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# Quantum Hall effect on the Grassmannians $\mathbf{Gr_2}(\mathbb{C}^N)$

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Quantum Hall effects on the complex Grassmann manifolds $\mathbf{Gr}_2(\mathbb{C}^N)$ are formulated. We set up the Landau problem in $\mathbf{Gr}_2(\mathbb{C}^N)$ and solve it using group theoretical techniques and provide the energy spectrum and the eigenstates in terms of the $SU(N)$ Wigner $\mathcal{D}$ functions for charged particles on $\mathbf{Gr}_2(\mathbb{C}^N)$ under the influence of Abelian and non-Abelian background magnetic monopoles or a combination of these. In particular, for the simplest case of $\mathbf{Gr}_2(\mathbb{C}^4)$, we explicitly write down the $U(1)$ background gauge field as well as the single- and many-particle eigenstates by introducing the Plücker coordinates and show by calculating the two-point correlation function that the lowest Landau level at filling factor $\nu = 1$ forms an incompressible fluid. Our results are in agreement with the previous results in the literature for the quantum Hall effect on $\mathbb{C}P^N$ and generalize them to all $\mathbf{Gr}_2(\mathbb{C}^N)$ in a suitable manner. Finally, we heuristically identify a relation between the $U(1)$ Hall effect on $\mathbf{Gr}_2(\mathbb{C}^4)$ and the Hall effect on the odd sphere $S^5$, which is yet to be investigated in detail, by appealing to the already-known analogous relations between the Hall effects on $\mathbb{C}P^3$ and $\mathbb{C}P^7$ and those on the spheres $S^4$ and $S^8$, respectively.

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## I. INTRODUCTION

Some time ago, Hu and Zhang introduced a four-dimensional generalization of the quantum Hall effect (QHE) [1]. They formulated and solved the Landau problem on $S^4$ for fermions carrying an additional $SU(2)$ degree of freedom and under the influence of an $SU(2)$ background gauge field. For the multiparticle problem in the lowest Landau level (LLL) with filling factor $\nu = 1$, it turns out that in the thermodynamic limit a finite spatial density is achieved only if the particles are in infinitely large irreducible representations of $SU(2)$ [i.e., they carry an infinitely large number of $SU(2)$ internal degrees of freedom]. In this limit, a two-point density correlation function immediately indicates the incompressibility property of this four-dimensional quantum Hall liquid, as the probability of finding two particles at a distance much shorter than the magnetic length in this model approaches zero. The appearance of massless chiral bosons at the edge of a two-dimensional quantum Hall droplet [2–5] also generalizes to this setting. Nevertheless, it is found that, among the edge excitations of this four-dimensional quantum Hall droplet, not only photons and gravitons but also other massless higher-spin states occur. The latter is essentially due to the presence of a large number of $SU(2)$ degrees of freedom attached to each particle, and, as such, it is not a desirable feature of the model. In a subsequent article [6], two equivalent effective Chern– Simons (CS) field theory descriptions, an Abelian CS theory in $6+1$ dimensions and a $SU(2)$ non-Abelian CS theory in $4+1$ dimensions, of the quantum Hall droplet on $S^4$ have been constructed as generalizations of the well-known Chern–Simons–Landau–Ginzburg model for the fractional QHE, which successfully captures the long-wavelength-limit structure of quantum Hall fluid as a topological field theory [2,7].

Other developments ensued in the ground-breaking work of Hu and Zhang. Several authors have addressed other higher-dimensional generalizations of the QHE to a variety of manifolds including complex projective spaces $\mathbb{C}P^N$, $S^8$, $S^3$, $\mathbb{R}^4$, the Flag manifold $\frac{SU(3)}{U(1)\times U(1)}$, and quantum Hall systems based on higher-dimensional fuzzy spheres [8–13]. Of particular interest to us is the work of Nair and Karabali on the formulation of the QHE problem on $\mathbb{C}P^N$ [8]. These authors solve the Landau problem on $\mathbb{C}P^N$ by appealing to the coset realization of $\mathbb{C}P^N$ over $SU(N+1)$ and performing a suitable restriction of the Wigner $\mathcal{D}$ functions on the latter. In this manner, wave functions for charged particles under the influence of both $U(1)$ Abelian and/or non-Abelian $SU(N)$ gauge backgrounds are obtained as sections of $U(1)$ and/or $SU(N)$ bundles over $\mathbb{C}P^N$. This formulation simultaneously permits the authors to give the energy spectrum of the Landau level (LL), where the degeneracy in each LL is identified with the dimension of the irreducible representation (IRR) to which the aforementioned restricted Wigner $\mathcal{D}$ functions belong. An important feature of these results is that the spatial density of particles remains finite without the need for infinitely large internal $SU(N)$ degrees of freedom, contrary to the situation encountered for the Hall effect on $S^4$. It also turns out that there is a close connection between the Hall effects on $\mathbb{C}P^3$ and $\mathbb{C}P^7$ with Abelian backgrounds and those on

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the spheres $S^4$ and $S^8$ with $SU(2)$ and $SO(8)$ backgrounds, respectively [8,9,13]. Effective actions for the edge dynamics in the limit of a large number of fermions at the LLL for $\nu = 1$ on $\mathbb{C}P^N$ were obtained also by Nair and Karabali in Refs. [14,15] for Abelian and non-Abelian backgrounds, respectively, and on $S^3$ with a non-Abelian background, which is taken as the spin connection, by Nair and Randjbar-Daemi [10]. These theories involve either Abelian bosonic fields or they are higher-dimensional generalizations of gauged Wess-Zumino-Witten models, which are chiral in a sense related to the geometry of these spaces. These investigations reveal that the effective edge action for the QHE on $S^4$ obtained from that of $\mathbb{C}P^3$ does not describe a relativistic field theory, although there are states that do satisfy the relativistic dispersion relation. Nevertheless, these models possess several features that make them interesting in their own right and worthy for further investigations.

In this paper, we focus on the formulation of the QHE on the complex Grassmannians $\mathbf{Gr}_k(\mathbb{C}^N)$, which are generalizations of complex projective spaces $\mathbb{C}P^N$ and share many of their nice features, such as being a Kähler manifold. Several of these features are effectively captured by their so-called Plücker embedding into $\mathbb{C}P^{\binom{N}{k}-1}$. For the case $k = 2$, to which we will be restricting ourselves in this paper, the Plücker embedding describes $\mathbf{Gr}_k(\mathbb{C}^N)$ as a projective algebraic hypersurface in $\mathbb{C}P^N$. For $\mathbf{Gr}_k(\mathbb{C}^4)$, this is the well-known Klein quadric in $\mathbb{C}P^5$ [16]. We use group theoretical techniques to solve the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ and provide the energy spectrum and the eigenfunctions in terms of $SU(N)$ Wigner $\mathcal{D}$ functions for charged particles on $\mathbf{Gr}_2(\mathbb{C}^N)$ under the influence of Abelian and/or non-Abelian background magnetic monopoles, where the latter are obtained as sections of bundles over $\mathbf{Gr}_2(\mathbb{C}^N)$. In addition to the developments regarding the QHE in higher dimensions, there are compelling reasons that motivate us to take up the formulation of the QHE problem on the Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$, which we explain in the ensuing paragraphs.

The Landau problem on two- and higher-dimensional spaces has close and striking connections to the physics of strings and D-branes, string-inspired matrix models, and to the structure of noncommutative or, to be more precise, fuzzy spaces such as the fuzzy sphere $S_F^2$ and fuzzy complex projective spaces $\mathbb{C}P_F^N$, which are studied at various levels of sophistication in the literature [17–19]. As it is well known, fuzzy spaces are quantized versions of their parent manifolds, and they are described by finite-dimensional matrix algebras that tend to the algebra of functions over the parent manifolds under a suitable mapping such as the diagonal coherent state map. Quantum field theories are formulated over fuzzy spaces as matrix models with finite degrees of freedom while preserving the symmetries of the parent space, which makes them appealing for quantum field theory applications (see Ref. [21] and references therein). Construction of fuzzy spaces using geometric quantization methods yields Hilbert spaces $\mathcal{H}_N$ of wave functions that are holomorphic sections of $U(1)$ bundles over the commutative parent manifold, and the matrix algebras $\mathrm{Mat}_N$ of linear transformations on $\mathcal{H}_N$'s form the fuzzy spaces [19]. Observables on the fuzzy spaces belong to this matrix algebra. It has been observed that the LLL in Landau problems over $S^2$, $\mathbb{C}P^N$ in $U(1)$ backgrounds define Hilbert spaces that are identical to $\mathcal{H}_N$ as they are also holomorphic sections of $U(1)$ bundles over these spaces.[1] Similar structural relations between $S_F^4$ and the QHE on $S^4$ also exist [19]. Building upon this connection, observables of the QHE problem are also contemplated as linear transformations in $\mathrm{Mat}_N$ acting on $\mathcal{H}_N$. From this angle, we see that there appears almost an immediate connection of our findings for the QHE problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ to fuzzy Grassmann spaces, which are discussed in some detail in the literature [29,32,33]. There is also a natural correspondence between the zero modes of Dirac operator with an Abelian background gauge field on $\mathbb{C}P^N$ and the degeneracy of the LLL (i.e., the dimension of $\mathcal{H}_N$) on $\mathbb{C}P^N$ in the Abelian background, which is discussed in some detail in Ref. [22] and which we think may be worthwhile to explore in our case too. Fuzzy spaces, such as $S_F^2$, $\mathbb{C}P_F^2$, and $\mathbb{C}P_F^3$ are also related to the Banks-Fischler-Shenker-Susskind [23] and Ishibashi-Kawai-Kitazawa-Tsuchiya [24] matrix models since they can appear as part of the vacuum solutions in these models or their certain deformations, signalling a somewhat indirect relation between the QHE problem and matrix models in string theory. These facts provide us good reasons for studying the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$, but, in fact, there is a stronger relationship to string physics as we discuss below.

The Landau problem on $S^2$, $S^4$ and in higher dimensions, which may be of interest in the context of string theory, has descriptions in terms of strings interacting with D-branes [17,18]. In the two-dimensional case, one considers a D2-brane wrapped around an $S^2$ and with $N$ D0-branes dissolved on it. A stack of $K$ D6-branes extending in directions perpendicular to the D2-brane are then moved to the center of the D2-brane. Because of the Hanany–Witten effect [20], $K$ fundamental strings stretch between a D2-brane and D6-branes. Each D0-brane provides a magnetic flux quantum over the world volume of the D2-brane, while the end points of the string on the D2-brane play the role of charged particles under the world volume gauge field. Low-energy excitations of this system are described by the QHE system on $S^2$ with $K$ playing the number of charged particles, $N$ being the magnetic flux, and the ratio $\frac{K}{N} = \nu$ being the filling factor. In this picture, the background

[1] It is also conjectured that QHE problems in non-Abelian backgrounds are related to vector bundles over the corresponding fuzzy spaces [19].

magnetic field may be described as the density of D0-branes on the D2-brane, and the D0-brane may be viewed to form an incompressible fluid. An alternative point of view is obtained by describing the background magnetic field in terms of a combination of D0-branes and flux due to a background 2-form field $B_{\mu\nu}$. In a similar manner, Fabinger [18] was able to argue that the QHE on $S^4$ describes the low-energy dynamics of a configuration of strings interacting with D-branes, in which one now wraps a stack of D4-branes on $S^4$ and spreads D0-branes on it. Moving flat infinite D4-branes to the center of $S^4$ gives once again fundamental strings connecting the branes at the center and those forming the $S^4$. Low-energy dynamics of this configuration turns out to be the QHE of Hu and Zhang on $S^4$. Alternatively, one may develop another interpretation of the latter in terms of a certain number of D0-branes expanded into a fuzzy 4-sphere $S_F^4$ [25]. We consider the possibility that these connections between string physics, fuzzy geometries, and QHE systems over two- and higher-dimensional compact manifolds may be further exploited to give a description of the QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ in terms of a strings–D-branes configuration, although it may prove very hard to address the stability of the latter. Nevertheless, we hope that our results may be preliminarily conceived as a low-energy limit of such a strings–D-branes configuration.

Our work in this paper is organized as follows. In Sec. III, we first treat the simplest and perhaps more interesting case of $\mathbf{Gr}_2(\mathbb{C}^4)$, in which the solution for the most general case of nonzero $U(1)$ and $SU(2)\times SU(2)$ backgrounds is given. In particular, we show that at the LLL with $\nu=1$, finite spatial densities are obtained at finite $SU(2)\times SU(2)$ internal degrees of freedom in agreement with the results of Ref. [8]. In Sec. IV, we generalize these results to all $\mathbf{Gr}_2(\mathbb{C}^N)$. The local structure of the solutions on $\mathbf{Gr}_2(\mathbb{C}^4)$ in the presence of a $U(1)$ background gauge field is presented in Sec. V. There, we give the single- and multiparticle wave functions by introducing the Plücker coordinates and show by calculating the two-point correlation function that the LLL at filling factor $\nu=1$ forms an incompressible fluid. The $U(1)$ gauge field, its associated field strength, and their properties are illustrated using the differential geometry on $\mathbf{Gr}_2(\mathbb{C}^4)$. We also briefly comment on the generalization of this local formulation to all $\mathbf{Gr}_2(\mathbb{C}^N)$. As we noted earlier, the QHE problem on $\mathbb{C}P^3$ with $U(1)$ is special because of its close connection to the QHE on $S^4$. Already, $\mathbb{C}P^3$ has the form $S^4\times S^2$ locally, but, in fact, $\mathbb{C}P^3$ is the projective twistor space, and it forms a nontrivial fiber bundle over $S^4$ with $S^2$ fibers [8,16]. Thus, another motivation for our work comes from the twistor correspondence between $\mathbb{C}P^3$ and $\mathbf{Gr}_2(\mathbb{C}^4)\equiv F_2$. The latter is also a twistor manifold, and together they form the base spaces for the double fibration from the Flag manifold $F_{12}$. We hope that our work may be taken as a first step toward an extensive study to reveal possible twistor correspondences between the QHE on these manifolds and also between QHE formulations on similarly related twistor spaces such as $F_{13}$ and $F_2$. We discuss our conclusions and some possible future directions of research in Sec. VI. In this section, we also discuss a heuristical correspondence between the $U(1)$ Hall effect on $\mathbf{Gr}_2(\mathbb{C}^4)$ and the Hall effect on the odd sphere $S^5$. An independent treatment of the latter, apart form our proposed suggestion in Sec. VI, is still missing in the literature, while we think that it can be formulated by generalizing the results of Ref. [10] on $S^3$.

## II. REVIEW OF THE QHE ON $\mathbb{C}P^1$ AND $\mathbb{C}P^2$

In this section, we provide a short account of the formulation of the quantum Hall problem on $\mathbb{C}P^1$ and $\mathbb{C}P^2$ for the purposes of orienting the developments in the subsequent sections and making the exposition self-contained. The formulation of the QHE on $\mathbb{C}P^1\equiv S^2$ is originally due to Haldane [26]. Karabali and Nair [8] have provided a reformulation of the QHE on $\mathbb{C}P^1$ in a manner that is adaptable to formulate the QHE on $\mathbb{C}P^N$. Here, we closely follow the discussion of Ref. [8], and while at it, we provide the Young diagram techniques for handling the QHE problem on $\mathbb{C}P^2$. In Secs. III and IV, we employ the latter to transparently handle the branching of the IRR of $SU(N)$ under the relevant subgroups appearing in the coset realizations of $\mathbf{Gr}_2(\mathbb{C}^N)$.

The Landau problem on $\mathbb{C}P^1$ can be viewed as electrons on a two-sphere under the influence of a Dirac monopole sitting at the center. Our task is to construct the Hamiltonian for a single electron under the influence of a monopole field. To this end, let us first point out that by the Peter–Weyl theorem the functions on the group manifold of $SU(2)\equiv S^3$ may be expanded in terms of the Wigner $\mathcal{D}$ functions $\mathcal{D}^{(j)}_{L_3R_3}(g)$, where $g$ is an $SU(2)$ group element and $j$ is an integral or a half-odd integral number labeling the IRR of $SU(2)$. The subscripts $L_3$ and $R_3$ are the eigenvalues of the third component of the left- and right-invariant vector fields on $SU(2)$.[2] The left- and right-invariant vector fields on $SU(2)$ satisfy

$$[L_i,L_j]=-\varepsilon_{ijk}L_k,\qquad [R_i,R_j]=\varepsilon_{ijk}R_k,\qquad [L_i,R_j]=0. \tag{2.1}$$

The harmonics as well as sections of bundles over $\mathbb{C}P^1$ may be obtained from the Wigner $\mathcal{D}$ functions on $SU(2)$ by a suitable restriction of the latter. The coset realization of $\mathbb{C}P^1$ is

$$\mathbb{C}P^1\equiv S^2=\frac{SU(2)}{U(1)}. \tag{2.2}$$

[2]Throughout the article, we sometimes denote the left- and right-invariant vector fields of $SU(N)$ and their eigenvalues by $L_i$ and $R_i$, respectively; which one is meant will be clear from the context.

This implies that the sections of $U(1)$ bundles over $\mathbb{C}P^1$ should fulfill

$$\mathcal{D}(ge^{iR_3\theta}) = e^{i\frac{n}{2}\theta}\mathcal{D}(g), \tag{2.3}$$

where $n$ is an integer. This condition is solved by the functions of the form $\mathcal{D}^{(j)}_{L_3\frac{n}{2}}(g)$. In fact, the eigenvalue $\frac{n}{2}$ of $R_3$ corresponds to the strength of the Dirac monopole at the center of the sphere, and $\mathcal{D}^{(j)}_{L_3\frac{n}{2}}(g)$ are the desired wave functions as will be made clear shortly. In particular, $\mathcal{D}^{(j)}_{L_3 0}(g)$ correspond to the spherical harmonics on $S^2$, which are the wave functions for electrons on a sphere with zero magnetic monopole background.

In the presence of a magnetic monopole field $B$, the Hamiltonian must involve covariant derivatives for which the commutator is proportional to the magnetic field. Let us take this commutator as $[D_+, D_-] = B$. It is now observed that the covariant derivatives $D_\pm$ may be identified by the right-invariant vector fields $R_\pm = R_1 \pm iR_2$ as

$$D_\pm = \frac{1}{\sqrt{2}\ell} R_\pm, \tag{2.4}$$

where $\ell$ denotes the radius of the sphere. Noting that $[R_+, R_-] = 2R_3$, for the eigenvalue $\frac{n}{2}$ of $R_3$, we have

$$B = \frac{n}{2\ell^2} \tag{2.5}$$

for the magnetic monopole with the strength $\frac{n}{2}$ in accordance with the Dirac quantization condition. The associated magnetic flux through the sphere is $2\pi n$.

The Hamiltonian may be expressed as

$$\begin{aligned} H &= \frac{1}{2M}(D_+D_- + D_-D_+) \\ &= \frac{1}{2M\ell^2}\left(\sum_{i=1}^{3} R_i^2 - R_3^2\right), \end{aligned} \tag{2.6}$$

where $M$ is the mass of the particle. We have that $\sum_{i=1}^3 R_i^2 = \sum_{i=1}^3 L_i^2 = j(j+1)$. To guarantee that $\frac{n}{2}$ occurs as one of the possible eigenvalues of $R_3$, we need to have $j = \frac{1}{2}n + q$ where $q$ is an integer. The spectrum of the Hamiltonian reads

$$\begin{aligned} E_{q,n} &= \frac{1}{2M\ell^2}\left(\left(\frac{n}{2}+q\right)\left(\frac{n}{2}+q+1\right) - \frac{n^2}{4}\right) \\ &= \frac{B}{2M}(2q+1) + \frac{q(q+1)}{2M\ell^2}. \end{aligned} \tag{2.7}$$

The associated eigenfunctions are $\mathcal{D}^{(j)}_{L_3\frac{n}{2}}(g)$ as noted earlier. In Eq. (2.7), $q$ is readily interpreted as the LL index. The ground state, which is the LLL, is at $q = 0$ and has the energy $\frac{B}{2M}$. The LLL is separated from the higher LL by finite energy gaps.

The degeneracy of the LL is controlled by the left-invariant vector fields $L_i$ since they commute with the covariant derivatives $[L_i, D_j] = 0$. Each LL is $(2j+1 = n+1+2q)$-fold degenerate. In other words, there are this many wave functions $\mathcal{D}^{(j)}_{L_3\frac{n}{2}}(g)$ at a given LL with $L_3$ eigenvalues ranging from $-j$ to $j$.

The local form of the wave functions may be written down by picking a suitable coordinate system. We omit this here and refer the reader to the original literature [8], in which this was done in detail. In particular, it is shown in Ref. [8] that the LLL forms an incompressible liquid by computing the two-point correlation function for the wave function density. We will address this crucial property of the LLL for our case in Sec. V.

Let us now briefly turn our attention to the formulation of the Landau problem on $\mathbb{C}P^2$. This and its generalization to $\mathbb{C}P^N$ are given in Ref. [8]. The coset realization of $\mathbb{C}P^2$ may be written as

$$\mathbb{C}P^2 \equiv \frac{SU(3)}{U(2)} \sim \frac{SU(3)}{SU(2)\times U(1)}. \tag{2.8}$$

Following a similar line of development as in the previous case, we can obtain the harmonics and local sections of bundles over $\mathbb{C}P^2$ from a suitable restriction of the Wigner $\mathcal{D}$ functions on $SU(3)$. Let $g \in SU(3)$, and let us denote the left- and the right-invariant vector fields on $SU(3)$ by $L_\alpha$ and $R_\alpha$ ($\alpha$: $1, \ldots, 8$); they fulfill the Lie algebra commutation relations for $SU(3)$. We can introduce the Wigner $\mathcal{D}$ functions on $SU(3)$ as

$$\mathcal{D}^{(p,q)}_{L,L_3,L_8;R,R_3,R_8}(g), \tag{2.9}$$

where $(p, q)$ label the irreducible representations of $SU(3)$ and the subscripts denote the relevant quantum numbers for the left and right rotations. In particular, the left and right generators of the $SU(2)$ subgroup are labeled by $L_i$ and $R_i$ ($i$: $1, 2, 3$) and $L_iL_i = L(L+1)$ and $R_iR_i = R(R+1)$.

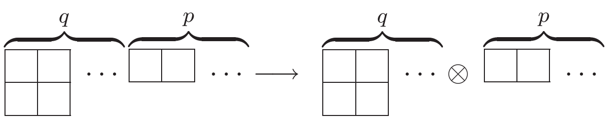
We note that the tangents along $\mathbb{C}P^2$ may be parametrized by the right-invariant fields, $R_\alpha$ ($\alpha$: $4, 5, 6, 7$). Consequently, the Hamiltonian on $\mathbb{C}P^2$ may be written down as

$$\begin{aligned} H &= \frac{1}{2M\ell^2}\sum_{\alpha=4}^{7} R_\alpha^2 \\ &= \frac{1}{2M\ell^2}(\mathcal{C}_2(p,q) - R(R+1) - R_8^2), \end{aligned} \tag{2.10}$$

where $\mathcal{C}_2(p, q)$ is the quadratic Casimir of $SU(3)$.

The coset realization of $\mathbb{C}P^2$ implies that there can be both Abelian and non-Abelian background gauge fields corresponding to the gauging of the $U(1)$ and $SU(2)$ subgroups, respectively.

Let us first obtain the wave functions with the $U(1)$ background gauge field. This means that our desired $\mathcal{D}^{(p,q)}$ should transform trivially under the $SU(2)$ and carry a $U(1)$ charge under the right actions of these groups. In other words, these wave functions must be singlets under $SU(2)$ with $R = 0$, $R_3 = 0$ and a nonzero $R_8$ eigenvalue. We can use the Young tableaux to see the branching of the $SU(3)$ IRR satisfying this requirement. The $SU(3)$ IRR labeled by $(p, q)$ may be assigned to a Young tableau with $p$ columns with one box each and $q$ columns with two boxes each. The branching $SU(3) \supset SU(2) \times U(1)$, which keeps the $SU(2)$ in the singlet representation, is therefore




(2.11)

where the diagram on the lhs of the arrow represents the generic $(p, q)$ IRR of $SU(3)$ and the first diagram on the rhs of the arrow represents the $SU(2)$ IRR, which is the singlet in this case. A general formula exits [27] for expressing the $U(1)$ charge of the branching $SU(3) \supset SU(2) \times U(1)$ [see Eq. (3.8) for a more general case],

$$n = \frac{1}{2}(J_1 - 2J_2), \quad n \in \mathbb{Z}, \tag{2.12}$$

where $J_1$ is the number of boxes in the tableau of $SU(2)$ and $J_2$ is the number of boxes in the rightmost tableau in the branching. Thus, for the tableaux given above, we conclude that $n = q - p$. To fix the relation between $R_8$ eigenvalues and the integer $n$, we use the fundamental representation (1,0) with the generators $\lambda_a$ fulfilling the normalization condition $\mathrm{Tr}(\lambda_a \lambda_b) = \frac{1}{2}\delta_{ab}$, and $\lambda_8 = \frac{1}{2\sqrt{3}}\mathrm{diag}(1, 1, -2)$, so that

$$R_8 = -\frac{n}{\sqrt{3}} = -\frac{p - q}{\sqrt{3}}. \tag{2.13}$$

It is useful to note that the flux of the $U(1)$ field strength corresponding to the background gauge field is proportional to the number $n$. We omit the details of this here and refer the reader to Ref. [8].

The spectrum of the Hamiltonian (2.10) may be given as

$$E_{q,n} = \frac{1}{2M\ell^2}(q(q + n + 2) + n), \tag{2.14}$$

where we have used the eigenvalue of the quadratic Casimir $\mathcal{C}_2(p, q)$ of the IRR $(p, q)$, which is

$$\mathcal{C}_2(p, q) = \frac{1}{3}(p(p + 3) + q(q + 3) + pq), \tag{2.15}$$

and expressed the energy levels in terms of $q$ and $n$ only. In Eq. (2.14), $q$ appears as the Landau level index; the ground-state energy may be obtained by setting $q = 0$, and that gives the LLL energy $E_{LLL} = \frac{n}{2M\ell^2}$.

The wave functions corresponding to this energy spectrum can be written in terms of the Wigner $\mathcal{D}$ functions as

$$\mathcal{D}^{(p,q)}_{L,L_3,L_8;0,0,-\frac{n}{\sqrt{3}}}(g). \tag{2.16}$$

The degeneracy of each Landau level $q$ is given by the dimension of the IRR $(p, q)$, which is

$$\dim(p, q) = \frac{(p + 1)(q + 1)(p + q + 2)}{2}. \tag{2.17}$$

This means that the set of quantum numbers $L$, $L_3$, and $L_8$ can take $\dim(p, q)$ different values.

It is also useful to note that the case $n = 0$ simply reduces the Wigner $\mathcal{D}$ functions to the harmonics on $\mathbb{C}P^2$, corresponding to the wave functions of a particle on $\mathbb{C}P^2$ with a vanishing monopole background.

Consider the case of filling factor $\nu = 1$; i.e., each of the LL states is occupied by one fermion. We therefore have that $p = n$, $q = 0$, and the number of fermions $\mathcal{N}$ is equal to $\dim(n, 0) = (n + 1)(n + 2)/2$. The density of particles $\rho$ is given by

$$\rho = \frac{\mathcal{N}}{\mathrm{vol}(\mathbb{C}P^2)}, \tag{2.18}$$

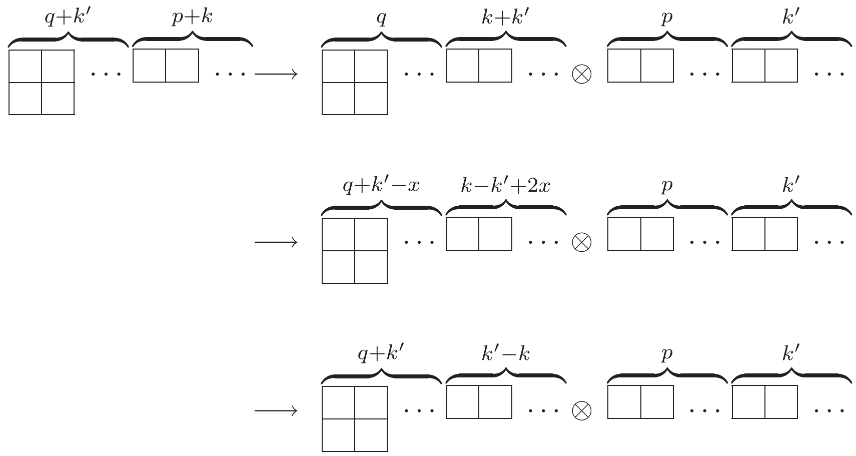
where $\mathrm{vol}(\mathbb{C}P^2) = 8\pi^2\ell^4$. In the thermodynamic limit $\ell \to \infty$ and $\mathcal{N} \to \infty$, this yields the finite result

$$\rho = \frac{\mathcal{N}}{8\pi^2\ell^4} \underset{\ell\to\infty,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^2}{16\pi^2\ell^4} = \left(\frac{B}{2\pi}\right)^2, \tag{2.19}$$

as first discussed in Ref. [8].

The wave functions can be expressed in suitable local coordinates, and, taking advantage of these functions, the multiparticle wave function for the filling factor $\nu = 1$ state can immediately be constructed. A straightforward calculation for the two-point correlation function for the wave function density that signals the incompressibility of the LLL may be given. We refer the reader to Ref. [8] for details.

The case of $SU(2)$ and $U(1)$ background gauge fields may be handled as follows. In this case, we allow for all possible right $SU(2)$ IRR labeled by spin $R$. It is possible to label $SU(3)$ representations in the form $(p + k, q + k')$. The branching $SU(3) \supset SU(2) \times U(1)$ may be represented by the Young tableaux:




(2.20)

These tableaux represent the maximum, generic, and minimum-spin $R$-value configurations that can result from the branching, and we have assumed without loss of generality that $k' > k$ and $k \geq x \geq 0$. Here, $x$ is an integer introduced to conveniently represent the generic case. From the tableaux, the range of the spin $R$ and $R_8$ eigenvalues may be easily obtained as follows:

$$R = \frac{|k-k'|}{2}, \ldots, \frac{k+k'}{2} \tag{2.21}$$

$$R_8 = \frac{1}{2\sqrt{3}}(-2(p-q) + (k-k')) = -\frac{n}{\sqrt{3}}. \tag{2.22}$$

Noting that $n$ is an integer restricts the spin $R$ to integer values. The spectrum of the Hamiltonian (2.10) is now

$$\begin{aligned} E &= \frac{1}{2M\ell^2}(\mathcal{C}_2(p+k, q+k') - R(R+1) - R_8^2) \\ &= \frac{1}{2M\ell^2}(q^2 + q(2k - m + n + 2) + n(k+1) \\ &\quad + k^2 + 2k + m^2 - m(k+1) - R(R+1)), \end{aligned} \tag{2.23}$$

where $k' = k - 2m$ and $m$ is an integer. As indicated in Eq. (2.21), there is an interval for the values of $R$. The LLL is obtained when we choose the maximum value for $R$,

$$R_{\max} = \frac{k+k'}{2} = k - m, \tag{2.24}$$

where $m$ should take only integer values within the interval $m = 0, \ldots, \frac{k}{2}$ if $k$ is even and $m = 0, \ldots, \frac{k-1}{2}$ if $k$ is odd. Using Eq. (2.24) in Eq. (2.23), the energy spectrum is expressed as

$$\begin{aligned} E = \frac{1}{2M\ell^2}(q^2 + q(2R + n + m + 2) + n(R + m + 1) \\ + (R+m)(m+1)). \end{aligned} \tag{2.25}$$

For fixed $n$, $R$, we observe from this expression that the LL is controlled by the two integers $q$ and $m$. The LLL is obtained for $q = 0$ and $m = 0$.

As discussed in Ref. [8], for the pure $SU(2)$ background, to ensure the finiteness of energy eigenvalues, $R$ should scale like $R \sim \ell^2$ in the thermodynamic limit. For $\nu = 1$, we have $\mathcal{N} = \dim(R, R) = \frac{1}{2}(R+1)(R+1)(2R+2)$, and this results in a finite density of particles:

$$\rho \sim \frac{\mathcal{N}}{(2R+1)\ell^4} \underset{\ell\to\infty, \mathcal{N}\to\infty}{\longrightarrow} \frac{R^3}{2R\ell^4}. \tag{2.26}$$

As for the case of both $U(1)$ and $SU(2)$ backgrounds, it is possible to pick either $n$ or $R$ to scale like $\ell^2$. Taking $n \sim \ell^2$ and $R$ to be finite as $\ell \to \infty$ gives again a finite spatial density

$$\rho \sim \frac{\dim(R+n, R)}{(2R+1)\ell^4} \underset{\ell\to\infty, \mathcal{N}\to\infty}{\longrightarrow} \frac{n^2}{4\ell^4}, \tag{2.27}$$

for $\nu = 1$ with $\dim(R+n, R) = \frac{1}{2}(n+R+1)(R+1) \times (n+2R+1)$.

## III. LANDAU PROBLEM ON THE GRASSMANNIAN $\mathbf{Gr}_2(\mathbb{C}^4)$

Starting in this section, we will consider the quantum Hall problem on the complex Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$. To set up the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$, it is necessary to list a few facts about the Grassmannians and their geometry.

The complex Grassmannians $\mathbf{Gr}_k(\mathbb{C}^N)$ are the set of all $k$-dimensional linear subspaces of the vector space $\mathbb{C}^N$ with the complex dimension $k(N-k)$. They are smooth and compact complex manifolds and admit Kähler structures. Grassmannians are homogeneous spaces and can therefore be realized as the cosets of $SU(N)$ as

$$\mathbf{Gr}_k(\mathbb{C}^N) = \frac{SU(N)}{S[U(N-k)\times U(k)]} \sim \frac{SU(N)}{SU(N-k)\times SU(k)\times U(1)}. \tag{3.1}$$

It is clear from this realization that $\mathbf{Gr}_1(\mathbb{C}^N) \equiv \mathbb{C}P^N$. $\mathbf{Gr}_2(\mathbb{C}^4)$ is therefore the simplest Grassmannian that is not a projective space. The coset space realization of the Grassmannians is the most suitable setting for group theoretical techniques that we will employ to formulate and solve the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^4)$ first and subsequently on all $\mathbf{Gr}_2(\mathbb{C}^N)$.

To set up and solve the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^4)$, we contemplate, following the ideas reviewed in the previous section, that $SU(4)$ Wigner $\mathcal{D}$ functions may be suitably restricted to obtain the harmonics and local sections of bundles over $\mathbf{Gr}_2(\mathbb{C}^4)$. Let $g\in SU(4)$, and let us denote the left- and the right-invariant vector fields on $SU(4)$ by $L_\alpha$ and $R_\alpha$ ($\alpha$: $1,\ldots,15$); they fulfill the Lie algebra commutation relations for $SU(4)$. We can introduce the Wigner $\mathcal{D}$ functions on $SU(4)$ as

$$g \to \mathcal{D}^{(p,q,r)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};R^{(1)}R_3^{(1)}R^{(2)}R_3^{(2)}R_{15}}(g), \tag{3.2}$$

where $(p,q,r)$ are three integers labeling the irreducible representations of $SU(4)$ and the subscripts denote the relevant quantum numbers for the left and right rotations. In particular, the left and right generators of the $SU(2)\times SU(2)$ subgroup are labeled by $L_\alpha \equiv (L_i^{(1)}, L_i^{(2)})$ and $R_\alpha \equiv (R_i^{(1)}, R_i^{(2)})$ ($i$: $1,2,3$, $\alpha$: $1,\ldots,6$) with corresponding $SU(2)\times SU(2)$ quadratic Casimirs $\mathcal{C}_2^L = L^{(1)}(L^{(1)}+1)+L^{(2)}(L^{(2)}+1)$, $\mathcal{C}_2^R = R^{(1)}(R^{(1)}+1)+R^{(2)}(R^{(2)}+1)$.

The real dimension of $\mathbf{Gr}_2(\mathbb{C}^4)$ is 8, and tangents along $\mathbf{Gr}_2(\mathbb{C}^4)$ may be parametrized by the eight right-invariant fields $R_\alpha$ ($\alpha$:$7,\ldots,14$). Consequently, the Hamiltonian on $\mathbf{Gr}_2(\mathbb{C}^4)$ may be written down as

$$H = \frac{1}{2M\ell^2}\sum_{\alpha=7}^{14} R_\alpha^2 = \frac{1}{2M\ell^2}(\mathcal{C}_2(p,q,r) - \mathcal{C}_2^R - R_{15}^2), \tag{3.3}$$

where $\mathcal{C}_2(p,q,r)$ is the quadratic Casimir of $SU(4)$ in the IRR $(p,q,r)$ with the eigenvalue

$$\mathcal{C}_2(p,q,r) = \frac{3}{8}(r^2+p^2) + \frac{1}{2}q^2 + \frac{1}{8}(2pr+4pq+4qr+12p+16q+12r). \tag{3.4}$$

The dimension of the IRR $(p,q,r)$ is

$$\dim(p,q,r) = \frac{1}{12}(p+q+2)(p+q+r+3)(q+r+2) \times (p+1)(q+1)(r+1). \tag{3.5}$$

The coset realization of $\mathbf{Gr}_2(\mathbb{C}^4)$ implies that there can be both Abelian and non-Abelian background gauge fields corresponding to the gauging of the $U(1)$ and one or both of the $SU(2)$ subgroups. We list these as three distinct cases:

(i) $U(1)$ background gauge fields only,
(ii) $U(1)$ background gauge field and a single $SU(2)$ background gauge field,
(iii) $U(1)$ background gauge field and $SU(2)\times SU(2)$ background gauge field.

It is useful to remark that the second case may be viewed as a certain restriction of the third. We will discuss these matters in detail in what follows.

Following Refs. [28,29], it is useful to list a few facts regarding the branching:

$$SU(N_1+N_2) \supset SU(N_1)\times SU(N_2)\times U(1). \tag{3.6}$$

We can embed $SU(N_1)\times SU(N_2)\times U(1)$ into $SU(N_1+N_2)$ as

$$\begin{pmatrix} e^{iN_2\phi}U_1 & 0 \\ 0 & e^{-iN_1\phi}U_2 \end{pmatrix}, \tag{3.7}$$

where $U_1\in SU(N_1)$ and $U_2\in SU(N_2)$. Let us denote the IRR of $SU(N_1)$ and $SU(N_2)$ with $\mathcal{J}_1$ and $\mathcal{J}_2$. We also let $J_a$ be the total number of boxes in the Young tableaux of $SU(N_a)$ ($a$: $1,2$). The $U(1)$ charge may thus be expressed as

$$n = \frac{1}{N_1N_2}(N_2J_1 - N_1J_2). \tag{3.8}$$

Clearly, the IRR of $U(1)$ is fixed by those of the $SU(N_a)$ factors, and the IRR content of the subgroup $SU(N_1)\times SU(N_2)\times U(1)$ may be denoted as $(\mathcal{J}_1,\mathcal{J}_2)_n$. The decomposition of a given IRR $\mathcal{J}$ of $SU(N_1+N_2)$ under this subgroup is expressed as

$$\mathcal{J} = \bigoplus_{\mathcal{J}_1,\mathcal{J}_2} m^{\mathcal{J}}_{\mathcal{J}_1,\mathcal{J}_2}(\mathcal{J}_1,\mathcal{J}_2)_n, \tag{3.9}$$

where $m^{\mathcal{J}}_{\mathcal{J}_1,\mathcal{J}_2}$ are the multiplicities of the IRR $(\mathcal{J}_1,\mathcal{J}_2)_n$ occurring in the direct sum. Further details may be found in the references [28,29] and in the original article of Hagen and Macfarlane [30].

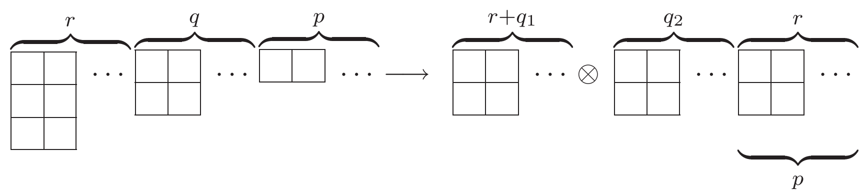
### A. $U(1)$ gauge field background

For the QHE problem on $\mathbf{Gr}_2(\mathbb{C}^4)$, we are concerned with the branching

$$SU(4) \supset SU(2) \times SU(2) \times U(1). \tag{3.10}$$

Obtaining the wave functions with the $U(1)$ background gauge field requires us to restrict $\mathcal{D}^{(p,q,r)}$ in such a way that they transform trivially under the right action of $SU(2) \times SU(2)$ and carry a right $U(1)$ charge; that is, they should be singlets under $SU(2) \times SU(2)$ with a $\mathcal{C}_2^R = 0$ eigenvalue and a nonzero $R_{15}$ eigenvalue.

We can use the Young tableaux to see the branching of the $SU(4)$ IRR fulfilling this requirement. The $SU(4)$ IRR labeled by $(p, q, r)$ may be denoted as a Young tableau with $p$ columns with one box on each, $q$ columns with two boxes on each, and $r$ columns with three boxes on each. The branching (3.10), which keeps the $SU(2) \times SU(2)$ in the singlet representation, is therefore




(3.11)

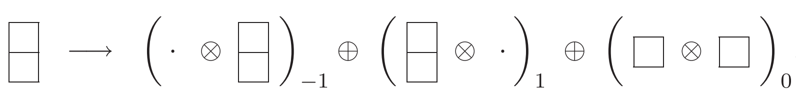
where we have introduced the splitting $q = q_1 + q_2$ in the representation in order to handle the partition of columns labeled by $q$ in the branching. It is important to realize that in the last row of the $SU(4)$ representation there are $r$ (fully symmetrized) boxes, which are moved as a whole under this branching to the second slot in the rhs, and the trivial representation of $SU(2) \times SU(2)$ is obtained if and only if $p$ is equal to $r$. Otherwise, we have a nontrivial representation for the second $SU(2)$ in the branching (3.10).

Using the formula (3.8), we compute the $U(1)$ charge as

$$n = \frac{1}{2}((2r + 2q_1) - (p + r + 2q_2)) = q_1 - q_2, \tag{3.12}$$

where we have used $p = r$.

To fix the relation between the eigenvalues of $R_{15}$ and the $U(1)$ charge $n$, we need to use the six-dimensional fundamental representation (0,1,0) (Young tableaux: $\begin{smallmatrix}\square\\\square\end{smallmatrix}$) of $SU(4)$. As opposed to $\mathbb{C}P^3 \approx SU(4)/SU(3) \times U(1)$, where the branching of the four-dimensional representations [i.e., (1,0,0) and (0,0,1)] of $SU(4)$ contain singlets of $SU(3)$, in the present case, the smallest $SU(4)$ IRR containing the singlet of $SU(2) \times SU(2)$ is (0,1,0), and it has the branching




(3.13)

where subscripts show the charge (3.12). Taking the generators $\lambda_a$ of $SU(4)$ fulfilling the normalization condition $\mathrm{Tr}(\lambda_a \lambda_b) = \frac{1}{2}\delta_{ab}$, in one of the four-dimensional IRR [(1, 0, 0) or (0, 0, 1)], it is possible to show that in the six-dimensional IRR[3] (0, 1, 0)

$$R_{15} = \frac{1}{\sqrt{2}} \operatorname{diag}(0, 0, 0, 0, -1, 1), \tag{3.14}$$

and therefore we in general have

$$R_{15} = \frac{n}{\sqrt{2}} = \frac{q_1 - q_2}{\sqrt{2}}. \tag{3.15}$$

It is now easy to give the energy spectrum corresponding to the Hamiltonian (3.3), using Eq. (3.4), $p = r$, $R_{15}$ taking the value in Eq. (3.15) and $\mathcal{C}_2^R = 0$:

$$E = \frac{1}{2M\ell^2}(p^2 + 3p + np + 2q_2^2 + 4q_2 + 2pq_2 + 2n(1 + q_2)). \tag{3.16}$$

The LLL energy at a fixed monopole background $n$ is obtained for $q_2 = p = 0$, and it is

$$E_{LLL} = \frac{n}{M\ell^2} = \frac{2B}{M}, \tag{3.17}$$

with the degeneracy $\dim(0, n, 0) = \frac{1}{12}(n+1)(n+2)^2(n+3)$. In Eq. (3.17), $B = \frac{n}{2\ell^2}$ is the field strength of the $U(1)$

[3] Generalization of this result to the $\frac{N(N-1)}{2}$-dimensional representations of $SU(N)$ is used in the subsequent sections. A proof is provided in Appendix A.

magnetic monopole. The gauge field associated to $B$ and related matters will be discussed in Sec. V.

The wave functions corresponding to this energy spectrum can be written in terms of the Wigner $\mathcal{D}$ functions as

$$\mathcal{D}^{(p,q_1+q_2,p)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};0,0,0,0,\frac{n}{\sqrt{2}}}(g) \equiv \mathcal{D}^{(p,[\frac{q+n}{2}]+[\frac{q-n}{2}],p)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};0,0,0,0,\frac{n}{\sqrt{2}}}(g). \tag{3.18}$$

The degeneracy of each Landau level is given by the dimension of the IRR $(p,q,p)$ in Eq. (3.5). This means that the set of left quantum numbers $\{L^{(1)}, L_3^{(1)}, L^{(2)}, L_3^{(2)}, L_{15}\}$ can take on $\dim(p, q_1+q_2, p)$ different values as a set.

For the many-body fermion problem in which all the states of the LLL are filled with the filling factor $\nu = 1$, in the thermodynamic limit $\ell \to \infty$, $\mathcal{N} \to \infty$, we obtain a finite spatial density of particles

$$\rho = \frac{\mathcal{N}}{\frac{\pi^4\ell^8}{12}} \underset{\ell\to\infty,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^4}{\pi^4\ell^8} = \left(\frac{2B}{\pi}\right)^4, \tag{3.19}$$

where we have used $\mathcal{N} = \dim(0,n,0) = \frac{1}{12}(n+1)\times(n+2)^2(n+3)$ for the number of fermions in the LLL with $\nu = 1$ and[4] $\mathrm{vol}(\mathbf{Gr}_2(\mathbb{C}^4)) = \frac{\pi^4\ell^8}{12}$.

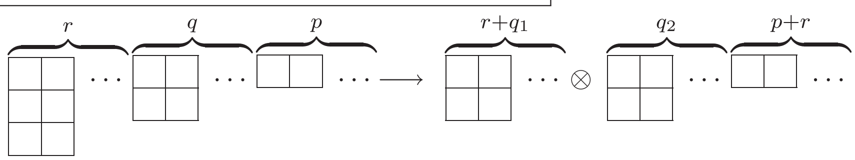
We note that the case $n = 0$ simply reduces the Wigner $\mathcal{D}$ functions to the harmonics on $\mathbf{Gr}_2(\mathbb{C}^4)$ corresponding to the wave functions of a particle on $\mathbf{Gr}_2(\mathbb{C}^4)$ with the vanishing monopole background.

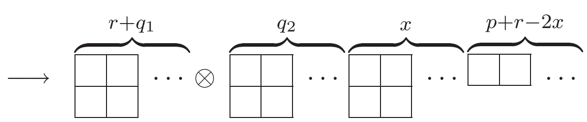
It is possible to interchange the Young tableaux of the two $SU(2)$'s in Eq. (3.11). This flips the sign of the $U(1)$ charge, $n \to -n$; in the formulas for the energy, degeneracy, etc., this fact can be compensated by substituting $|n|$ for $n$.

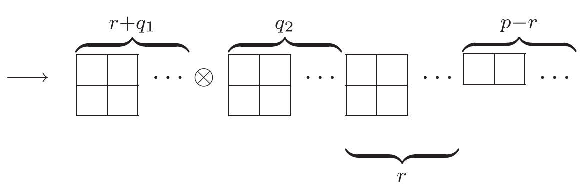
In Sec. V, we give the single- and many-particle wave functions (for the filling factor $\nu = 1$ state) in terms of the Plücker coordinates for $\mathbf{Gr}_2(\mathbb{C}^4)$ and use the latter to obtain the two-point correlation function for the wave-function density signaling the incompressibility of the LLL. An account of the $U(1)$ gauge field is also provided for illustrative purposes.

### B. Single $SU(2)$ gauge field and $U(1)$ gauge field background

In this case, we need to restrict to $\mathcal{D}^{(p,q,r)}$, which transform as a singlet under one or the other $SU(2)$ in the right action of $SU(2) \times SU(2)$ and carry a $U(1)$ charge. Therefore, we have a range of possibilities within the branching (3.10) as given in the following Young tableaux decomposition:


(3.22)


(3.23)


(3.24)

[4]It may be useful to state that this volume is computed with the help of the repeated iteration of (special) unitary group manifolds in terms of the odd-dimensional spheres,

$$\begin{aligned} SU(N) &\approx \frac{SU(N)}{SU(N-1)} \times \frac{SU(N-1)}{SU(N-2)} \times \cdots \times \frac{SU(3)}{SU(2)} \times SU(2) \\ &\cong S^{2N-1} \times S^{2N-3} \times \cdots \times S^5 \times S^3 \end{aligned} \tag{3.20}$$

(for $N \geq 3$), where $\approx$ means "locally equal to" and $\cong$ indicates isomorphism. Considering this local expression, we can expand all the special unitary groups in Eq. (3.1) and employ the volume formula for spheres to obtain an approximation for the volume of the Grassmannians [31], namely,

$$\mathrm{vol}(\mathbf{Gr}_k(\mathbb{C}^N)) = \frac{1!2!\cdots(k-1)!}{(N-1)!(N-2)!\ldots(N-k)!}(\pi\ell^2)^{k(N-k)}, \tag{3.21}$$

which produces the factor $\frac{1}{12}$ for $k = 2$ and $N = 4$. This factor is in general subject to change upon using other methods. Since this is immaterial for our purposes, we will stick to the approximation (3.21) throughout this paper.

We have assumed that $p > r$ and split $q_1 + q_2 = q$. We have introduced the integer $x$ $(0 \leq x \leq r)$ to conveniently represent the generic case. From the tableaux, $R_{15}$ eigenvalues may be easily obtained as

$$n = \frac{1}{2}(2(q_1 - q_2) - (p - r)), \tag{3.25}$$

and we observe that the first $SU(2)$ in the branching remains a singlet, while the second may take on values over a range;

$$R^{(1)} = 0, \qquad R^{(2)} = \frac{p-r}{2}, \ldots, \frac{r+p}{2}. \tag{3.26}$$

Since $n$ is an integer, we must have that $p - r$ is an even integer. This condition restricts the spin $R^{(2)}$ to integer values.

Using $\mathcal{C}_2^R = R^{(2)}(R^{(2)} + 1)$, the energy spectrum corresponding to the Hamiltonian (3.3) is given as

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2(p,q,r) - R^{(2)}(R^{(2)}+1) - \frac{n^2}{2}\right). \tag{3.27}$$

This can be rewritten in terms of $q_2$, $n$, $p$ using Eqs. (3.4) and (3.25), assuming $p > r$ and introducing $m$ via $r = p - 2m$ ($m = 0, \ldots, \frac{p}{2}$ if $p$ is even and $m = 0, \ldots, \frac{p-1}{2}$ if $p$ is odd) as

$$E = \frac{1}{2M\ell^2}(2q_2^2 + 2q_2(n + p + 2) + n(p+2) + p^2 + 3p + m^2 - m(p+1) - R^{(2)}(R^{(2)}+1)). \tag{3.28}$$

To obtain the lowest energy, we have to take the maximum value of the spin $R^{(2)}_{\max} = \frac{r+p}{2} = p - m$. Then, the energy spectrum becomes

$$E = \frac{1}{2M\ell^2}(2q_2^2 + 2q_2(n + R^{(2)} + m + 2) + n(R^{(2)} + m + 2) + (R^{(2)} + m)(2 + m)). \tag{3.29}$$

The LLL energy at fixed background fields $R^{(2)}$ and $n$ is obtained for $q_2 = m = 0$ as follows:

$$E_{LLL} = \frac{1}{2M\ell^2}(n(R^{(2)} + 2) + 2R^{(2)}). \tag{3.30}$$

The wave functions in the present case can be written in the form

$$\mathcal{D}^{(p,q_1+q_2,r)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};0,0,R^{(2)},R_3^{(2)},\frac{n}{\sqrt{2}}}(g), \tag{3.31}$$

where $R^{(2)}$ is given in Eq. (3.26).

To have finite energy eigenvalues in the thermodynamic limit $\ell \to \infty$, $\mathcal{N} \to \infty$, the scales of $n$ and $R^{(2)}$ in terms of the powers of $\ell$ have to be determined. For a pure $SU(2)$ background ($n = 0$, $R^{(1)} = 0$, $R^{(2)} \neq 0$), $R^{(2)}$ should scale in the thermodynamic limit as $R^{(2)} \sim \ell^2$. The number of fermions in the LLL with $\nu = 1$ is

$$\mathcal{N} = \dim(R^{(2)}, 0, R^{(2)}) = \frac{1}{12}(R^{(2)} + 2)^2(2R^{(2)} + 3)(R^{(2)} + 1)^2 \underset{R^{(2)}\to\infty}{\longrightarrow} \frac{(R^{(2)})^5}{6}, \tag{3.32}$$

and the corresponding spatial density is

$$\rho \sim \frac{\mathcal{N}}{\frac{\pi^4\ell^8}{12}(2R^{(2)} + 1)} \underset{\ell\to\infty,\mathcal{N}\to\infty}{\longrightarrow} \frac{(R^{(2)})^4}{\pi^4\ell^8}, \tag{3.33}$$

which is finite.

When both $U(1)$ and $SU(2)$ backgrounds are present (i.e., $n \neq 0$, $R^{(1)} = 0$, $R^{(2)} \neq 0$), just like the case of $\mathbb{C}P^2$ reviewed in the previous section, we may choose either one of $n$ or $R^{(2)}$ to scale like $\ell^2$. Taking $n \sim \ell^2$ and $R^{(2)}$ to be finite in thermodynamic limit, we again get a finite spatial density

$$\rho \sim \frac{\mathcal{N}}{\frac{\pi^4\ell^8}{12}(2R^{(2)} + 1)} \underset{\ell\to\infty,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^4}{2\pi^4\ell^8 R^{(2)}}, \tag{3.34}$$

where we have the number of fermions $\mathcal{N}$ in the LLL with $\nu = 1$ given in this case as

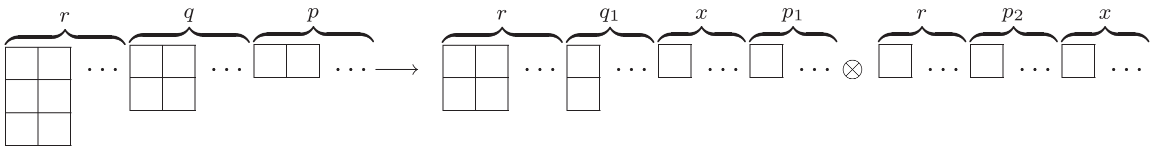
$$\dim(R^{(2)}, n, R^{(2)}) = \frac{1}{12}(R^{(2)} + n + 2)^2(2R^{(2)} + n + 3)(R^{(2)} + 1)^2(n + 1) \underset{n\to\infty,R^{(2)}\to\text{finite}}{\longrightarrow} \frac{n^4}{12}. \tag{3.35}$$

Before closing this subsection, we note that interchanging the Young tableaux of two $SU(2)$'s amounts to interchanging $R^{(1)}$ and $R^{(2)}$ in Eq. (3.26) and also a flip in the sign of the $U(1)$ charge. In the relevant formulas above, one can compensate for these changes by replacing $R^{(2)}$ with $R^{(1)}$ and substituting $|n|$ for $n$.

### C. $SU(2) \times SU(2)$ gauge field background

Now, we need to restrict $\mathcal{D}^{(p,q,r)}$ to those wave functions that transform as an IRR $(R^{(1)}, R^{(2)})$ of $SU(2) \times SU(2)$ and carry a $U(1)$ charge. It is useful to partition the IRR of $SU(4)$ as $(p_1 + p_2, q_1 + q_2 + x, r)$. There are now two classes of branchings, which differ in their $U(1)$ charge as given in terms of $p_1$, $p_2$, $q_1$, $q_2$, and $r$ below.

If $q_2 = 0$, the branching with maximal $R^{(2)}$ value is

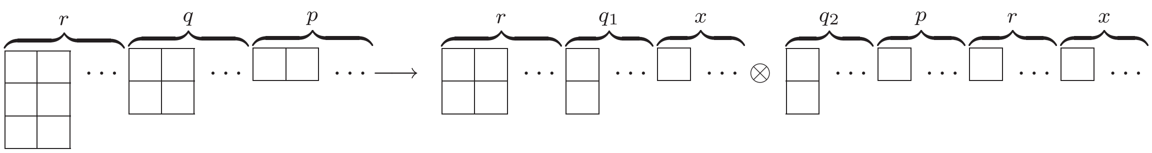


(3.36)

As $R^{(2)}$ decreases down from its maximal value $R^{(2)} = \frac{r+p_2+x}{2}$ in increments of 1, the total number of boxes in each $SU(2)$ does not vary, so we have, with $q = q_1 + x$,

$$n = \frac{1}{2}(2q_1 - (p_2 - p_1 - r)). \tag{3.37}$$

Suppose now that $q_2 \neq 0$. This may happen only if all $p$ boxes are already in the tableaux of the second $SU(2)$ in the branching; thus, we must have that $p_1 = 0$. Once again, we have the branching with the maximal $R^{(2)}$ value as



(3.38)

and the $U(1)$ charge is now (with $p = p_2$)

$$n = \frac{1}{2}(2(q_1 - q_2) - (p_2 - r)). \tag{3.39}$$

Using both of the tableaux, we observe that the first $SU(2)$ in the branching takes the value

$$R^{(1)} = \frac{p_1 + x}{2}, \qquad 0 \leq x \leq q, \qquad 0 \leq p_1 \leq p. \tag{3.40}$$

For this value of $R^{(1)}$, the second $SU(2)$ takes on values between $R^{(2)}_{\max} = \frac{S}{2}$ and $R^{(2)}_{\min} = \frac{|2\mathcal{M}-S|}{2}$,

$$\frac{|2\mathcal{M} - S|}{2} \leq R^{(2)} \leq \frac{S}{2}, \qquad S = p_2 + x + r, \tag{3.41}$$

where $\mathcal{M}$ is defined as the largest among the integers $p_2$, $x$, and $r$.

We consider the cases $q_2 = 0$ and $q_2 \neq 0$ with the $U(1)$ charges given in Eqs. (3.37) and (3.39) separately to determine the energy spectrum corresponding to the Hamiltonian (4.3). We have that

$$E = \frac{1}{2M\ell^2}\bigg(\mathcal{C}_2(p,q,r) - R^{(1)}(R^{(1)} + 1) - R^{(2)}(R^{(2)} + 1) - \frac{n^2}{2}\bigg). \tag{3.42}$$

For the case $q_2 = 0$, we have the condition that

$$m \coloneqq \frac{p_2 - p_1 - r}{2} \tag{3.43}$$

is an integer to ensure that $n$ is so. Let us assume that $p_2 > p_1 + r$ so that $m$ is positive.

To obtain the lowest-energy eigenvalues, we use Eq. (3.40) together with the maximum value of $R^{(2)}$ as given in Eq. (3.41). Next, we eliminate $p_2$, $q_1$, $x$, and $r$ in favor of $n$, $R^{(1)}$, $R^{(2)}$, $p_1$, and $m$ (explicitly, we have $p_2 = R^{(2)} - R^{(1)} + p_1 + m,\ q_1 = n + m,\ x = 2R^{(1)} - p_1$, and $r = R^{(2)} - R^{(1)} - 2m$) to get

$$\begin{aligned} E &= \frac{1}{2M\ell^2}\bigg(\mathcal{C}_2(R^{(2)} - R^{(1)} + 2p_1 + m, n + m + 2R^{(1)} - p_1, R^{(2)} - R^{(1)} - m) - R^{(1)}(R^{(1)} + 1) - R^{(2)}(R^{(2)} + 1) - \frac{n^2}{2}\bigg) \\ &= \frac{1}{2M\ell^2}(p_1^2 + p_1(m + R^{(2)} - R^{(1)} + 1) + m^2 + m(R^{(1)} + R^{(2)} + n + 2) + n(R^{(1)} + R^{(2)} + 2) + 2R^{(2)}), \end{aligned} \tag{3.44}$$

where $R^{(2)} > R^{(1)}$ due to the assumption $p_2 > p_1 + r$. For fixed $R^{(1)}$, $R^{(2)}$, and $n$, Landau levels are controlled by the two integers $p_1$ and $m$. Taking $p_1 = m = 0$ results in the LLL energy

$$E_{LLL} = \frac{1}{2M\ell^2}(n(R^{(1)} + R^{(2)} + 2) + 2R^{(2)}). \quad (3.45)$$

We note that assuming $p_2 < p_1 + r$ flips the sign of $m$, and in Eq. (3.44), $m \to -m$.[5]

It is also important to remark that for $R^{(1)} = R^{(2)} = R$ we have $p_1 = p_2 + r$ and thus

$$\tilde{m} := \frac{p_1 + r - p_2}{2} = r, \quad (3.46)$$

and the energy levels are given by

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2(2p_1 - r, n - r + 2R - p_1, r) - 2R(R+1) - \frac{n^2}{2}\right) = \frac{1}{2M\ell^2}(2R + p_1(1 + p_1 - \tilde{m}) + (n - \tilde{m})(2 + 2R - \tilde{m})). \quad (3.47)$$

The energy values here are positive since $p_1 \geq \tilde{m}$, $n \geq \tilde{m}$, and $2R - \tilde{m} \geq 0$ by construction. The LLL energy is given by $p_1 = \tilde{m} = 0$, which is indeed the same as the one obtained from Eq. (3.45) when $R := R^{(1)} = R^{(2)}$.

The case $p_1 = 0$ may be treated along similar lines. We have that

$$m := \frac{p - r}{2} \quad (3.48)$$

is an integer for the same reason that $n$ is so. Let us assume $p > r$ so that $m$ is positive. In this case, we can write $p$, $q_1$, $x$, and $r$ in terms of $n$, $R^{(1)}$, $R^{(2)}$, $q_2$ and $m$. Hence, we find for the lowest -energy eigenvalues

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2(R^{(2)} - R^{(1)} + m, 2q_2 + 2R^{(1)} + n + m, R^{(2)} - R^{(1)} - m) - R^{(1)}(R^{(1)} + 1) - R^{(2)}(R^{(2)} + 1) - \frac{n^2}{2}\right)$$
$$= \frac{1}{2M\ell^2}(2q_2^2 + 2q_2(n + R^{(1)} + R^{(2)} + m + 2) + n(R^{(1)} + R^{(2)} + 2) + m^2 + m(R^{(1)} + R^{(2)} + n + 2) + 2R^{(2)}). \quad (3.49)$$

We note that here we do have the condition $R^{(2)} > R^{(1)}$ as well. In this case, $q_2$ and $m$ specify the Landau levels. We take $q_2 = m = 0$ in Eq. (3.49) to obtain the LLL energy, and this yields the same result given in Eq. (3.45) as expected.

The LLL energy for $R^{(2)} < R^{(1)}$ can be found by interchanging $R^{(1)}$ and $R^{(2)}$ in Eq. (3.45) and taking $n$ to $-n$, where now $n < 0$. This gives

$$E_{LLL} = \frac{1}{2M\ell^2}(-n(R^{(2)} + R^{(1)} + 2) + 2R^{(1)}). \quad (3.50)$$

We do have two distinct cases to consider in the thermodynamic limit. For a pure $SU(2) \times SU(2)$ background, $n = 0$, $R^{(1)} \neq 0$, $R^{(2)} \neq 0$, both $R^{(1)}$ and $R^{(2)}$ should scale in the thermodynamic limit as $\ell^2$. The number of fermions in the LLL with $\nu = 1$ is

$$\dim(R^{(2)} - R^{(1)}, 2R^{(1)}, R^{(2)} - R^{(1)}) \sim 4R^{(1)5}R^{(2)}, \quad (3.51)$$

and the corresponding spatial density in this limit is

$$\rho \sim \frac{4R^{(1)^5}R^{(2)}}{\pi^4\ell^8(2R^{(1)} + 1)(2R^{(2)} + 1)} \underset{\ell\to\infty,\mathcal{N}\to\infty}{\longrightarrow} \text{finite}. \quad (3.52)$$

For the nonzero background $n \neq 0$, $R^{(1)} \neq 0$, $R^{(2)} \neq 0$, we have three parameters $n$, $R^{(1)}$, and $R^{(2)}$. We can choose, say, $n$ to scale like $\ell^2$ and the others to remain finite in thermodynamic limit. For $\nu = 1$, we get

$$\dim(R^{(2)} - R^{(1)}, 2R^{(1)} + n, R^{(2)} - R^{(1)}) \longrightarrow n^4, \quad (3.53)$$

and the spatial density is

$$\rho \sim \frac{n^4}{\pi^4\ell^8(2R^{(1)} + 1)(2R^{(2)} + 1)} \longrightarrow \text{finite}. \quad (3.54)$$

## IV. LANDAU PROBLEM ON $\mathbf{Gr}_2(\mathbb{C}^N)$

We are now ready to generalize the results of the previous section to all Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$. It is useful to write down the coset realization

$$\mathbf{Gr}_2(\mathbb{C}^N) = \frac{SU(N)}{S[U(N-2) \times U(2)]} \sim \frac{SU(N)}{SU(N-2) \times SU(2) \times U(1)}. \quad (4.1)$$

[5]The energy levels are still, of course, positive as can easily be checked.

The $SU(N)$ Wigner $\mathcal{D}$ functions for $g \in SU(N)$,

$$\mathcal{D}^{(P_1,P_2,P_3,\ldots,P_{N-2},P_{N-1})}_{L^{SU(N-2)},L,L_3,L_{N^2-1},R^{SU(N-2)},R,R_3,R_{N^2-1}}(g), \tag{4.2}$$

carrying the IRR $(P_1, P_2, P_3, \ldots, P_{N-2}, P_{N-1})$ labeled by $N-1$ non-negative integers, may be appropriately restricted to obtain the harmonics and local sections of bundles over $\mathbf{Gr}_2(\mathbb{C}^N)$. Let us denote the left- and the right-invariant vector fields on $SU(N)$ by $L_\alpha$ and $R_\alpha$ ($\alpha$: $1, \ldots, N^2-1$); they satisfy the Lie algebra commutation relations for $SU(N)$. In Eq. (4.2), $L^{SU(N-2)}$ and $R^{SU(N-2)}$ stand for the suitable sets of left and right quantum numbers, which we will not need in what follows.

The real dimension of $\mathbf{Gr}_2(\mathbb{C}^N)$ is $4N-8$, and tangents along $\mathbf{Gr}_2(\mathbb{C}^N)$ may be parametrized by the $4N-8$ right-invariant fields, $R_\alpha$, ($\alpha$: $N^2-4N+7, \ldots, N^2-2$). Consequently, the Hamiltonian may be written as

$$\begin{aligned} H &= \frac{1}{2m\ell^2} \sum_{\alpha=N^2-4N+7}^{N^2-2} R_\alpha^2 \\ &= \frac{1}{2m\ell^2}\left(\mathcal{C}_2^{SU(N)} - \mathcal{C}_2^{SU(N-2)} - \mathcal{C}_2^{SU(2)} - R_{N^2-1}^2\right). \end{aligned} \tag{4.3}$$

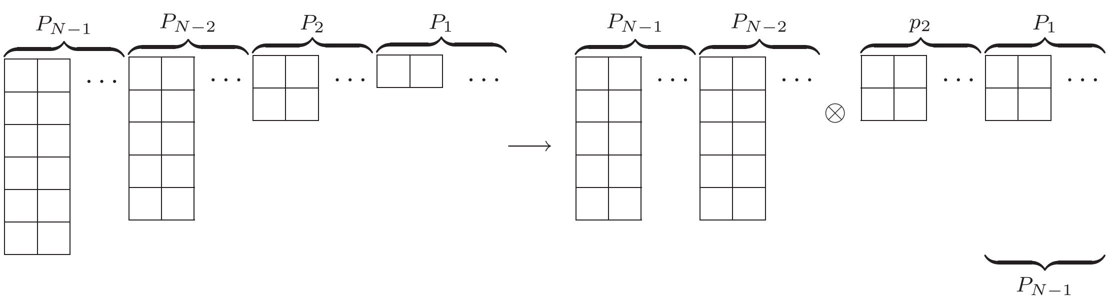
Here, for future use, we give the eigenvalue of $\mathcal{C}_2^{SU(N)}$ in the IRR $(P_1, P_2, 0, \ldots, 0,\ P_{N-2}, P_{N-1})$, which reads

$$\begin{aligned} &\mathcal{C}_2(P_1, P_2, 0, \ldots, 0, P_{N-2}, P_{N-1}) \\ &\quad = \left(\frac{N-1}{2N}\right)P_1^2 + \left(\frac{N-2}{N}\right)P_2^2 + \left(\frac{N-2}{N}\right)P_{N-2}^2 \\ &\qquad + \left(\frac{N-1}{2N}\right)P_{N-1}^2 + \left(\frac{N-2}{N}\right)P_1P_2 + \frac{2}{N}P_1P_{N-2} \\ &\qquad + \frac{1}{N}P_1P_{N-1} + \frac{4}{N}P_2P_{N-2} + \frac{2}{N}P_2P_{N-1} \\ &\qquad + \left(\frac{N-2}{N}\right)P_{N-2}P_{N-1} + \left(\frac{N-1}{2}\right)P_1 + (N-2)P_2 \\ &\qquad + (N-2)P_{N-2} + \left(\frac{N-1}{2}\right)P_{N-1}, \end{aligned} \tag{4.4}$$

and the dimension of this representation is given in Appendix B.

To obtain the wave functions with only a $U(1)$ background gauge field, we consider those $\mathcal{D}$ functions that transform trivially under the right action of $SU(N-2)$ and $SU(2)$ and carry a right $U(1)$ charge. This means these wave functions remain singlets under $SU(N-2)$ and $SU(2)$ with nonzero $\mathcal{C}_2^{SU(N-2)}$, $\mathcal{C}_2^{SU(2)}$ eigenvalues and a nonzero $R_{N^2-1}$ eigenvalue.

The branching $SU(N) \supset SU(N-2) \times SU(2) \times U(1)$ may be used for this purpose. To have both $SU(N-2)$ and $SU(2)$ as singlets in the branching, we must require all $P_i$ except $P_1$, $P_2$, $P_{N-2}$, and $P_{N-1}$ to vanish, and also $P_{N-1} = P_1$. In terms of Young tableaux, this branching can be shown by



(4.5)

where the tableaux on the lhs represent the IRR $(P_1, P_2, 0, \ldots, 0, P_{N-2}, P_1)$ of $SU(N)$. The tableaux on the rhs are those of $SU(N-2)$ and $SU(2)$, respectively, and both are singlets in this case.

From Eq. (3.8), we compute the $U(1)$ charge as

$$\begin{aligned} n &= \frac{1}{2(N-2)}(2J_1 - (N-2)J_2) \\ &= P_{N-2} - P_2. \end{aligned} \tag{4.6}$$

The relation between eigenvalues of $R_{N^2-1}$ and $n$ is found to be (see Appendix A)

$$R_{N^2-1} = -\sqrt{1-\frac{2}{N}}n. \tag{4.7}$$

The energy spectrum of the Hamiltonian is

$$\begin{aligned} E &= \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(N)} - \left(1-\frac{2}{N}\right)n^2\right) \\ &= \frac{1}{2M\ell^2}\left(P_1^2 + \left(2-\frac{4}{N}\right)P_2^2 + (N-1+2n)P_1 \right. \\ &\quad \left. + 2\left(n+N-2+\frac{2}{N}\right)P_2 + 4P_1P_2 + n(N-2)\right), \end{aligned} \tag{4.8}$$

where we have used Eq. (4.4) with $P_{N-1} = P_1$ and $P_{N-2} = P_2 + n$. The integers $P_1$ and $P_2$ are in fact considered to be the Landau level indices. The LLL energy can be obtained by setting $P_1 = P_2 = 0$, which is

$$E_{LLL} = \frac{Nn - 2n}{2M\ell^2}. \tag{4.9}$$

The corresponding wave functions may be expressed by

$$\mathcal{D}^{(P_1,P_2,0,\ldots,0,P_{n-2}=P_2+n,P_{n-1}=P_1)}_{L^{SU(N-2)},L,L_3,L_{N^2-1},0,0,0,-\sqrt{1-\frac{2}{N}}n}(g). \tag{4.10}$$

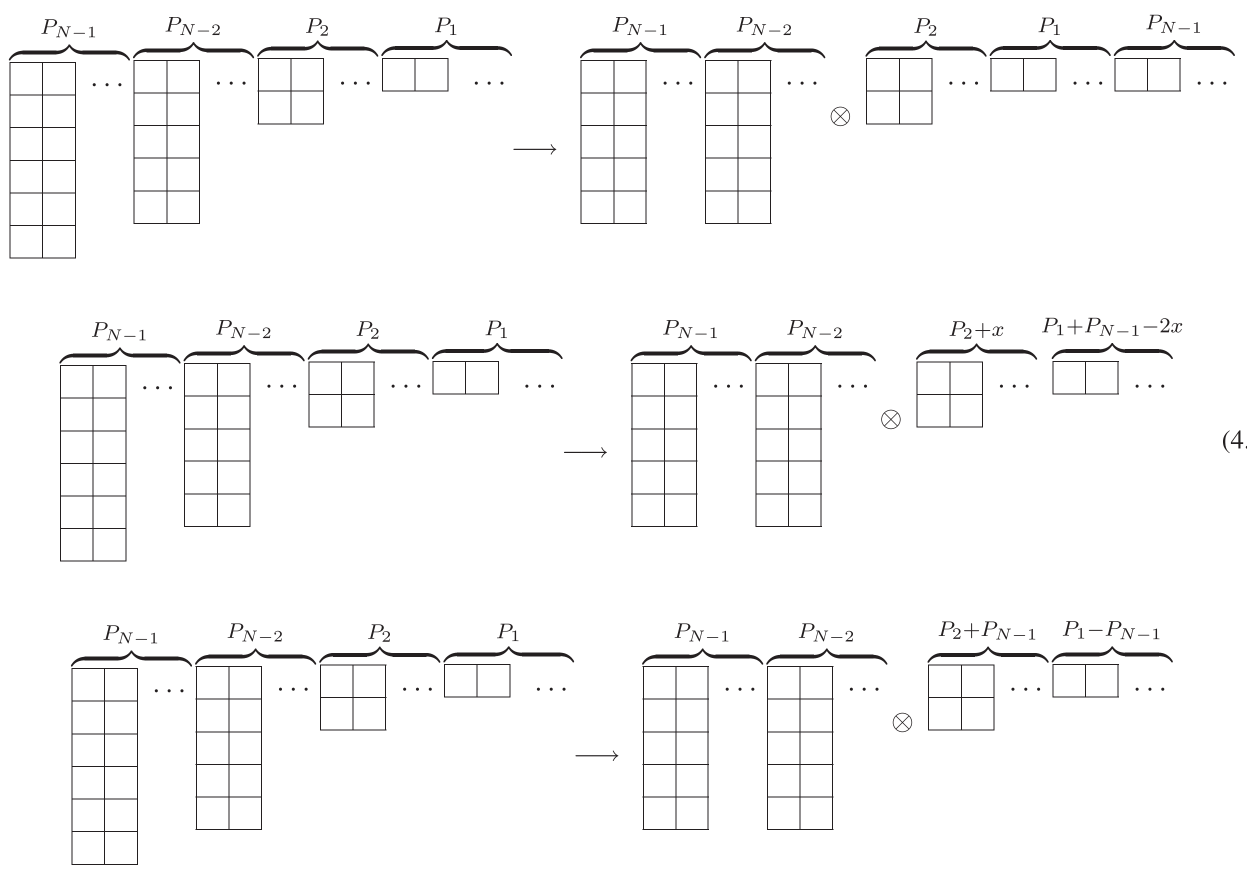
The spatial density of particles in the thermodynamic limit is computed in a manner analogous to those for the case of $\mathbf{Gr}_2(\mathbb{C}^4)$. We have $\mathrm{vol}(\mathbf{Gr}_2(\mathbb{C}^N)) = \frac{\pi^{2(N-2)}}{(N-2)!(N-1)!}\ell^{4N-8}$ from Eq. (3.21) and in the LLL, $P_{N-2} = n$, $P_1 = P_2 = P_{N-1} = 0$, with $\nu = 1$. Correspondingly, the dimension formula (B1) for the LLL with $\nu = 1$ reduces to

$$\mathcal{N} = \dim(0, 0, \ldots, n, 0) = \frac{(n+N-3)!(n+N-4)!(n+N-2)^2(n+N-1)(n+N-3)}{(N+1)!(N-2)!n!(n+1)!}. \tag{4.11}$$

In the thermodynamic limit ($\ell \to \infty$ and $\mathcal{N} \to \infty$), the density of the states takes the form

$$\rho = \frac{\mathcal{N}}{\frac{\pi^{2(N-2)}}{(N-2)!(N-1)!}\ell^{4N-8}} \longrightarrow \frac{n^{2N-4}}{\ell^{4N-8}} = \left(\frac{B}{2\pi}\right)^{2N-4}. \tag{4.12}$$

For the case of both $SU(2)$ and $U(1)$ background gauge fields, the spectrum of the Hamiltonian and the wave functions are obtained in a similar manner. We still have to demand all $P_i$ except $P_1$, $P_2$, $P_{N-2}$, $P_{N-1}$ to vanish but no longer impose the condition $P_{N-1} = P_1$. The relevant branching of $SU(N)$ is now given by the Young tableaux below:




(4.13)

where the branching rule for maximum, generic, and minimum $SU(2)$ spin are given, respectively, and $0 \le x \le P_{N-1}$. We have assumed that $P_1 \ge P_{N-1}$. The $SU(2)$ spin interval is then

$$R = \frac{P_1 - P_{N-1}}{2}, \ldots, \frac{P_{N-1} + P_1}{2}, \tag{4.14}$$

and the $U(1)$ charge is given by

$$n = \frac{1}{2}(P_{N-1} + 2(P_{N-2} - P_2) - P_1). \tag{4.15}$$

By the Dirac quantization condition, $n$ should be an integer, so we must have that

$$m := \frac{P_1 - P_{N-1}}{2} \tag{4.16}$$

is an integer taking values within the interval $m = 0, \ldots, \frac{P_1}{2}$ if $P_1$ is even and $m = 0, \ldots, \frac{P_1 - 1}{2}$ if $P_1$ is odd. The energy spectrum corresponding to the Hamiltonian (4.3) reads

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(N)} - R(R+1) - \left(1 - \frac{2}{N}\right)n^2\right). \tag{4.17}$$

This equation can be rewritten in terms of $P_2$, $P_1$, $m$, and $n$ by using Eqs. (4.17), (4.15), and (4.16):

$$\begin{aligned} E = \frac{1}{2M\ell^2}\Bigg(&\left(\frac{N-1}{2N}\right)P_1^2 + \left(\frac{N-2}{N}\right)P_2^2 + \left(\frac{N-2}{N}\right)(n^2 + m^2 + 2nm + P_2^2 + 2nP_2 + 2mP_2) \\ &+ \left(\frac{N-1}{2N}\right)(4m^2 + P_1^2 - 4mP_1) + \left(\frac{N-2}{N}\right)P_1P_2 + \frac{2}{N}P_1(n + m + P_2) - \frac{1}{N}(2mP_1 - P_1^2) + \frac{4}{N}P_2(n + m + P_2) \\ &- \frac{2}{N}P_2(2m - P_1) - \left(\frac{N-2}{N}\right)(2m - P_1)(n + m + P_2) + \left(\frac{N-1}{2}\right)P_1 + (N-2)P_2 + (N-2)P_{N-2} \\ &+ \left(\frac{N-1}{2}\right)(-2m + P_1) - \left(\frac{N-2}{N}\right)n^2 - R(R+1)\Bigg). \end{aligned} \tag{4.18}$$

Taking the maximum value of the spin $R$,

$$R = \frac{P_{N-1} + P_1}{2} = P_1 - m, \tag{4.19}$$

the lowest energy becomes

$$\begin{aligned} E = \frac{1}{2M\ell^2}\Bigg(&\left(\frac{N-1}{2N}\right)(2R^2 + 2m^2) \\ &+ \frac{N-2}{N}(2P_2^2 + mn + 2nP_2 + 2RP_2 + 2mP_2 + Rn + Rm) \\ &+ \frac{1}{N}(2Rn + 4RP_2 + 2mn + R^2 + m^2 + 2Rm \\ &+ 4P_2n + 4P_2m + 4P_2^2) \\ &+ \left(\frac{N-1}{2}\right)(2R) + (N-2)(2P_2 + n + m) - R(R+1)\Bigg). \end{aligned} \tag{4.20}$$

Once again, the LLL at fixed background charges $n$ and $R$ are controlled by two integers, $m$ and $P_2$. The LLL is found by putting $P_2 = m = 0$. This gives the energy eigenvalue

$$E_{LLL} = \frac{1}{2M\ell^2}(nR + (N-2)(n + R)), \tag{4.21}$$

which collapses to Eq. (3.30) for $N = 4$ as expected. More generally, to match the formulas of this section to those for $N = 4$, we note that the correspondence for the IRR labels is determined to be

$$\begin{aligned} &(p, q = q_1 + q_2, r) \\ &\longrightarrow (P_1, P_2 = q_2, 0, \ldots, P_{N-2} = q_1, P_{N-1}). \end{aligned} \tag{4.22}$$

For a pure $SU(2)$ background $n = 0$, $R \neq 0$, $R$ should scale in the thermodynamic limit as $R^{(2)} \sim \ell^2$. The number of fermions in the LLL with $\nu = 1$ is $\mathcal{N} = \dim(R, 0, \ldots, 0, R)$, where

$$\begin{aligned} \dim(R, 0, \ldots, 0, R) &= \frac{1}{(N-1)!(N-2)!(N-3)!(R+1)!R!} \\ &\times ((R + N - 3)!(N-4)!(R + N - 3)! \\ &\times (R + N - 2)(R+1)(2R + N - 1)(N-3) \\ &\times (R + N - 2)), \end{aligned}$$

and the corresponding spatial density is

$$\rho \sim \frac{\mathcal{N}}{\ell^{4N-8}(2R+1)} \longrightarrow \frac{R^{2N-3}}{k\ell^{4N-8}(2R+1)} \longrightarrow \text{finite}. \tag{4.23}$$

For both $U(1)$ and $SU(2)$ backgrounds $n \neq 0$, $R \neq 0$, we can choose the scaling $n \sim \ell^2$ and keep $R$ finite in thermodynamic limit. The $\mathcal{N}$ in the LLL with $\nu = 1$ is

$$\mathcal{N} = \dim(R, 0, \ldots, n, R) \longrightarrow n^{2N-4}, \tag{4.24}$$

and the spatial density reads

$$\rho \sim \frac{\mathcal{N}}{\ell^{4N-8}(2R+1)} \longrightarrow \frac{n^{2N-4}}{k\ell^{4N-8}(2R+1)} \longrightarrow \text{finite}. \tag{4.25}$$

Before ending this section, let us briefly list a few of the results of our analysis for the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^5)$. Labeling the IRR of $SU(5)$ with $(p, q, r, s)$, we find that the energy spectrum due to only an Abelian monopole background is

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(5)} - \frac{3}{5}n^2\right) = \frac{1}{2M\ell^2}(p^2 + 2q^2 + 2nq + 2qp + pn + 4p + 6q + 3n), \tag{4.26}$$

where we have used $p = s$ and $r = n + q$ in $\mathcal{C}_2^{SU(5)}$. The numbers $p$ and $q$ play the role of Landau level indices. So the ground-state energy is obtained by letting $p = q = 0$, which yields

$$E_{LLL} = \frac{3n}{2M\ell^2}, \tag{4.27}$$

and wave functions take the form

$$\mathcal{D}^{(p,q,n+q,p)}_{L^{SU(3)},L,L_3,L_{24},0,0,0,-\sqrt{\frac{3}{5}}n}(g). \tag{4.28}$$

With reference to Eq. (B1) the dimension of the $(0, 0, n, 0)$ representation gives the degeneracy of the LLL as follows:

$$\dim(0, 0, n, 0) = \frac{(n+2)!(n+1)!(n+3)^2(n+4)(n+2)}{4!3!n!(n+1)!}. \tag{4.29}$$

Finally, the spatial density of fermions is readily computed to be

$$\rho \longrightarrow \frac{n^6}{\ell^{12}} = \left(\frac{B}{2\pi}\right)^6. \tag{4.30}$$

For $SU(2)$ and $U(1)$ backgrounds together, the energy spectrum reads

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(5)} - R(R+1) - \frac{3}{5}n^2\right), \tag{4.31}$$

where $SU(2)$ has the spin range

$$R = \frac{p-s}{2}, \ldots, \frac{s+p}{2}, \tag{4.32}$$

assuming that $p > s$. The $U(1)$ charge now reads $n = \frac{1}{2}(s + 2(r-q) - p)$. Setting $s = p - 2m$, the maximal $SU(2)$ charge $R = p - m$ gives the energy eigenvalues

$$E = \frac{1}{2M\ell^2}(m^2 + 2q^2 + mn + 2qn + 2Rq + 2mq + Rn + Rm + 3R + 6q + 3n + 3m). \tag{4.33}$$

Here, applying the LLL condition gives the lowest energy as

$$E_{LLL} = \frac{1}{2M\ell^2}(n(R+3) + 3R). \tag{4.34}$$

## V. LOCAL FORM OF THE WAVE FUNCTIONS AND THE GAUGE FIELDS

In this section, we first provide the local form of the wave functions for solutions of the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^4)$. For this purpose, we will use the well-known Plücker coordinates for $\mathbf{Gr}_2(\mathbb{C}^4)$.

The Plücker coordinates for $\mathbf{Gr}_k(\mathbb{C}^N)$ are constructed out of a projective embedding, the so-called Plücker embedding $\mathbf{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbf{P}(\bigwedge^k \mathbb{C}^n)$, which provides a one-to-one map between the set of $k$-dimensional subspaces of $\mathbb{C}^n$ [i.e., the Grassmannian $\mathbf{Gr}_k(\mathbb{C}^N)$] and a subset of the projective space of the $k$th exterior power of the vector space $\mathbb{C}^n$, where the latter is denoted as $\mathbf{P}(\bigwedge^k \mathbb{C}^n)$. This subset of $\mathbf{P}(\bigwedge^k \mathbb{C}^n)$ is a projective variety characterized by the intersection of quadrics induced by all possible relations between generalized Plücker coordinates. In what follows, we focus on the Plücker embedding of $\mathbf{Gr}_2(\mathbb{C}^4)$; more details and general discussions can be found in Refs. [16,35].

For $\mathbf{Gr}_2(\mathbb{C}^4)$, this construction entails the projective space $\mathbf{P}(\mathbb{C}^4 \wedge \mathbb{C}^4) \equiv \mathbb{C}P^5$. Introducing two sets of complex coordinates $v_\alpha$, $w_\alpha$ $(\alpha = 1, \ldots, 4)$, which is one set for each $\mathbb{C}^4$, a fully antisymmetric basis for the exterior product space $\mathbb{C}^4 \wedge \mathbb{C}^4$ would be given in the form of

$$P_{\alpha\beta} = \frac{1}{\sqrt{2}}(v_\alpha w_\beta - v_\beta w_\alpha). \tag{5.1}$$

$P_{\alpha\beta}$ may be contemplated as the homogenous coordinates on $\mathbb{C}P^5$ with the identification $P_{\alpha\beta} \sim \lambda P_{\alpha\beta}$, where $\lambda \in U(1)$ and $\sum_{\alpha,\beta}^4 |P_{\alpha\beta}|^2 = 1$.

The Plücker embedding of $\mathbf{Gr}_2(\mathbb{C}^4)$ in $\mathbb{C}P^5$ is given by the homogeneous condition

$$\varepsilon_{\alpha\beta\gamma\delta} P_{\alpha\beta} P_{\gamma\delta} = P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0, \tag{5.2}$$

defining the Klein quadric $Q_4$ in $\mathbb{C}P^5$, which is complex analytically equivalent to $\mathbf{Gr}_2(\mathbb{C}^4)$. The homogeneous equation $\varepsilon_{\alpha\beta\gamma\delta}P_{\alpha\beta}P_{\gamma\delta}=0$ is nothing but the restriction to a projective hypersurface of degree 2, which is the quadric $Q_4$.

It is possible to employ $P_{\alpha\beta}$ to parametrize the columns of $g\in SU(4)$ in the IRR (0,1,0); we choose a parametrization of the form

$$g := \left(\begin{array}{c|c|c|c|cc} \vdots & \vdots & \vdots & \vdots & P^*_{34} & P_{12} \\ \vdots & \vdots & \vdots & \vdots & -P^*_{24} & P_{13} \\ \vdots & \vdots & \vdots & \vdots & P^*_{23} & P_{14} \\ \vdots & \vdots & \vdots & \vdots & P^*_{14} & P_{23} \\ \vdots & \vdots & \vdots & \vdots & -P^*_{13} & P_{24} \\ \vdots & \vdots & \vdots & \vdots & P^*_{12} & P_{34} \end{array}\right) \tag{5.3}$$

where the orthogonality of the columns follows from the Plücker relation in Eq. (5.2). For a shorthand notation, we will employ $g_{N6}=P_N := P_{\alpha\beta}$, $g_{N5}=\varepsilon_{NM}P^*_M=\varepsilon_{\alpha\beta\gamma\delta}P^*_{\gamma\delta}$ with $N\equiv[\alpha\beta]$, $N=1,\ldots,6$ and $\alpha\beta=(12,13,14,23,24,34)$.

The wave functions in the $U(1)$ background gauge field, $\mathcal{D}^{(0,q_1+q_2,0)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};0,0,0,0,\frac{n}{\sqrt{2}}}(g)$, are the sections of the $U(1)$ bundle over $\mathbf{Gr}_2(\mathbb{C}^4)$, which fulfill the gauge transformation property

$$\begin{aligned}\mathcal{D}^{(0,q_1+q_2,0)}(gh) &= \mathcal{D}^{(0,q_1+q_2,0)}(ge^{i\lambda_{15}\theta}) \\ &= \mathcal{D}^{(0,q_1+q_2,0)}(g)e^{i\frac{n}{\sqrt{2}}\theta}.\end{aligned} \tag{5.4}$$

Using Eq. (3.14) for $\lambda_{15}$ and Eq. (5.3), this yields immediately

$$\mathcal{D}^{(0,1,0)}(g)\sim P_{\alpha\beta}. \tag{5.5}$$

We point out that the $(0,q,0)$ IRR is the $q$-fold symmetric tensor product of the (0,1,0) representation; to wit, $(0,q,0)\equiv\prod_{\otimes q}(0,1,0)$. This can be shown by the symmetric tensor product $(\otimes_S)$ of $\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}$ tableaux as

$$\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array} \otimes_S \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array} \otimes_S \cdots \otimes_S \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array} \longrightarrow \overbrace{\begin{array}{|c|c|c|}\hline \, & \, & \, \\ \hline \, & \, & \, \\ \hline\end{array}\cdots\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}}^{q}. \tag{5.6}$$

We infer that

$$\mathcal{D}^{(0,q_1+q_2,0)}(g)\sim P_{\alpha_1\beta_1}P_{\alpha_2\beta_2}\cdots P_{\alpha_{q_1}\beta_{q_1}}P^*_{\gamma_1\delta_1}P^*_{\gamma_2\delta_2}\cdots P^*_{\gamma_{q_2}\delta_{q_2}}. \tag{5.7}$$

So the LLL wave functions are those with $q_2=0$,

$$\mathcal{D}^{(0,q_1,0)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};0,0,0,0,\frac{n}{\sqrt{2}}}(g)\sim P_{\alpha_1\beta_1}P_{\alpha_2\beta_2}\cdots P_{\alpha_{q_1}\beta_{q_1}}, \tag{5.8}$$

which are holomorphic in the Plücker coordinates.

Another useful point to mention here is that, although the right-invariant vector fields on $SU(4)$ cannot be easily written down, the left-invariant vector fields can be easily given as [33]

$$\begin{aligned}L_k = &-v_j(\lambda_k)_{ij}\frac{\partial}{\partial v_i} - w_j(\lambda_k)_{ij}\frac{\partial}{\partial w_i} + v_i^*(\lambda_k)_{ij}\frac{\partial}{\partial v_j^*} \\ &+ w_i^*(\lambda_k)_{ij}\frac{\partial}{\partial w_j^*},\end{aligned} \tag{5.9}$$

where $\lambda_k$ $(k=1,\ldots,15)$ are the Gell-Mann matrices for $SU(4)$. Choosing complex vectors $\mathbf{v}$ and $\mathbf{w}$ to satisfy the orthonormality conditions

$$v_iw_i^*=0, \qquad |\mathbf{v}|^2=|\mathbf{w}|^2=1, \tag{5.10}$$

and using the identity

$$\sum_{k=1}^{N^2-1}\lambda^k_{ij}\lambda^k_{mn}=\frac{1}{2}\delta_{in}\delta_{jm}-\frac{1}{2N}\delta_{ij}\delta_{mn}, \tag{5.11}$$

for $N=4$, the Casimir $\mathcal{C}_2^{SU(4)}$ may be realized as the differential operator,

$$\begin{aligned}\mathcal{C}_2^{SU(4)} = &\frac{15}{8}\left(v_i\frac{\partial}{\partial v_i}+w_i\frac{\partial}{\partial w_i}+v_i^*\frac{\partial}{\partial v_i^*}+w_i^*\frac{\partial}{\partial w_i^*}\right) \\ &+\frac{3}{8}\left(v_iv_j\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}+w_iw_j\frac{\partial}{\partial w_i}\frac{\partial}{\partial w_j}+\text{c.c.}\right) \\ &-\frac{2}{8}\left(v_iw_j\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_j}-v_iw_j^*\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_j^*}+\text{c.c}\right) \\ &+\frac{1}{8}\left(v_iv_j^*\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j^*}+w_iw_j^*\frac{\partial}{\partial w_i}\frac{\partial}{\partial w_j^*}+\text{c.c.}\right) \\ &+v_iw_j\frac{\partial}{\partial v_j}\frac{\partial}{\partial w_i}+v_i^*w_j^*\frac{\partial}{\partial v_j^*}\frac{\partial}{\partial w_i^*}-\frac{\partial}{\partial v_j}\frac{\partial}{\partial v_j^*}-\frac{\partial}{\partial w_j}\frac{\partial}{\partial w_j^*},\end{aligned} \tag{5.12}$$

which clearly generates the eigenvalues $\frac{q^2}{2}+2q$ when applied to the wave functions (5.7).

The LLL with filling factor $\nu=1$ has $\mathcal{N}=\dim(0,1,0)=\frac{1}{12}(n+1)(n+2)^2(n+3)$ number of particles. Its multiparticle wave function is given in terms of the Slater determinant as

$$\Psi_{\rm MP} = \frac{1}{\sqrt{\mathcal{N}!}} \det \begin{pmatrix} \Psi_{\Lambda_1}(P^{\mathbf{1}}) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^{\mathbf{1}}) \\ \Psi_{\Lambda_1}(P^{\mathbf{2}}) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^{\mathbf{2}}) \\ \vdots & \ddots & \vdots \\ \Psi_{\Lambda_1}(P^{\mathcal{N}}) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^{\mathcal{N}}) \end{pmatrix} = \frac{1}{\sqrt{\mathcal{N}!}} \varepsilon^{\Lambda_1\Lambda_2\cdots\Lambda_n} \Psi_{\Lambda_1}(P^{(1)})\Psi_{\Lambda_2}(P^{(2)})\cdots\Psi_{\Lambda_N}(P^{(N)}). \tag{5.13}$$

Here, $P^{\mathbf{i}}$ denotes the $i$th position fixed in the Hall fluid, and correspondingly $\Psi_{\Lambda_j}(P^{\mathbf{i}})$ refers to the wave function of the $j$th particle located at the position $P^{\mathbf{i}}$. Now, let us calculate the two-point correlation function in this fluid in the presence of only a $U(1)$ background. For a one-particle wave function in Eq. (5.5) (with $n = 1$), our notation transcribes as

$$\Psi_{\Lambda_i}(P^{\mathbf{i}}) \equiv \Psi^{\mathbf{i}}_{\alpha\beta} \sim P^{\mathbf{i}}_{\alpha\beta}. \tag{5.14}$$

The LLL wave function given in Eq. (5.8) may now be denoted by

$$\Psi_{\Lambda_i}(P^{\mathbf{i}}) \equiv \Psi^{\mathbf{i}}_{\Lambda_i} \sim (P^{\mathbf{i}}_{\alpha\beta})^n. \tag{5.15}$$

The general form of the correlation function between a pair of particles, say 1 and 2, on a manifold $\mathcal{M}$ is given by

$$\Omega(1,2) = \int_{\mathcal{M}} |\Psi_{\rm MP}|^2 d\mu(3)d\mu(4)\cdots d\mu(\mathcal{N}), \tag{5.16}$$

with $d\mu(i)$ being the measure of integration on $\mathcal{M}$ in the coordinates of the $i$th particle, and $\Psi_{\rm MP}$ represents the multiparticle wave function of the Hall fluid on the manifold $\mathcal{M}$. Expanding the determinant formula (5.13) and using some algebra, one can show that $\Omega(1,2)$ can be simplified as

$$\Omega(1,2) = \int_{\mathcal{M}} |\Psi_{\rm MP}|^2 d\mu(3)d\mu(4)\cdots d\mu(\mathcal{N}) = |\Psi^{\mathbf{1}}|^2|\Psi^{\mathbf{2}}|^2 - |\Psi^{*\mathbf{1}}_{\Lambda}\Psi^{\mathbf{2}}_{\Lambda}|^2. \tag{5.17}$$

To compute Eq. (5.17) for our case, we take the normalized coordinate chart $\gamma_i := \frac{P_{\alpha\beta}}{P_{12}}$, where $P_{12} \neq 0$,

$$\mathcal{P} = \frac{1}{\sqrt{1+|\gamma_a|^2}}(1,\gamma_1,\ldots,\gamma_5)^T := \frac{1}{\sqrt{1+|\gamma_a|^2}}(1,\vec{\gamma}), \tag{5.18}$$

on the Grassmannian $\mathbf{Gr}_2(\mathbb{C}^4)$. In this coordinate patch, Eq. (5.15) becomes $\Psi^{\mathbf{i}}_\alpha \sim (\mathcal{P}^{\mathbf{i}}_\alpha)^n$. Inserting this into Eq. (5.17) yields

$$\begin{aligned}\Omega(1,2) &= 1 - |\mathcal{P}^{*\mathbf{1}}_{\Lambda}\mathcal{P}^{\mathbf{2}}_{\Lambda}|^n \\ &= 1 - \left[\frac{\gamma^{*\mathbf{1}}_a\gamma^{\mathbf{2}}_a\gamma^{\mathbf{1}}_b\gamma^{*\mathbf{2}}_b}{1+|\gamma^{\mathbf{1}}_a|^2+|\gamma^{\mathbf{2}}_a|^2+|\gamma^{\mathbf{1}}_a|^2|\gamma^{\mathbf{2}}_a|^2}\right]^n \\ &= 1 - \left[1 - \frac{|\vec{\gamma}^{\mathbf{1}}-\vec{\gamma}^{\mathbf{2}}|^2}{1+|\gamma^{\mathbf{1}}_a|^2+|\gamma^{\mathbf{2}}_a|^2+|\gamma^{\mathbf{1}}_a|^2|\gamma^{\mathbf{2}}_a|^2}\right]^n. \end{aligned} \tag{5.19}$$

Let us set $\vec{X} = \vec{\gamma}\ell$. In the thermodynamic limit $\mathcal{N} \to \infty$ and $n \to \infty$, Eq. (5.19) takes the form

$$\begin{aligned}\Omega(1,2) &= 1 - [1 - |\vec{X}^{\mathbf{1}} - \vec{X}^{\mathbf{2}}|^2 \\ &\quad \times [\ell^2 + |\vec{X}^{\mathbf{1}}|^2 + |\vec{X}^{\mathbf{2}}|^2 + \ell^{-2}|\vec{X}^1|^2|\vec{X}^{\mathbf{2}}|^2]^{-1}]^n \\ &\to 1 - \left[1 - \frac{2B}{n}|\vec{X}^{\mathbf{1}} - \vec{X}^{\mathbf{2}}|^2\right]^n \\ &\to 1 - e^{-2B|\vec{X}^{\mathbf{1}}-\vec{X}^{\mathbf{2}}|^2} \\ &= 1 - e^{-2B(\vec{x}^{\mathbf{1}}-\vec{x}^{\mathbf{2}})^2} e^{-2B\ell^2(\det\Gamma^{\mathbf{1}}-\det\Gamma^{\mathbf{2}})^2}, \end{aligned} \tag{5.20}$$

where we have used $n = 2B\ell^2$ and where

$$\Gamma^{\mathbf{i}} := \begin{pmatrix} \gamma^{\mathbf{i}}_2 & \gamma^{\mathbf{i}}_1 \\ \gamma^{\mathbf{i}}_4 & \gamma^{\mathbf{i}}_3 \end{pmatrix}. \tag{5.21}$$

Note that the last line of Eq. (5.20) shows the two-point function of the particles located at the positions $\vec{x}^{\mathbf{1}}$, $\vec{x}^{\mathbf{2}}$ on $\mathbf{Gr}_2(\mathbb{C}^4)$ and is extracted from that of the particles on $\mathbb{C}P^5$ at the positions $\vec{X}^{\mathbf{1}}$, $\vec{X}^{\mathbf{2}}$ by a restriction of these particles to the algebraic variety determined by $X^{\mathbf{i}}_5 \equiv \ell \det\Gamma^{\mathbf{i}}$, as expected. It is apparent from this function that the probability of finding two particles at the same point goes to zero. This result indicates the incompressibility of the Hall fluid.

Turning our attention to the $U(1)$ gauge field, we may write

$$A = -\frac{in}{\sqrt{2}}\,\mathrm{Tr}(\lambda^{15}_{(6)}g^{-1}dg). \tag{5.22}$$

With the help of Eqs. (5.3) and (3.15), one can express $A$ in terms of the Plücker coordinates as

$$\begin{aligned} A &= -\frac{in}{\sqrt{2}}(\lambda^{15}_{(6)})_{LM}(g^{-1})_{MN}(dg)_{NL} \\ &= -\frac{in}{2}(-(g^{-1})_{5N}(dg)_{N5} + (g^{-1})_{6N}(dg)_{N6}) \\ &= -\frac{in}{2}(-g^*_{N5}(dg)_{N5} + g^*_{N6}(dg)_{N6}) \\ &= -\frac{in}{2}(-P_N dP^*_N + P^*_N dP_N) \\ &= -inP^*_N dP_N, \end{aligned} \tag{5.23}$$

where use has been made of the notational conventions stated below Eq. (5.3) and where we have noted the fact that

$d(P_N^* P_N) = 0$ due to Eq. (5.2). Under $U(1)$ gauge transformations, $A$ transforms to $A + d(\frac{n\theta}{\sqrt{2}})$, which is consistent with the transformation of the wave functions given in Eq. (5.4).

Let us introduce the notation $\tilde{\mathcal{P}} \equiv (P_1, \ldots P_6)^T$, where $T$ stands for transpose, and define a nonhomogeneous coordinate chart $\mathcal{Q} \equiv \frac{\tilde{\mathcal{P}}}{P_1}$ with $P_1 \neq 0$ on $\mathbf{Gr}_2(\mathbb{C}^4)$ as

$$\mathcal{Q} \equiv (1, \gamma_1, \ldots, \gamma_5)^T, \tag{5.24}$$

subject to the Plücker relation (5.2), which, in terms of the (affine coordinates) $\gamma_i$, takes the form

$$\gamma_5 = \gamma_2\gamma_3 - \gamma_1\gamma_4. \tag{5.25}$$

Without Eq. (5.25), $\mathcal{Q}$ is a nonhomogenous coordinate chart in $\mathbb{C}P^5$. We can express our gauge potential as

$$\begin{aligned} A &= -in\mathcal{P}^\dagger d\mathcal{P} \\ &= -in|P_1|^2\mathcal{Q}^\dagger d\mathcal{Q} - inP_1^*|\mathcal{Q}|^2 dP_1 \\ &= -in|\mathcal{Q}|^{-2}\mathcal{Q}^\dagger d\mathcal{Q} - inP_1^*|P_1|^{-2}dP_1 \\ &= -in|\mathcal{Q}|^{-2}\mathcal{Q}^\dagger d\mathcal{Q} - inP_1^{-1}dP_1 \\ &= -in\partial\ln(|\mathcal{Q}|^2) - ind\ln(P_1) \\ &= -in\partial K - ind\ln(P_1), \end{aligned} \tag{5.26}$$

where $K$ is the $\mathbb{C}P^5$ Kähler potential given by

$$K = \ln|\mathcal{Q}|^2 \equiv \ln(1 + |\gamma_i|^2) \tag{5.27}$$

and subject to the condition (5.25).

The field strength is calculated via

$$\begin{aligned} F = dA &= -\frac{in}{\sqrt{2}}\mathrm{Tr}(\lambda_{(6)}^{15} g^{-1}dg \wedge g^{-1}dg) \\ &= -indP_N^* \wedge dP_N. \end{aligned} \tag{5.28}$$

We note that $F$ is an antisymmetic, gauge invariant, and closed 2-form on $\mathbf{Gr}_2(\mathbb{C}^4)$, and as such it is proportional to the Kähler 2-form $\Omega$ over $\mathbf{Gr}_2(\mathbb{C}^4)$. This fact can be readily verified using Eq. (5.26) and writing

$$F = dA = in\partial\partial^* K = n\Omega, \tag{5.29}$$

where $\partial$, $\partial^*$ are the Dolbeault operators in the coordinates $\gamma_i$ and $\gamma_i^*$, respectively, and $d = \partial + \partial^*$. The relation (5.29) with Eq. (5.27) leads to the form of the field strength [35]

$$F = -in\left(\frac{d\gamma_i^* \wedge d\gamma_i}{1 + |\gamma|^2} - \frac{\gamma_i d\gamma_i^* \wedge \gamma_j^* d\gamma_j}{(1 + |\gamma|^2)^2}\right), \tag{5.30}$$

being subject to the Plücker relation (5.25). Let us associate with each index $i$ a dual index $\hat{i}$ in the sense that $i$ is dual to $\hat{i}$ if $\gamma_i\gamma_{\hat{i}}$ appears in the Plücker relation. Hence, 1,4 and 2,3 are dual to one another. Expanding $\gamma_5$ in Eq. (5.25) results in the Hermitian components for the Kähler form $\Omega$ as

$$\begin{aligned} \Omega_{ii^*} &= iN_\gamma\left(1 + \prod_{\alpha=1,\alpha\neq i,\hat{i}}^{4}|\gamma_\alpha|^2 + (1 + |\gamma_{\hat{i}}|^2)\sum_{\alpha=1,\alpha\neq i}^{4}|\gamma_\alpha|^2\right), \\ \Omega_{ij^*} &= -iN_\gamma(1 + |\gamma_{\hat{i}}|^2 + |\gamma_{\hat{j}}|^2)(\gamma_i^*\gamma_j + \gamma_{\hat{i}}\gamma_{\hat{j}}^*), \quad i < j, \quad j \neq \hat{i}, \\ \Omega_{\hat{i}i^*} &= -iN_\gamma\left(\gamma_i^*\gamma_{\hat{i}}\left(\sum_{\alpha=1}^{4}(|\gamma_\alpha|^2 - |\gamma_i|^2 - |\gamma_{\hat{i}}|^2\right)\right. \\ &\quad \left. - \frac{1}{2}(\gamma_i^*)^2\prod_{j\neq i,\hat{i}}\gamma_j\gamma_{\hat{j}} - \frac{1}{2}(\gamma_{\hat{i}})^2\prod_{j\neq i,\hat{i}}\gamma_j^*\gamma_{\hat{j}}^*\right), \quad i < \hat{i}, \end{aligned} \tag{5.31}$$

where $N_\gamma = (1 + \sum_{a=1}^{5}|\gamma_a|^2)^{-2}$. In these formulas, Einstein summation convention is not in use.

It is known from very general considerations [36] that the integral of $F$ over a noncontractable 2-surface $\Sigma$ in $\mathbf{Gr}_2(\mathbb{C}^4)$ is an integral multiple of $2\pi$:

$$\frac{1}{2\pi}\int_\Sigma F = n. \tag{5.32}$$

In the present context, this result signals an analog of the Dirac quantization condition with $\frac{n}{2}$ identified as the magnetic monopole charge. Therefore, we do have that the magnetic field is $B = \frac{n}{2\ell^2}$.

A number of remarks is in order. The generalization of our results to all higher-dimensional Grassmannians is fairly straightforward. Taking $\mathbf{Gr}_2(\mathbb{C}^N)$, the only difference is that now both the vector potential $A$ and field strength $F$ are subject to the Plücker relations

$$\gamma_{ik}\gamma_{jl} = \gamma_{ij}\gamma_{kl} - \gamma_{il}\gamma_{kj}, \qquad 1 \leq i < k < j < l \leq 2(N-2), \tag{5.33}$$

in terms of the nonhomogeneous coordinates $\gamma_{ij} \coloneqq P_{ij}/P_{12}$ in the patch where $P_{12} \neq 0$. The parametrization in Eq. (5.3) can be generalized to $N(N-1)/2$-dimensional fundamental representations of the $SU(N)$ group by means of these Plücker relations. Let us also note that the Grassmannians have a nontrivial algebraic topological structure that, for the best of our purposes here, is reflected in their second cohomology group, which is nonzero, or more precisely $H^2(\mathbf{Gr}_k(\mathbb{C}^N)) = \mathbb{Z}$ [37]. This is the reason why the integral of the first Chern character in Eq. (5.32) is an integer. Similarly, one may consider the integral of the $d$th $(d = 2(N-2))$-order Chern character for the Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$ [38],

$$\frac{1}{d!(2\pi)^d \mathrm{vol}(\mathbf{Gr}_2(\mathbb{C}^N))} \int_{\mathbf{Gr}_2(\mathbb{C}^N)} F \wedge \Omega \cdots \wedge \Omega = n, \tag{5.34}$$

for $F = n\Omega$.

## VI. CONCLUSIONS AND OUTLOOK

In this paper, we have given a formulation of the QHE on $\mathbf{Gr}_2(\mathbb{C}^N)$. We solved the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ using group theoretical techniques and gave the energy spectra and the wave functions of charged particles on $\mathbf{Gr}_2(\mathbb{C}^N)$ in the background of both Abelian and non-Abelian magnetic monopoles. For the Abelian monopole background, using the local description of wave functions in terms of Plücker coordinates on $\mathbf{Gr}_2(\mathbb{C}^4)$, we showed that the LLL at filling factor $\nu = 1$ forms an incompressible fluid and indicated how this result generalizes to all $\mathbf{Gr}_2(\mathbb{C}^N)$.

We want to make the following observations about the QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ with $U(1)$ background. Because of the isomorphisms $\mathrm{Spin}(6) \cong SU(4)$ and $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$, the Stiefel manifold $\mathbf{St}_2(\mathbb{R}^6) \equiv \frac{\mathrm{Spin}(6)}{\mathrm{Spin}(4)}$ forms the principal $U(1)$ fibration [34]

$$U(1) \longrightarrow \mathbf{St}_2(\mathbb{R}^6) \longrightarrow \mathbf{Gr}_2(\mathbb{C}^4). \tag{6.1}$$

Let us also make note of the family of fibrations $\mathbf{St}_{k-1}(\mathbb{R}^{n-1}) \longrightarrow \mathbf{St}_k(\mathbb{R}^n) \longrightarrow S^{n-1}$, which for $k = 2$ and $n = 6$ is

$$S^4 \longrightarrow \mathbf{St}_2(\mathbb{R}^6) \longrightarrow S^5. \tag{6.2}$$

Together, these facts imply that $\mathbf{Gr}_2(\mathbb{C}^4)$ has the local structure $\frac{S^5 \times S^4}{U(1)}$. We therefore propose that the QHE on $S^5$ with the $S^4$ fibers may be seen as a QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ with a $U(1)$ background gauge field. We expect that the $S^4$ fibers will be associated to a $SO(5)$ gauge field background. In fact, the formulation of the QHE on the 3-sphere [10],

$$S^3 = \frac{SU(2) \times SU(2)}{SU(2)_{\mathrm{diag}}} \cong \frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)}, \tag{6.3}$$

selects the constant background gauge field as the spin connection, and in a construction generalizing this to the QHE on $S^5$,

$$S^5 = \frac{SO(6)}{SO(5)} = \frac{\mathrm{Spin}(6)}{\mathrm{Spin}(5)}, \tag{6.4}$$

one should be selecting a constant $SO(5)$ background gauge field, taking it again as the spin connection. Such a choice of the gauge field appears to be consistent with our heuristic argument. Our observation is inspired by and bears a resemblance to the relation between the QHE on $\mathbb{C}P^7$ and $S^8$. The former can be realized locally as $\frac{S^8 \times S^7}{U(1)}$, while the latter forms the base of the third Hopf map $S^7 \longrightarrow S^{15} \longrightarrow S^8$, and $S^{15}$ is a $U(1)$ bundle over $\mathbb{C}P^7$ [9].

A number of future directions for further research is foreseen. First, the aforementioned relation with the QHE on $S^5$ could be made more concrete by developing the latter along the lines discussed here and adapting the approach of Refs. [9] and [10]. It may be possible to develop Chern–Simons-type effective field theories along the lines of Ref. [6] to shed more light on the structure of the QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ in particular. Formulation of the edge states may also be investigated building upon the ideas of Refs. [14,15]. We hope to report on the progress of any of these topics elsewhere.

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## APPENDIX A

In this short appendix, we provide a derivation of the normalization coefficient of $R_{N^2-1}$ in the $\frac{N(N-1)}{2}$-dimensional IRR of $SU(N)$ for $N \geq 3$. Let $T_a^{(D)}$ label the $N^2 - 1$ generators of $SU(N)$ in the defining $N$-dimensional representation. Let us choose their trace normalization to be

$$\mathrm{Tr}(T_a^{(D)} T_b^{(D)}) = \frac{1}{2}\delta_{ab}. \tag{A1}$$

It is a well-known fact in the representation theory of Lie groups that such a choice fixes the trace normalization of the generators in all the IRR [39]. We can proceed to write the trace normalization in an IRR $R$ of $SU(N)$ as

$$\mathrm{Tr}(T_a^{(R)} T_b^{(R)}) = \kappa_{ab}, \tag{A2}$$

where $\kappa_{ab}$ is a rank-2 tensor invariant under $SU(N)$ transformations. Since the only rank-2 invariant $SU(N)$ tensor is a Kronecker delta, $\delta_{ab}$, we have

$$\kappa_{ab} = X_{(R)}\delta_{ab}, \tag{A3}$$

where $X_{(R)}$, commonly known as the *Dynkin index* of the representation $R$ of the group $SU(N)$, is given by [39]

$$X_{(R)} = \frac{\dim(R)}{\dim(SU(N))} \mathcal{C}_2(R). \tag{A4}$$

We have that $\dim(SU(N))$ is equal to $N^2-1$ and $\mathcal{C}_2^R$ is the quadratic Casimir of the IRR $R$. For either of the $\frac{N(N-1)}{2}$-dimensional IRRs, $(0,1,0,\ldots,0,0)$ or $(0,0,\ldots,1,0)$ of $SU(N)$, this gives, using Eq. (4.4),

$$X_{(R)} = \frac{N-2}{N}, \tag{A5}$$

and the trace formula (A2) then reads

$$\mathrm{Tr}(T_a T_b) = \frac{N-2}{N}\delta_{ab} \tag{A6}$$

in either of the $\frac{N(N-1)}{2}$-dimensional IRRs. Our aim is to find the coefficient of $R_{N^2-1}$ in these representations. In terms of the Young diagrams, the branching of, say, $(0,1,0,\ldots,0,0)$ representation under $SU(N-2)\times SU(2)\times U(1)$ gives

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} = \left(\cdot \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right)_{-1} \oplus \left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \otimes \cdot\right)_{\frac{2}{N-2}} \oplus \left(\square \otimes \square\right)_{\frac{4-N}{2(N-2)}}, \tag{A.7}$$

where the subscripts give the $U(1)$ charge (3.8). Considering the dimension of each representation in this branching, we find

$$R_{N^2-1} = \zeta \mathrm{diag}\Bigg(\underbrace{\frac{N-4}{2(N-2)},\ldots,\frac{N-4}{2(N-2)}}_{2(N-2)},\underbrace{\frac{-2}{N-2},\ldots,\frac{-2}{N-2}}_{\frac{(N-2)(N-3)}{2}},1\Bigg), \tag{A8}$$

where $\zeta$ represents the coefficient of $R_{N^2-1}$ and the dimensions of the IRR in the branching (A7) are given in the underbraces. Finally, using Eq. (A8) in Eq. (A6) gives

$$\zeta = \sqrt{\frac{N-2}{N}}. \tag{A9}$$

## AAPPENDIX B

The dimension of the $(P_1,P_2,P_3,\ldots,P_{N-2},P_{N-1})$ representation may be written as

$$\begin{aligned}\dim(P_1,P_2,0,\ldots,0,P_{N-2},P_{N-1}) = \frac{1}{j}&((P_{N-2}+P_{N-1}+N-3)!(P_{N-2}+N-4)!(P_2+N-4)!(P_1+P_2+N-3)!\\ &\times(P_{N-2}+P_{N-1}+P_2+N-2)(P_{N-1}+1)(P_1+P_2+P_{N-2}+P_{N-1}+N-1)\\ &\times(P_1+1)(P_{N-2}+P_2+N-3)(P_1+P_2+P_{N-2}+N-2)),\end{aligned} \tag{B1}$$

where $j$ is

$$j = (N-1)!(N-2)!(N-3)!(N-4)!P_2!P_{N-2}!(P_{N-2}+P_{N-1}+1)!(P_1+P_2+1)!. \tag{B2}$$


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