Hard nonlogarithmic corrections of order $m\alpha^7$
to hyperfine splitting in positronium

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We consider hard three-loop nonlogarithmic corrections of order $m\alpha^7$ to hyperfine splitting in positronium. All these contributions are generated by the graphs with photon and/or electron loop radiative insertions in the two-photon exchange diagrams. We calculate contributions of six gauge-invariant sets of diagrams. The total result for all these diagrams is $\Delta E = -1.2917(1)m\alpha^7/\pi^3 = -5.672$ kHz.

Experimental and theoretical research on hyperfine splitting (HFS) in positronium has a long and distinguished history. The experimental research started with the discovery of positronium [1] and the first HFS measurement [2], both in 1951. Results comparable in accuracy were obtained in the 1980s [3–5],

$$\Delta E_{exp} = 203388.5(1.0) \text{ MHz,} \qquad (1)$$

and much later in 2013 [6],

$$\Delta E_{exp} = 203394.2(1.6)_{stat}(1.3)_{sys} \text{ MHz.} \qquad (2)$$

An even later but less accurate result in Ref. [7] is compatible with this last number. We see that the recent result in Ref. [6] is about three standard deviations higher than the earlier results. In this situation, new high-precision measurements of the positronium HFS are warranted.

Theoretical work started with calculation of the leading contribution of order $m\alpha^4$ to positronium HFS splitting at the end of the 1940s and the beginning of the 1950s [8–10]. The full quantum electrodynamic theoretical expression for the positronium HFS splitting is a series in the fine structure constant α with the coefficients that are polynomials in $\ln \alpha$. During the years, many corrections to the leading contribution were calculated. Nowadays, all terms up to and including $m\alpha^7 \ln \alpha$ are already calculated; see the state-of-the-art theoretical expression in Ref. [11]. Calculation of the nonlogarithmic corrections of order $m\alpha^7$ is the next theoretical goal. First results for these corrections were published recently [11,12].

The current theoretical uncertainty of the positronium HFS theory can be estimated by comparison with the known results for HFS in muonium. There are two major differences between HFS in muonium and positronium.

First, an additional annihilation channel arises in the case of positronium, and, second, the masses of constituents coincide in the case of positronium. After calculation of the one-photon annihilation contribution of order $m\alpha^7$ in Ref. [11], the dominant contribution to the theoretical uncertainty of HFS in positronium is generated by the unknown nonlogarithmic terms of order $m\alpha^7$ that are similar to the terms of order $\alpha(Z\alpha)^2E_F$ in muonium. These corrections in muonium are generated by the one-loop radiative insertions in the electron line. Structurally, they are similar to the classic Lamb contributions and are represented by the series in $\ln(Z\alpha)$, $[c_1\ln^2(Z\alpha) + c_2\ln(Z\alpha) + c_3]\alpha(Z\alpha)^2E_F$; see, e.g., reviews in Refs. [13,14]. The nonlogarithmic term with the coefficient c_3 arises from the ultrasoft momenta of order $m(Z\alpha)^2$ and is about $2 \times 10^{-6}E_F$ [15,16] in muonium. Similar ultrasoft terms in positronium arise due to one-loop radiative insertions in each of the fermion lines and also due to an exchange of an ultrasoft photon between the fermion lines. In this situation, we accept $2 \times 10^{-6}E_F^{Ps} \sim 250$ kHz as a fair estimate of the ultrasoft nonlogarithmic contribution in positronium. This term is still uncalculated, and, after the recent calculation of the one-photon annihilation contribution in Ref. [11], it can be used as an estimate of the theoretical uncertainty of the current positronium HFS theory.

There are other sources of nonlogarithmic corrections of order $m\alpha^7$ besides ultrasoft nonlogarithmic contributions. Hard nonlogarithmic corrections of order $m\alpha^7$ are generated by seven gauge-invariant sets of nonannihilation diagrams; six of them are presented in Figs. 2–4 and in Figs. 6–8[1]. One more set of diagrams with two-photon exchanges and two radiative photon insertions in one and the same fermion line also generates hard nonlogarithmic corrections of order $m\alpha^7$. The contributions to HFS in

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[1]We systematically omit diagrams with crossed exchanged photons in all figures.

FIG. 1.   Diagrams with two-photon exchanges.

FIG. 2.   Diagrams with two one-loop polarization insertions.

positronium produced by these diagrams are similar to the contributions of orders $\alpha^2(Z\alpha)E_F$, $\alpha^2(Z\alpha)(m/M)E_F$, and $\alpha(Z^2\alpha)(Z\alpha)(m/M)E_F$ in muonium; see reviews in Refs. [13,14,17] and more recent results in Refs. [18–22]. We report below the results of calculations of gauge-invariant contributions to HFS in positronium generated by the diagrams in Figs. 2–4 and in Figs. 6–8[2]. All these diagrams can be obtained by radiative insertions in the diagrams with two-photon exchanges in Fig. 1. Because of radiative insertions, the characteristic integration momenta in all these diagrams are or order of the electron mass m, much larger than the characteristic bound state momenta of order $m\alpha$. As a result, all these contributions can be calculated in the scattering approximation with the on-shell external electron (positron) lines, and the result should be multiplied by the Schrödinger—Coulomb wave function squared at the origin. To keep control of the positronium calculations, we derived general expressions for contributions of order $\alpha^2(Z\alpha)E_F$ to HFS for a system with constituents with an arbitrary mass ratio. We have checked that these expressions reproduce the contributions to HFS in muonium obtained earlier as series in the small mass ratio [13,14,17–22]. Then, we used the same expressions for the calculation of the hard nonlogarithmic corrections of order $m\alpha^7$ in positronium.

We start the calculations with the infrared-divergent contribution to HFS in positronium generated by the two-photon exchange diagrams in Fig. 1 calculated in the scattering approximation. It can be written in the form

$$\Delta E = -\frac{\alpha}{\pi}E_F^{\text{Ps}}(2m^2)\int\frac{d^4q}{i\pi^2q^4}(2q^2+q_0^2)L_{skel}(q)L_{skel}(-q),\tag{3}$$

where $E_F^{\text{Ps}} = m\alpha^4/3$ is the leading nonannihilation contribution to HFS in positronium and the factor L_{skel} is defined by the skeleton electron line factor

$$L_{skel}^{\mu\nu}(q) \equiv -\frac{2q^2}{q^4-4m^2q_0^2}\gamma^\mu\hat{q}\gamma^\nu = 2L_{skel}\gamma^\mu\hat{q}\gamma^\nu.\tag{4}$$

Explicitly, after the Wick rotation, we obtain

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[2]The contribution of the diagrams with the light-by-light scattering insertions in Fig. 7 was recently obtained in Ref. [12].
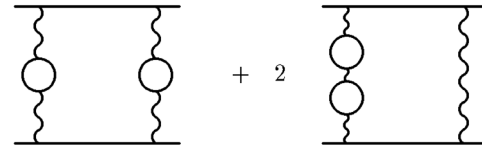
$$\Delta E = \frac{\alpha}{\pi}E_F^{\text{Ps}}\frac{4m^2}{\pi}\int_0^\pi d\theta\sin^2\theta\int_0^\infty dq^2\frac{2+\cos^2\theta}{(q^2+4m^2\cos^2\theta)^2}$$
$$\equiv \frac{\alpha}{\pi}E_F^{\text{Ps}}\int_0^\infty dq^2 f_p(q).\tag{5}$$

Radiative insertions in Figs. 2–4 and in Figs. 6–8 make these diagrams infrared convergent and justify the validity of the scattering approximation for their calculation. All corrections calculated below are obtained by some modifications of the basic integrals in Eqs. (3) and (5).

Consider first the diagrams in Fig. 2 with two one-loop polarization insertions:

$$\frac{\alpha}{\pi}I_1(q) = \frac{\alpha}{\pi}\int_0^1 dv\frac{v^2(1-\frac{v^2}{3})}{4m^2+q^2(1-v^2)}.\tag{6}$$

The contribution of the diagrams in Fig. 2 is obtained by insertion of the one-loop photon polarization squared $(\alpha/\pi)^2q^4I_1^2(q)$ in the integrand in Eq. (5). After calculations, we obtain

$$\Delta E_1 = 3\frac{\alpha^3}{\pi^3}E_F^{\text{Ps}}\int_0^\infty dq^2 f_p(q)q^4I_1^2(q)$$
$$= \left(\frac{6\pi^2}{35}-\frac{8}{9}\right)\frac{\alpha^3}{\pi^3}E_F^{\text{Ps}}$$
$$= 0.803043294\frac{\alpha^3}{\pi^3}E_F^{\text{Ps}},\tag{7}$$

where the factor 3 before the integral accounts for the multiplicity of the diagrams.

Similarly the contribution of the two-loop vacuum polarization in Fig. 3 can be obtained by the insertion of the two-loop photon polarization $(\alpha^2/\pi^2)q^2I_2(q)$ [23,24] in the integrand in Eq. (5),

$$\Delta E_2 = 2\frac{\alpha^3}{\pi^3}E_F^{\text{Ps}}\int_0^\infty dq^2 f_p(q)q^2I_2(q),\tag{8}$$

where 2 is the combinatorial factor. The integral representation [24] for I_2 is too cumbersome to put it down here.
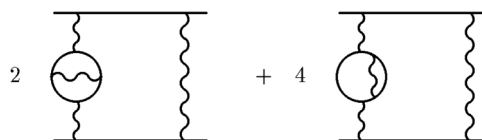

FIG. 3.   Diagrams with two-loop polarization insertions.
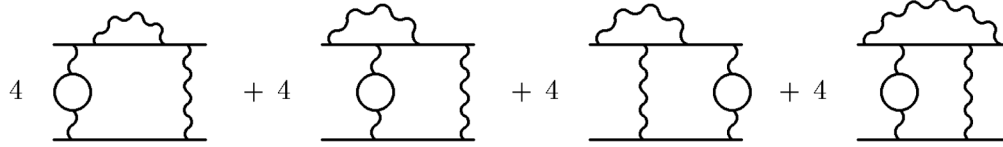
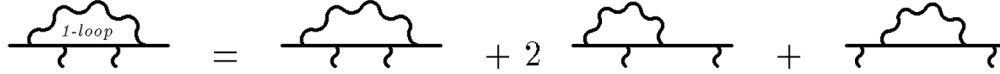FIG. 4.   Diagrams with one-loop polarization and radiative photon insertions.

FIG. 5.   One-loop fermion factor.

Nevertheless, it admits an analytic calculation of the integral above, and we obtain

$$\Delta E_2 = \left[-\frac{217}{30}\zeta(3) + \frac{28\pi^2}{15}\ln 2 + \frac{\pi^2}{675} + \frac{403}{360} \right] \frac{\alpha^3}{\pi^3} E_F^{\text{Ps}}$$
$$= 5.209219614 \frac{\alpha^3}{\pi^3} E_F^{\text{Ps}}. \tag{9}$$

The diagrams in Fig. 4 are obtained from the skeleton diagrams in Fig. 1 by one-loop radiative insertions in one of the exchanged photons and one of the fermion lines. Because of the one-loop radiative insertions in the fermion line in Fig. 4, effectively the skeleton fermion line in Fig. 1 is replaced by the one-loop fermion factor $L_{\mu\nu}$ in Fig. 5. This corresponds to the substitution

$$L_{skel}^{\mu\nu}(q) \to L^{\mu\nu}(q)$$
$$= 2\frac{\alpha}{4\pi} \left\{ \gamma^\mu \hat{q} \gamma^\nu \tilde{L}_I(q^2, q_0^2) \right.$$
$$\left. + q_0 \left[\gamma^\mu \gamma^\nu - \frac{q^\mu \hat{q} \gamma^\nu + \gamma^\mu \hat{q} q^\nu}{q^2} \right] \tilde{L}_{\text{II}}(q^2, q_0^2) \right\}, \tag{10}$$

in the integral in Eq. (3), where $\tilde{L}_{\text{I(II)}}(q^2, q_0^2)$ are scalar form factors. We derived explicit integral representations for these form factors a long time ago, calculating contributions to HFS in muonium [25,26].

Then, we are ready to calculate the contribution of the diagrams in Fig. 4 to HFS:

$$\Delta E_3 = \frac{\alpha^3}{\pi^3} E_F^{Ps} \frac{4m^2}{\pi} \int_0^\infty dq^2 q^2 I_1(q)$$
$$\times \int_0^\pi d\theta \sin^2\theta L_{skel}[(2 + \cos^2\theta)\tilde{L}_I - 3\cos^2\theta \tilde{L}_{\text{II}}]$$
$$= -1.28709(1) \frac{\alpha^3}{\pi^3} E_F^{Ps}. \tag{11}$$

Next, we turn to the diagrams in Fig. 6 with the one-loop polarization insertions in the radiative photon. Effectively,

these diagrams contain a radiatively corrected electron factor in Eq. (10). A photon line with the insertion of a one-loop polarization operator has a natural interpretation as a massive photon propagator, with the mass squared $\lambda^2 = 4m^2/(1 - v^2)$. This propagator should be integrated over v with the weight $(\alpha/\pi)v^2(1 - v^2/3)/(1 - v^2)$; compare Eq. (6). To obtain the electron factor necessary for calculation of the diagrams in Fig. 6, we restored the photon mass in the one-loop electron factor in Eq. (10) and made the substitution above. In this way, we obtained an explicit integral representation for this radiatively corrected electron factor. This factor is similar to the radiatively corrected electron factor used in our earlier calculations of the respective contributions to HFS in muonium in Refs. [20,27]. All entries in a two-loop fermion factor except the two-loop anomalous magnetic moment decrease at least as q^2 when $q^2 \to 0$. As a result, the term with the two-loop anomalous magnetic moments leads to an infrared-divergent contribution in the integral for the diagrams in Fig. 6. This linear infrared divergence indicates the existence of a contribution to HFS of the previous order in α that is already accounted for. To get rid of this spurious divergence, we subtract the term with the two-loop anomalous magnetic moment from the two-loop electron factor. Then, the contribution of the diagrams in Fig. 6 to HFS can be written in the form

$$\Delta E_4 = \frac{\alpha^3}{\pi^3} E_F^{Ps} \frac{2m^2}{\pi} \int_0^\infty dq^2$$
$$\times \int_0^\pi d\theta \sin^2\theta L_{skel}[(2 + \cos^2\theta)L_I^p - 3\cos^2\theta L_{\text{II}}^p], \tag{12}$$
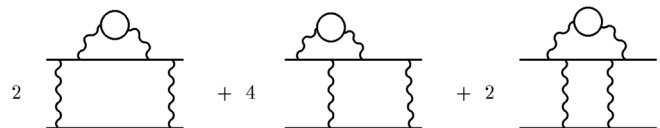

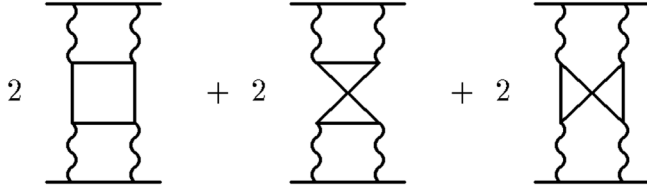FIG. 6.   Diagrams with one-loop polarization insertions in radiative photons.

FIG. 7.   Diagrams with light-by-light scattering insertions.

FIG. 8.   Diagrams with one-loop radiative photon insertions in both fermion lines.

where L_I^p and L_{II}^p are the two-loop form factors [20,27] similar to the one-loop form factors in Eq. (10) but with the subtracted anomalous magnetic moment terms. After calculations, we obtain

$$\Delta E_4 = -3.15441(1)\frac{\alpha^3}{\pi^3}E_F^{Ps}. \tag{13}$$

The calculation of the light-by-light scattering contribution in Fig. 7 proceeds exactly like the calculation of the respective nonlogarithmic radiative-recoil correction to HFS in muonium in Ref. [22]. The only difference is that we need to let the muon mass be equal to the electron mass and restore the terms of higher order in the recoil factor m/M omitted terms in Ref. [22]. This can be easily achieved by restoring the factor $1/(q^2 + 4m^2\cos^2\theta)$ instead of $1/q^2$ in the integrand in Eq. (32) of Ref. [22]. Then, the integral for the light-by-light diagrams in Fig. 8 acquires the form

$$\Delta E_5 = \frac{\alpha^3}{\pi^3}E_F^{Ps}\frac{3m^2}{32\pi}\int_0^\infty dq^2\int_0^\pi d\theta\sin^2\theta\frac{T(q^2,\cos^2\theta)}{(q^2+4m^2\cos^2\theta)^2}. \tag{14}$$

The explicit integral representation for the function $T(q^2,\cos^2\theta)$ can be found in Ref. [22]. Calculating this integral, we obtain

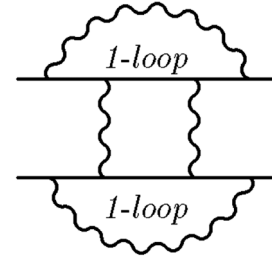$$\Delta E_5 = -0.70627(5)\frac{\alpha^3}{\pi^3}E_F^{Ps}, \tag{15}$$

which coincides with the result obtained recently in Ref. [12].

Consider now the diagrams in Fig. 8. These diagrams contain one-loop fermion factors in Eq. (10) in both fermion lines. As we mentioned above in discussion of the diagrams in Fig. 6, all entries in a fermion factor except the one-loop anomalous magnetic moment decrease at least as q^2 when $q^2 \to 0$. As a result, the product of anomalous magnetic moments leads to an infrared-divergent contribution in the integral for the diagrams in Fig. 8. This linear infrared divergence indicates the existence of a contribution to HFS of the previous order in α. We need to subtract this lower-order contribution. To facilitate this substraction, we write the fermion factors (after the Wick rotation) in the form

$$\tilde{L}_I = L_1 + L_A, \qquad \tilde{L}_{II} = L_{II} - L_A, \tag{16}$$

where we have separated the contribution of the anomalous magnetic moment

$$L_A = 2L_{skel} = \frac{2}{q^2 + 4m^2\cos^2\theta} \tag{17}$$

to the scalar form factors.

After subtraction of the infrared-divergent part, we calculate the finite integral for the contribution of the diagrams in Fig. 8 to HFS:

$$\Delta E_6 = \frac{\alpha^3}{\pi^3}E_F^{Ps}\left\{\frac{m^2}{4\pi}\int_0^\infty dq^2\sin^2\theta d\theta\left[(2+\cos^2\theta)(L_IL_I + L_AL_I + L_IL_A)\right.\right.$$
$$- 3\cos^2\theta(L_IL_{II} + L_AL_{II} - L_IL_A + L_{II}L_I - L_AL_I + L_{II}L_A)$$
$$\left.\left.+ \cos^2\theta(1+2\cos^2\theta)(L_{II}L_{II} - L_AL_{II} - L_{II}L_A)\right] + \frac{9}{16}\right\}$$
$$= -4.73955(40)\frac{\alpha^3}{\pi^3}E_F^{Ps}. \tag{18}$$

Next, we collect the results in Eqs. (7), (9), (11), (13), (15), and (18) and obtain the total hard contribution to HFS of order $m\alpha^7$ generated by the diagrams in Figs. 2–4 and in Figs. 6–8:

$$\Delta E = -3.8750(4)\left(\frac{\alpha}{\pi}\right)^3 E_F^{Ps}$$
$$= -1.2917(1)\frac{m\alpha^7}{\pi^3}$$
$$= -5.672 \text{ kHz.} \qquad (19)$$

Then, the total state-of-the-art theoretical prediction for HFS in positronium with account of all theoretical contributions known today [11,12] is

$$\Delta E_{theor} = 203391.90(25) \text{ MHz}, \qquad (20)$$

to be compared with the experimental numbers in Eqs. (1) and (2). Clearly, further reduction of both the experimental and theoretical uncertainties is warranted. On the theoretical side, calculation of the still unknown ultrasoft and hard nonlogarithmic contributions of order $m\alpha^7$ is the next goal. Work on the calculation of the remaining hard correction of order $m\alpha^7$ is now in progress, and we hope to report its results in the near future.

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