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Exact classical sphere-plate Casimir interaction in $(D+1)$-dimensional spacetime

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We consider the high-temperature limit of the Casimir interaction between a Dirichlet sphere and a Dirichlet plate due to the vacuum fluctuations of a scalar field in $(D+1)$-dimensional Minkowski spacetime. The high-temperature leading term of the Casimir free interaction energy is known as the classical term, since it does not depend on the Planck constant $\hbar$. From the functional representation of the zero-temperature Casimir interaction energy, we use Matsubara formalism to derive the finite-temperature Casimir free energy and obtain the classical term. It can be expressed as a weighted sum over logarithms of determinants. Using similarity transforms of matrices, we reexpress this classical term as an infinite series. This series is then computed exactly using a generalized Abel-Plana summation formula. From this, we deduce the short-distance asymptotic expansions of the classical Casimir interaction force. As expected, the leading term agrees with the proximity force approximation. The next two terms in the asymptotic expansion are also computed. It is observed that the ratio of the next-to-leading-order term to the leading-order term is proportional to the dimension of spacetime. Hence, a larger correction to the proximity force approximation is expected in spacetime with higher dimensions. This is similar to a previous result deduced for the zero-temperature case.

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I. INTRODUCTION

Casimir interactions between objects of nontrivial geometries have been under active study in the past ten years. This is partly motivated by the advent of nanotechnology which explores physics and technology in a length scale that renders Casimir interaction highly non-negligible. Another motivation comes from Casimir experiments where Casimir force is usually measured for the sphere-plate configuration due to the absence of alignment problems [1].

In the past few years, multiple scattering formalism has been used to cook up a recipe for computing the exact functional representation for the Casimir interaction energy between any two objects [2–17] in $(3+1)$-dimensional spacetime. In principle, one has to compute the scattering matrices of the objects in specific coordinate systems and the translation matrices between different coordinate frames. For objects with additional symmetries, such as planes, cylinders, and spheres, there are special coordinate systems available, and the problem is tractable. Given the explicit formula for the Casimir interaction energy, one can then explore its properties numerically or analytically. Of particular interest are the small-separation and large-separation limits. The computation of the large-separation limit is usually straightforward. In the small-separation regime, which is of more concern to nanotechnology, it has been long believed that the leading Casimir interaction agrees with the proximity force approximation, and this has been verified for various geometric configurations [11,12,18–27]. However, to better reflect the actual strength of the Casimir force and for comparison to Casimir measurements, there is a need to go beyond proximity force approximation. The computation of the next-to-leading-order term in the small-separation asymptotic expansion is not an easy task [11,12,18–32]. A scheme based on derivative expansion has been proposed [33,34] but is yet to be verified.

Most of the above mentioned works only dealt with the zero-temperature interaction. Nonetheless, the finite-temperature effect cannot be neglected. Of particular appeal is the limit of the Casimir interaction in the high-temperature regime. It has long been known that the high-temperature leading term is linear in the temperature, given by the term with zero Matsubara frequency. This term is known as the classical term, since it does not depend on the Planck constant $\hbar$. The asymptotic expansion of this classical term in the small-separation regime is not much known. However, it was shown in Ref. [35] that the classical Casimir interaction between a sphere and a plate with Dirichlet boundary conditions can be computed exactly, which can then be used to derive the full small-separation asymptotic expansion.

Studying physics in higher-dimensional spacetime has become a norm rather than an exception. The finite-temperature Casimir effect inside a rectangular cavity in $(D+1)$-dimensional Minkowski spacetime has been explored in Ref. [36] more than thirty years ago. Since then, there have been quite a number of works that have considered the finite-temperature Casimir effect in higher-dimensional spacetime [37–54]. As a first step to understand the Casimir effect between two nontrivial objects in

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higher-dimensional spacetime, we studied the zero-temperature Casimir effect between a sphere and a plate [55] and between two spheres [56] in $(D+1)$-dimensional Minkowski spacetime. In this work, we extend the work to the finite-temperature regime and consider the high-temperature limit. As in the three-dimensional case considered in Ref. [35], we expect that the classical term can be computed exactly. Unlike Ref. [35], we do not make use of bispherical coordinates but use similarity transformation of matrices directly to obtain the result. An important tool in our computation is the generalized Abel-Plana summation formula [57–59]. From the exact formula for the classical Casimir interaction, we derive the small-separation asymptotic expansion. In the case $D=3$, we recover the result obtained in Ref. [35].

II. THE CASIMIR FREE ENERGY BETWEEN A SPHERE AND A PLATE

In Ref. [55], we showed that when $D\geq 4$, the zero-temperature Casimir interaction energy between a Dirichlet sphere of radius R and a Dirichlet plate can be written as

$$E_{\text{Cas}}^{T=0}=\frac{\hbar c}{2\pi}\int_0^\infty d\kappa\sum_{m=0}^\infty\frac{(2m+D-3)(m+D-4)!}{(D-3)!m!}\text{Tr}_m\ln\left(\mathbb{I}-\mathbb{M}_m(\kappa)\right),\tag{1}$$

where the elements $M_{m;l,l'}$ of $\mathbb{M}_m$ are

$$M_{m;l,l'}=(-1)^{l+l'}2^{2m+D-3}\Gamma\left(m+\frac{D-2}{2}\right)^2\sqrt{\frac{(l+\frac{D-2}{2})(l'+\frac{D-2}{2})(l-m)!(l'-m)!}{(l+m+D-3)!(l'+m+D-3)!}}\frac{I_{l+\frac{D-2}{2}}(\kappa R)}{K_{l+\frac{D-2}{2}}(\kappa R)}$$
$$\times\int_0^\infty d\theta(\sinh\theta)^{2m+D-2}C_{l-m}^{m+\frac{D-2}{2}}(\cosh\theta)C_{l'-m}^{m+\frac{D-2}{2}}(\cosh\theta)e^{-2\kappa L\cosh\theta}.\tag{2}$$

Here L is the distance from the center of the sphere to the plate. For fixed m, the trace Tr_m is

$$\sum_{l=m}^\infty.$$

When $D=3$, we can also represent the Casimir interaction energy by Eq. (1), provided that the summation $\sum_{m=0}^\infty$ is replaced by the summation $\sum_{m=0}^\infty{}'$, where the prime $'$ indicates that the term $m=0$ is summed with weight $1/2$.

Using Matsubara formalism, the finite-temperature Casimir free interaction energy between a Dirichlet sphere and a Dirichlet plate can be obtained by replacing the integration over κ by summation over

$$\kappa_p=\frac{2\pi pk_BT}{\hbar c}$$

for p from $-\infty$ to ∞. Namely,

$$E_{\text{Cas}}=k_BT\sum_{p=0}^\infty{}'\sum_{m=0}^\infty\frac{(2m+D-3)(m+D-4)!}{(D-3)!m!}\text{Tr}_m\ln\left(1-\mathbb{M}_m(\kappa_p)\right).\tag{3}$$

When κ is large, $M_{m;l,l'}(\kappa)$ decays exponentially. Hence, in the high-temperature regime where $1\ll RT<LT$, the contribution to the Casimir free interaction energy from those terms with $p\neq 0$ in Eq. (3) is exponentially small. The high-temperature limit of the free energy is given by the $p=0$ term in Eq. (3), which is called the classical term, since it is independent of the Planck constant $\hbar$. Namely,

$$E_{\text{Cas}}^{\text{classical}}=\frac{k_BT}{2}\sum_{m=0}^\infty\frac{(2m+D-3)(m+D-4)!}{(D-3)!m!}\lim_{\kappa\to 0}\text{Tr}\ln\left(1-\mathbb{M}_m(\kappa)\right).\tag{4}$$

Here we do not directly set $\kappa=0$, since $M_{m;l,l'}(\kappa)$ might not be well defined when $\kappa=0$. Nevertheless, the limit

$$\lim_{\kappa\to 0}\text{Tr}\ln\left(1-\mathbb{M}_m(\kappa)\right)$$

should be well defined, and this is what we are going to compute.

As $z\to 0$,

$$I_\nu(z)\sim\frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu,\qquad K_\nu(z)\sim\frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}.\tag{5}$$

Hence, we find that as $\kappa\to 0$,

$$\frac{I_{l+\frac{D-2}{2}}(\kappa R)}{K_{l+\frac{D-2}{2}}(\kappa R)}\sim\frac{1}{2^{2l+D-3}\Gamma(l+\frac{D-2}{2})\Gamma(l+\frac{D}{2})}(\kappa R)^{2l+D-2}. \tag{6}$$

By making a change of variables $u=\kappa\cosh\theta$, we have

$$\begin{aligned}&\int_0^\infty d\theta(\sinh\theta)^{2m+D-2}C_{l-m}^{m+\frac{D-2}{2}}(\cosh\theta)C_{l'-m}^{m+\frac{D-2}{2}}(\cosh\theta)e^{-2\kappa L\cosh\theta}\\&=\frac{1}{\kappa^{2m+D-2}}\int_\kappa^\infty du(u^2-\kappa^2)^{m+\frac{D-3}{2}}C_{l-m}^{m+\frac{D-2}{2}}\left(\frac{u}{\kappa}\right)C_{l'-m}^{m+\frac{D-2}{2}}\left(\frac{u}{\kappa}\right)e^{-2Lu}.\end{aligned} \tag{7}$$

To obtain the leading behavior of this integral when $\kappa\to 0$, we need to find the leading term of the Gegenbauer polynomials $C_n^\nu(z)$ when z is large. From Rodrigues's formula for the Gegenbauer polynomial [60,61],

$$C_n^\nu(z)=\frac{1}{2^n}\frac{\Gamma(2\nu+n)\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)\Gamma(\nu+\frac{1}{2}+n)}\frac{(z^2-1)^{\frac{1}{2}-\nu}}{n!}\frac{d^n}{dz^n}(z^2-1)^{n+\nu-\frac{1}{2}}, \tag{8}$$

we find that the leading term of the polynomial $C_n^\nu(z)$ is

$$C_n^\nu(z)=\frac{1}{2^n n!}\frac{\Gamma(2\nu+2n)\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)\Gamma(\nu+\frac{1}{2}+n)}z^n+\cdots. \tag{9}$$

Hence, as $\kappa\to 0$, the leading term of the integral in Eq. (7) is given by

$$\begin{aligned}&\frac{1}{\kappa^{l+l'+D-2}}\frac{1}{2^{l+l'-2m}(l-m)!(l'-m)!}\frac{\Gamma(2l+D-2)\Gamma(2l'+D-2)\Gamma(m+\frac{D-1}{2})^2}{\Gamma(2m+D-2)^2\Gamma(l+\frac{D-1}{2})\Gamma(l'+\frac{D-1}{2})}\int_0^\infty du u^{l+l'+D-3}e^{-2Lu}\\&=\frac{1}{\kappa^{l+l'+D-2}}\frac{1}{2^{2l+2l'-2m+D-2}(l-m)!(l'-m)!}\frac{\Gamma(2l+D-2)\Gamma(2l'+D-2)\Gamma(m+\frac{D-1}{2})^2}{\Gamma(2m+D-2)^2\Gamma(l+\frac{D-1}{2})\Gamma(l'+\frac{D-1}{2})}\frac{\Gamma(l+l'+D-2)}{L^{l+l'+D-2}}\\&=\frac{1}{\kappa^{l+l'+D-2}}\frac{1}{2^{2m+D-2}(l-m)!(l'-m)!}\frac{\Gamma(l+\frac{D-2}{2})\Gamma(l'+\frac{D-2}{2})}{\Gamma(m+\frac{D-2}{2})^2}\frac{\Gamma(l+l'+D-2)}{L^{l+l'+D-2}}.\end{aligned} \tag{10}$$

In the last row, we have used the identity

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{11}$$

From Eqs. (6) and (10), one can deduce that as $\kappa\to 0$,

$$M_{m;l,l'}(\kappa)\sim\kappa^{l-l'}. \tag{12}$$

Since the trace of a matrix is not changed if the matrix is replaced by a similar matrix, we define

$$\tilde{\mathbb{M}}_m=\mathbb{P}_1^{-1}\mathbb{M}_m\mathbb{P}_1, \tag{13}$$

where $\mathbb{P}_1$ is a diagonal matrix with elements

$$\mathbb{P}_{1;l,l'}=(-1)^l\sqrt{\frac{(l+m+D-3)!}{(l+\frac{D-2}{2})(l-m)!}}\frac{1}{\Gamma(l+\frac{D-2}{2})}\left(\frac{\kappa R}{2}\right)^l\delta_{l,l'}. \tag{14}$$

Then

$$\tilde{M}_{m;l,l'}(\kappa)=(-1)^{-l+l'}\sqrt{\frac{(l+\frac{D-2}{2})(l'+m+D-3)!(l-m)!}{(l'+\frac{D-2}{2})(l+m+D-3)!(l'-m)!}}\frac{\Gamma(l+\frac{D-2}{2})}{\Gamma(l'+\frac{D-2}{2})}\left(\frac{\kappa R}{2}\right)^{-l+l'}M_{m;l,l'}(\kappa), \tag{15}$$

and it follows that

$$\tilde{M}_{m;l,l'}(0) = \frac{(l+l'+D-3)!}{(l+m+D-3)!(l'-m)!}\left(\frac{R}{2L}\right)^{l+l'+D-2}. \tag{16}$$

Hence, the classical Casimir interaction energy is

$$E_{\mathrm{Cas}}^{\mathrm{classical}} = \frac{k_B T}{2}\sum_{m=0}^{\infty}\frac{(2m+D-3)(m+D-4)!}{(D-3)!m!} \times \ln\det(\mathbb{I}-\mathbb{N}_m), \tag{17}$$

where

$$N_{m;l,l'} = \frac{(l+l'+D-3)!}{(l+m+D-3)!(l'-m)!}\left(\frac{R}{2L}\right)^{l+l'+D-2}, \quad l,l' \geq m. \tag{18}$$

Since $L = R + d$, where d is the distance from the sphere to the plate, the determinant $\det(\mathbb{I}-\mathbb{N}_m)$ is finite. In the next section, we will derive an alternative expression for this determinant.

III. ALTERNATIVE EXPRESSION FOR THE CLASSICAL CASIMIR INTERACTION ENERGY

In this section, we use similarity transforms of matrices to find an alternative expression for the classical Casimir interaction energy [Eq. (17)]. Essentially, we transform the matrices $\mathbb{N}_m$ to upper triangular matrices and use the fact that the determinant of an upper triangular matrix is equal to the product of its diagonal elements.

Let

$$x = \frac{R}{2L} = \frac{1}{2(1+\varepsilon)}, \qquad \varepsilon = \frac{d}{R}. \tag{19}$$

Let $\mathbb{P}_2$ be a lower triangular matrix with elements

$$(P_2)_{l,l'} = \begin{cases} \frac{y^{l-l'}}{(l-l')!}\frac{(l-m)!}{(l'-m)!}, & l \geq l', \\ 0, & l < l', \end{cases} \tag{20}$$

where $0 < y < 1$ is such that

$$y + y^{-1} = \frac{1}{x}. \tag{21}$$

One can check that the inverse $\mathbb{P}_2^{-1}$ has elements

$$(P_2^{-1})_{l,l'} = \begin{cases} (-1)^{l-l'}\frac{y^{l-l'}}{(l-l')!}\frac{(l-m)!}{(l'-m)!}, & l \geq l' \\ 0 & l < l' \end{cases}. \tag{22}$$

Using the fact that

$$\frac{1}{(1-v)^{n+1}} = \sum_{j=0}^{\infty}\frac{(n+j)!}{n!j!}v^j, \tag{23}$$

we find that

$$\begin{aligned}
(\mathbb{P}_2^{-1}\mathbb{N}_m\mathbb{P}_2)_{l,l'} &= \sum_{l_1=m}^{l}\sum_{l_2=l'}^{\infty}(-1)^{l-l_1}\frac{y^{l-l_1}}{(l-l_1)!}\frac{(l-m)!}{(l_1-m)!}\frac{(l_1+l_2+D-3)!}{(l_1+m+D-3)!(l_2-m)!}x^{l_1+l_2+D-2}\frac{y^{l_2-l'}}{(l_2-l')!}\frac{(l_2-m)!}{(l'-m)!} \\
&= \sum_{l_1=m}^{l}\sum_{l_2=0}^{\infty}(-1)^{l-l_1}\frac{y^{l-l_1}}{(l-l_1)!}\frac{(l-m)!}{(l_1-m)!}\frac{(l_1+l_2+l'+D-3)!}{(l_1+m+D-3)!}x^{l_1+l_2+l'+D-2}\frac{y^{l_2}}{l_2!}\frac{1}{(l'-m)!} \\
&= \sum_{l_1=m}^{l}(-1)^{l-l_1}\frac{y^{l-l_1}}{(l-l_1)!}\frac{(l-m)!}{(l_1-m)!}\frac{(l_1+l'+D-3)!}{(l_1+m+D-3)!}\left(\frac{x}{1-xy}\right)^{l_1+l'+D-2}\frac{1}{(l'-m)!} \\
&= \sum_{l_1=m}^{l}(-1)^{l-l_1}\frac{y^{l+l'+D-2}}{(l-l_1)!}\frac{(l-m)!}{(l_1-m)!}\frac{(l_1+l'+D-3)!}{(l_1+m+D-3)!}\frac{1}{(l'-m)!} \\
&= y^{l+l'+D-2}\times\text{coefficient of } v^{l+m+D-3}\text{ in }(1-v)^{l-m}(1-v)^{-l'+m-1} \\
&= \begin{cases} \frac{y^{l+l'+D-2}}{(l'-l)!}\frac{(l'+m+D-3)!}{(l+m+D-3)!}, & l' \geq l \\ 0 & l' < l \end{cases}.
\end{aligned} \tag{24}$$

Notice that $\mathbb{P}_2^{-1}\mathbb{N}_m\mathbb{P}_2$ is an upper triangular matrix, and the diagonal elements are

$$(\mathbb{P}_2^{-1}\mathbb{N}_m\mathbb{P}_2)_{l,l} = y^{2l+D-2}. \tag{25}$$

Hence,

$$\det(\mathbb{I}-\mathbb{N}_m) = \sum_{l=m}^{\infty}\ln(1-y^{2l+D-2}). \tag{26}$$

In fact, using a suitable matrix $\mathbb{P}_3$, one can transform $\mathbb{P}_2^{-1}\mathbb{N}_m\mathbb{P}_2$ into a diagonal matrix. We leave it to the reader to check that if $\mathbb{P}_3$ is the upper triangular matrix with elements

$$(P_3)_{l,l'} = \begin{cases} \frac{z^{l-l'}}{(l'-l)!}\frac{(l'+m+D-3)!}{(l+m+D-3)!}, & l' \geq l \\ 0 & l' < l \end{cases}, \qquad (27)$$

where

$$z = y - y^{-1},$$

then

$$(\mathbb{P}_3^{-1}\mathbb{P}_2^{-1}\mathbb{N}_m\mathbb{P}_2\mathbb{P}_3)_{l,l'} = \delta_{l,l'}y^{2l+D-2}. \qquad (28)$$

Returning to the classical Casimir interaction energy, we obtain from Eqs. (17) and (26) that

$$\begin{aligned} E_{\rm Cas}^{\rm classical} &= \frac{k_BT}{2}\sum_{m=0}^\infty \frac{(2m+D-3)(m+D-4)!}{(D-3)!m!}\sum_{l=m}^\infty \ln(1-y^{2l+D-2}) \\ &= \frac{k_BT}{2}\sum_{l=0}^\infty\sum_{m=0}^l \frac{(2m+D-3)(m+D-4)!}{(D-3)!m!}\ln(1-y^{2l+D-2}) \\ &= \frac{k_BT}{2}\sum_{l=0}^\infty \frac{(2l+D-2)(l+D-3)!}{(D-2)!l!}\ln(1-y^{2l+D-2}). \end{aligned} \qquad (29)$$

Notice that when $D = 3$, Eq. (29) gives

$$E_{\rm Cas}^{\rm classical} = \frac{k_BT}{2}\sum_{l=0}^\infty (2l+1)\ln(1-y^{2l+1}), \qquad (30)$$

which is exactly the result derived in Ref. [35] using bispherical coordinates.

From the definitions in Eqs. (21) and (19), we find that

$$y = 1+\varepsilon-\sqrt{\varepsilon^2+2\varepsilon} = \frac{L}{R} - \sqrt{\left(\frac{L}{R}\right)^2 - 1}. \qquad (31)$$

When $L \gg R$,

$$y \sim \frac{1}{2}\frac{R}{L}, \qquad (32)$$

which shows that the large-separation leading term of the classical Casimir interaction energy comes from the $l = 0$ term in Eq. (29) and is given by

$$E_{\rm Cas}^{\rm classical} \sim -\frac{k_BTR^{D-2}}{2^{D-1}L^{D-2}}. \qquad (33)$$

IV. SMALL-SEPARATION EXACT FORMULA OF THE CLASSICAL CASIMIR INTERACTION FORCE

In this section, we use a generalized Abel-Plana summation formula to compute the classical Casimir interaction force. We need to consider the case when D is even and the case when D is odd separately.

Let

$$y = e^{-\mu},$$

where

$$\mu = -\ln(1+\varepsilon-\sqrt{\varepsilon^2+2\varepsilon}) > 0. \qquad (34)$$

First, we want to rewrite Eq. (29). When D is even, let

$$\tilde{l} = l + \frac{D-2}{2}.$$

Then

$$E_{\rm Cas}^{\rm classical} = \frac{k_BT}{2}\sum_{\tilde{l}=\frac{D-2}{2}} \frac{2}{(D-2)!}\tilde{l}\left(\tilde{l}+\frac{D-4}{2}\right)\left(\tilde{l}+\frac{D-6}{2}\right)\ldots(\tilde{l}+1)\tilde{l}(\tilde{l}-1)\ldots\left(\tilde{l}-\frac{D-4}{2}\right)\ln(1-e^{-2\tilde{l}\mu}). \qquad (35)$$

Notice that the summand is zero when $\tilde{l} = 1, 2, \ldots, (D-4)/2$. This allows us to start the summation from $\tilde{l} = 1$ instead of $(D-2)/2$. Moreover,

$$\tilde{l}\left(\tilde{l}+\frac{D-4}{2}\right)\left(\tilde{l}+\frac{D-6}{2}\right)\dots(\tilde{l}+1)\tilde{l}(\tilde{l}-1)\dots\left(\tilde{l}-\frac{D-4}{2}\right)$$

is a polynomial of degree $D-2$ in $\tilde{l}$ which can be written as

$$\sum_{j=1}^{D-2}x_{D;j}\tilde{l}^j,$$

with $x_{D;j}=0$ when j is odd. Hence,

$$E_{\mathrm{Cas}}^{\mathrm{classical}}=\frac{k_BT}{(D-2)!}\sum_{\tilde{l}=1}^{\infty}\sum_{j=1}^{D-2}x_{D;j}\tilde{l}^j\ln(1-e^{-2\tilde{l}\mu}). \tag{36}$$

When D is odd, let

$$\tilde{l}=l+\frac{D-3}{2}.$$

We find in the same way that

$$\begin{aligned}E_{\mathrm{Cas}}^{\mathrm{classical}}&=\frac{k_BT}{2}\sum_{\tilde{l}=\frac{D-3}{2}}\frac{2}{(D-2)!}\left(\tilde{l}+\frac{1}{2}\right)\left(\tilde{l}+\frac{1}{2}+\frac{D-4}{2}\right)\left(\tilde{l}+\frac{1}{2}+\frac{D-6}{2}\right)\dots\left(\tilde{l}+\frac{1}{2}+\frac{1}{2}\right)\left(\tilde{l}+\frac{1}{2}-\frac{1}{2}\right)\dots\left(\tilde{l}+\frac{1}{2}-\frac{D-4}{2}\right)\\&\quad\times\ln(1-e^{-(2\tilde{l}+1)\mu})\\&=\frac{k_BT}{(D-2)!}\sum_{\tilde{l}=0}^{\infty}\sum_{j=1}^{D-2}x_{D;j}\left(\tilde{l}+\frac{1}{2}\right)^j\ln(1-e^{-(2\tilde{l}+1)\mu}),\end{aligned} \tag{37}$$

where now

$$\sum_{j=1}^{D-2}x_{D;j}\left(\tilde{l}+\frac{1}{2}\right)^j=\left(\tilde{l}+\frac{1}{2}\right)\left(\tilde{l}+\frac{1}{2}+\frac{D-4}{2}\right)\left(\tilde{l}+\frac{1}{2}+\frac{D-6}{2}\right)\dots\left(\tilde{l}+\frac{1}{2}+\frac{1}{2}\right)\left(\tilde{l}+\frac{1}{2}-\frac{1}{2}\right)\dots\left(\tilde{l}+\frac{1}{2}-\frac{D-4}{2}\right), \tag{38}$$

with $x_{D;j}=0$ when j is even.

As a function of the complex variable z, $\ln(1-e^{-az})$ does not have good analytic properties. So instead of considering the classical Casimir energy, we consider the classical Casimir force. Since

$$\mu'(\varepsilon)=\frac{1}{\sqrt{\varepsilon^2+2\varepsilon}}, \tag{39}$$

we find that when D is even,

$$F_{\mathrm{Cas}}^{\mathrm{classical}}=-\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\frac{2}{(D-2)!}\sum_{\tilde{l}=0}^{\infty}\sum_{j=1}^{D-2}x_{D;j}\frac{\tilde{l}^{j+1}}{e^{2\tilde{l}\mu}-1}; \tag{40}$$

and when D is odd,

$$F_{\mathrm{Cas}}^{\mathrm{classical}}=-\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\frac{2}{(D-2)!}\sum_{\tilde{l}=0}^{\infty}\sum_{j=1}^{D-2}x_{D;j}\frac{(\tilde{l}+\frac{1}{2})^{j+1}}{e^{(2\tilde{l}+1)\mu}-1}. \tag{41}$$

Notice that $x_{D;j}\neq0$ only if D and j have the same parity.

Now we have to deal with functions of the form

$$\frac{z^n}{e^{az}-1},$$

which is not analytic but is meromorphic. We cannot apply the Abel-Plana summation formula, but instead we can apply the generalized Abel-Plana summation formula [57–59], which says that if $f(z)$ is a meromorphic function that only has poles on the imaginary axis,

$$\frac{1}{2}f(0)+\sum_{p=1}^{\infty}f(p)=\int_0^{\infty}f(x)dx+i\int_0^{\infty}\frac{f(iy)-f(-iy)}{e^{2\pi y}-1}dy+\pi i\sum_{y>0}\frac{\mathrm{Res}_{z=iy}f(z)-\mathrm{Res}_{z=-iy}f(z)}{e^{2\pi y}-1},\tag{42}$$

$$\sum_{p=0}^{\infty}f(2p+1)=\frac{1}{2}\int_0^{\infty}f(x)dx-\frac{i}{2}\int_0^{\infty}\frac{f(iy)-f(-iy)}{e^{\pi y}+1}dy-\frac{\pi i}{2}\sum_{y>0}\frac{\mathrm{Res}_{z=iy}f(z)-\mathrm{Res}_{z=-iy}f(z)}{e^{\pi y}+1}.\tag{43}$$

When D is even, we apply Eq. (42) with

$$f(z)=\frac{z^{j+1}}{e^{2\mu z}-1},\tag{44}$$

where $j\geq 1$ is even. $f(z)$ has poles at $z=\pm i\pi n/\mu$, $n=1,2,\ldots$, and

$$\mathrm{Res}_{z=\pm\frac{i\pi n}{\mu}}f(z)=\pm\frac{i^{j+1}\pi^{j+1}n^{j+1}}{2\mu^{j+2}}.\tag{45}$$

Hence,

$$\pi i\sum_{y>0}\frac{\mathrm{Res}_{z=iy}f(z)-\mathrm{Res}_{z=-iy}f(z)}{e^{2\pi y}-1}=\frac{(-1)^{\frac{j}{2}+1}\pi^{j+2}}{\mu^{j+2}}\sum_{n=1}^{\infty}\frac{n^{j+1}}{e^{\frac{2\pi^2 n}{\mu}}-1}.\tag{46}$$

This sum goes to zero exponentially fast when $\mu\to 0$. On the other hand,

$$\int_0^{\infty}f(x)dx=\int_0^{\infty}\frac{x^{j+1}}{e^{2\mu x}-1}dx=\frac{\Gamma(j+2)}{2^{j+2}\mu^{j+2}}\zeta(j+2),\tag{47}$$

$$i\int_0^{\infty}\frac{f(iy)-f(-iy)}{e^{2\pi y}-1}dy=(-1)^{\frac{j}{2}}\int_0^{\infty}\frac{y^{j+1}}{e^{2\pi y}-1}dy=(-1)^{\frac{j}{2}}\frac{\Gamma(j+2)}{2^{j+2}\pi^{j+2}}\zeta(j+2).\tag{48}$$

Here $\zeta(s)=\sum_{n=1}^{\infty}1/n^s$ is the Riemann zeta function. Since $f(0)=0$, the generalized Abel-Plana summation formula [Eq. (42)] implies that

$$\frac{\tilde{l}^{j+1}}{e^{2\tilde{l}\mu}-1}=\frac{\Gamma(j+2)}{2^{j+2}\mu^{j+2}}\zeta(j+2)+(-1)^{\frac{j}{2}}\frac{\Gamma(j+2)}{2^{j+2}\pi^{j+2}}\zeta(j+2)+\frac{(-1)^{\frac{j}{2}+1}\pi^{j+2}}{\mu^{j+2}}\sum_{n=1}^{\infty}\frac{n^{j+1}}{e^{\frac{2\pi^2 n}{\mu}}-1}.\tag{49}$$

From this, we obtain the exact expression for the classical Casimir interaction force:

$$F_{\mathrm{Cas}}^{\mathrm{classical}}=-\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\frac{2}{(D-2)!}\sum_{j=1}^{D-2}x_{D;j}\left\{\frac{\Gamma(j+2)}{2^{j+2}\mu^{j+2}}\zeta(j+2)+(-1)^{\frac{j}{2}}\frac{\Gamma(j+2)}{2^{j+2}\pi^{j+2}}\zeta(j+2)+\frac{(-1)^{\frac{j}{2}+1}\pi^{j+2}}{\mu^{j+2}}\sum_{n=1}^{\infty}\frac{n^{j+1}}{e^{\frac{2\pi^2 n}{\mu}}-1}\right\}.\tag{50}$$

When D is odd, we apply the generalized Abel-Plana summation formula [Eq. (43)] with

$$f(z)=\frac{1}{2^{j+1}}\frac{z^{j+1}}{e^{\mu z}-1},\tag{51}$$

where $j\geq 1$ is odd. Then $f(z)$ has poles at $z=\pm 2i\pi n/\mu$ with

$$\mathrm{Res}_{z=\pm\frac{2i\pi n}{\mu}}f(z)=\frac{i^{j+1}\pi^{j+1}n^{j+1}}{\mu^{j+2}}.\tag{52}$$

Consequently,

$$-\frac{\pi i}{2}\sum_{y>0}\frac{\mathrm{Res}_{z=iy}f(z)-\mathrm{Res}_{z=-iy}f(z)}{e^{\pi y}+1}=0.\tag{53}$$

On the other hand,

$$\frac{1}{2}\int_0^{\infty}f(x)dx=\frac{1}{2^{j+2}}\int_0^{\infty}\frac{x^{j+1}}{e^{\mu x}-1}dx=\frac{\Gamma(j+2)}{2^{j+2}\mu^{j+2}}\zeta(j+2),\tag{54}$$

$$f(iy)-f(-iy)=\frac{i^{j+1}}{2^{j+1}}\frac{y^{j+1}}{e^{i\mu y}-1}-\frac{i^{j+1}}{2^{j+1}}\frac{y^{j+1}}{e^{-i\mu y}-1}=\frac{i^j}{2^{j+1}}y^{j+1}\cot\frac{\mu y}{2},\tag{55}$$

which gives

$$-\frac{i}{2}\int_0^\infty \frac{f(iy)-f(-iy)}{e^{\pi y}+1}dy = \frac{(-1)^{\frac{j-1}{2}}}{2^{j+2}}\int_0^\infty \frac{y^{j+1}\cot\frac{\mu y}{2}}{e^{\pi y}+1}dy. \tag{56}$$

From these, we obtain the exact formula for the classical Casimir interaction force:

$$F_{\rm Cas}^{\rm classical} = -\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\frac{2}{(D-2)!}\sum_{j=1}^{D-2}x_{D;j}\left\{\frac{\Gamma(j+2)}{2^{j+2}\mu^{j+2}}\zeta(j+2)+\frac{(-1)^{\frac{j-1}{2}}}{2^{j+2}}\int_0^\infty \frac{y^{j+1}\cot\frac{\mu y}{2}}{e^{\pi y}+1}dy\right\}. \tag{57}$$

From the definition of μ [Eq. (34)], we find that as $\varepsilon \ll 1$,

$$\mu \sim \sqrt{2\varepsilon}. \tag{58}$$

Hence, Eqs. (50) and (57) are ideal for studying the small-separation asymptotic behavior of the Casimir interaction force. In particular, we find that when $\mu \ll 1$,

$$F_{\rm Cas}^{\rm classical} = -\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\frac{2}{(D-2)!}\sum_{j=1}^{D-2}x_{D;j}\left\{\frac{\Gamma(j+2)}{2^{j+2}\mu^{j+2}}\zeta(j+2)+(-1)^{\frac{j}{2}}\frac{\Gamma(j+2)}{2^{j+2}\pi^{j+2}}\zeta(j+2)\right\}+O(\mu) \tag{59}$$

if D is even; and

$$F_{\rm Cas}^{\rm classical} = -\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\frac{2}{(D-2)!}\sum_{j=1}^{D-2}x_{D;j}\left\{\frac{\Gamma(j+2)}{2^{j+2}\mu^{j+2}}\zeta(j+2)+\frac{(-1)^{\frac{j-1}{2}}}{2^{j+1}\mu}\int_0^\infty \frac{y^j}{e^{\pi y}+1}dy\right\}+O(\mu) \tag{60}$$

if D is odd. In the latter, we have used the fact that

$$\cot\frac{\mu y}{2}\sim\frac{2}{\mu y}+O(\mu) \tag{61}$$

as $\mu \ll 1$.

V. COMPARISON TO PROXIMITY FORCE APPROXIMATION

The proximity force approximation approximates the Casimir interaction force between two objects by summing the local Casimir force density between two planes over the surfaces. In $(D+1)$-dimensional Minkowski spacetime, the classical Casimir force density between two parallel plates both subject to Dirichlet boundary conditions is given by [62]

$$\mathcal{F}_{\rm Cas}^{\rm classical,\parallel}(d) = -k_BT\frac{(D-1)\Gamma(\frac{D}{2})\zeta(D)}{2^D\pi^{\frac{D}{2}}}\frac{1}{d^D}=\frac{b_D}{d^D}, \tag{62}$$

where d is the distance between the two plates.

As in Ref. [55], we find that the proximity force approximation to the classical Casimir interaction force between a Dirichlet sphere and a Dirichlet plate in $(D+1)$-dimensional spacetime is

$$\begin{aligned} F_{\rm Cas}^{\rm classical,PFA} &= R^{D-1}b_D\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}\int_0^\pi \frac{d\theta_1\sin^{D-2}\theta_1}{(d+R(1-\cos\theta_1))^D}\\ &\sim b_D\frac{\pi^{\frac{D}{2}}}{2^{\frac{D-1}{2}}\Gamma(\frac{D}{2})}\frac{1}{R\varepsilon^{\frac{D+1}{2}}}\\ &= -k_BT\frac{(D-1)}{2^{\frac{3D-1}{2}}}\frac{\zeta(D)}{R\varepsilon^{\frac{D+1}{2}}}. \end{aligned} \tag{63}$$

From Eqs. (50) and (57), we find that when $\varepsilon \ll 1$, the leading term of the classical Casimir interaction force comes from the term with $j = D-2$. Since $x_{D;D-2}=1$, we have

$$\begin{aligned} F_{\rm Cas}^{\rm classical} &\sim -\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\frac{D-1}{2^{D-1}\mu^D}\zeta(D)\\ &\sim -k_BT\frac{(D-1)}{2^{\frac{3D-1}{2}}}\frac{\zeta(D)}{R\varepsilon^{\frac{D+1}{2}}}, \end{aligned} \tag{64}$$

which agrees with the proximity force approximation [Eq. (63)].

VI. SMALL-SEPARATION ASYMPTOTIC EXPANSION

In this section, we derive the small-separation asymptotic expansion of the classical Casimir interaction force from Eqs. (50) and (57) in terms of $\varepsilon = d/R$.

As $\varepsilon \ll 1$,

$$\frac{1}{\sqrt{\varepsilon^2+2\varepsilon}}=\frac{1}{\sqrt{2\varepsilon}}\left(1-\frac{\varepsilon}{4}+\frac{3}{32}\varepsilon^2+\cdots\right), \tag{65}$$

$$\mu=\sqrt{2\varepsilon}\left(1-\frac{\varepsilon}{12}+\frac{3\varepsilon^2}{160}+\cdots\right), \tag{66}$$

which gives

$$\frac{1}{\mu^D}=\frac{1}{2^{\frac{D}{2}}\varepsilon^{\frac{D}{2}}}\left(1+\frac{D\varepsilon}{12}+\frac{D(5D-22)}{1440}\varepsilon^2+\cdots\right). \tag{67}$$

For the constants $x_{D;j}$, it is straightforward to show that

$$\begin{aligned}
x_{D;D-2}&=1,\\
x_{D;D-3}&=0,\\
x_{D;D-4}&=-\frac{(D-2)(D-3)(D-4)}{24},\\
x_{D;D-5}&=0,\\
x_{D;D-6}&=\frac{(D-2)(D-3)(D-4)(D-5)(D-6)(5D-8)}{5760},\\
&\vdots
\end{aligned} \tag{68}$$

On the other hand,

$$\cot\frac{\mu y}{2}=\frac{2}{\mu y}-\frac{\mu y}{6}+\cdots. \tag{69}$$

Hence,

$$\begin{aligned}
\frac{(-1)^{\frac{j-1}{2}}}{2^{j+2}}\int_0^\infty\frac{y^{j+1}\cot\frac{\mu y}{2}}{e^{\pi y}+1}dy&=\frac{(-1)^{\frac{j-1}{2}}}{2^{j+2}}\int_0^\infty\frac{y^{j+1}(\frac{2}{\mu y}-\frac{\mu y}{6}+\cdots)}{e^{\pi y}+1}dy\\
&=\frac{(-1)^{\frac{j-1}{2}}}{2^{j+1}}\frac{1}{\mu}\frac{\Gamma(j+1)}{\pi^{j+1}}(1-2^{-j})\zeta(j+1)+\frac{(-1)^{\frac{j+1}{2}}}{3\times 2^{j+3}}\mu\frac{\Gamma(j+3)}{\pi^{j+3}}(1-2^{-j-2})\zeta(j+3)+O(\mu^3).
\end{aligned} \tag{70}$$

When $D=3$, Eqs. (57), (70), (65), and (67) give

$$\begin{aligned}
F_{\mathrm{Cas}}^{\mathrm{classical}}&=-\frac{2k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\left\{\frac{1}{4\mu^3}\zeta(3)+\frac{1}{8}\int_0^\infty\frac{y^4\cot\frac{\mu y}{2}}{e^{\pi y}+1}dy\right\}\\
&=-\frac{2k_BT}{R\sqrt{2\varepsilon}}\left(1-\frac{\varepsilon}{4}+\frac{3}{32}\varepsilon^2+\cdots\right)\\
&\quad\times\left(\frac{1}{8\sqrt{2}\varepsilon^{\frac{3}{2}}}\zeta(3)\left(1+\frac{\varepsilon}{4}-\frac{7\varepsilon^2}{480}+\cdots\right)+\frac{1}{48\sqrt{2\varepsilon}}\left(1+\frac{\varepsilon}{12}+\cdots\right)-\frac{7}{5760}\sqrt{2\varepsilon}+\cdots\right)\\
&=-\frac{k_BT}{8R\varepsilon^2}\zeta(3)\left(1+\frac{1}{6\zeta(3)}\varepsilon+\left(\frac{1}{60}-\frac{17}{360\zeta(3)}\right)\varepsilon^2+\cdots\right).
\end{aligned} \tag{71}$$

When $D=4$, Eqs. (50), (65), and (67) give

$$\begin{aligned}
F_{\mathrm{Cas}}^{\mathrm{classical}}&=-\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\left(\frac{3}{8\mu^4}\zeta(4)-\frac{3}{8\pi^4}\zeta(4)+\cdots\right)\\
&=-k_BT\frac{3\zeta(4)}{32\sqrt{2}R\varepsilon^{\frac{5}{2}}}\left(1-\frac{\varepsilon}{4}+\frac{3}{32}\varepsilon^2+\cdots\right)\left(1+\frac{\varepsilon}{3}-\frac{\varepsilon^2}{180}-\frac{4\varepsilon^2}{\pi^4}+\cdots\right)\\
&=-k_BT\frac{3\zeta(4)}{32\sqrt{2}R\varepsilon^{\frac{5}{2}}}\left(1+\frac{\varepsilon}{12}+\left(\frac{7}{1440}-\frac{4}{\pi^4}\right)\varepsilon^2+\cdots\right).
\end{aligned} \tag{72}$$

When $D=5$, Eqs. (57), (70), (65), and (67) give

$$\begin{aligned}F_{\mathrm{Cas}}^{\mathrm{classical}}&=-\frac{k_BT}{3R\sqrt{\varepsilon^2+2\varepsilon}}\left\{\frac{3}{4\mu^5}\zeta(5)-\frac{1}{32}\int_0^\infty\frac{y^4\cot\frac{\mu y}{2}}{e^{\pi y}+1}dy-\frac{1}{4}\left(\frac{1}{4\mu^3}\zeta(3)+\frac{1}{8}\int_0^\infty\frac{y^2\cot\frac{\mu y}{2}}{e^{\pi y}+1}dy\right)\right\}\\&=-\frac{k_BT}{3R\sqrt{2\varepsilon}}\left(1-\frac{\varepsilon}{4}+\frac{3}{32}\varepsilon^2+\cdots\right)\\&\quad\times\left(\frac{3\zeta(5)}{16\sqrt{2}\varepsilon^{\frac{5}{2}}}\left(1+\frac{5\varepsilon}{12}+\frac{\varepsilon^2}{96}+\cdots\right)-\frac{7}{1920}\frac{1}{\sqrt{2\varepsilon}}-\frac{\zeta(3)}{32\sqrt{2}\varepsilon^{\frac{3}{2}}}\left(1+\frac{\varepsilon}{4}+\cdots\right)-\frac{1}{192}\frac{1}{\sqrt{2\varepsilon}}+\cdots\right)\\&=-\frac{k_BT}{32R\varepsilon^3}\zeta(5)\left(1+\frac{\varepsilon}{6}-\frac{\zeta(3)}{6\zeta(5)}\varepsilon-\frac{17}{360\zeta(5)}\varepsilon^2+\cdots\right).\end{aligned}\tag{73}$$

When $D\geq 6$, we do not need to take into account the term Eq. (70) in Eq. (57), or the second and third terms in Eq. (50). Equations (50), (57), (65), (67), and (68) give

$$\begin{aligned}F_{\mathrm{Cas}}^{\mathrm{classical}}&=-\frac{k_BT}{R\sqrt{\varepsilon^2+2\varepsilon}}\frac{2}{(D-2)!}\left\{\frac{\Gamma(D)}{2^D\mu^D}\zeta(D)-\frac{(D-2)(D-3)(D-4)}{24}\frac{\Gamma(D-2)}{2^{D-2}\mu^{D-2}}\zeta(D-2)\right.\\&\quad\left.+\frac{(D-2)(D-3)(D-4)(D-5)(D-6)(5D-8)}{5760}\frac{\Gamma(D-4)}{2^{D-4}\mu^{D-4}}\zeta(D-4)+\cdots\right\}\\&=-k_BT\frac{(D-1)\zeta(D)}{2^{\frac{3D-1}{2}}R\varepsilon^{\frac{D+1}{2}}}\left(1-\frac{\varepsilon}{4}+\frac{3}{32}\varepsilon^2+\cdots\right)\left(1+\frac{D\varepsilon}{12}-\frac{(D-3)(D-4)}{3(D-1)}\frac{\zeta(D-2)}{\zeta(D)}\varepsilon\right.\\&\quad\left.+\frac{D(5D-22)}{1440}\varepsilon^2-\frac{(D-2)(D-3)(D-4)}{36(D-1)}\frac{\zeta(D-2)}{\zeta(D)}\varepsilon^2+\frac{(D-5)(D-6)(5D-8)}{90(D-1)}\frac{\zeta(D-4)}{\zeta(D)}\varepsilon^2+\cdots\right)\\&=-k_BT\frac{(D-1)\zeta(D)}{2^{\frac{3D-1}{2}}R\varepsilon^{\frac{D+1}{2}}}\left(1+\frac{(D-3)}{12}\varepsilon-\frac{(D-3)(D-4)}{3(D-1)}\frac{\zeta(D-2)}{\zeta(D)}\varepsilon\right.\\&\quad\left.+\frac{(D-5)(5D-27)}{1440}\varepsilon^2-\frac{(D-3)(D-4)(D-5)}{36(D-1)}\frac{\zeta(D-2)}{\zeta(D)}\varepsilon^2+\frac{(D-5)(D-6)(5D-8)}{90(D-1)}\frac{\zeta(D-4)}{\zeta(D)}\varepsilon^2+\cdots\right).\end{aligned}\tag{74}$$

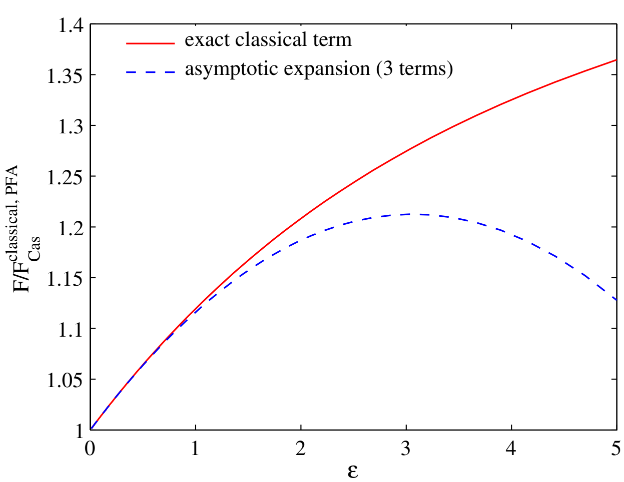

FIG. 1 (color online). Comparison between the exact Casimir interaction force and the asymptotic expansion in Eq. (71) when $D=3$. Both quantities are normalized by the proximity force approximation.

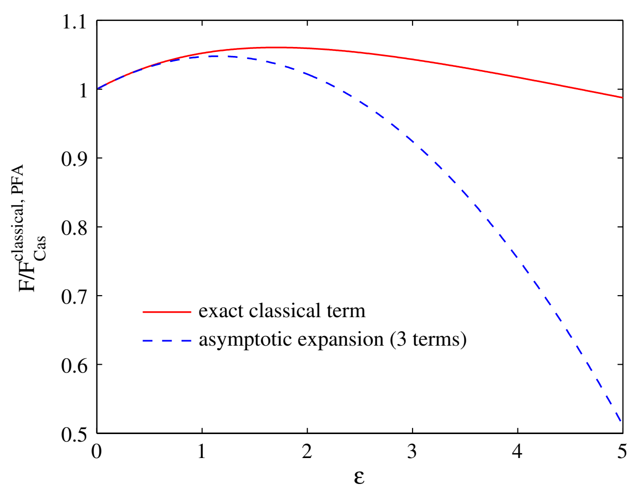

FIG. 2 (color online). Comparison between the exact Casimir interaction force and the asymptotic expansion in Eq. (72) when $D=4$. Both quantities are normalized by the proximity force approximation.

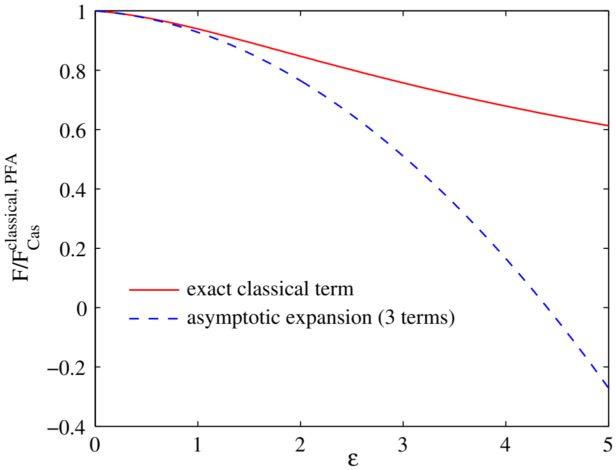

FIG. 3 (color online). Comparison between the exact Casimir interaction force with the asymptotic expansion in Eq. (73) when $D = 5$. Both quantities are normalized by the proximity force approximation.

In Figs. 1, 2, 3, and 4, we compare the exact classical Casimir interaction force to the three-term asymptotic expansions derived in Eqs. (71), (72), (73), and (74) when $D = 3, 4, 5, 6$. Both quantities are normalized by the proximity force approximation. It is observed that when $\varepsilon = 1$, there is a considerable amount of correction to the proximity force approximation, but the three-term asymptotic expansion still gives a quite good approximation to the exact classical Casimir term. However, the three-term approximation will break down when ε is larger.

In Fig. 5, we plot the dependence of the three-term asymptotic expansion [Eq. (74)], normalized by the proximity force approximation, on the normalized distance ε and dimension D. It is observed that the correction to the proximity force approximation becomes larger when D is larger. In fact, from Eq. (74), we find that when D is large,

$$F_{\mathrm{Cas}}^{\mathrm{classical}} \sim F_{\mathrm{Cas}}^{\mathrm{classical,\,PFA}} \left(1 - \frac{D}{4}\varepsilon + \frac{D^2}{32}\varepsilon^2 + \cdots\right). \tag{75}$$

This shows that the proximity force approximation becomes less accurate in spacetime with higher dimensions.

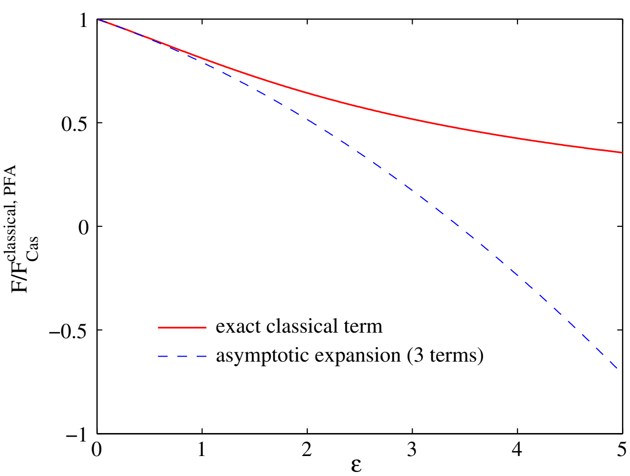

FIG. 4 (color online). Comparison between the exact Casimir interaction force and the asymptotic expansion in Eq. (74) when $D = 6$. Both quantities are normalized by the proximity force approximation.

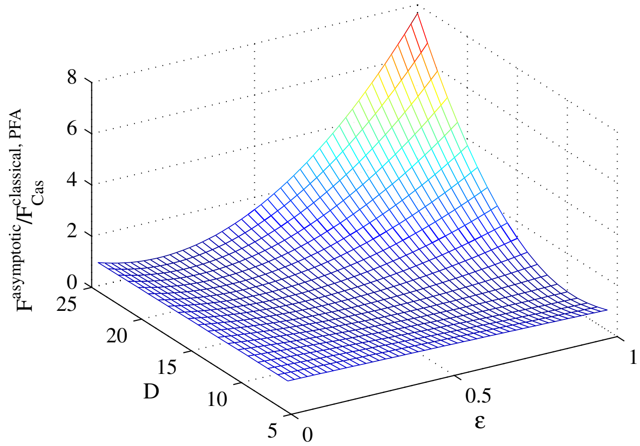

FIG. 5 (color online). Dependence of the asymptotic expansion in Eq. (74) on ε and D for $0 \leq \varepsilon \leq 1$ and $6 \leq D \leq 25$. The asymptotic expansion is normalized by the proximity force approximation.

VII. CONCLUSION

In this work, we have computed the high-temperature limit for the Casimir free interaction energy between a Dirichlet sphere and a Dirichlet plate in $(D+1)$-dimensional Minkowski spacetime. This high-temperature limit is known as the classical term, since it does not depend on the Planck constant. It comes from the term with Matsubara frequency zero in the functional representation of the Casimir free energy and can be expressed as a weighted sum of logarithms of determinants. We derive two alternative exact expressions for this classical term. First, we express the logarithms of the determinants as sums of the logarithms of the eigenvalues. We then use the generalized Abel-Plana summation formula to rewrite this sum so that one can deduce the small-separation asymptotic behaviors of the classical interaction force. The first three terms of the small-separation asymptotic expansion are derived explicitly. As expected, the leading term agrees with the proximity force approximation and is proportional to $d^{-\frac{D+1}{2}}$, where d is the distance between the sphere and the plate. The dimension dependence of the next two terms in the expansion is studied, and it is found that in higher dimensions, proximity force approximation becomes less accurate.

In this work, we only study the classical term of the finite-temperature Casimir free interaction energy, which is the limit of the Casimir interaction when $1 \ll RT < LT$, where R is the radius of the sphere and L is the distance from the center of the sphere to the plane. The small-separation asymptotic behavior is the asymptotic behavior of the Casimir free interaction energy when $RT \gg 1$ and $d \ll R$, where $d = L - R$ is the distance between the sphere and the plate. When studying finite-temperature Casimir interaction, there are three zones of interest: i.e., $RT \ll LT \ll 1$, $RT \ll 1 \ll LT$, and $1 \ll RT < LT$. The first one is dominated by the zero-temperature behavior studied in Ref. [55]. The last one is the high-temperature region studied in this work. The intermediate region $RT \ll 1 \ll LT$ is the transition from the first one to the last one. To study the small-separation asymptotic behavior of the Casimir free interaction energy in this region is quite challenging, and we leave it for a future work.

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