frac{1}{4}\eta^2[1-C_E+\ln 2-\ln(\varsigma\eta)+O((\varsigma\eta)^2)], \quad (15)$$

where $C_E = 0.5772\ldots$ is the Euler constant.

Considering the contribution of the region $\chi < \varsigma$ Molière introduced the notation of the screening angle

$$-\ln\chi_a = \lim_{\varsigma\to\infty}\left[\int_0^{\varsigma}\frac{d\chi}{\chi}q(\chi) + \frac{1}{2} - \ln\varsigma\right]. \quad (16)$$

He also proposed a simple functional form for $q(\chi)$

$$q(\chi) = \chi^4/(\chi^2+\chi_a^2)^2, \quad (17)$$

which satisfies (16) and is similar to $q(\chi) = \chi^4/(\chi^2+\chi_0^2)^2$ as χ_0 and χ_a practically do not differ from each other at very small angles.

Using (17) in (12), substituting the obtained solution, together with (15), back into (11) and taking into account the definition

$$b = \ln(\chi_c/\chi_a')^2 \equiv \ln(\chi_c/\chi_a)^2 + 1 - 2C_E, \quad (18)$$

one can get Molière's expression

$$\ln g(\eta,t) = \frac{1}{4}(\chi_c\eta)^2\ln\left(\frac{\eta\chi_a'}{2}\right)^2 \quad (19)$$

for the exponent $N(\eta,t) - N_0(0,t) = \ln g(\eta,t)$ of (4). In the next section this relation between $\ln g(\eta,t)$ and $\ln(\chi_a')$ will be used to obtain an exact expression for the Coulomb correction to the screening angle.

Next, in order to obtain an expression for the angular distribution valid for all angles, Molière defined a new parameter B by the transcendental equation

$$B - \ln B = b. \quad (20)$$

Through (18), the parameter B depends only on the χ_a. The distribution function can be written then as

$$W(\theta,B) = \frac{1}{\bar{\theta^2}}\int_0^{\infty} ydyJ_0(\theta y)e^{-y^2/4}\exp\left[\frac{y^2}{4B}\ln\left(\frac{y^2}{4}\right)\right]$$

with $y = \chi_c\eta$ and the mean square scattering angle $\bar{\theta^2}$.

The Molière expansion method is to consider the term $y^2\ln(y^2/4)/4B$ as a small parameter. This allows expansion of the angular distribution function in a power series in $1/B$:

$$W(\theta,B) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{B^n}W_n(\theta,t),$$

$$W_n(\theta,t) = \frac{1}{\bar{\theta^2}}\int_0^{\infty} ydyJ_0\left(\frac{\theta}{\sqrt{\bar{\theta^2}}}y\right)e^{-y^2/4}\left[\frac{y^2}{4}\ln\left(\frac{y^2}{4}\right)\right]^n,$$

$$\bar{\theta^2} = \chi_c^2 B, \qquad \sqrt{\bar{\theta^2}} = \chi_c\sqrt{B} = \theta_e, \quad (21)$$

where the first term (a Gaussian) with the width θ_e (i.e., the angle at which the intensity falls by $1/e$) dominates in the small-angle scattering, and the second and the third terms, etc., become important only at large angles. This method is valid for $B \geqslant 4.5$ ($b \sim \ln N_0 \geqslant 3$) and $\bar{\theta^2} < 1$.

One of the most important results of the Molière theory is that the scattering is described by a single parameter, the screening angle χ_a (χ'_a)

$$\chi'_a = \sqrt{1.167}\chi_a = [\exp(C_{\rm E} - 0.5)]\chi_a \approx 1.080\chi_a$$

as the angular distribution depends only on the logarithmic ratio b of the characteristic angle χ_c describing the foil thickness (8) to the screening angle χ_a (16), which characterizes the scattering atom.

To find a result valid for large ξ, Molière used the WKB method and a rather rough approximation in describing the screening angle

$$\chi_a^M = \chi_a^B \sqrt{1 + 3.34\xi^2}. \tag{22}$$

This formula is determined only up to second order in ξ; its coefficient in the second term is found approximately using an interpolation scheme (see critical remarks on its deviation in [21,25]).

Below we will use the eikonal approximation to obtain an exact analytical expression for the Coulomb correction to the Born screening angle $\chi_a^B = \sqrt{1.13}\chi_0$. The accuracy of the eikonal approximation used below is the accuracy of the small-angle approximation [26], i.e., $1 + O(\chi_0/\chi_c) = 1 + O(10^{-2})$, which is better than one percent.

III. COULOMB CORRECTIONS TO THE PARAMETERS OF THE MOLIÈRE THEORY

Recall now the relations for the scattering amplitude in the eikonal approximation (see, e.g., [26,27]):

$$A(\vec{q}) = \frac{1}{2\pi i}\int d^2\rho \exp\left(\frac{-i\vec{q}\,\vec{\rho}}{\hbar}\right) S(\rho), \tag{23}$$

$$S(\rho) = \exp\left(-i\frac{\Phi(\rho)}{\hbar}\right) - 1, \tag{24}$$

$$\Phi(\rho) = \frac{Ze^2}{\beta}\int_{-\infty}^{\infty} dz \frac{1}{r}\exp\left(-\frac{r}{a}\right), \tag{25}$$

$$d\sigma(q) = |A(\vec{q})|^2 d^2q, \qquad r = \sqrt{\rho^2 + z^2}, \tag{26}$$

where $\vec{q}$ is the momentum transfer, $(z, \vec{\rho})$ are the longitudinal and transverse coordinates, respectively, and $\Phi(\rho)$ is the eikonal phase in the case of the screened Coulomb potential with the Thomas-Fermi atom radius a.

It is convenient to introduce a two-dimensional screened potential $V(\rho)$ that appears in the Landau-Pomeranchuk-Migdal effect theory when solving a transport equation (see Appendix A in [12]):

$$V(\rho) = n\int [1 - \exp(i\vec{q}\,\vec{\rho})] d\sigma(q) \tag{27}$$

$$= n\int d^2[S(x)S^*(x) - S(\vec{\rho} + \vec{x})S(x)], \tag{28}$$

where $\vec{x} = \gamma\vec{\rho}$, and γ is the usual relativistic factor of the scattered particle.

Using (24), (25), and (28) we find the following expression for the difference $\Delta V(\rho)$ between the Born and the eikonal approximations of this potential, i.e., for the Coulomb correction $\Delta_{CC}[V(\rho)]$:

$$\begin{aligned}\Delta V(\rho) &= -\Delta_{CC}[V(\rho)] \equiv -[V(\rho) - V^B(\rho)] \\ &= n\int d^2x \Big\{\exp\{i[\Phi(|\vec{\rho} + \vec{x}|) - \Phi(x)]\} \\ &\quad - 1 + \frac{1}{2}[\Phi(|\vec{\rho} + \vec{x}|) - \Phi(x)]^2\Big\},\end{aligned} \tag{29}$$

$$\Phi(x) = 2\frac{Ze^2}{\beta}K_0\left(\frac{x}{a}\right), \tag{30}$$

where $K_0(x/a)$ is the modified Bessel function.

Note that the equation for the potential $V(\rho)$ (27) can be written (after performing the angular integration) as

$$\frac{V(\rho)}{2\pi n} = \int [1 - J_0(q\rho)] d\sigma(q). \tag{31}$$

Comparing this result with

$$-\frac{\ln g(\eta)}{nt} = \int [1 - J_0(\eta\chi)] d\sigma(\chi), \tag{32}$$

we obtain the similarity with (31) when accepting $q\rho = \eta\chi$, $q = p\eta$, $\rho = \chi/p$. So the problem of the deviation of the potential $V(\rho)$ from the Born one (29) is similar to problem of deviation of the quantity $-\ln g(\eta)$ in the eikonal approximation from its Born value.

Let us notice also that $K_0(x/a)$ is large only for $x/a \ll 1$, and the main contribution in (29) gives the region $x \ll a$, in which

$$\Phi(|\vec{\rho} + \vec{x}|) - \Phi(x) = \xi \ln[x^2/(\vec{\rho} + \vec{x})^2]. \tag{33}$$

Substituting (33) into (29), we obtain the expression for the difference $\Delta[-\ln g(\eta)]$ between the Born and the eikonal approximations of this quantity:

$$\begin{aligned}\Delta[-\ln g(\eta)] &= \Delta_{CC}[\ln g(\eta)] \equiv \ln g(\eta) - \ln g^B(\eta) \\ &= (\chi_c\eta)^2 \frac{1}{4\pi}\int d^2x \left[\left(\frac{(\vec{x} + \vec{\rho})^2}{x^2}\right)^{i\xi} \right. \\ &\quad \left. - 1 + \frac{\xi^2}{2}\ln^2\frac{(\vec{x} + \vec{\rho})^2}{x^2}\right] \equiv \frac{1}{2}(\chi_c\eta)^2 f(\xi).\end{aligned} \tag{34}$$

The accuracy of transformations in going from (31) to (34) coincides with the accuracy of the eikonal approximation.

For further integration, it is convenient to introduce new variables. Putting $\vec{x}/x^2 = \vec{u}$ and $\vec{\kappa} = \vec{u} + \vec{\rho}$, one obtains

$$f(\xi) = \frac{1}{2\pi}\int \frac{d^2\kappa}{(\vec{\kappa}-1)^4}\left[(\kappa^2)^{i\xi} - 1 + \frac{\xi^2}{2}\ln^2(\kappa^2)\right]$$

for the function $f(\xi)$ defined by (34). Taking the integral over the azimuthal angle

$$\int_0^{2\pi}\frac{d\phi}{(\kappa^2 - 2\kappa\cos\phi + 1)^2} = \frac{2\pi(1+\kappa^2)}{|1-\kappa^2|^3},$$

setting $\kappa^2 = w$, integrating by parts, and introducing the variable $w = e^{-u}$ (see details in [12]), one can get

$$f(\xi) = \mathrm{Re}\int_0^\infty \frac{e^{-u}du}{(1-e^{-u})^2}[-i\xi e^{-i\xi u} + \xi^2 u].$$

After performing another integration by parts and using the standard representation of the digamma function ψ, the following universal function of the Born parameter ξ can be finally obtained:

$$f(\xi) = \mathrm{Re}[\psi(1+i\xi) - \psi(1)] \equiv \xi^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+\xi^2)}. \quad (35)$$

It is known also as the Bethe-Maximon function.

Using (19) in (34), we arrive at the relation

$$\Delta_{CC}[\ln g(\eta)] = \frac{1}{2}(\chi_c\eta)^2\Delta_{CC}[\ln(\chi'_a)] = \frac{1}{2}(\chi_c\eta)^2 f(\xi)$$

and, consequently, with account of $C_E = -\psi(1)$, we get

$$\Delta_{CC}[\ln(\chi'_a)] = f(\xi) = \mathrm{Re}[\psi(1+i\xi)] + C_E. \quad (36)$$

Here we used the smallness of the ratios $x/a \ll 1$, $\rho \sim x \ll a$ and applied the relevant asymptotes of the Bessel function $K_0(z) = C - \ln(z/2) + O(z^2)$. The main reason of such derivation of relations (34) and (36) is the significantly different regions of contributions of the screening effects and the Coulomb corrections. Really, the last ones play the main role in the region of small impact parameters, where the number of atom electrons is small and the screening effects are negligible. These results are valid in the ultra-relativistic case considered in [12]. They can also be obtained by using the technique developed in [6].

In order to calculate in ξ the exact absolute correction $\Delta_{CC}[\ln(\chi'_a)] = f(\xi)$ and exact relative correction $\delta_{CC}[\chi_a]$ to the Born screening angle

$$\delta_{CC}[\chi_a] = \frac{\chi_a - \chi_a^B}{\chi_a^B} = \exp[f(\xi)] - 1, \quad (37)$$

we must first calculate the values of the function $f(\xi) = \mathrm{Re}[\psi(1+i\xi)] + C_E$.

The digamma series

$$\psi(1+\xi) = 1 - C_E - \frac{1}{1+\xi} + \sum_{n=2}^{\infty}(-1)^n[\zeta(n-1)]\xi^{n-1},$$

where ζ is the Riemann zeta function and $|\xi| < 1$, leads to the corresponding power series for $\mathrm{Re}[\psi(1+i\xi)] = \mathrm{Re}[\psi(i\xi)]$ and $|\xi| < 2$:

$$\mathrm{Re}[\psi(i\xi)] = 1 - C_E - \frac{1}{1+\xi^2} + \sum_{n=1}^{\infty}(-1)^{n+1}[\zeta(2n+1) - 1]\xi^{2n}.$$

The function $f(\xi) = \xi^2\sum_{n=1}^{\infty}[n(n^2+\xi^2)]^{-1}$ can be represented in this case as [28]

$$f(\xi) = 1 - \frac{1}{1+\xi^2} + \sum_{n=1}^{\infty}(-1)^{n+1}[\zeta(2n+1)-1]\xi^{2n} = 1 - \frac{1}{1+\xi^2} + 0.2021\xi^2 - 0.0369\xi^4 + 0.0083\xi^6 - \cdots. \quad (38)$$

An equivalent way to estimate $f(\xi)$ (35) to four decimal figures is to present the sum $\sum_{n=1}^{\infty}[n(n^2+\xi^2)]^{-1}$ in the following form [3]:

$$\sum = (1+\xi^2)^{-1} + \sum_{n=1}^{\infty}(-\xi^2)^{n-1}[\zeta(2n+1)-1] = (1+\xi^2)^{-1} + 0.2021 - 0.0369\xi^2 + 0.0083\xi^4 - 0.0020\xi^6 + \cdots. \quad (39)$$

Equation (39) is sufficient to evaluate this sum up to $\xi \sim 0.67$, which corresponds to uranium.

The calculation results for $\sum$ (39), function $f(\xi)$ (38), which is equal to $\Delta_{CC}[\ln(\chi'_a)]$, and the relative Coulomb correction $\delta_{CC}[\chi_a] = \exp[f(\xi)] - 1$ (37) at $\beta = 1$ and $z = 1$ are given in Table I. Some results from Table I are illustrated in Fig. 1. It can be seen from Fig. 1 that the magnitude of $\delta_{CC}[\chi_a]$ is about 50% for $Z \sim 95$. So a large $\delta_{CC}[\chi_a]$ value is not surprising as the Coulomb corrections can become very large even at high energies [8]. The positive relative CCs are also obtained in [5,9].

The $f(Z\alpha)$ values computed on the basis of (38) and (39) coincide up to four decimal digits and show good agreement with the corresponding values of this function from paper [29]. So $f(Z\alpha) = 0.3129$ [29] and $f(Z\alpha) = 0.3125$ (Table I) for $Z = 79$; $f(Z\alpha) = 0.3318$ [29] and $f(Z\alpha) = 0.3316$ (Table I) for $Z = 82$.

TABLE I. The Z dependence of the Coulomb corrections and differences defined by Eqs. (37)–(43), (46), and (47) for $z = 1$, $\beta = 1$, and $B^B = 8.46$ [24].

M	Z	$\sum$	$f(\xi)$	$\delta_{CC}[\chi_a]$	$\Delta_{CCM}[\delta_{CC}]$	$\delta_{CCM}[\delta_{CC}]$	$\delta_{CCM}[\chi_a]$	$\delta_{CC}[\bar{\theta^2}]$	$\Delta_{CC}[b]$	$\Delta_{CC}[B]$	θ_e/θ_e^B
C	6	1.2001	0.0023	0.0023	−0.0009	−0.2816	−0.0009	−0.0003	−0.0023	−0.0026	0.9998
Al	13	1.1928	0.0107	0.0108	−0.0041	−0.2764	−0.0040	−0.0014	−0.0107	−0.0121	0.9993
Ti	22	1.1758	0.0303	0.0308	−0.0114	−0.2701	−0.0109	−0.0041	−0.0303	−0.0344	0.9979
Ni	28	1.1602	0.0487	0.0499	−0.0179	−0.2646	−0.0168	−0.0064	−0.0487	−0.0552	0.9968
Mo	42	1.1127	0.1046	0.1103	−0.0360	−0.2459	−0.0314	−0.0140	−0.1046	−0.1186	0.9930
Sn	50	1.0799	0.1436	0.1545	−0.0473	−0.2345	−0.0396	−0.0192	−0.1436	−0.1628	0.9904
Ta	73	0.9710	0.2758	0.3175	−0.0784	−0.1981	−0.0562	−0.0370	−0.2758	−0.3128	0.9817
Pt	78	0.9467	0.3067	0.3590	−0.0840	−0.1895	−0.0582	−0.0411	−0.3067	−0.3478	0.9797
Au	79	0.9414	0.3125	0.3670	−0.0850	−0.1880	−0.0585	−0.0419	−0.3125	−0.3545	0.9793
Pb	82	0.9262	0.3316	0.3930	−0.0890	−0.1846	−0.0600	−0.0445	−0.3316	−0.3760	0.9780
U	92	0.8761	0.3951	0.4845	−0.0985	−0.1689	−0.0622	−0.0530	−0.3951	−0.4481	0.9738

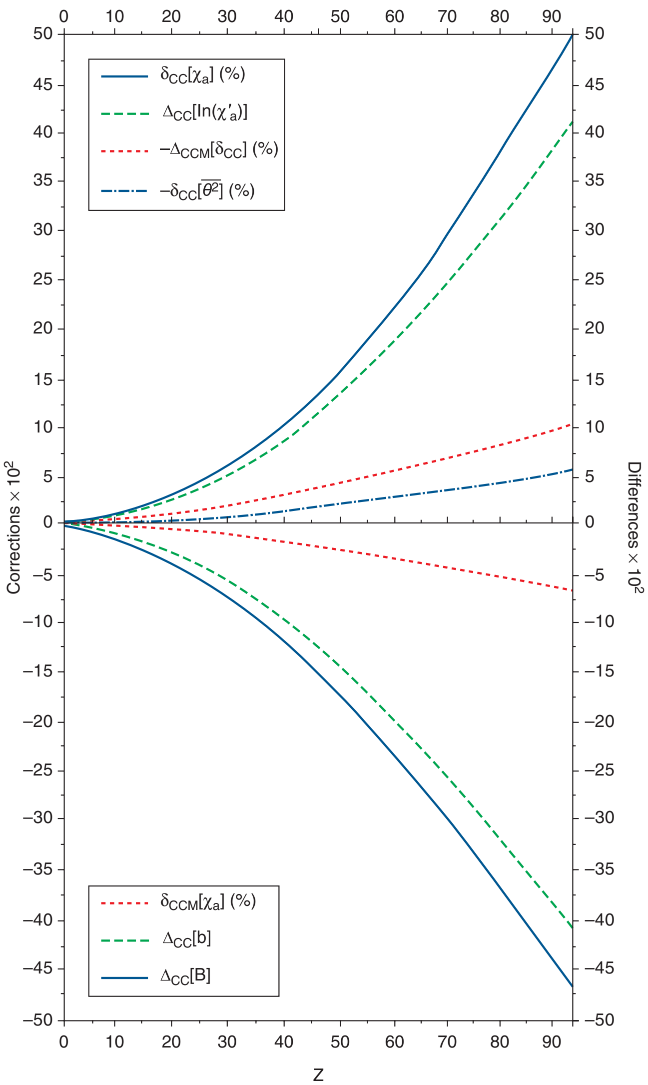

FIG. 1 (color online). The Z dependence of the Coulomb corrections Δ_{CC}, δ_{CC} to some parameters of the Molière theory and the differences Δ_{CCM}, δ_{CCM} between exact and approximate results.

During our analysis, we omit systematically the contribution of an order of α compared with that of an order of 1. We emphasize that only the ultrarelativistic case is considered during our numerical calculations, so $\beta = v/c = 1$.

We can also compare (37) with the Molière result $\delta_M[\chi_a]$:

$$\delta_{CCM}[\delta_{CC}] \equiv \frac{\delta_{CC}[\chi_a] - \delta_M[\chi_a]}{\delta_M[\chi_a]} = \frac{\Delta_{CCM}[\delta_{CC}]}{\delta_M[\chi_a]},$$
$$\delta_M[\chi_a] \equiv \frac{\chi_a^M - \chi_a^B}{\chi_a^B} = \sqrt{1 + 3.34\xi^2} - 1. \quad (40)$$

In order to obtain the relative difference between the approximate χ_a^M and exact χ_a results

$$\delta_{CCM}[\chi_a] \equiv (\chi_a - \chi_a^M)/\chi_a^M = \chi_a/\chi_a^M - 1,$$

we rewrite (37) and (40) as follows:

$$\delta_{CC}[\chi_a] + 1 = \chi_a/\chi_a^B, \qquad \delta_M[\chi_a] + 1 = \chi_a^M/\chi_a^B$$

and obtain the expression

$$\delta_{CCM}[\chi_a] = \frac{\delta_{CC}[\chi_a] + 1}{\delta_M[\chi_a] + 1} - 1. \quad (41)$$

We calculate also the Coulomb corrections to other important parameters of the Molière theory. Inserting (18) into (20) and differentiating the latter, we arrive at

$$\Delta_{CC}[b] \equiv b - b^B = -f(\xi) = (1 - 1/B^B) \cdot \Delta_{CC}[B]. \quad (42)$$

Together with (20), this gives the expressions for the Coulomb correction $\Delta_{CC}[B]$ to the width parameter B^B:

$$\Delta_{CC}[B] \equiv B - B^B = f(\xi)/(1/B^B - 1), \quad (43)$$

$$\Delta_{CC}[B] = \delta_{CC}[B] - f(\xi) = \Delta_{CC}[\ln(B)] - f(\xi). \quad (44)$$

The corrections for $\bar{\theta^2} = \chi_c^2 B$ and $\theta_e = \chi_c\sqrt{B}$ are

$$\Delta_{CC}[\bar{\theta^2}] = \chi_c^2 \cdot \Delta_{CC}[B], \qquad \delta_{CC}[\bar{\theta^2}] = \delta_{CC}[B],$$
$$2\Delta_{CC}[\ln(\theta_e)] \equiv 2\ln(\theta_e/\theta_e^B) = \Delta_{CC}[\ln(B)]. \qquad (45)$$

After employing of (43) and (44), they finally become

$$2\Delta_{CC}[\ln(\theta_e)] = \delta_{CC}[\bar{\theta^2}] = \frac{f(\xi)}{1 - B^B}, \qquad (46)$$

and the ratio of the widths θ_e and θ_e^B in (45) reads

$$\theta_e/\theta_e^B = \exp(\delta_{CC}[\bar{\theta^2}]/2). \qquad (47)$$

The Z dependence of the corrections (37)–(39), (42), (43), (46), the relative differences (40), (41), and the ratio (47) are presented in Table I (see also Fig. 1).

Table I shows that while the modulus of $\delta_{CC}[\bar{\theta^2}]$ value reaches about 5% for high-Z targets, the maximal $\delta_{CC}[\chi_a]$ value is an order of magnitude higher and amounts approximately to 50% for $Z = 92$.

It is also obvious that whereas the relative difference δ_{CCM} between exact and approximate results varies between 17% and 30% over the range $6 \leqslant Z \leqslant 92$ for $\delta_{CC}[\chi_a]$ (40), it only reaches about 6% for the screening angle χ_a itself (41) at $Z = 92$.

As can be seen from Table I, modules of the corrections to the parameters b and B reach large values for heavy target elements. So $-\Delta_{CC}[B] \sim 0.45$, $-\Delta_{CC}[b] \sim 0.40$, such as $\Delta_{CC}[\ln(\chi'_a)] \sim 0.40$ for $Z = 92$.

Let us notice also that the sizes of the corrections $-\delta_{CC}[\bar{\theta^2}]$ and $-\Delta_{CC}[B]$ depending on the parameter B^B increase from 0.053 and 0.448 to 0.112 and 0.551 ($Z = 92$), respectively, with decreasing $B^B = 8.46$ [24,30] to a minimal B^B value 4.5.

It is well known [14,21,22,25,31] that the predictions of a Gaussian shape with the corresponding value of $\bar{\theta^2}$ are not in good agreement with experimental measurements. The correction $\Delta_{CC}[\ln(\theta_e)]$ can explain the fact that Molière's theory yields too great a width θ_e^B compared with the experimental value θ_e^{ex} [14,21,25,31].

So the obtained results give $\theta_e^{ex}/\theta_e = 1.0036$ instead of $\theta_e^{ex}/\theta_e^B = 0.9857$ for $Z = 79$ and $B^B = 10.30$ ([25], Table II), $\theta_e^{ex}/\theta_e = 1.0031$ instead of $\theta_e^{ex}/\theta_e^B = 0.981$ for $Z = 82$, and $\theta_e^{ex}/\theta_e = 1.026$, which coincides with $\theta_e^{ex}/\theta_e^B = 1.026$ for $Z = 6$ ([32], Table V).

Thus, in the description of experiments with nuclear targets such corrections to the parameters of the Molière expansion method as $\Delta_{CC}[\ln(\chi'_a)]$, $\delta_{CC}[\chi_a]$, $-\Delta_{CC}[b]$, $-\Delta_{CC}[B]$, $\delta_{CC}[\bar{\theta^2}]$, and $\Delta_{CC}[\ln(\theta_e)]$ become significant and should be considered, e.g., in the Monte-Carlo calculations of the angular distributions for their adequate description. It is necessary in many cases [14–18,24,30,33] and plays a crucial role, particularly, in the DIRAC experiment at CERN [14].

IV. SUMMARY AND OUTLOOK

Within the eikonal approach, we have obtained exact analytical results for the Coulomb corrections to the parameters $\chi'_a, \chi_a, b, B, \bar{\theta^2}$, and θ_e of the Molière expansion method. We also estimated numerically these Coulomb corrections to the parameters of the Molière theory for homogeneous absorbers with no energy loss and ultra-relativistic charged projectiles in the range of nuclear charge of target atoms from $Z = 6$ to $Z = 92$ at $\beta = 1$ and studied their Z dependence.

We found that the corrections $\Delta_{CC}[\ln(\chi'_a)]$, $\delta_{CC}[\chi_a]$, $-\Delta_{CC}[b]$, and $-\Delta_{CC}[B]$ have large values that increase up to 0.4–0.5 for $Z \sim 95$. For instance, the magnitude of $\delta_{CC}[\chi_a]$ ranges from around 10% for $Z \sim 40$ up to 40%–50% for $Z \sim 80 - 90$. The contribution of such corrections is larger than experimental errors in the most high-energy experiments whose measurement accuracy is of order of a few percent, and these corrections should be appropriately considered in experimental data processing.

We evaluated the difference and the relative difference between our results for the screening angle and those of Molière over the range $6 \leqslant Z \leqslant 92$, and we found that while the values of $\delta_{CCM}[\chi_a]$ and $\Delta_{CCM}[\delta_{CC}]$ increase with Z up only to 6% and 10%, respectively, the relative difference $\delta_{CCM}[\delta_{CC}]$ varies between 28% and 17% over this range of Z. Thus, we can conclude that these corrections to the approximate Molière result must also be borne in mind for a rather accurate description of high-energy experiments with nuclear targets (MUSCAT, MUCOOL experiments [16], the DIRAC experiment [14], etc).

The further development of this approach involves the use of the Coulomb corrections found in the present work for the calculation of the Coulomb corrections to the quantities of the classical Migdal theory of the Landau-Pomeranchuk-Migdal (LPM) effect [34]. The obtained CCs to the parameters of this theory and its analogue for a thin layer of matter allow one to eliminate the known discrepancy between the predictions of the Migdal LPM effect theory and SLAC E-146 experiment results [30] at least for high-Z targets as well as to improve the agreement between the predictions of the LPM effect theory analogue for thin targets and experimental data [33].

The developed approach can be useful for the analysis of electromagnetic processes in strong crystalline fields at high energies (CERN-NA63 experiment) [35], in cosmic-ray neutrino experiments [18–20], where some data have appeared, in which LPM suppression is important (e.g., IceCubes neutrino-induced showers) [19,20]. The Coulomb corrections to the parameters of the quantum LPM effect theory to describe the shower production at the energies exceeding 10^4 GeV [36], especially to the Migdal functions $G(s)$ and $\Phi(s)$, are of special interest there [19]. The corresponding results for these corrections will be the subject of a separate publication [37].

APPENDIX: DERIVATION OF THE TRANSPORT EQUATION FOR THE BESSEL-TRANSFORMED DISTRIBUTION FUNCTION

We put here the details of inferring Eq. (3). We apply first the integration operation $\int_0^\infty \theta d\theta J_0(\eta\theta)$ to both sides of Eq. (2). Using the definition of the Bessel transform of the probability distribution (2), we obtain

$$\frac{\partial g(\eta,t)}{\partial t} = -ng(\eta,t)\int_0^\infty \sigma(\chi)\chi d\chi + n\int_0^\infty \sigma(\chi)\chi d\chi I(\eta,\chi), \quad \text{(A1)}$$

$$I(\eta,\chi) = \int_0^{2\pi}\frac{d\phi}{2\pi}\int \theta d\theta J_0(\eta\theta)W(|\vec{\theta}-\vec{\chi}|,t). \quad \text{(A2)}$$

Applying the opposite Bessel transform to the probability (2), we get for the last integral

$$I(\eta,\chi) = \int_0^{2\pi}\frac{d\phi}{2\pi}\int \theta d\theta J_0(\eta\theta) \times \int_0^\infty \eta_1 d\eta_1 J_0(\eta_1|\vec{\theta}-\vec{\chi}|)g(\eta_1,t),$$

where the integration over θ can be performed using the folding theorem:

$$\int_0^{2\pi}\frac{d\phi}{2\pi}J_0(\eta_1|\vec{\theta}-\vec{\chi}|) = J_0(\eta_1\theta)J_0(\eta_1\chi). \quad \text{(A3)}$$

With the means of the orthogonality relation for the Bessel functions

$$\int_0^\infty xdxJ_0(xa)J_0(xb) - \frac{1}{a}\delta(a-b), \quad \text{(A4)}$$

we get for $I(\eta,\chi)$:

$$I(\eta,\chi) = g(\eta,t)J_0(\eta\chi). \quad \text{(A5)}$$

Inserting (A5) into (A1), we immediately arrive at a final result:

$$\frac{\partial g(\eta,t)}{\partial t} = -ng(\eta,t)\int_0^\infty \sigma(\chi)\chi d\chi[1-J_0(\eta\chi)].$$

To prove the folding theorem (A3), we use the series expansion for the Bessel function

$$J_0(z) = 1 - \frac{(z^2/4)}{(1!)^2} + \frac{(z^2/4)^2}{(2!)^2} - \cdots, \quad z^2 = \eta^2[\theta^2+\chi^2-2\theta\chi\cos\phi], \quad \text{(A6)}$$

and perform the integration over ϕ:

$$\frac{1}{2\pi}\int_0^{2\pi}(\cos\phi)^{2n}d\phi = \frac{(2n-1)!!}{(2n)!!}. \quad \text{(A7)}$$

[1] H. V. Koch and J. W. Motz, Rev. Mod. Phys. **31**, 920 (1959).
[2] H. A. Bethe, Z. Phys. **76**, 293 (1932); F. Bloch, Ann. Phys. (Berlin) **408**, 285 (1933).
[3] H. A. Bethe and W. Heitler, Proc. R. Soc. London, Sect. A **146**, 83 (1934); H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954).
[4] H. Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. **93**, 788 (1954).
[5] I. Øverbø, K. J. Mork, and H. A. Olsen, Phys. Rev. **175**, 1978 (1968); Phys. Rev. A **8**, 668 (1973).
[6] O. Voskresenskaya, S. Gevorkyan, and A. Tarasov, Phys. At. Nucl. **61**, 1517 (1998).
[7] A. Aste, K. Hencken, J. Jourdan, I. Sick, and D. Trautmann, Nucl. Phys. **A743**, 259 (2004); A. Aste, Nucl. Phys. **A806**, 191 (2008); J. Arrington, J. Phys. G **40**, 115003 (2013).
[8] A. Aste, K. Hencken, and D. Trautmann, Eur. Phys. J. A **21**, 161 (2004).
[9] B. Kopeliovich, A. Tarasov, and O. Voskresenskaya, Eur. Phys. J. A **11**, 345 (2001).
[10] D. Ivanov and K. Melnikov, Phys. Rev. D **57**, 4025 (1998); D. Y. Ivanov, E. A. Kuraev, A. Schiller, and V. G. Serbo, Phys. Lett. B **442**, 453 (1998).
[11] A. J. Baltz, F. Gelis, L. McLerran, and A. Peshier, Nucl. Phys. **A695**, 395 (2001).
[12] V. N. Baier and V. M. Katkov, Phys. Rev. D **57**, 3146 (1998).
[13] G. Molière, Z. Naturforsch. A **2a**, 133 (1947); Z. Naturforsch. A **3a**, 78 (1948).
[14] A. Dudarev, V. Kruglov, L. Kruglova, and M. Nikitin, DIRAC Note No. 2005-02; B. Adeva *et al.* (DIRAC-Collaboration), Phys. Lett. B **619**, 50 (2005); Phys. Lett. B **704**, 24 (2011); arXiv:1403.0845.
[15] A. P. Radlinski, E. Radlińska, M. Agamalian, G. Wignall, P. Lindner, and O. Randl, Phys. Rev. Lett. **82**, 3078 (1999); H. Takeshita, T. Kanaya, K. Nishida, K. Kaji, T. Takahashi, and M. Hashimoto, Phys. Rev. E **61**, 2125 (2000); N. O. Elyutin, D. V. Lvov, E. V. Rakshun, and A. N. Tyulyusov, J. Surf. Invest.: X-Ray, Synchrotron Neutron Tech. **4**, 908 (2010).

[16] D. Attwood *et al.* (MuScat Collaboration), Nucl. Instrum. Methods Phys. Res., Sect. B **251**, 41 (2006); C. M. Ankenbrandt *et al.* (MUCOOL Collaboration), Proposal of the MUCOOL Collaboration, 2012 (See also http://nfmcc-docdb.fnal.gov/cgi-bin/DocumentDatabase).
[17] S. I. Striganov, Radiation Protection Dosimetry **116**, 293 (2005).
[18] A. V. Butkevich, R. P. Kokoulin, G. V. Matushko, and S. P. Mikheyev, Nucl. Instrum. Methods Phys. Res., Sect. A **488**, 282 (2002).
[19] S. R. Klein, private communication.
[20] M. G. Aartsen *et al.* (IceCube Collaboration), Phys. Rev. D **88**, 112008 (2013); L. Gerhardt and S. R. Klein, Phys. Rev. D **82**, 074017 (2010); S. R. Klein, Nucl. Phys. Proc. Supl. **229-232**, 284 (2012).
[21] W. T. Scott, Rev. Mod. Phys. **35**, 231 (1963).
[22] H. A. Bethe, Phys. Rev. **89**, 1256 (1953).
[23] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).
[24] N. F. Shul'ga and S. P. Fomin, ZhETF **113**, 58 (1998); A. I. Akhiezer and N. F. Shul'ga, *High-Energy Electrodynamics in Matter* (Gordon and Breach, Amsterdam, 1996).
[25] B. P. Nigam, M. K. Sundaresan, and T. Y. Wu, Phys. Rev. **115**, 491 (1959).
[26] V. N. Baier, V. M. Katkov, and V. S. Fadin, *Radiation from Relativistic Electrons* (Atomizdat, Moscow, 1973).
[27] O. O. Voskresenskaya and A. V. Tarasov, J. Phys. G **40**, 095106 (2013).
[28] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).
[29] U. D. Jentschura, K. Hencken, and V. G. Serbo, Eur. Phys. J. C **58**, 281 (2008).
[30] P. L. Anthony *et al.* (SLAC E-146 Collaboration), Phys. Rev. Lett. **75**, 1949 (1995); Phys. Rev. Lett. **76**, 3550 (1996); Phys. Rev. D **56**, 1373 (1997); S. R. Klein, Rev. Mod. Phys. **71**, 1501 (1999).
[31] X. A. Li and D. V. O. Rogers, Med. Phys. **22**, 531 (1995).
[32] G. Shen, C. Ankenbrandt, M. Atac *et al.*, Phys. Rev. D **20**, 1584 (1979).
[33] E. A. Kuraev, O. O. Voskresenskaya, and H. T. Torosyan, arXiv:1309.7946; H. T. Torosyan and O. O. Voskresenskaya, Universal Journal of Physics and Application **2**, 226 (2014).
[34] A. B. Migdal, Proc. of SSSR Akad. of Science **96**, 49 (1954); L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. of Science of SSSR **92**, 535 (1953) [*The Collected Papers of L. D. Landau* (Pergamon Press, New-York, 1965)].
[35] K. K. Andersen, J. Esberg, H. Knudsen, H. D. Thomsen, U. I. Uggerhøj, P. Sona, A. Mangiarotti, T. J. Ketel, A. Dizdar, and S. Ballestrero (CERN-NA63 Collaboration), Phys. Rev. D **86**, 072001 (2012).
[36] A. B. Migdal, Phys. Rev. **103**, 1811 (1956); JETP **32**, 633 (1957); N. V. Laskin, A. S. Mazmanishvili, and N. F. Shul'ga, Proc. of SSSR Akad. of Science **277**, 850 (1984).
[37] O. O. Voskresenskaya (to be published).