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Closed-time-path approach to the Casimir energy in real media

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The closed-time-path (CTP) formalism is applied, in the framework of open quantum systems, to study the time evolution of the expectation value of the energy-momentum tensor of a scalar field in the presence of real materials. We analyze quantum (Casimir) fluctuations in a fully nonequilibrium scenario, when the scalar field is interacting with the polarization degrees of freedom of matter, described as quantum Brownian particles (harmonic oscillators coupled to a bath) at each point of space. A generalized analysis is done for two types of couplings between the field and the polarization degrees of freedom. On the one hand, we consider a bilinear coupling between the field and the polarization degrees of freedom, and on the other hand, a (more realistic) current-type coupling as in the case of the electromagnetic field interacting with matter. We successfully compute the CTP generating functional for the field through calculating the corresponding influence functionals. We consider the high-temperature limit for the field, keeping arbitrary temperatures for each part of the material's volume elements. We obtain a closed form for the Hadamard propagator, which allows us to study the dynamical evolution of the expectations values of the energy-momentum tensor components from the initial time when the interactions are turned on. We show that two contributions always take place in the transient evolution: one is associated with the material, and the other is only associated with the field. Transient features are studied and the long-time limit is derived in several cases. We prove that in the steady situation of a field in $n + 1$ dimensions, the material always contributes unless it is nondissipative. Conversely, the proper field contribution vanishes unless the material is nondissipative or—at least for the $1 + 1$ case—if there are regions without material. We finally conclude that any steady quantization scheme in $1 + 1$ dimensions must consider both contributions and, on the other hand, we argue why these results are physically expected from a dynamical point of view, and also could be valid for higher dimensions based on the expected continuity between the nondissipative and real-material cases.

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I. INTRODUCTION

The study of the Casimir forces in the framework of open quantum systems holds the possibility of analyzing nonequilibrium effects, such as the Casimir force between objects at different temperatures [1], the power of heat transfer between bodies [2], and the inclusion of time-dependent evolutions until reaching a stationary situation. Although the celebrated Lifshitz formula [3] describes the forces between dielectrics in a steady situation in terms of their macroscopic electromagnetic properties, it is not derived from a first-principles quantum framework. The original derivation of this very general formula is based on a macroscopic approach, starting from stochastic Maxwell equations and using thermodynamical properties for the stochastic fields. As pointed out in several papers, the connection between this approach and an approach based on a fully quantized model is not completely clear. Moreover, some doubts have been raised about the applicability of the Lifshitz formula to lossy dielectrics [4–6].

Moreover, from a conceptual point of view, the theoretical calculations for mirrors with general electromagnetic properties, including absorption, is not a completely settled issue [4–6]. Since dissipative effects imply the possibility of energy interchanges between different parts of the full system (mirrors, vacuum field, and environment), the theory of open quantum systems [7] is the natural approach to clarify the role of dissipation in Casimir physics. Indeed, in this framework dissipation and noise appear in the effective theory of the relevant degrees of freedom (the electromagnetic field) after integration of the matter and other environmental degrees of freedom.

The quantization in the steady situation (steady quantization scheme) can be performed starting from the macroscopic Maxwell equations and by including noise terms to account for absorption [8]. In this approach a canonical quantization scheme is not possible unless one couples the electromagnetic field to a reservoir (see Ref. [5]), following the standard route to include dissipation in simple quantum-mechanical systems. Another possibility is to establish a first-principles model in which the slabs are described through their microscopic degrees of freedom, which are coupled to the electromagnetic field. In these

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kinds of models, losses are also incorporated by considering a thermal bath to allow for the possibility of absorption of light. There is a large body of literature on the quantization of the electromagnetic field in dielectrics. Regarding microscopic models, the fully canonical quantization of the electromagnetic field in dispersive and lossy dielectrics has been performed by Huttner and Barnett (HB) [9]. In the HB model, the electromagnetic field is coupled to matter (the polarization field), and the matter is coupled to a reservoir that is included in the model to describe the losses. In the context of the theory of quantum open systems, one can regard the HB model as a composite system in which the relevant degrees of freedom belong to two subsystems (the electromagnetic field and the matter), and the matter degrees of freedom are in turn coupled to an environment (the thermal reservoir). The indirect coupling between the electromagnetic field and the thermal reservoir is responsible for the losses. As we will comment below, this will be our starting point to compute the Casimir force between absorbing media.

In a previous work [10], we have followed a steady canonical quantization program similar to that of Ref. [11], generalizing it by considering a general and well-defined open quantum system. In this work, we will work with two simplified models analogous to that of HB, both of which assume that the dielectric atoms in the slabs are quantum Brownian particles, and that they are subjected to fluctuations (noise) and dissipation due to the coupling to an external thermal environment. We will use a general spectral density to specify the bath to which the atoms are coupled. In this way, we are generalizing the constant dissipation model, as it was done in Ref. [10]. Indeed, after integration of the environmental degrees of freedom, it will be possible to obtain the dissipation and noise kernels that modify the unitary equation of motion of the dielectric atoms.

The difference between both models lies in their couplings to the field. On the one hand, the first model—which we will call the bilinear coupling model—consists in a direct coupling between the field and the atom's polarization degree of freedom at each point of space. On the other hand, the current-type coupling model consists in a coupling between the field time derivative and the atom's polarization degree of freedom. The former model is more suitable for the development of the calculations, while the latter is more realistic in the sense that is closer to the real coupling between the electromagnetic (EM) field and the matter. However, both models are of interest and can be studied together in a compact way to obtain general conclusions about the nonequilibrium thermodynamics and transient-time evolution of quantum fields in the framework of open quantum systems.

With this aim, we use the Schwinger-Keldysh [or closed-time-path (CTP), i.e., in-in] formalism to provide the theoretical framework, which is based on the original papers by Schwinger [12] and Keldysh [13] and is particularly useful for nonequilibrium quantum field theory (see also Refs. [14–16]). According to the CTP formalism, the expectation value of an operator and its correlation functions can be derived from an in-in generating functional in a path integral representation [17], in a similar way as it happens in the well-known in-out formalism [18] but by doubling the fields and connecting them by a CTP boundary condition, which ensures that the functional derivatives of the generating functional give expectation values of the field operator. In this scheme, the open quantum systems framework is totally integrated through the concept of the influence action [19], resulting from a partial trace over the environment degrees of freedom, which gives the effective dynamics for the system through a coarse graining of the environments. Influence actions have been calculated in different contexts in the literature; for example, specific models assume that during cosmological inflation the UV (or sub-Hubble) modes of a field, once integrated out, decohere the IR (or super-Hubble) modes because the former modes are observationally inaccessible. In these models, the CTP formalism applied to cosmological perturbations aims to describing the transition between the quantum nature of the initial density inhomogeneities as a consequence of inflation and the classical stochastic behavior [20].

This paper is organized as follows. In the next section we introduce the bilinear model. In Sec. III, we fully develop the CTP formalism for the open quantum system to obtain the generating functional for the field and the influence actions that result after each functional integration, identifying the dissipation and noise kernels in each influence action. In Sec. IV, we extend—via a few modifications—the calculation of the generating functional done in Sec. III to the current-type model, calculating the new dissipation and noise kernels. In Sec. V, we derive a closed form for the expectation values of the energy-momentum tensor components in terms of the Hadamard propagators for each coupling model. Then, in Sec. VI we study different scenarios of interest where our general results give different transient-time behaviors and different conclusions about the steady situations in each coupling case. Finally, Sec. VII summarizes our findings. The Appendix contains some details of intermediate calculations.

II. BILINEAR COUPLING

In order to include the effects of dissipation and noise (fluctuations) in the calculation of the energy density of the electromagnetic field in interactions with real media, we will develop a full CTP approach to the problem.

Therefore, we will consider a composite system consisting of two parts: the field—which we will consider to be a real massless scalar field—and the real media, which in turn are modeled by continuous sets of quantum Brownian particles localized in certain regions of space. With this, we represent the polarization density degrees of freedom.

These degrees of freedom are basically harmonic oscillators coupled to the field at each point and that can be associated with the material's atoms. The composite system (field and material atoms) is also coupled to an external bath of harmonic oscillators through the interaction between the atoms in the material and the thermal environment.

Then, the total action for the whole system is given by

$$S[\phi, r, q_n] = S_0[\phi] + S_0[r] + \sum_n S_0[q_n] + S_{\rm int}[\phi, r] + \sum_n S_{\rm int}[r, q_n], \tag{1}$$

where each term is given by

$$S_0[\phi] = \int d\mathbf{x} \int_{t_0}^{t_f} d\tau \frac{1}{2} \partial_\mu \phi \partial^\mu \phi, \tag{2}$$

$$S_0[r] = \int d\mathbf{x} \int_{t_0}^{t_f} d\tau 4\pi \eta_{\mathbf{x}} g(\mathbf{x}) \frac{m_{\mathbf{x}}}{2} (\dot{r}_{\mathbf{x}}^2(\tau) - \omega_{\mathbf{x}}^2 r_{\mathbf{x}}^2(\tau)), \tag{3}$$

$$S_0[q_n] = \int d\mathbf{x} \int_{t_0}^{t_f} d\tau 4\pi \eta_{\mathbf{x}} g(\mathbf{x}) \frac{m_{n,\mathbf{x}}}{2} \times (\dot{q}_{n,\mathbf{x}}^2(\tau) - \omega_{n,\mathbf{x}}^2 q_{n,\mathbf{x}}^2(\tau)), \tag{4}$$

$$S_{\rm int}[\phi, r] = \int d\mathbf{x} \int_{t_0}^{t_f} d\tau 4\pi \eta_{\mathbf{x}} g(\mathbf{x}) \lambda_{0,\mathbf{x}} \phi(\mathbf{x}, \tau) r_{\mathbf{x}}(\tau), \tag{5}$$

$$S_{\rm int}[r, q_n] = \int d\mathbf{x} \int_{t_0}^{t_f} d\tau 4\pi \eta_{\mathbf{x}} g(\mathbf{x}) \frac{\lambda_{n,\mathbf{x}}}{\sqrt{2 m_{n,\mathbf{x}} \omega_{n,\mathbf{x}}}} r_{\mathbf{x}}(\tau) q_{n,\mathbf{x}}(\tau), \tag{6}$$

where the subindex $\mathbf{x}$ denotes the fact that the oscillators (and its properties) at each point of the space are independent of one another. In other words, we have to regard the set of oscillators r associated to the polarization density that form the material as a continuum of independent quantum degrees of freedom (with density $\eta_{\mathbf{x}}$), where each polarization degree of freedom has its own material properties (mass $m_{\mathbf{x}}$, frequency $\omega_{\mathbf{x}}$, and coupling $\lambda_{0,\mathbf{x}}$), where $\mathbf{x}$ is only a label when appearing as a subindex (we are assuming that the material can be inhomogeneous). Analogously, we consider the respective properties of each thermal bath interacting with the polarization degrees of freedom represented by the sets of oscillators $\{q_{n,\mathbf{x}}\}$ at each spatial point.

On the other hand, the matter distribution $g(\mathbf{x})$ defines the regions of material and is $g = 1$ for these regions and $g = 0$ outside them.

It is also worth noting that the scalar field seems to be one of the electromagnetic field components interacting with matter. In this first model we consider, for simplicity, a bilinear coupling between the field and the polarization degree of freedom.

Finally, we will assume that the total system is initially uncorrelated, and thus the initial density matrix is written as a direct product of each part, which we also suppose to be initially in thermal equilibrium at proper characteristic temperatures ($\beta_\phi, \beta_{r_{\mathbf{x}}}, \beta_{B,\mathbf{x}}$; the material can also be thermally inhomogeneous),

$$\hat{\rho}(t_0) = \hat{\rho}_\phi(t_0) \otimes \hat{\rho}_{r_{\mathbf{x}}}(t_0) \otimes \hat{\rho}_{\{q_{n,\mathbf{x}}\}}(t_0). \tag{7}$$

III. GENERATING AND INFLUENCE FUNCTIONALS

Our goal in this section is to compute the expectation value of the field quantum correlation function. We will employ the in-in formalism by means of a CTP to write the field's generating functional, after integrating out the environment by generalizing the procedure known from, for example, Refs. [17,21],

$$Z[J, J'] = \int d\phi_{\rm f} \int d\phi_0 d\phi_0' \int_{\phi(\mathbf{x},t_0)=\phi_0(\mathbf{x})}^{\phi(\mathbf{x},t_{\rm f})=\phi_{\rm f}(\mathbf{x})} \mathcal{D}\phi \int_{\phi'(\mathbf{x},t_0)=\phi_0'(\mathbf{x})}^{\phi'(\mathbf{x},t_{\rm f})=\phi_{\rm f}(\mathbf{x})} \mathcal{D}\phi' \rho_\phi(\phi_0, \phi_0', t_0) e^{i(S_0[\phi]-S_0[\phi'])} \mathcal{F}[\phi, \phi'] \times e^{i \int d\mathbf{x} \int_{t_0}^{t_f} d\tau (J(\mathbf{x},\tau)\phi(\mathbf{x},\tau) - J'(\mathbf{x},\tau)\phi'(\mathbf{x},\tau))}, \tag{8}$$

where the field's functional $\mathcal{F}$ is known as the influence functional [19], which is related to the field's influence action $S_{\rm IF}[\phi, \phi']$ generated by the material degrees of freedom (atoms plus baths).

Since the material is modeled as a continuum of spatially independent oscillators, each of which interacts with its own bath, the influence functional clearly factorizes in the spatial label, which gives

$$\mathcal{F}[\phi, \phi'] = e^{iS_{\rm IF}[\phi,\phi']} = \prod_{\mathbf{x}} \int dr_{{\rm f},\mathbf{x}} \int dr_{0,\mathbf{x}} dr_{0,\mathbf{x}}' \int_{r_{\mathbf{x}}(t_0)=r_{0,\mathbf{x}}}^{r_{\mathbf{x}}(t_{\rm f})=r_{{\rm f},\mathbf{x}}} \mathcal{D}r_{\mathbf{x}} \int_{r_{\mathbf{x}}'(t_0)=r_{0,\mathbf{x}}'}^{r_{\mathbf{x}}'(t_{\rm f})=r_{{\rm f},\mathbf{x}}} \mathcal{D}r_{\mathbf{x}}' \rho_{r_{\mathbf{x}}}(r_{0,\mathbf{x}}, r_{0,\mathbf{x}}', t_0) \times e^{i(S_0[r_{\mathbf{x}}]-S_0[r_{\mathbf{x}}'])} e^{i4\pi\eta_{\mathbf{x}} g(\mathbf{x}) S_{\rm QBM}[r_{\mathbf{x}}, r_{\mathbf{x}}']} e^{i(S_{\rm int}[\phi, r_{\mathbf{x}}] - S_{\rm int}[\phi', r_{\mathbf{x}}'])}, \tag{9}$$

where $S_{\mathrm{QBM}}[r_{\mathbf{x}}, r'_{\mathbf{x}}] = \int_{t_0}^{t_f} d\tau \int_{t_0}^{t_f} d\tau' \Delta r_{\mathbf{x}}(\tau)(-2D_{\mathrm{QBM},\mathbf{x}} \times (\tau - \tau')\Sigma r_{\mathbf{x}}(\tau') + \frac{i}{2} N_{\mathrm{QBM},\mathbf{x}}(\tau - \tau')\Delta r_{\mathbf{x}}(\tau'))$ is the well-known influence action for the quantum Brownian motion (QBM) theory [22,23], which represents the influence of a bath at $\mathbf{x}$ (given by the set $\{q_{n,\mathbf{x}}\}$) over the polarization degrees of freedom $r_{\mathbf{x}}$ at the same spatial point. It is worth noting that in this expression the scalar fields ϕ and ϕ' appear as additional external sources just as J and J' do for the field. It is worth noting that we have set $\Delta r_{\mathbf{x}} = r'_{\mathbf{x}} - r_{\mathbf{x}}$ and $\Sigma r_{\mathbf{x}} = (r_{\mathbf{x}} + r'_{\mathbf{x}})/2$, and that the QBM's influence action is clearly the analogous result of the CTP expression for the influence functional of the field of Eq. (9), where the trace has been taken over the bath's degree of freedom $\{q_{n,\mathbf{x}}\}$ and they are considered to be in a thermal state.

The kernels $N_{\mathrm{QBM},\mathbf{x}}$ and $D_{\mathrm{QBM},\mathbf{x}}$ in S_{QBM} are nothing more than the QBM noise and dissipation kernels, respectively [17,23]. It is clear that the expression of the influence action is quite general and applies to all types of baths (characterized by the spectral density being sub-Ohmic, Ohmic, or supra-Ohmic [7,23]) characterized by a particular temperature. In the same way, it turns out that the noise kernel $N_{\mathrm{QBM},\mathbf{x}}$ corresponds to the sum of the Hadamard propagators for the bath oscillators at the point $\mathbf{x}$, while the dissipation kernel $D_{\mathrm{QBM},\mathbf{x}}$ corresponds to the sum of the retarded propagators at the same point, which clearly shows a causal behavior $[D_{\mathrm{QBM},\mathbf{x}}(\tau,\tau') \propto \Theta(\tau - \tau')]$.

A. The field's influence functional

At this point, we have to compute the influence functional for the field $\mathcal{F}$ of Eq. (9). For this purpose, we have to evaluate each factor in the product. The result of this type of CTP integral can be found in Ref. [21]. We present a generalization and consider the degrees of freedom of the polarization density, which is straightforward. [The polarization or bath degrees of freedom must contain a dimensional normalization factor $\frac{1}{4\pi\eta_{\mathbf{x}}}$ (see Ref. [10]) and we also have to take into account that the matter distribution satisfies $g^2(\mathbf{x}) = g(\mathbf{x})$.] Therefore, we obtain

$$\mathcal{F}[\phi,\phi'] = \prod_{\mathbf{x}} \langle e^{-i4\pi\eta_{\mathbf{x}} g(\mathbf{x})\lambda_{0,\mathbf{x}} \int_{t_0}^{t_f} d\tau \Delta\phi(\mathbf{x},\tau)\mathcal{R}_{0,\mathbf{x}}(\tau)} \rangle_{r_{0,\mathbf{x}},p_{0,\mathbf{x}}} e^{-2\pi\eta_{\mathbf{x}} g(\mathbf{x}) \int_{t_0}^{t_f} d\tau \int_{t_0}^{t_f} d\tau' \Delta\phi(\mathbf{x},\tau)\mathcal{N}_{B,\mathbf{x}}(\tau,\tau')\Delta\phi(\mathbf{x},\tau')} \times e^{-i4\pi\eta_{\mathbf{x}} g(\mathbf{x}) \int_{t_0}^{t_f} d\tau \int_{t_0}^{t_f} d\tau' \Delta\phi(\mathbf{x},\tau) 2\mathcal{D}_{\mathbf{x}}(\tau,\tau')\Sigma\phi(\mathbf{x},\tau')}, \tag{10}$$

where $\Delta\phi = \phi' - \phi$, $\Sigma\phi = (\phi + \phi')/2$, and $\mathcal{D}_{\mathbf{x}}(\tau,\tau') \equiv \mathcal{D}_{\mathbf{x}}(\tau - \tau') = \frac{\lambda_{0,\mathbf{x}}^2}{2} G_{\mathrm{Ret},\mathbf{x}}(\tau - \tau')$ is the dissipation kernel over the field, where $G_{\mathrm{Ret},\mathbf{x}}$ is the retarded Green function and $\mathcal{R}_{0,\mathbf{x}}$ is the solution with initial conditions $\{r_{0,\mathbf{x}}, p_{0,\mathbf{x}}\}$ associated with the semiclassical equation of motion, which results from the homogeneous equation

$$\left.\frac{\delta S_{\mathrm{CTP}}[r_{\mathbf{x}}, r'_{\mathbf{x}}]}{\delta r_{\mathbf{x}}}\right|_{r_{\mathbf{x}}=r'_{\mathbf{x}}} = \left.\frac{\delta S_{\mathrm{CTP}}[\Delta r_{\mathbf{x}}, \Sigma r_{\mathbf{x}}]}{\delta \Delta r_{\mathbf{x}}}\right|_{\Delta r_{\mathbf{x}}=0} = 0,$$
$$\ddot{r}_{\mathbf{x}} + \omega_{\mathbf{x}}^2 r_{\mathbf{x}} - \frac{2}{m_{\mathbf{x}}} \int_{t_0}^{t} d\tau D_{\mathrm{QBM},\mathbf{x}}(t - \tau) r_{\mathbf{x}}(\tau) = 0, \tag{11}$$

where $S_{\mathrm{CTP}}[r_{\mathbf{x}}, r'_{\mathbf{x}}] = S_0[r_{\mathbf{x}}] - S_0[r'_{\mathbf{x}}] + S_{\mathrm{QBM}}[r_{\mathbf{x}}, r'_{\mathbf{x}}]$, and

$$\mathcal{R}_{0,\mathbf{x}}(\tau) = r_{0,\mathbf{x}} \dot{G}_{\mathrm{Ret},\mathbf{x}}(\tau - t_0) + \frac{p_{0,\mathbf{x}}}{m_{\mathbf{x}}} G_{\mathrm{Ret},\mathbf{x}}(\tau - t_0). \tag{12}$$

On the other hand, the kernel $\mathcal{N}_{B,\mathbf{x}}$ is the part of the noise kernel associated to the baths that acts on the field [there is another part associated to the first factor on the right-hand side of Eq. (10)],

$$\mathcal{N}_{B,\mathbf{x}}(\tau,\tau') = \lambda_{0,\mathbf{x}}^2 \int_{t_0}^{t_f} ds \int_{t_0}^{t_f} ds' G_{\mathrm{Ret},\mathbf{x}}(\tau - s) \times N_{\mathrm{QBM},\mathbf{x}}(s - s') G_{\mathrm{Ret},\mathbf{x}}(\tau' - s'). \tag{13}$$

Finally, the first factor on the right-hand side of Eq. (10) is given by (see Ref. [21])

$$\langle e^{-i4\pi\eta_{\mathbf{x}} g(\mathbf{x})\lambda_{0,\mathbf{x}} \int_{t_0}^{t_f} d\tau \Delta\phi(\mathbf{x},\tau)\mathcal{R}_{0,\mathbf{x}}(\tau)} \rangle_{r_{0,\mathbf{x}},p_{0,\mathbf{x}}} = \int dr_{0,\mathbf{x}} \int dp_{0,\mathbf{x}} e^{-i4\pi\eta_{\mathbf{x}} g(\mathbf{x})\lambda_{0,\mathbf{x}} \int_{t_0}^{t_f} d\tau \Delta\phi(\mathbf{x},\tau)\mathcal{R}_{0,\mathbf{x}}(\tau)} \times W_{r_{\mathbf{x}}}(r_{0,\mathbf{x}}, p_{0,\mathbf{x}}, t_0), \tag{14}$$

where $W_{r_{\mathbf{x}}}(r_{0,\mathbf{x}}, p_{0,\mathbf{x}}, t_0)$ is the Wigner functional associated to the density matrix of the polarization degrees of freedom $\hat{\rho}_{r_{\mathbf{x}}}(t_0)$. This functional can be written by generalizing the expression found in Ref. [21],

$$W_{r_{\mathbf{x}}}(r_{0,\mathbf{x}}, p_{0,\mathbf{x}}, t_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\Gamma e^{i4\pi\eta_{\mathbf{x}} g(\mathbf{x}) p_{0,\mathbf{x}}\Gamma} \rho_{r_{\mathbf{x}}}\left(r_{0,\mathbf{x}} - \frac{\Gamma}{2}, r_{0,\mathbf{x}} + \frac{\Gamma}{2}, t_0\right). \tag{15}$$

Considering thermal initial states for each part of the total composite system, we take the density matrices for the polarization degrees of freedom to be Gaussian functions. Therefore, Eq. (14) is also Gaussian since the Wigner function is Gaussian in $r_{0,\mathbf{x}}$ and $p_{0,\mathbf{x}}$. This way, by considering Eq. (12), we can easily calculate the first factor on the right-hand side of Eq. (10) as

$$\langle e^{-i4\pi\eta_{\mathbf{x}}g(\mathbf{x})\lambda_{0,\mathbf{x}}\int_{t_0}^{t_f}d\tau\Delta\phi(\mathbf{x},\tau)\mathcal{R}_{0,\mathbf{x}}(\tau)}\rangle_{r_{0,\mathbf{x}},p_{0,\mathbf{x}}}=\frac{1}{4\pi\eta_{\mathbf{x}}g(\mathbf{x})2\sinh(\frac{\beta_{r_{\mathbf{x}}}\omega_{\mathbf{x}}}{2})}\times e^{-2\pi\eta_{\mathbf{x}}g(\mathbf{x})\int_{t_0}^{t_f}d\tau\int_{t_0}^{t_f}d\tau'\Delta\phi(\mathbf{x},\tau)\mathcal{N}_{r,\mathbf{x}}(\tau,\tau')\Delta\phi(\mathbf{x},\tau')},\tag{16}$$

with

$$\mathcal{N}_{r,\mathbf{x}}(\tau,\tau')=\frac{\lambda_{0,\mathbf{x}}^2}{2m_{\mathbf{x}}\omega_{\mathbf{x}}}\coth\left(\frac{\beta_{r_{\mathbf{x}}}\omega_{\mathbf{x}}}{2}\right)[\dot{G}_{\mathrm{Ret},\mathbf{x}}(\tau-t_0)\dot{G}_{\mathrm{Ret},\mathbf{x}}(\tau'-t_0)+\omega_{\mathbf{x}}^2G_{\mathrm{Ret},\mathbf{x}}(\tau-t_0)G_{\mathrm{Ret},\mathbf{x}}(\tau'-t_0)],\tag{17}$$

which is the other part of the noise kernel that acts on the field. This is associated to the influence generated by the polarization degrees of freedom (it carries a global thermal factor containing the temperature of the polarization degrees of freedom $\beta_{r_{\mathbf{x}}}$).

Hence, after the normalization procedure of $Z[J,J']$, Eq. (10) finally reads

$$\mathcal{F}[\phi,\phi']=e^{iS_{\mathrm{IF}}[\phi,\phi']},\tag{18}$$

with

$$\begin{aligned}S_{\mathrm{IF}}[\phi,\phi']&=\int d\mathbf{x}\int_{t_0}^{t_f}d\tau\int_{t_0}^{t_f}d\tau'4\pi\eta_{\mathbf{x}}g(\mathbf{x})\Delta\phi(\mathbf{x},\tau)\left[-2\mathcal{D}_{\mathbf{x}}(\tau-\tau')\Sigma\phi(\mathbf{x},\tau')+\frac{i}{2}\mathcal{N}_{\mathbf{x}}(\tau,\tau')\Delta\phi(\mathbf{x},\tau')\right]\\&=\int d^4x\int d^4x'\Delta\phi(x)\left[-2\mathcal{D}(x,x')\Sigma\phi(x')+\frac{i}{2}\mathcal{N}(x,x')\Delta\phi(x')\right],\end{aligned}\tag{19}$$

where in the last line $\mathcal{D}(x,x')\equiv4\pi\eta_{\mathbf{x}}g(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\mathcal{D}_{\mathbf{x}}(\tau-\tau')$ and $\mathcal{N}(x,x')\equiv4\pi\eta_{\mathbf{x}}g(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\mathcal{N}_{\mathbf{x}}(\tau,\tau')$ [with $\mathcal{N}_{\mathbf{x}}(\tau,\tau')=\mathcal{N}_{r,\mathbf{x}}(\tau,\tau')+\mathcal{N}_{B,\mathbf{x}}(\tau,\tau')$] for the dissipation and noise kernels, respectively. The four-dimensional translational symmetry is broken by the spatial coordinates, because the $n+1$ field is interacting with $0+1$ fields, i.e., the polarization degrees of freedom. This means that the temporal and spatial coordinates are not on equal footing.

As expected for linear couplings, the influence action for the field has the same form as S_{QBM} obtained after the bath's integration but for a field in four dimensions over all space. (This is not only true for bilinear couplings between the coordinates; it is also true for bilinear couplings between a coordinate and a momenta, but logically the kernels change, as we will see in next sections).

B. CTP generating functional

We have achieved an exact result for the influence functional and, consequently, for the influence action. Thus, going back to Eq. (8) we can note that these CTP integrals are of the form of Eq. (10), where the degree of freedom is replaced by a scalar field. Generalizing the result found in Ref. [21] for fields, we get

$$Z[J,J']=\langle e^{-i\int d^4xJ_\Delta(x)\Phi_0(x)}\rangle_{\phi_0(\mathbf{x}),\Pi_0(\mathbf{x})}e^{-\frac{1}{2}\int d^4x\int d^4x'\int d^4y'\int d^4yJ_\Delta(x)\mathcal{G}_{\mathrm{Ret}}(x,x')\mathcal{N}(x',y')\mathcal{G}_{\mathrm{Ret}}(y,y')J_\Delta(y)}\times e^{-i\int d^4x\int d^4yJ_\Delta(x)\mathcal{G}_{\mathrm{Ret}}(x,y)J_\Sigma(y)},\tag{20}$$

where $\phi_0(\mathbf{x})=\phi(\mathbf{x},t_0)$ and $\Pi_0(\mathbf{x})=\dot{\phi}(\mathbf{x},t_0)$ are the initial conditions for the field, while $J_\Delta=J'-J$ and $J_\Sigma=(J+J')/2$.

Analogously to the integration performed in the last section, $\mathcal{G}_{\mathrm{Ret}}$ is a retarded Green function, this time associated to the field's semiclassical equation that results from the homogeneous equation of motion for the CTP effective action for the field: $S_{\mathrm{CTP}}[\phi,\phi']=S_0[\phi]-S_0[\phi']+S_{\mathrm{IF}}[\phi,\phi']$,

$$\begin{aligned}\left.\frac{\delta S_{\mathrm{CTP}}[\phi,\phi']}{\delta\phi}\right|_{\phi=\phi'}&=\left.\frac{\delta S_{\mathrm{CTP}}[\Delta\phi,\Sigma\phi]}{\delta\Delta\phi}\right|_{\Delta\phi=0}=0,\\\partial_\mu\partial^\mu\phi-2\int d^4x'\mathcal{D}(x,x')\phi(x')&=0.\end{aligned}\tag{21}$$

In the same way, $\Phi_0(x)$ is the solution of the last equation that satisfies the initial condition $\{\phi_0(\mathbf{x}),\Pi_0(\mathbf{x})\}$, i.e.,

$$\Phi_0(x)=\int d\mathbf{x}'\dot{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{x},\mathbf{x}',t-t_0)\phi_0(\mathbf{x}')+\int d\mathbf{x}'\mathcal{G}_{\mathrm{Ret}}(\mathbf{x},\mathbf{x}',t-t_0)\Pi_0(\mathbf{x}').\tag{22}$$

To calculate the first factor involving the average over the initial conditions, we use

$$\langle e^{-i\int d^4x J_\Delta(x)\Phi_0(x)}\rangle_{\phi_0(\mathbf{x}),\Pi_0(\mathbf{x})} = \int \mathcal{D}\phi_0(\mathbf{x}')\int \mathcal{D}\Pi_0(\mathbf{x}')W_\phi[\phi_0(\mathbf{x}'),\Pi_0(\mathbf{x}'),t_0] \times e^{-i\int d\mathbf{x}'\int d^4x J_\Delta(x)[\dot{\mathcal{G}}_{\rm Ret}(\mathbf{x},\mathbf{x}',\tau-t_0)\phi_0(\mathbf{x}')+\mathcal{G}_{\rm Ret}(\mathbf{x},\mathbf{x}',\tau-t_0)\Pi_0(\mathbf{x}')]}, \tag{23}$$

where $W_\phi[\phi_0(\mathbf{x}'),\Pi_0(\mathbf{x}'),t_0]$ plays the same role as the Wigner function in Ref. [24].

1. Initial-state contribution of the field

Once we have calculated the Wigner functional for the field in a thermal state [Eq. (A11)], we can go back to Eq. (23) to finally calculate the first factor on the right-hand side of Eq. (20).

For an arbitrary value for the field's temperature, the factor—which in principle is a functional integral over the field $\phi_0(\mathbf{x})$ and its associated momentum $\Pi_0(\mathbf{x})$—splits in each functional integration because the exponent is also separated in each of the variables; therefore,

$$\langle e^{-i\int d^4x J_\Delta(x)\Phi_0(x)}\rangle_{\phi_0(\mathbf{x}),\Pi_0(\mathbf{x})} = \int \mathcal{D}\phi_0(\mathbf{x}) e^{-\frac{\beta_\phi}{2}\int d\mathbf{x}\int d\mathbf{x}'\Delta_{\beta_\phi}(\mathbf{x}-\mathbf{x}')\nabla\phi_0(\mathbf{x})\cdot\nabla\phi_0(\mathbf{x}')} e^{\beta_\phi\int d\mathbf{x}\mathcal{J}_\phi(\mathbf{x})\phi_0(\mathbf{x})} \times \int \mathcal{D}\Pi_0(\mathbf{x}) e^{-\frac{\beta_\phi}{2}\int d\mathbf{x}\int d\mathbf{x}'\Delta_{\beta_\phi}(\mathbf{x}-\mathbf{x}')\Pi_0(\mathbf{x})\Pi_0(\mathbf{x}')} e^{\beta_\phi\int d\mathbf{x}\mathcal{J}_\Pi(\mathbf{x})\Pi_0(\mathbf{x})}, \tag{24}$$

where

$$\mathcal{J}_\phi(\mathbf{x}) \equiv -\frac{i}{\beta_\phi}\int d^4x' J_\Delta(x')\dot{\mathcal{G}}_{\rm Ret}(\mathbf{x}',\mathbf{x},t'-t_0), \tag{25}$$

$$\mathcal{J}_\Pi(\mathbf{x}) \equiv -\frac{i}{\beta_\phi}\int d^4x' J_\Delta(x')\mathcal{G}_{\rm Ret}(\mathbf{x}',\mathbf{x},t'-t_0). \tag{26}$$

Both functional integrals will define the contribution of the first factor to the generating functional of Eq. (A11). In fact, it will define the contribution of the initial state of the field to the dynamical evolution, relaxation, and steady situation of the system.

2. High-temperature limit

First of all, to continue the calculation we can explore the high-temperature limit for the field, which seems to be the easier case in which to solve the functional integrals in Eq. (24). The high-temperature approximation is given by $\frac{\beta_\phi|\mathbf{p}|}{2} \ll 1$ on the thermal weight in momentum space [Eq. (A10)]. Then, $\tanh(\frac{\beta_\phi|\mathbf{p}|}{2}) \approx \frac{\beta_\phi|\mathbf{p}|}{2}$, and the thermal weight in momentum space is approximately 1. Thus, in coordinate space

$$\Delta_{\beta_\phi}(\mathbf{x}'-\mathbf{x}'') \approx \int \frac{d\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x}'')} \equiv \delta(\mathbf{x}'-\mathbf{x}''). \tag{27}$$

In this approximation, Eq. (24) simplifies because one integral in the exponents is straightforwardly evaluated. In this limit, both functional integrals are easily calculated, and in fact the integration over the momentum $\Pi_0(\mathbf{x})$ is simply a Gaussian,

$$\int \mathcal{D}\Pi_0(\mathbf{x}) e^{-\frac{\beta_\phi}{2}\int d\mathbf{x}\Pi_0(\mathbf{x})\Pi_0(\mathbf{x})} e^{\beta_\phi\int d\mathbf{x}\mathcal{J}_\Pi(\mathbf{x})\Pi_0(\mathbf{x})} = e^{-\frac{1}{2\beta_\phi}\int d^4y\int d^4y' J_\Delta(y)[\int d\mathbf{x}\mathcal{G}_{\rm Ret}(\mathbf{y},\mathbf{x},\tau-t_0)\mathcal{G}_{\rm Ret}(\mathbf{y}',\mathbf{x},\tau'-t_0)]J_\Delta(y')}, \tag{28}$$

where in this notation $y=(\tau,\mathbf{y})$, $y'=(\tau',\mathbf{y}')$ and we are discarding any normalization constant that will eventually go away in the normalization of the generating functional.

At this point, it is interesting that the high-temperature approximation seems to suggest that the result does not depend on the number of spatial dimensions. This can be noted by the fact that in this limit the thermal weight turns out to be the Dirac delta function in all the coordinates in question—independently of the spatial dimensionality—and thus all the possible differences due to the different functional forms of the thermal weight on a given number of dimensions seem to disappear. However, the dimensionality again appears to create differences when the functional integral over $\phi_0(\mathbf{x})$ has to be solved. That integral is a Gaussian functional integral as well. Then, we can proceed by integrating by parts the exponent involving gradients and discard terms involving the vanishing asymptotic decay of the field at infinity [$\phi_0(x_i=\pm\infty)\to 0$]. Therefore, the functional integral over $\phi_0(\mathbf{x})$ in the high-temperature limit is a simple Gaussian functional integral,

$$\int \mathcal{D}\phi_0(\mathbf{x}) e^{-\frac{\beta_\phi}{2}\int d\mathbf{x}\nabla\phi_0(\mathbf{x})\cdot\nabla\phi_0(\mathbf{x})} e^{\beta_\phi\int d\mathbf{x}\mathcal{J}_\phi(\mathbf{x})\phi_0(\mathbf{x})} \propto e^{\frac{1}{2}\int d\mathbf{x}\int d\mathbf{x}'\beta_\phi^2\mathcal{J}_\phi(\mathbf{x}')K(\mathbf{x},\mathbf{x}')\mathcal{J}_\phi(\mathbf{x}')}$$
$$= e^{-\frac{1}{2}\int d^4y\int d^4y' J_\Delta(y)[\int d\mathbf{x}\int d\mathbf{x}'\dot{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{y},\mathbf{x},\tau-t_0)K(\mathbf{x}-\mathbf{x}')\dot{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{y}',\mathbf{x}',\tau'-t_0)]J_\Delta(y')}, \quad (29)$$

where $K(\mathbf{x},\mathbf{x}') = (-\beta_\phi\nabla^2)^{-1}$ is the inverse of the Laplace operator, i.e., the Green function defined by

$$-\beta_\phi\nabla^2 K(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}'). \quad (30)$$

It is clear that the kernel has to depend on $\mathbf{x}-\mathbf{x}'$.

Since the equation is analogous to the one for the Green function of a point charge in free space (although the thermal factor appears as a constant permittivity), we can solve the equation by taking the Fourier transform,

$$K(\mathbf{x}-\mathbf{x}') = \int \frac{d\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\bar{K}(\mathbf{p}), \quad (31)$$

where

$$\bar{K}(\mathbf{p}) = \frac{1}{\beta_\phi|\mathbf{p}|^2}. \quad (32)$$

It is worth noting that the kernel $K(\mathbf{x}-\mathbf{x}')$ strongly depends on the dimensionality of the problem, so the number of dimensions in the problem could modify the final results.

Finally, the first factor in the generating functional in the high-temperature limit is

$$\langle e^{-i\int d^4x J_\Delta(x)\Phi_0(x)}\rangle_{\phi_0(\mathbf{x}),\Pi_0(\mathbf{x})} = e^{-\frac{1}{2}\int d^4y\int d^4y' J_\Delta(y)[\mathcal{A}(y,y')+\mathcal{B}(y,y')]J_\Delta(y')}, \quad (33)$$

where the kernels are

$$\mathcal{A}(y,y') \equiv \frac{1}{\beta_\phi}\int d\mathbf{x}\mathcal{G}_{\mathrm{Ret}}(\mathbf{y},\mathbf{x},\tau-t_0)\mathcal{G}_{\mathrm{Ret}}(\mathbf{y}',\mathbf{x},\tau'-t_0), \quad (34)$$

$$\mathcal{B}(y,y') \equiv \int d\mathbf{x}\int d\mathbf{x}'\dot{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{y},\mathbf{x},\tau-t_0)K(\mathbf{x}-\mathbf{x}') \times \dot{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{y}',\mathbf{x}',\tau'-t_0). \quad (35)$$

The result is symmetric, i.e., $\mathcal{A}(y,y') = \mathcal{A}(y',y)$ and $\mathcal{B}(y,y') = \mathcal{B}(y',y)$, and we can clearly note that both kernels depend linearly on the field's initial temperature, as we expected from the high-temperature approximation. The presence of the kernels $\mathcal{A}(y,y')$ and $\mathcal{B}(y,y')$ is one of the main results of this article. We will remark on their role in the Casimir energy density and their contribution to the energy in the long-time regime.

All in all, we now can finally write the normalized generating functional for the field in the high-temperature limit by inserting Eq. (33) into Eq. (20),

$$Z[J,J'] = e^{-\frac{1}{2}\int d^4y\int d^4y' J_\Delta(y)[\mathcal{A}(y,y')+\mathcal{B}(y,y')+\int d^4x\int d^4x'\mathcal{G}_{\mathrm{Ret}}(y,x)\mathcal{N}(x,x')\mathcal{G}_{\mathrm{Ret}}(y',x')]J_\Delta(y')} \times e^{-i\int d^4y\int d^4y' J_\Delta(y)\mathcal{G}_{\mathrm{Ret}}(y,y')J_\Sigma(y')}, \quad (36)$$

where it is worth noting that the first factor on the right-hand side is accompanied by two J_Δ's, whereas the second factor is accompanied by one J_Δ and one J_Σ. This difference means that the first and third exponents will contribute to the energy while the second one will not.

Finally, we have calculated the field generating functional in a fully dynamical scenario in the high-temperature limit for the field. This was done by keeping the polarization degrees of freedom volume elements and its baths, which retain their own properties and temperatures. However, the model contains a bilinear interaction between the matter and the field. In the next section, we will see how to straightforwardly obtain the generating functional for the case of a more realistic model, i.e., a current-type interaction.

IV. CURRENT-TYPE COUPLING

At this point, we have calculated the generating functional for a massless scalar field interacting with matter modeled as Brownian particles. It is clear that in the calculation done in the previous sections, the field and the polarization degrees of freedom are coupled linearly, i.e., the coupling is directly to the quantum degrees of freedom. Therefore, that model is not a scalar version of one of the electromagnetic field components interacting with matter, since the interaction is not a current-type interaction. Therefore, in this section we will show how to extend the calculation to the case of a current-type interaction between the matter and the field, which is closer to a realistic electromagnetic model.

We have to start by replacing the interaction action $S_{\rm int}[\phi, r]$ between the field and the matter in Eq. (5) by a current-type interaction term,

$$\tilde{S}_{\rm int}[\phi, r] = \int d\mathbf{x} \int_{t_0}^{t_{\rm f}} d\tau 4\pi\eta_{\mathbf{x}} g(\mathbf{x}) \lambda_{0,\mathbf{x}} \dot{\phi}(\mathbf{x}, \tau) r_{\mathbf{x}}(\tau) \equiv S_{\rm int}[\dot{\phi}, r], \tag{37}$$

where $\lambda_{0,\mathbf{x}}$ effectively plays the role of the electric charge in the electromagnetic model. It is also worth noting that we write the time derivative as acting on the polarization degree of freedom instead of on the field. Both choices lead to the same equations of motion for the composite system so they are physically equivalent. In fact, all the calculations of the last section (devoted to calculating the field influence action) are formally the same, and we can obtain them in principle by simply replacing ϕ by $\dot{\phi}$. Therefore, the influence action on the field in this case reads

$$\tilde{S}_{\rm IF}[\phi, \phi'] \equiv S_{\rm IF}[\dot{\phi}, \dot{\phi}'] = \int d\mathbf{x} \int_{t_0}^{t_{\rm f}} d\tau \int_{t_0}^{t_{\rm f}} d\tau' 4\pi\eta_{\mathbf{x}} g(\mathbf{x}) \Delta\dot{\phi}(\mathbf{x}, \tau) \left[-2\mathcal{D}_{\mathbf{x}}(\tau - \tau') \Sigma\dot{\phi}(\mathbf{x}, \tau') + \frac{i}{2} \mathcal{N}_{\mathbf{x}}(\tau, \tau') \Delta\dot{\phi}(\mathbf{x}, \tau') \right]$$
$$= \int d^4x \int d^4x' \Delta\dot{\phi}(x) \left[-2\mathcal{D}(x, x') \Sigma\dot{\phi}(x') + \frac{i}{2} \mathcal{N}(x, x') \Delta\dot{\phi}(x') \right]. \tag{38}$$

Now, to continue the calculation as in the last section, and to identify the noise and dissipation kernels of the present model, we integrate by parts in both time variables to obtain an influence action that depends on the sum and difference of the fields instead of their time derivatives. Therefore, as in Ref. [25], we obtain

$$\tilde{S}_{\rm IF}[\phi, \phi'] = \int d\mathbf{x} \int_{t_0}^{t_{\rm f}} d\tau \int_{t_0}^{t_{\rm f}} d\tau' 4\pi\eta_{\mathbf{x}} g(\mathbf{x}) \Delta\phi(\mathbf{x}, \tau) \left[-2\partial^2_{\tau\tau'} \mathcal{D}_{\mathbf{x}}(\tau - \tau') \Sigma\phi(\mathbf{x}, \tau') + \frac{i}{2} \partial^2_{\tau\tau'} \mathcal{N}_{\mathbf{x}}(\tau, \tau') \Delta\phi(\mathbf{x}, \tau') \right]. \tag{39}$$

Since the dissipation kernel $\mathcal{D}$ involves the product of two distributions [because $\mathcal{D}(\tau - \tau')$ contains $\Theta(\tau - \tau')$ times an accompanying function of the time difference] the kernel is not well defined [25]. By differentiating the kernel twice—first with respect to τ' and second with respect to τ—gives us

$$\partial^2_{\tau\tau'} \mathcal{D}_{\mathbf{x}}(\tau - \tau') = -\delta(\tau - \tau') \dot{\mathcal{D}}_{\mathbf{x}}(\tau - \tau') - \ddot{\mathcal{D}}_{\mathbf{x}}(\tau - \tau'), \tag{40}$$

where dots over the kernels represent time derivatives involving differentiation over the accompanying function of the time difference, which avoids the differentiation of the Heaviside function contained in the kernel. Without confusion, it is remarkable that in the first term the Dirac delta function comes from the differentiation of the Heaviside function, but the actual notation means that the Heaviside function contained in $\dot{\mathcal{D}}_{\mathbf{x}}$ is superfluous and meaningless. On the other hand, we shall also note that we have exploited the fact that the dissipation kernel $\mathcal{D}$ depends on the time difference $\tau - \tau'$, which gives $\partial_{\tau'} \mathcal{D} = -\partial_{\tau} \mathcal{D} = -\dot{\mathcal{D}}$. However, this is unnecessary for the kernel $\mathcal{N}_{\mathbf{x}}$.

By inserting Eq. (40) into the influence action Eq. (39), and considering that from its definition $\dot{\mathcal{D}}_{\mathbf{x}}(0^+) = \lambda^2_{0,\mathbf{x}}/2$ for the first term of Eq. (40), we clearly obtain

$$\tilde{S}_{\rm IF}[\phi, \phi'] = \int d\mathbf{x} \int_{t_0}^{t_{\rm f}} d\tau 4\pi\eta_{\mathbf{x}} \lambda^2_{0,\mathbf{x}} g(\mathbf{x}) \Delta\phi(\mathbf{x}, \tau) \Sigma\phi(\mathbf{x}, \tau)$$
$$+ \int d\mathbf{x} \int_{t_0}^{t_{\rm f}} d\tau \int_{t_0}^{t_{\rm f}} d\tau' 4\pi\eta_{\mathbf{x}} g(\mathbf{x}) \Delta\phi(\mathbf{x}, \tau) \left[2\ddot{\mathcal{D}}_{\mathbf{x}}(\tau - \tau') \Sigma\phi(\mathbf{x}, \tau') + \frac{i}{2} \partial^2_{\tau\tau'} \mathcal{N}_{\mathbf{x}}(\tau, \tau') \Delta\phi(\mathbf{x}, \tau') \right], \tag{41}$$

where the first term is a finite renormalization position-dependent mass term for the scalar field which will be meaningless in the determination of the Green function, as will see in the following sections. These renormalization mass terms also appear in the QBM theory, but in general they are divergent due to the fact that the bath is a set of infinite harmonic oscillators, each one contributing to the mass renormalization. In our case, the field is coupled at each spacial point $\mathbf{x}$ to a unique harmonic oscillator represented by the polarization degree of freedom located at $\mathbf{x}$, so the renormalization term is only 1, and therefore finite.

It is worth noting that from the second term we have what we shall call the current-dissipation kernel and the current-noise kernel; these are the derivatives of the dissipation and noise kernels of the bilinear model,

respectively, i.e., the current-dissipation kernel is $-\ddot{\mathcal{D}}_{\mathbf{x}}$, while the current-noise kernel is $\partial^2_{\tau\tau'}\mathcal{N}_{\mathbf{x}}$. To avoid confusion, henceforth we shall use the prefix "current" for the kernels when referring to the current-type model, keeping the terms "dissipation kernel" and "noise kernel" for $\mathcal{D}_{\mathbf{x}}$ and $\mathcal{N}_{\mathbf{x}}$, respectively.

All in all, and having written the influence action of Eq. (41) formally as a renormalization mass term plus a nonlocal term [identical to Eq. (19) but with different kernels], we can continue with the procedure done for the bilinear coupling in the last section.

Despite the renormalization mass term, the CTP functional integral over the field variables can be done as in the last section. Therefore, the generating functional is formally identical to Eq. (36). However, in the present case the current-noise and current-dissipation kernels are different, so the first one will define the contribution due to the matter fluctuations, while the second one will contribute to the definition of the retarded Green function by appearing in the field's semiclassical equation obtained from the CTP effective action for the current-type model. This equation can be derived similarly to Eq. (21),

$$\partial_\mu\partial^\mu\phi + 4\pi\eta_{\mathbf{x}}\lambda^2_{0,\mathbf{x}}g(\mathbf{x})\phi(\mathbf{x},t) + 8\pi\eta_{\mathbf{x}}g(\mathbf{x})\int_{t_0}^{t} d\tau\ddot{\mathcal{D}}_{\mathbf{x}}(t-\tau)\phi(\mathbf{x},\tau) = 0, \quad (42)$$

where the scalar field has a well-defined (positive) position-dependent mass $2\sqrt{\pi\eta_{\mathbf{x}}}|\lambda_{0,\mathbf{x}}|$ at every point $\mathbf{x}$ where there is material [so $g(\mathbf{x}) = 1$], while it is massless in the free regions. This last equation is in agreement with the one obtained from a canonical quantization scheme (see for example Ref. [10]) and it is in fact its generalization.

V. ENERGY-MOMENTUM TENSOR AND FIELD CORRELATION

At this point, we have obtained the field CTP generating functional for both coupling models after tracing out all the material degrees of freedom (polarization plus thermal baths). Now we are interested in evaluating the expectation value of the symmetric energy-momentum tensor operator $\langle\hat{T}_{\mu\nu}\rangle$, which gives the energy density and radiation pressure associated to the field. It is defined by [18,26]

$$\hat{T}_{\mu\nu}(x_1) \equiv -\eta_{\mu\nu}\frac{1}{2}\partial_\gamma\hat{\phi}(x_1)\partial^\gamma\hat{\phi}(x_1) + \partial_\mu\hat{\phi}(x_1)\partial_\nu\hat{\phi}(x_1), \quad (43)$$

where $\eta_{\mu\nu}$ is the Minkowski metric ($\eta_{00} = -\eta_{ii} = 1$ for the nonvanishing elements).

We can proceed with the point-splitting technique, employing the field correlation function as

$$\langle\hat{T}_{\mu\nu}(x_1)\rangle = \lim_{x_2\to x_1}\left(-\eta_{\mu\nu}\frac{1}{2}\partial_{\gamma_1}\partial^{\gamma_2} + \partial_{\mu_1}\partial_{\nu_2}\right)\langle\hat{\phi}(x_1)\hat{\phi}(x_2)\rangle, \quad (44)$$

where the notation implies $\partial_{\gamma_1}\partial^{\gamma_2} \equiv \partial_{t_1}\partial_{t_2} - \nabla_1\cdot\nabla_2$ and so on for $\partial_{\mu_1}\partial_{\nu_2}$.

Therefore, we need the field correlation function to know the expectation value of every energy-momentum tensor component. In fact, we need the correlation to be finite, so we have to insert a regularized expression for the correlation function. From the generating functional in Eq. (36), this is straightforward [17]. We will evaluate the field correlation at two different points x_1 and x_2 (where there is no specific relation between the points because they are in different branches of the CTP). Then, we have four alternatives depending on the relation between x_1 and x_2; however, in the coincidence limit this is not relevant,

$$\langle\hat{\phi}(x_1)\hat{\phi}(x_2)\rangle = \left.\frac{\delta^2 Z}{\delta J'(x_1)\delta J(x_2)}\right|_{J=J'=0}. \quad (45)$$

Because the generating functional has a simple form in Eq. (36) we can easily compute its functional derivatives by taking advantage of the symmetry kernel's properties,

$$\langle\hat{\phi}(x_1)\hat{\phi}(x_2)\rangle = \mathcal{A}(x_1,x_2) + \mathcal{B}(x_1,x_2) + \int d^4x\int d^4x'\mathcal{G}_{\mathrm{Ret}}(x_1,x)\times\mathcal{N}(x,x')\mathcal{G}_{\mathrm{Ret}}(x_2,x') + \frac{1}{2}\mathcal{G}_{\mathrm{Jordan}}(x_1,x_2), \quad (46)$$

where $\mathcal{G}_{\mathrm{Jordan}}(x_1,x_2) \equiv i(\mathcal{G}_{\mathrm{Ret}}(x_2,x_1) - \mathcal{G}_{\mathrm{Ret}}(x_1,x_2))$ is the Jordan propagator [17]. Then, the kernels are the ones in Eqs. (19), (34), and (35) for the case of the bilinear model, where the retarded Green function is defined from the semiclassical equation of motion in Eq. (21). On the other hand, to obtain the result for the current-type model we have to take into account that the retarded Green function is defined from the corresponding semiclassical equation of motion for the field in this model [given by Eq. (42)], but the formal expressions for the kernels $\mathcal{A}$ and $\mathcal{B}$ are unchanged. To finish, we have to replace the noise kernel $\mathcal{N}$ in Eq. (46) by the current-noise kernel $\partial^2_{\tau\tau'}\mathcal{N}$ associated to the field's influence action for this model [Eq. (41)]. All in all, a smart and compact notation can be achieved by including a parameter α encompassing both models; therefore, we can write the generalized noise kernel as $\partial^{2\alpha}_{\tau\tau'}\mathcal{N}$, with $\alpha = 0$, 1 for the bilinear and current-type models, respectively.

It is worth noting that the correlation function in Eq. (46) corresponds to the Wightman function for the field in this

open system. In fact, considering that $\mathcal{G}_{\rm Ret}$ is real, it is clear that the correlation is a complex quantity, and its imaginary part is given by $\mathcal{G}_{\rm Jordan}$, whereas the real part consists of the other three terms. If we want to match the Wightman propagator with the typical relations for propagators, the Hadamard propagator is given by

$$\mathcal{G}_{\rm H}(x_1,x_2)\equiv 2\left[\mathcal{A}(x_1,x_2)+\mathcal{B}(x_1,x_2)+\int d^4x\int d^4x'\mathcal{G}_{\rm Ret}(x_1,x)\partial^{2\alpha}_{\tau\tau'}[\mathcal{N}(x,x')]\mathcal{G}_{\rm Ret}(x_2,x')\right]. \tag{47}$$

On the other hand, we want to calculate energy-momentum tensor expectation values, so the result must be real. This is apparently not the case because the correlation function is complex and its imaginary part is given by $\mathcal{G}_{\rm Jordan}$. However, to compute the expectation values we will have to proceed in a symmetric way in both coordinates $x_{1,2}$ and then we will have to calculate the coincidence limit when $x_2\to x_1$. Due to the definition of the Jordan propagator, this operation (symmetric derivation plus the coincidence limit) makes the contribution vanish. Therefore, the expectation values effectively turn out to be real numbers, as is expected.

Finally, the expectation value of the energy-momentum tensor can be written as

$$\langle\hat{T}_{\mu\nu}(x_1)\rangle=\frac{1}{2}\lim_{x_2\to x_1}\left(-\eta_{\mu\nu}\frac{1}{2}\partial_{\gamma_1}\partial^{\gamma_2}+\partial_{\mu_1}\partial_{\nu_2}\right)\mathcal{G}_{\rm H}(x_1,x_2), \tag{48}$$

where the Hadamard propagator must be a well-defined (nondivergent) propagator.

It is important to note that the full nonequilibrium dynamics—both time evolution and thermodynamical nonequilibrium—is contained in this result.

Due to the structure of the noise kernel $\partial^{2\alpha}_{\tau\tau'}\mathcal{N}$ for each model, this term accounts for the influence generated by the material (polarization degrees of freedom plus thermal baths) from the initial time when the interaction with the field is turned on, which describes the relaxation process of the material forming the contours. Both parts—the polarization degrees of freedom and the thermal baths—can have different initial temperatures, implying thermal nonequilibrium. In fact, each volume element in the material can have its own properties.

On the other hand, there are two terms that are proportional to the field's initial temperature, namely, the kernels $\mathcal{A}$ and $\mathcal{B}$ (and their derivatives in the contribution of the expectation values). These terms account for the dynamical evolution and change of the field in the presence of the material contours when the interaction is turned on. Therefore, these terms must be entirely related to the modified normal modes that appear in a (steady situation) canonical quantization scheme as a vacuum contribution [10,11].

VI. NONEQUILIBRIUM BEHAVIOR FOR DIFFERENT CONFIGURATIONS OF THE COMPOSITE SYSTEM

A. 0 + 1 field

As a first example, we consider the case of a scalar field in $0+1$ dimensions, i.e., we take the field ϕ as a quantum harmonic oscillator degree of freedom of unit mass. Therefore, to adapt our results to this situation a few changes are needed. In this case the spatial notion is erased, and the volume element concept is meaningless, so the composite system is a harmonic oscillator ($0+1$ field) coupled to another one (polarization degree of freedom) which is also coupled to a set of harmonic oscillators (thermal bath).

The spatial label $\mathbf{x}$ will be unnecessary and the quantum degree of freedom will be characterized by a frequency Ω (which plays the role of the effect that the spatial derivative has on the field in $n+1$ dimensions, with $n>0$). The initial action for the field [Eq. (2)] must be replaced by the straightforward expression

$$S_0[\phi]=\int d\mathbf{x}\int_{t_0}^{t_{\rm f}}d\tau\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \longrightarrow\int_{t_0}^{t_{\rm f}}d\tau\frac{1}{2}\left[\left(\frac{d\phi}{d\tau}\right)^2-\Omega^2\phi(\tau)\right]. \tag{49}$$

Equations (3)–(6) can also be simply adapted by discarding the spatial integrals, the labels, the density η (together with the factor 4π), and the distribution function g in all the actions. All the integrations and traces can be performed in the same way without further modifications until the functional integration over the field. The influence action in Eq. (19) is still valid. In this way, the generating functional from Eq. (8) to Eq. (20) is formally the same. However, in this case the calculation of the first factor—which involves the initial state of the $0+1$ field $\phi(t)$—implies a Wigner function and not a functional, so the factor results from the same formal calculation done for the polarization degree of freedom r in Eq. (16), but the kernel obtained is clearly different. Therefore, we have for both models ($\alpha=0,\,1$)

$$Z[J,J'] = e^{-\frac{1}{2}\int_{t_0}^{t_f} d\tau \int_{t_0}^{t_f} ds J_\Delta(\tau)[\mathcal{A}(\tau,s)+\mathcal{B}(\tau,s)]J_\Delta(s)} e^{-\frac{1}{2}\int_{t_0}^{t_f} d\tau \int_{t_0}^{t_f} d\tau' \int_{t_0}^{t_f} ds' \int_{t_0}^{t_f} ds J_\Delta(\tau)\mathcal{G}^\Omega_{\rm Ret}(\tau,\tau')\partial^{2\alpha}_{\tau's'}[\mathcal{N}(\tau',s')]\mathcal{G}^\Omega_{\rm Ret}(s,s')J_\Delta(s)}$$
$$\times e^{-i\int_{t_0}^{t_f} d\tau \int_{t_0}^{t_f} ds J_\Delta(\tau)\mathcal{G}^\Omega_{\rm Ret}(\tau,s)J_\Sigma(s)}, \tag{50}$$

where the sum of the kernels $\mathcal{A}$ and $\mathcal{B}$ results from the ordinary integration over the initial values of the field $\phi_0 \equiv \phi(t_0)$ and $\Pi_0 \equiv \Pi(t_0)$, and it has the same form as Eq. (17) [indeed, the high-temperature limit of this expression has exactly the form as Eq. (17) if we discard its spatial features],

$$\mathcal{A}(\tau,s)+\mathcal{B}(\tau,s) \equiv \frac{1}{2\Omega}\coth\left(\frac{\beta_\phi\Omega}{2}\right)[\Omega^2\mathcal{G}^\Omega_{\rm Ret}(\tau-t_0)\mathcal{G}^\Omega_{\rm Ret}(s-t_0)+\dot{\mathcal{G}}^\Omega_{\rm Ret}(\tau-t_0)\dot{\mathcal{G}}^\Omega_{\rm Ret}(s-t_0)], \tag{51}$$

where $\mathcal{G}^\Omega_{\rm Ret}$ is the retarded Green function [which is a function of the time difference, i.e., $\mathcal{G}^\Omega_{\rm Ret}(t,s) \equiv \mathcal{G}^\Omega_{\rm Ret}(t-s)$, as we can infer from its equation of motion] associated to Eqs. (21) and (42) for each model, respectively, in the $0+1$ case. These can be combined to give

$$\frac{d^2\phi}{dt^2}+(\Omega^2+\alpha\lambda_0^2)\phi(t)-(-1)^\alpha 2\int_{t_0}^{t} d\tau\partial^{2\alpha}_{tt}[\mathcal{D}(t-\tau)]\phi(\tau)=0, \tag{52}$$

where in this case the (finite) mass term appears as a frequency renormalization term, and $\partial^{2\alpha}_{tt}[\mathcal{D}(t-\tau)]$ is the generalized dissipation kernel.

Therefore, the $0+1$ counterpart of the Hadamard propagator of Eq. (47) is

$$\mathcal{G}^\Omega_{\rm H}(t_1,t_2) \equiv \frac{1}{\Omega}\coth\left(\frac{\beta_\phi\Omega}{2}\right)[\Omega^2\mathcal{G}^\Omega_{\rm Ret}(t_1-t_0)\mathcal{G}^\Omega_{\rm Ret}(t_2-t_0)+\dot{\mathcal{G}}^\Omega_{\rm Ret}(t_1-t_0)\dot{\mathcal{G}}^\Omega_{\rm Ret}(t_2-t_0)]$$
$$+2\int_{t_0}^{t_f} d\tau\int_{t_0}^{t_f} d\tau'\mathcal{G}^\Omega_{\rm Ret}(t_1-\tau)\partial^{2\alpha}_{\tau\tau'}[\mathcal{N}(\tau,\tau')]\mathcal{G}^\Omega_{\rm Ret}(t_2-\tau'), \tag{53}$$

where the noise kernel consists of two contributions $\mathcal{N}(\tau,\tau') = \mathcal{N}_B(\tau,\tau')+\mathcal{N}_r(\tau,\tau')$, each of which is characterized by its own temperature $\beta_{r,B}$ given in Eqs. (13) and (17), respectively. In fact, we can relate the temperature value associated to the term to the contribution of that part of the total system, i.e., the terms carrying the field's temperature β_ϕ are associated to the proper (influenced) system contribution, while each part of the noise kernel $\mathcal{N}$ has one term associated to the polarization degree of freedom (denoted by containing the temperature β_r) and another term associated to the bath (denoted by containing the temperature β_B).

It is clear now that the energy-momentum tensor is simply the energy of the $0+1$ field, where the evolution of the expectation value can be easily written in terms of the Hadamard propagator [as in Eq. (48)],

$$\langle E(t_1)\rangle \equiv \frac{1}{2}\lim_{t_2\to t_1}\left(\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}+\Omega^2\right)\mathcal{G}^\Omega_{\rm H}(t_1,t_2). \tag{54}$$

Finally, we have written the mean value of the energy as a function of time from the initial conditions for the composite system. It is clear that the dynamics depends on the retarded Green functions $\mathcal{G}^\Omega_{\rm Ret}, G_{\rm Ret}$ [where $G_{\rm Ret}$ is contained in the field's noise kernels for each model through Eqs. (13) and (17)] from each part of the system and the QBM noise kernel $N_{\rm QBM}$ (which depends on the type of bath we are considering) after tracing out the degrees of freedom that influence its dynamics. Since we are interested in the field dynamics, the traces are performed by taking the field ϕ as the system and doing them in sequential steps, namely, the partial traces over each part of the complex environment formed by the polarization degree of freedom r and the bath $\{q_n\}$ [19].

Therefore, the transient-time behavior of the energy expectation value and its relaxation to a steady state will depend on the fluctuations of each part of the environment through the noise kernels, and how the system evolves to the steady situation depends on its own Green function $\mathcal{G}^\Omega_{\rm Ret}$, as is clear from Eq. (60).

Then, for the long-time limit ($t_0 \to -\infty$) we need to know the long-time behavior of each retarded Green function $\mathcal{G}^\Omega_{\rm Ret}$, $G_{\rm Ret}$. Thus, we must focus on the specific Green functions that we have in our system, which are determined by each equation of motion we have obtained at each stage of the tracing.

The retarded Green function for the polarization degree of freedom r is determined by the equation of motion of the polarization degree of freedom [Eq. (11)]. The associated equation for the Green function $G_{\rm Ret}$ can be solved by Laplace transforming the equation subject to the initial

conditions $G_{\rm Ret}(0)=0$, $\dot{G}_{\rm Ret}(0)=1$ (see Ref. [7]). It is straightforward to prove that, for every type of bath, the Laplace transform of the retarded Green function is given by

$$\tilde{G}_{\rm Ret}(z)=\frac{1}{(z^2+\omega^2-2\tilde{D}_{\rm QBM}(z))}, \tag{55}$$

where $\tilde{D}_{\rm QBM}$ is the Laplace transform of the QBM's dissipation kernel contained in $S_{\rm QBM}$ [22,23].

The analyticity properties of the Laplace transform $\tilde{G}_{\rm Ret}$ and the location of its poles define the time evolution and the asymptotic behavior of the Green function $G_{\rm Ret}$. In this way, causality implies—by Cauchy's theorem—that the poles of $\tilde{G}_{\rm Ret}$ should be located in the left half of the complex z plane, i.e., the poles' real parts must be negative or zero. Assuming that $\omega\neq 0$ and that the modeled bath includes a cutoff function in frequencies (see Ref. [7]), the discussion given in Ref. [10] implies that all the poles are simple and have negative real parts. Using Mellin's formula and the residue theorem to retransform to the time-dependent function [27], we easily obtain that the Green function formally reads

$$G_{\rm Ret}(t)=\Theta(t)\sum_j Res[\tilde{G}_{\rm Ret}(z),z_j]e^{z_j t}. \tag{56}$$

Since $\mathrm{Re}[z_j]<0$, it is clear that in the long-time limit, when $t_0\to-\infty$, we have $G_{\rm Ret}(t-t_0)\to 0$, and similarly for its time derivatives.

Indeed, this asymptotic behavior defines the long-time contribution of the polarization degree of freedom to the field's energy density at the steady situation. Since the Green function goes to zero, we have also that the part of the field's noise kernels directly associated to the polarization degree of freedom $\mathcal{N}_r$ goes to zero. This means that the polarization degree of freedom does not contribute through its thermal state to the energy at the steady situation in either of the two coupling models. Although the dependence on the temperature β_r is erased in the long-time regime [due to the asymptotic decay of the retarded Green function $G_{\rm Ret}(t-t_0)$], this function also appears in the bath's contribution $\mathcal{N}_B$. This term is characterized, of course, by the bath's temperature β_B.

All in all, in the long-time limit ($t_0\to-\infty$) we have that the (generalized) noise kernel contribution (polarization degree of freedom plus bath) in Eq. (60) is

$$\begin{aligned}&\int_{t_0}^{t_{\rm f}}d\tau\int_{t_0}^{t_{\rm f}}d\tau'\mathcal{G}^{\Omega}_{\rm Ret}(t_1-\tau)\partial^{2\alpha}_{\tau\tau'}[\mathcal{N}(\tau,\tau')]\mathcal{G}^{\Omega}_{\rm Ret}(t_2-\tau')\\&\longrightarrow\int_{-\infty}^{t_{\rm f}}d\tau\int_{-\infty}^{t_{\rm f}}d\tau'\mathcal{G}^{\Omega}_{\rm Ret}(t_1-\tau)\\&\quad\times\partial^{2\alpha}_{\tau\tau'}[\mathcal{N}_B(\tau,\tau')]\mathcal{G}^{\Omega}_{\rm Ret}(t_2-\tau'),\end{aligned} \tag{57}$$

where the QBM noise kernel does not depend on t_0, and thus the bath contribution does not vanish in the steady situation.

Finally, we have to analyze the behavior of the contribution associated to the proper field system. Thus we have to study the retarded Green function $\mathcal{G}^{\Omega}_{\rm Ret}$. We then proceed as in the case of the polarization degree of freedom to study $G_{\rm Ret}$ by considering the same initial conditions $[\mathcal{G}^{\Omega}_{\rm Ret}(0)=0,\ \dot{\mathcal{G}}^{\Omega}_{\rm Ret}(0)=-1]$. From the equation of motion for the Green function $\mathcal{G}^{\Omega}_{\rm Ret}$ associated to the field in both models [Eq. (52)], we can easily obtain an expression analogous to that in the first case for the Laplace transform,

$$\tilde{\mathcal{G}}^{\Omega}_{\rm Ret}(z)=\frac{-1}{(z^2+\Omega^2-\lambda_0^2(-z^2)^{\alpha}\tilde{G}_{\rm Ret}(z))}, \tag{58}$$

where it is worth noting that this compact expression is due to the fact that the renormalization (mass) frequency term cancels out with a term coming from the derivative of the dissipation kernel $\mathcal{D}$ at the initial time.

Analyticity properties of this Laplace transform define the asymptotic behavior of the proper contribution of the field. For λ_0, Ω, $\omega\neq 0$ and an Ohmic bath, it is easy to show that the Laplace transform for both models has four simple poles with negative real parts, verifying the causality requirement. We assume that the general case gives the same features and that the poles are simple and have negative real parts. From this, in the time domain it follows that

$$\mathcal{G}^{\Omega}_{\rm Ret}(t)=\Theta(t)\sum_l Res[\tilde{\mathcal{G}}^{\Omega}_{\rm Ret}(z),z_l]e^{z_l t}. \tag{59}$$

Therefore, since $\mathrm{Re}[z_l]<0$ we clearly have in the long-time limit ($t_0\to-\infty$) that $\mathcal{G}^{\Omega}_{\rm Ret}(t-t_0)\to 0$, and similarly for its time derivatives.

The long-time limit of the Hadamard propagator $\mathcal{G}^{\Omega}_{\rm H}$ is given only by the bath's long-time contribution,

$$\begin{aligned}\mathcal{G}^{\Omega}_{\rm H}(t_1,t_2)\to 2\int_{-\infty}^{t_{\rm f}}d\tau\int_{-\infty}^{t_{\rm f}}d\tau'\mathcal{G}^{\Omega}_{\rm Ret}(t_1-\tau)\\\times\mathcal{N}_B(\tau,\tau')\mathcal{G}^{\Omega}_{\rm Ret}(t_2-\tau'),\end{aligned} \tag{60}$$

corresponding to the steady situation with the bath's fluctuation at temperature β_B, as the fluctuation-dissipation theorem asserts.

Finally, to summarize this section, for a $0+1$ field in both types of coupling models, the energy density at the steady situation only has contributions from the bath, while the polarization degree of freedom and the proper field contributions go to zero through the time evolution.

Now, let us see how these calculations apply for the case of a field in $n+1$ dimensions with a homogeneous material over all space.

B. Field in infinite material

Let us now consider a scalar field in $n+1$ dimensions (with $n \neq 0$) with no boundaries, i.e., this is the case of a homogeneous material that appears over all space at the initial time t_0. In this situation, $g(\mathbf{x}) \equiv 1$ for every $\mathbf{x}$ and we have to eliminate the spatial label due to the homogeneity of the problem. Then, Eqs. (21) and (42) can be written together through their generalized form as

$$\partial_\mu \partial^\mu \phi + 4\pi\eta\lambda_0^2 \alpha \phi - (-1)^\alpha 8\pi\eta \times \int_{t_0}^{t} d\tau \partial_{tt}^{2\alpha}[\mathcal{D}(t-\tau)]\phi(\mathbf{x},\tau) = 0, \tag{61}$$

which is basically a wave-type equation for the field in a dissipative medium.

Therefore, the associated equation for the retarded Green function $\mathcal{G}_{\mathrm{Ret}}$ is straightforward and it is subjected to the typical wave-equation initial conditions,

$$\mathcal{G}_{\mathrm{Ret}}(\mathbf{x},\mathbf{x}',0) = 0, \qquad \dot{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{x},\mathbf{x}',0) = -\delta(\mathbf{x}-\mathbf{x}'). \tag{62}$$

Due to the translational symmetry of the problem, $\mathcal{G}_{\mathrm{Ret}}(\mathbf{x},\mathbf{x}',t) = \mathcal{G}_{\mathrm{Ret}}(\mathbf{x}-\mathbf{x}',t)$, the Fourier transform satisfies

$$\partial_{tt}^2 \bar{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{k},t) + (k^2 + 4\pi\eta\lambda_0^2\alpha)\bar{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{k},t) - (-1)^\alpha 8\pi\eta \times \int_0^t d\tau \partial_{tt}^{2\alpha}[\mathcal{D}(t-\tau)]\bar{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{k},\tau) = 0, \tag{63}$$

where, as in the last section, $\mathcal{D}(t-\tau) = \lambda_0^2 G_{\mathrm{Ret}}(t-\tau)$, $k \equiv |\mathbf{k}|$, and the initial conditions are

$$\bar{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{k},0) = 0, \qquad \dot{\bar{\mathcal{G}}}_{\mathrm{Ret}}(\mathbf{k},0) = -1. \tag{64}$$

Equations (63) and (64) are equivalent to the field equation and initial conditions for the retarded Green function for the $0+1$ field, i.e., each field mode behaves as a $0+1$ field of natural frequency k and the dynamics are equivalent. Then, the Fourier transform of the retarded Green function is closely related to the retarded Green function in the last section; in fact, we have

$$\bar{\mathcal{G}}_{\mathrm{Ret}}(\mathbf{k},t) \equiv \bar{\mathcal{G}}^k_{\mathrm{Ret}}(t), \tag{65}$$

where $\mathcal{G}^k_{\mathrm{Ret}}$ is the retarded function of a $0+1$ field of frequency k.

We can write

$$\mathcal{G}_{\mathrm{Ret}}(\mathbf{x}-\mathbf{x}',t) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}^k_{\mathrm{Ret}}(t). \tag{66}$$

In order to study the behavior of the contributions to the expectation value of the energy-momentum tensor $\langle \hat{T}_{\mu\nu} \rangle$ in Eq. (48), let us first consider the contributions of the polarization degrees of freedom and the thermal baths at each point $\mathbf{x}$ in the last term of Eq. (47). As we are considering a homogeneous material where all the polarization degrees of freedom have the same temperature β_r, then all the baths at each point have the same temperature β_B (note that this does not imply thermal equilibrium because each part of the material can have a different temperature, i.e., we can still have the situation in which $\beta_r \neq \beta_B$). In the present case $\mathcal{N}(x,x') = 4\pi\eta\delta(\mathbf{x}-\mathbf{x}')\mathcal{N}(\tau,\tau')$.

If we use the Fourier representation of $\mathcal{G}_{\mathrm{Ret}}$ to write the last term of Eq. (47), it is straightforward that

$$\int d^4x \int d^4x' \mathcal{G}_{\mathrm{Ret}}(x_1,x)\partial_{\tau\tau'}^{2\alpha}[\mathcal{N}(x,x')]\mathcal{G}_{\mathrm{Ret}}(x_2,x') = 4\pi\eta \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)} \int_{t_0}^{t_f} d\tau \int_{t_0}^{t_f} d\tau' \bar{\mathcal{G}}^k_{\mathrm{Ret}}(t_1-\tau)\partial_{\tau\tau'}^{2\alpha}[\mathcal{N}(\tau,\tau')]\bar{\mathcal{G}}^k_{\mathrm{Ret}}(t_2-\tau'), \tag{67}$$

where it is remarkable that both integrals over τ and τ' and the integrand are exactly one half of the last term in Eq. (60) (the contribution of the polarization degree of freedom and the bath in the last section, i.e., for the $0+1$ field of frequency Ω). This is clear because, as we have inferred from the equation for the Fourier-transformed Green function, each field $\mathbf{k}$ mode is matched to a $0+1$ field of frequency $k = |\mathbf{k}|$.

Then, we have for each field mode the same time evolution as for a $0+1$ field of natural frequency k in any coupling model. Considering the analysis done in the last section regarding the Green function G_{Ret}, we can easily conclude that the long-time regime ($t_0 \to -\infty$) of this contribution is given by

$$\int d^4x \int d^4x' \mathcal{G}_{\mathrm{Ret}}(x_1,x)\partial_{\tau\tau'}^{2\alpha}[\mathcal{N}(x,x')]\mathcal{G}_{\mathrm{Ret}}(x_2,x') \longrightarrow 4\pi\eta \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)} \int_{-\infty}^{t_f} d\tau \int_{-\infty}^{t_f} d\tau' \bar{\mathcal{G}}^k_{\mathrm{Ret}}(t_1-\tau)\partial_{\tau\tau'}^{2\alpha}[\mathcal{N}_B(\tau,\tau')]\bar{\mathcal{G}}^k_{\mathrm{Ret}}(t_2-\tau'), \tag{68}$$

where, as in the last section, we have that the polarization degrees of freedom do not contribute to the steady situation of the $n+1$ field in a homogeneous material.

On the other hand, for the proper contribution of the field—contained in the kernels $\mathcal{A}$ and $\mathcal{B}$ of Eqs. (34) and (35)—we can again exploit the Fourier representation,

$$\mathcal{A}(x_1,x_2)+\mathcal{B}(x_1,x_2)=\int\frac{d\mathbf{k}}{(2\pi)^3}e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\left[\frac{1}{\beta_\phi}\bar{\mathcal{G}}^k_{\mathrm{Ret}}(t_1-t_0)\bar{\mathcal{G}}^k_{\mathrm{Ret}}(t_2-t_0)+\bar{K}(k)\dot{\bar{\mathcal{G}}}^k_{\mathrm{Ret}}(t_1-t_0)\dot{\bar{\mathcal{G}}}^k_{\mathrm{Ret}}(t_2-t_0)\right]. \tag{69}$$

Therefore, considering the analysis done in the last section regarding the retarded Green function $\mathcal{G}^{\Omega}_{\mathrm{Ret}}$, in the long-time limit ($t_0\to-\infty$) the Fourier transform of the retarded Green function vanishes, i.e., $\bar{\mathcal{G}}^k_{\mathrm{Ret}}(t-t_0)\to 0$; this also causes the proper contribution to vanish at the steady situation.

All in all, as in the $0+1$ field, the long-time regime is defined by the bath contribution to the Hadamard propagator, and it is expected to satisfy the fluctuation-dissipation theorem in the steady situation in either coupling model,

$$\mathcal{G}_{\mathrm{H}}(x_1,x_2)\to 8\pi\eta\int\frac{d\mathbf{k}}{(2\pi)^3}e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\int_{-\infty}^{t_f}d\tau\int_{-\infty}^{t_f}d\tau'\bar{\mathcal{G}}^k_{\mathrm{Ret}}(t_1-\tau)\partial^{2\alpha}_{\tau\tau'}[\mathcal{N}_B(\tau,\tau')]\bar{\mathcal{G}}^k_{\mathrm{Ret}}(t_2-\tau'). \tag{70}$$

Finally, the energy density at the steady situation will also depend only on the baths' fluctuations in the long-time regime for either of the coupling models. This conclusion is not necessarily true if the material is not homogeneous or if there are temperature gradients, whether between the polarization degrees of freedom or between the baths. In fact, in the next sections we will show that the conclusion could be different if, on the one hand, we consider nondissipative (constant-permittivity) media or, on the other hand, there are regions where the field fluctuates freely, i.e., regions where there is no material [$g(\mathbf{x})=0$] and the field is subjected to the presence of boundaries.

C. Constant dielectric permittivity limit

In previous sections we have analyzed two situations (a field in $0+1$ dimensions and a field in $n+1$ dimensions in the presence if an infinite material) where we have shown that—beyond the transient-time evolution of the system—the steady regime is described only by the fluctuations of the thermal baths which are in contact with the polarization degrees of freedom of the material, as is expected from a formalism based only on the fluctuation-dissipation theorem. This result can be seen from the final temperature dependence of the Hadamard propagator, which in the analyzed cases was β_B. On the other hand, we have shown that the kernels $\mathcal{A}$ and $\mathcal{B}$—associated to the proper contribution of the field—and the contribution from the polarization degrees of freedom vanish at the steady situation (due to the dissipative dynamics of the field at every point of the space and of the polarization degrees of freedom modeled as Brownian particles).

It is clear that these conclusions are due physically to the dissipative dynamics of the field in contact with reservoirs comprising the real material, which generates the damping and the absorption dominating the steady situation through its fluctuations.

We will now assume that the material is a nondissipative dielectric, i.e., a constant-permittivity material which presents no absorption and that is not dispersive because the permittivity function in the complex-frequency domain is real and it is not a smooth function over the imaginary frequency axis. It is worth noting that this verifies the Kramers-Kronig relations for the complex permittivity function in the frequency domain even though the function is real. In fact, the Kramers-Kronig relations are not satisfied by dispersive and real permittivity functions on the imaginary frequency axis. Therefore, our calculations must include this scenario as a limiting case.

As a first step, if we clearly turn off the dissipation provided by the baths in each point of the material, we have to set $D_{\mathrm{QBM}}\equiv 0$. From the fluctuation-dissipation theorem it is straightforward that $N_{\mathrm{QBM}}\equiv 0$. Therefore, this directly implies that the noise kernel also vanishes, i.e., $\mathcal{N}_{B,\mathbf{x}}\equiv 0$. In this way the bath contribution is erased from the result.

However, this is not enough because it leaves a material formed by harmonic oscillators without damping, i.e., which do not relax to a steady situation. This can be seen from the Laplace transform of the retarded Green function of the polarization degrees of freedom, which, through Eq. (55), turns out to be $\tilde{G}_{\mathrm{Ret},\mathbf{x}}(z)=1/(z^2+\omega^2_{\mathbf{x}})$, which presents purely imaginary poles at $z=\pm i\omega_{\mathbf{x}}$, so the retarded Green function in the time domain will be comprised sinusoidal functions. This means in principle that the contribution coming from the polarization degrees of freedom does not vanish, i.e., $\mathcal{N}_{r,\mathbf{x}}$ does not necessarily vanish.

Nevertheless, since the dissipation kernel is $\mathcal{D}_{\mathbf{x}}=\frac{\lambda^2_{0,\mathbf{x}}}{2}G_{\mathrm{Ret},\mathbf{x}}$, the generalized dissipation kernel $\partial^{2\alpha}_{tt}[\mathcal{D}_{\mathbf{x}}(t-\tau)]$ that acts over the field and forms the dielectric function through its Laplace transform will give a dispersive and real permittivity function for purely imaginary frequencies. Thus, it does not verify the Kramers-Kronig relations. Then, a vanishing bath dissipation is not enough to achieve the constant dielectric limit and, in fact, it is a nonphysical model.

To get a clue about how this limit can be taken, we can use the equation of motion for the field for arbitrary shapes

of material boundaries, which in both coupling models can be written as

$$\partial_\mu\partial^\mu\phi + 4\pi\eta_{\mathbf{x}}\lambda_{0,\mathbf{x}}^2 g(\mathbf{x})\alpha\phi - (-1)^\alpha 8\pi\eta_{\mathbf{x}} g(\mathbf{x}) \times \int_{t_0}^{t} d\tau\partial_{tt}^{2\alpha}[\mathcal{D}_{\mathbf{x}}(t-\tau)]\phi(\mathbf{x},\tau) = 0. \quad (71)$$

Going to the complex-frequency domain, we Laplace transform the associated equation for the Green function, imposing the same initial conditions as in the last section,

$$\nabla^2\tilde{\mathcal{G}}_{\mathrm{Ret}} - z^2\left[1 - (-1)^\alpha 4\pi\eta_{\mathbf{x}}\lambda_{0,\mathbf{x}}^2 g(\mathbf{x})\frac{z^{2(\alpha-1)}}{(z^2+\omega_{\mathbf{x}}^2)}\right]\tilde{\mathcal{G}}_{\mathrm{Ret}} = \delta(\mathbf{x}-\mathbf{x}'). \quad (72)$$

If we now consider the equation of motion of the retarded Green function corresponding to a field subjected to the same initial conditions and with boundaries of constant dielectric permittivity $\epsilon(\mathbf{x})$ from the very beginning, we will obtain

$$\nabla^2\tilde{\mathcal{G}}_{\mathrm{Ret}} - z^2\epsilon(\mathbf{x})\tilde{\mathcal{G}}_{\mathrm{Ret}} = \delta(\mathbf{x}-\mathbf{x}'), \quad (73)$$

which is analogous to what is found from a steady canonical quantization scheme of a field with constant-permittivity dielectric boundaries [28].

Comparing the equations, it is clear that in our case the permittivity function given by the expression in brackets should not depend on z, i.e., we have to replace it by a constant. So, we can try replacing it by its zeroth-order approximation. On the one hand, this is not possible in a simple way for the bilinear model ($\alpha = 0$) because it diverges for $z = 0$. On the other hand, the current-type model ($\alpha = 1$) gives a finite zeroth-order approximation, allowing us to find a feasible replacement,

$$\nabla^2\tilde{\mathcal{G}}_{\mathrm{Ret}} - z^2\left[1 + \frac{4\pi\eta_{\mathbf{x}}\lambda_{0,\mathbf{x}}^2}{\omega_{\mathbf{x}}^2} g(\mathbf{x})\right]\tilde{\mathcal{G}}_{\mathrm{Ret}} = \delta(\mathbf{x}-\mathbf{x}'), \quad (74)$$

where it is clear that the permittivity function is $\epsilon(\mathbf{x}) \equiv 1 + \frac{4\pi\eta_{\mathbf{x}}\lambda_{0,\mathbf{x}}^2}{\omega_{\mathbf{x}}^2} g(\mathbf{x})$, which correctly satisfies the Kramers-Kronig relations and is constant in time.

In fact, when this replacement is used from the very beginning, we remove all the dynamics of the polarization degrees of freedom and set them equal to the steady situation in the scenario including dissipation by the evaluation at $z = 0$. We clearly have that $G_{\mathrm{Ret}} \equiv 0$. All in all, it gives that the terms corresponding to the contribution of the material (polarization degrees of freedom and baths) vanishes since $\mathcal{N} \equiv 0$.

Therefore, the Hadamard propagator in Eq. (47) depends on the kernels $\mathcal{A}$ and $\mathcal{B}$, which in this case clearly do not vanish. In fact, as long as there is a Casimir force between constant dielectric boundaries due to the modification of the vacuum modes, these kernels should not vanish at the steady situation. This is in complete agreement with many results that have been found for nondissipative media boundaries obtained from a steady canonical quantization scheme (see, for example, Ref. [28]), where quantization is carried out by only considering a Hilbert space associated to the field and developing the Heisenberg canonical operator method in terms of creation and annihilation field mode operators. Thus, we can write the long-time limit ($t_0 \to -\infty$) as

$$\mathcal{G}_{\mathrm{H}}(x_1,x_2) \longrightarrow 2(\mathcal{A}(x_1,x_2) + \mathcal{B}(x_1,x_2)). \quad (75)$$

It is worth noting that—as it follows from steady canonical quantization schemes—the field's state must be taken as a thermal one, as this an additional requirement of consistency that results in the correct thermal global factors for the correlation and Green functions in the steady situation. However, our approach naturally gives the correct thermal dependence at least when an initial high-temperature state is considered for the field, which is in agreement with the high-temperature limit of the canonical quantization or in-out formalism schemes [10].

Finally, we have shown a first and simplest example where the kernels $\mathcal{A}$ and $\mathcal{B}$ do not vanish at the steady situation, and in fact they define the long-time regime in this case. However, this is not totally new because we clearly know that a Casimir force exists between constant dielectric boundaries due to the modified vacuum modes. Anyway, we have just proven that our approach correctly reproduce this situation as a limiting case. In the next section, we will study another situation where these terms do not vanish, but neither of them completely define the long-time regime.

D. Field and material boundaries

In this section we study a particular situation concerning the presence of boundaries. At this point, we have already seen that for the case of a $0+1$-dimension field of frequency Ω, the long-time regime is defined by the bath's contribution, while the polarization degree of freedom and the proper field contributions vanish at the steady situation. We have also seen that a $n+1$-dimension scalar field interacting with homogeneous material over all space can be reduced to an infinite set of $0+1$ fields with frequency k, representing the field modes that evolve in time due to the sudden appearance of the material. We have shown that in the long-time limit, as in the $0+1$ case, the only contribution to the energy-momentum tensor that survives is also the one associated to the baths. The polarization degrees of freedom and the field have vanishing contributions at the steady situation.

Although we were tempted to assume that the result of the last two sections is quite general and always valid, we

have presented a limiting case where the reverse is true and the removal of the dissipation causes the kernels $\mathcal{A}$ and $\mathcal{B}$ to be responsible for the Casimir force between nondissipative boundaries in the long-time regime.

As we pointed out before, this is not the only case where these kernels contribute to the steady situation. If the material is inhomogeneous or there exist regions without material (i.e., vacuum regions that define material boundaries) the same could be true. Therefore, this section gives a simple example of the presence of boundaries and the analysis of the steady situation.

1. The retarded Green function

Returning to the field equation for general boundaries given in Eq. (71) for both coupling models, we can again Laplace transform the associated equation for the retarded Green function with appropriate initial conditions to obtain

$$\nabla^2\tilde{\mathcal{G}}_{\mathrm{Ret}} - z^2[1-(-1)^\alpha 4\pi\eta_{\mathbf{x}}\lambda_{0,\mathbf{x}}^2 g(\mathbf{x}) z^{2(\alpha-1)}\tilde{G}_{\mathrm{Ret},\mathbf{x}}(z)]\tilde{\mathcal{G}}_{\mathrm{Ret}} = \delta(\mathbf{x}-\mathbf{x}'), \tag{76}$$

where $\tilde{\mathcal{G}}_{\mathrm{Ret}}$ is the inverse of the differential operator $\nabla^2 - z^2[1-(-1)^\alpha 4\pi\eta_{\mathbf{x}}\lambda_{0,\mathbf{x}}^2 g(\mathbf{x}) z^{2(\alpha-1)}\tilde{G}_{\mathrm{Ret},\mathbf{x}}(z)]$, i.e., it is the Green function associated to this operator.

We will consider a single homogeneous Dirac delta plate located at $x_\perp = 0$ ($x_\perp$, $\mathbf{x}_\parallel$ are the orthogonal and parallel coordinates, respectively, to the plate of a given space point $\mathbf{x}$), which is described by the material distribution $g(\mathbf{x}) \equiv \delta(x_\perp)$. Thus, Eq. (76) gives

$$\nabla^2\tilde{\mathcal{G}}_{\mathrm{Ret}} - z^2[1-(-1)^\alpha 4\pi\eta\lambda_0^2\delta(x_\perp) z^{2(\alpha-1)}\tilde{G}_{\mathrm{Ret}}(z)]\tilde{\mathcal{G}}_{\mathrm{Ret}} = \delta(\mathbf{x}-\mathbf{x}'). \tag{77}$$

It is clear that the last equation imposes translational invariance on the parallel coordinates $\mathbf{x}_\parallel$, so the Green function must depend on $\mathbf{x}_\parallel - \mathbf{x}'_\parallel$. Then,

$$\frac{\partial^2\tilde{\mathcal{G}}_{\mathrm{Ret}}}{\partial x_\perp^2} - (z^2+k_\parallel^2)\tilde{\mathcal{G}}_{\mathrm{Ret}} + (-1)^\alpha 4\pi\eta\lambda_0^2\delta(x_\perp) z^{2\alpha}\tilde{G}_{\mathrm{Ret}}(z)\tilde{\mathcal{G}}_{\mathrm{Ret}} = \delta(x_\perp - x'_\perp), \tag{78}$$

where $k_\parallel = |\mathbf{k}_\parallel|$ and $\tilde{\mathcal{G}}_{\mathrm{Ret}} \equiv \tilde{\mathcal{G}}_{\mathrm{Ret}}(x_\perp, x'_\perp, k_\parallel, z)$.

It is worth noting that the last equation turns out to be a Sturn-Liouville equation for the Green function, so it can be calculated by the technique described in Ref. [29], where it was constructed as

$$\tilde{\mathcal{G}}_{\mathrm{Ret}}(x_\perp, x'_\perp, k_\parallel, z) = \frac{\Phi^{(L)}(x_<)\Phi^{(R)}(x_>)}{W(x'_\perp)}, \tag{79}$$

where $x_<$ ($x_>$) is the smaller (bigger) of $x_\perp$ and $x'_\perp$, $W(x) = \Phi^{(L)}(x)\frac{d\Phi^{(R)}}{dx} - \frac{d\Phi^{(L)}}{dx}\Phi^{(R)}(x)$ is the Wronskian (which has to be a constant function) of the solutions $\{\Phi^{(L)}, \Phi^{(R)}\}$, which are two homogeneous solutions $\Phi^{(L,R)}$ that satisfy the associated homogeneous equation

$$\frac{\partial^2\Phi}{\partial x_\perp^2} - (z^2+k_\parallel^2)\Phi + (-1)^\alpha 4\pi\eta\lambda_0^2\delta(x_\perp) z^{2\alpha}\tilde{G}_{\mathrm{Ret}}(z)\Phi = 0, \tag{80}$$

and the boundary condition at one of the two range endpoints, i.e., Φ^L (Φ^R) satisfies the boundary condition at the left (right) endpoint of the interval. In our case, this boundary condition requires outgoing waves in the corresponding region including the respective endpoint.

The presence of a Dirac delta function in one of the terms of the equation means that we will obtain the solution in two regions with positive and negative coordinates $x_\perp$, respectively; on the other hand, it gives a boundary condition that gives a jolt to the derivative, which can be obtained from the equation itself by integrating over an interval containing the root of the delta function and then taking its length to zero around the root, clearly obtaining

$$\left.\frac{\partial\Phi}{\partial x_\perp}\right|_{x_\perp=0^+} - \left.\frac{\partial\Phi}{\partial x_\perp}\right|_{x_\perp=0^-} = (-1)^\alpha 4\pi\eta\lambda_0^2 z^{2\alpha}\tilde{G}_{\mathrm{Ret}}(z)\Phi(0), \tag{81}$$

which goes together with the continuity of the solution.

Therefore, in each region the solutions are plane waves; therefore, after imposing the boundary conditions, both solutions are

$$\Phi^{(L)}(x_\perp) = \begin{cases} te^{\sqrt{z^2+k_\parallel^2}x_\perp} & \text{for } x_\perp < 0, \\ e^{\sqrt{z^2+k_\parallel^2}x_\perp} + re^{-\sqrt{z^2+k_\parallel^2}x_\perp} & \text{for } 0 < x_\perp, \end{cases} \tag{82}$$

$$\Phi^{(R)}(x_\perp) = \begin{cases} e^{-\sqrt{z^2+k_\parallel^2}x_\perp} + re^{\sqrt{z^2+k_\parallel^2}x_\perp} & \text{for } x_\perp < 0, \\ te^{-\sqrt{z^2+k_\parallel^2}x_\perp} & \text{for } 0 < x_\perp, \end{cases} \tag{83}$$

where r and t are the reflection and transmission coefficients for one plate, respectively, and are given by

$$r = -(-1)^\alpha 2\pi\eta\lambda_0^2\frac{z^{2\alpha}}{\sqrt{z^2+k_\parallel^2}}\tilde{G}_{\mathrm{Ret}}(z)t,$$
$$t = \frac{1}{(1+(-1)^\alpha 2\pi\eta\lambda_0^2\frac{z^{2\alpha}}{\sqrt{z^2+k_\parallel^2}}\tilde{G}_{\mathrm{Ret}}(z))}, \tag{84}$$

where it is clear that $t = 1 + r$.

Then, the Laplace-Fourier transform of the retarded Green function for a field point $x_\perp < 0$ is

$$\tilde{\mathcal{G}}_{\mathrm{Ret}}(x_\perp, x'_\perp, k_\parallel, z) = -\frac{1}{2\sqrt{z^2+k_\parallel^2}} \begin{cases} e^{\sqrt{z^2+k_\parallel^2}x'_\perp}(e^{-\sqrt{z^2+k_\parallel^2}x_\perp} + re^{\sqrt{z^2+k_\parallel^2}x_\perp}) & \text{for } x'_\perp < x_\perp < 0, \\ (e^{-\sqrt{z^2+k_\parallel^2}x'_\perp} + re^{\sqrt{z^2+k_\parallel^2}x'_\perp})e^{\sqrt{z^2+k_\parallel^2}x_\perp} & \text{for } x_\perp < x'_\perp < 0, \\ te^{\sqrt{z^2+k_\parallel^2}(x_\perp - x'_\perp)} & \text{for } x_\perp < 0 < x'_\perp. \end{cases} \tag{85}$$

To simplify the calculations, we continue with the one-dimensional version of the problem, i.e., the case of a $1+1$ field where the only dimension of interest clearly is the one associated to the perpendicular coordinate $x_\perp$, which we now call x. Therefore, to obtain the results for this case we also have to discard everything related to the parallel dimensions. We can do this simply by setting $k_\parallel$ equal to 0 in all the results. This simplifies all the expressions, and the Laplace transform of the retarded Green function in Eq. (85) is

$$\tilde{\mathcal{G}}_{\mathrm{Ret}}(x, x', z) = -\frac{1}{2z} \begin{cases} e^{zx'}(e^{-zx} + re^{zx}) & \text{for } x' < x < 0, \\ (e^{-zx'} + re^{zx'})e^{zx} & \text{for } x < x' < 0, \\ te^{z(x-x')} & \text{for } x < 0 < x', \end{cases} \tag{86}$$

where the reflection and transmission coefficients are now given by

$$r = -(-1)^\alpha 2\pi\eta\lambda_0^2 z^{2\alpha-1}\tilde{G}_{\mathrm{Ret}}(z)t, \qquad t = \frac{1}{(1+(-1)^\alpha 2\pi\eta\lambda_0^2 z^{2\alpha-1}\tilde{G}_{\mathrm{Ret}}(z))}. \tag{87}$$

We can Laplace transform back using Mellin's formula and the residue theorem [27], assuming that the poles of the Laplace transform of the retarded Green function have nonpositive real parts. It is important to remark that (following the discussion in Ref. [10]) for both coupling models this can always be ensured by introducing an appropriate cutoff function in the Laplace transform of the dissipation kernel $\tilde{D}_{\mathrm{QBM}}$ (in fact, any spectral density that characterizes the environment has a physical cutoff function). Moreover, assuming that the dissipation (represented by D_{QBM}), the frequency ω, and coupling constant λ_0 are not zero, the only pole with a vanishing real part is the one at $z=0$, which appears (for each coupling case) in different terms of the Laplace transform of the retarded Green function, resulting in different behaviors of the retarded Green function. This can be seen by working out the last expression of the reflection coefficient r in each model. However, this pole does not change the conclusions of the present section, so we will continue the analysis without losing generality.

Therefore, the Green function can be formally written as

$$\mathcal{G}_{\mathrm{Ret}}(x, x', t) = -\frac{1}{2} \begin{cases} \Theta(x'-x+t) + \Theta(x+x'+t)\left[\alpha - 1 + \sum_{z_j} R_j e^{z_j(x+x'+t)}\right] & \text{for } x' < x < 0, \\ \Theta(x-x'+t) + \Theta(x+x'+t)\left[\alpha - 1 + \sum_{z_j} R_j e^{z_j(x+x'+t)}\right] & \text{for } x < x' < 0, \\ \Theta(x-x'+t)\left[\alpha + \sum_{z_j} R_j e^{z_j(x-x'+t)}\right] & \text{for } x < 0 < x', \end{cases} \tag{88}$$

where z_j are all the poles of r with negative real parts, i.e., the pole at $z=0$ is calculated explicitly in each model. For the others poles we have $R_j \equiv \mathrm{Res}[\frac{r}{z}, z_j] = \mathrm{Res}[\frac{t}{z}, z_j]$. Given that the retarded Green function must be real, its poles must come in pairs (i.e., if z_j is a pole then its conjugate z_j^* is a pole too) unless z_j is real.

This expression, however, can be worked out by rearranging the terms and combining their Heaviside functions to obtain a suitable closed form for the retarded Green function in each model for a field point $x < 0$,

$$\mathcal{G}_{\rm Ret}(x,x',t)=\mathcal{G}^0_{\rm Ret}(x,x',t)+\frac{(1-\alpha)}{2}\Theta(-x)\Theta(x+x'+t)\Theta(x-x'+t)$$
$$-\frac{\Theta(-x)}{2}\sum_{z_j}R_j e^{z_j(x+t)}[e^{z_j x'}\Theta(-x')\Theta(x+x'+t)+e^{-z_j x'}\Theta(x')\Theta(x-x'+t)], \quad (89)$$

where $\mathcal{G}^0_{\rm Ret}(x,x',t)\equiv-\frac{\Theta(-x)}{2}\Theta(x'-x+t)\Theta(x-x'+t)$ is the retarded Green function in free space for a field point $x<0$.

It is worth noting that the second term is an extra term only for the bilinear model due to the presence of the plate, but it is independent of the material properties. On the other hand, the third term is directly and entirely related to the presence of the plate and it contains all the information about the material contribution to the transient evolution (i.e., relaxation) and the new steady situation that the field will achieve. It is clear that it implicitly depends on the coupling model because the poles z_j depend on it.

From Eq. (89) it can be easily proven—by looking carefully at the products of distributions—that the time derivative of the retarded Green function has a simple form given by

$$\dot{\mathcal{G}}_{\rm Ret}(x,x',t)=\dot{\mathcal{G}}^0_{\rm Ret}(x,x',t)-\frac{\Theta(-x)}{2}\sum_{z_j}z_j R_j e^{z_j(x+t)}[e^{z_j x'}\Theta(-x')\Theta(x+x'+t)+e^{-z_j x'}\Theta(x')\Theta(x-x'+t)], \quad (90)$$

where in this expression the only difference between the coupling models relies on the poles z_j.

2. The long-time regime

Using the retarded Green function for the present problem, we can proceed to study some dynamical aspects and features of the steady situation.

As we just found in previous sections, our interest is the Hadamard propagator given in Eq. (47), which we can use to calculate the expectation value of the energy-momentum components through Eq. (48). As is shown by Eq. (47), the Hadamard propagator has several contributions that can be divided into two parts: one comes from the field generated by all the components of the material (polarization degrees of freedom and baths), which is represented by the noise kernel $\mathcal{N}$, and the other comes from the field generated by the vacuum fluctuations subjected to the actual boundary conditions, which is represented by the kernels $\mathcal{A}$ and $\mathcal{B}$ and will imply a modification of the field modes through a transient evolution from the initial free field to the new steady field.

Let us study firstly the material contribution. As we have proven in Sec. VI B, when the material is modeled as Brownian particles interacting with the field using both coupling models, the material contribution at the steady situation only consists of contributions coming from the baths, while the particles merely act as a bridge connecting the field with the baths and no contribution in the long-time regime due to their dissipative Brownian dynamics. This was basically contained in the fact that in the long-time limit ($t_0\to-\infty$) we clearly have that $\mathcal{N}\to\mathcal{N}_B$. In the present case, although there are regions without material, the result is still valid. It is clear that in this case the Green function is given by Eq. (89) but the formal expression is the same. In fact, it is also worth noting that the material distribution g will define the range of integration, which has no contribution from the points outside the material.

On the other hand, we have the contribution to the field generated by the vacuum fluctuations represented by $\mathcal{A}$ and $\mathcal{B}$. We are tempted to assume that, as in Sec. VI B, these contributions vanish at the steady situation, giving only a transient behavior. However (as we just pointed out at the end of that section) this could not be true when there are vacuum regions where the field fluctuates freely.

Therefore, considering the kernel $\mathcal{A}$ in Eq. (34) and the expression for the retarded Green function given in Eq. (89), it is clear that the product $\mathcal{G}_{\rm Ret}(x_1,x,t_1-t_0)\,\mathcal{G}_{\rm Ret}(x_2,x,t_2-t_0)$ will have at most nine terms (depending on which coupling model we are considering) due to all of the possible combinations of the three separated terms, which then have to be integrated over x. Using the symmetry of the kernel, the number of integrals to calculate is six at most. The complication in the full calculation is due to the fact that each integral involves products of distributions that contain the integration variable and both field points (x_1,t_1) and (x_2,t_2); results will depend on multiple relations between the coordinates of the field points.

On the other hand, the full calculation of kernel $\mathcal{B}$ is as complicated as that for the previous kernel. In the one-dimensional case, it is easy to calculate the kernel K in Eq. (31) through the residue theorem, which gives $K(x-x')=-\frac{|x-x'|}{2\beta_\phi}$, which can be written as two terms. Then, kernel $\mathcal{B}$ involves a double integration (over x and x')

of the triple product $\dot{\mathcal{G}}_{\rm Ret}(x_1,x,t_1-t_0)K(x-x')\dot{\mathcal{G}}_{\rm Ret}(x_2,x',t_2-t_0)$. From Eq. (90) it is clear that the derivative of the retarded Green function has two terms, so to obtain $\mathcal{B}$ the number of integrals to perform is in principle eight. Due to the symmetry, the final number of double integrals decreases to six for this kernel too.

As we are interested in general features about the transient-time evolution and the steady situation, we will not proceed to a complete calculation of the terms, but we will show that there are steady terms associated to the contribution of these kernels.

We should note that the terms in the kernels $\mathcal{A}$ and $\mathcal{B}$ associated to the products of the free-field retarded Green function $\mathcal{G}^0_{\rm Ret}$ and its derivative [i.e., the terms $\mathcal{G}^0_{\rm Ret}(x_1,x,t_1-t_0)\mathcal{G}^0_{\rm Ret}(x_2,x,t_2-t_0)$ in $\mathcal{A}$ and $\dot{\mathcal{G}}^0_{\rm Ret}(x_1,x,t_1-t_0)K(x-x')\dot{\mathcal{G}}^0_{\rm Ret}(x_2,x',t_2-t_0)$ in $\mathcal{B}$] are the ones that will be removed by the Casimir prescription (subtraction with the free-field case), so we do not have to calculate them.

We should also note that the crossed terms (i.e., terms combining different terms of the Green function) will be transient terms, since these integrations will generate constant terms that will vanish in the derivatives and through the limit needed to calculate the expectation values of the energy-momentum tensor, terms that will exponentially decay in the long-time limit, or divergent terms that must be subtracted to define a correct (nondivergent) Hadamard propagator. As we are interested now in the steady situation, we will not calculate them.

Independently of which model we are considering, to study the long-time regime the products involving two sums over poles will be the ones that result in steady contributions. As a first case, we consider the corresponding term found in the kernel $\mathcal{A}$ for a field point $x_{1,2}<0$,

$$\begin{aligned}\mathcal{A}(x_1,x_2,t_1,t_2) &= (\text{free-field terms}) + (\text{crossed terms}) + \frac{\Theta(-x_1)\Theta(-x_2)}{4\beta_\phi}\sum_{j,l}R_jR_l e^{z_j(x_1+t_1-t_0)}e^{z_l(x_2+t_2-t_0)}\\ &\times\int_{-\infty}^{+\infty}dx[e^{z_jx}\Theta(-x)\Theta(x_1+x+t_1-t_0)+e^{-z_jx}\Theta(x)\Theta(x_1-x+t_1-t_0)]\\ &\times[e^{z_lx}\Theta(-x)\Theta(x_2+x+t_2-t_0)+e^{-z_lx}\Theta(x)\Theta(x_2-x+t_2-t_0)].\end{aligned}\tag{91}$$

Considering that $\Theta(x)\Theta(-x)\equiv 0$ and $\Theta(\pm x)\Theta(\pm x)\equiv\Theta(\pm x)$, there are vanishing integrals in the expression. Then, making a substitution $x\to -x$ in one of the two resulting terms, all of the integrals are the same,

$$\begin{aligned}\mathcal{A}(x_1,x_2,t_1,t_2) &= (\text{free-field terms}) + (\text{crossed terms}) + \frac{\Theta(-x_1)\Theta(-x_2)}{2\beta_\phi}\sum_{j,l}R_jR_l e^{z_j(x_1+t_1-t_0)}e^{z_l(x_2+t_2-t_0)}\\ &\times\int_{-\infty}^{+\infty}dx e^{(z_j+z_l)x}\Theta(-x)\Theta(x_1+x+t_1-t_0)\Theta(x_2+x+t_2-t_0).\end{aligned}\tag{92}$$

Considering that $\Theta(x_1+x+t_1-t_0)\Theta(x_2+x+t_2-t_0)=\Theta(x_1-x_2+t_1-t_2)\Theta(x_2+x+t_2-t_0)+\Theta(x_2-x_1+t_2-t_1)\Theta(x_1+x+t_1-t_0)$, the last integral can be easily calculated,

$$\begin{aligned}\mathcal{A}(x_1,x_2,t_1,t_2) &= (\text{free-field terms}) + (\text{crossed terms}) + \frac{\Theta(-x_1)\Theta(-x_2)}{2\beta_\phi}\sum_{j,l}\frac{R_jR_l}{(z_j+z_l)}\times[(\Theta(x_1-x_2+t_1-t_2)\Theta(x_2+t_2-t_0)\\ &+\Theta(x_2-x_1+t_2-t_1)\Theta(x_1+t_1-t_0))e^{z_j(x_1+t_1-t_0)}e^{z_l(x_2+t_2-t_0)} - \Theta(x_1-x_2+t_1-t_2)\\ &\times\Theta(x_2+t_2-t_0)e^{z_j(x_1-x_2+t_1-t_2)}-\Theta(x_2-x_1+t_2-t_1)\Theta(x_1+t_1-t_0)e^{z_l(x_2-x_1+t_2-t_1)}].\end{aligned}\tag{93}$$

On the other hand, the kernel $\mathcal{B}$ presents a more complicated structure because it involves two integrations (one over x and another over x') and an extra kernel $K(x-x')$ which couples both integrations, preventing a separate calculation. Following the same train of thought, we focus on the terms involving two sums over poles. Therefore, kernel $\mathcal{B}$ reads

$$\mathcal{B}(x_1,x_2,t_1,t_2) = (\text{free-field terms}) + (\text{crossed terms}) - \frac{\Theta(-x_1)\Theta(-x_2)}{8\beta_\phi}\sum_{j,l} z_j z_l R_j R_l e^{z_j(x_1+t_1-t_0)} e^{z_l(x_2+t_2-t_0)}$$
$$\times \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' |x-x'| [e^{z_j x}\Theta(-x)\Theta(x_1+x+t_1-t_0) + e^{-z_j x}\Theta(x)\Theta(x_1-x+t_1-t_0)]$$
$$\times [e^{z_l x'}\Theta(-x')\Theta(x_2+x'+t_2-t_0) + e^{-z_l x'}\Theta(x')\Theta(x_2-x'+t_2-t_0)]. \quad (94)$$

The integration over x' can be done first by writing $|x-x'| = \Theta(x-x')(x-x') + \Theta(x'-x)(x'-x)$. Working out the integral, we obtain that the result can be separated again into terms that will be part of the transient evolution and will vanish in the long-time regime and terms that will give steady results. In fact, the integral can be written as

$$\int_{-\infty}^{+\infty} dx' |x-x'| [e^{z_l x'}\Theta(-x')\Theta(x_2+x'+t_2-t_0) + e^{-z_l x'}\Theta(x')\Theta(x_2-x'+t_2-t_0)]$$
$$= \frac{2}{z_l^2}[\Theta(-x)\Theta(x_2+x+t_2-t_0)e^{z_l x} + \Theta(x)\Theta(x_2-x+t_2-t_0)e^{-z_l x}] + (\text{transient terms}). \quad (95)$$

Thus, kernel $\mathcal{B}$ reads

$$\mathcal{B}(x_1,x_2,t_1,t_2) = (\text{free-field terms}) + (\text{crossed terms}) + (\text{transient terms}) - \frac{\Theta(-x_1)\Theta(-x_2)}{4\beta_\phi}\sum_{j,l}\frac{z_j}{z_l} R_j R_l$$
$$\times e^{z_j(x_1+t_1-t_0)} e^{z_l(x_2+t_2-t_0)} \int_{-\infty}^{+\infty} dx [e^{z_j x}\Theta(-x)\Theta(x_1+x+t_1-t_0) + e^{-z_j x}\Theta(x)\Theta(x_1-x+t_1-t_0)]$$
$$\times [\Theta(-x)\Theta(x_2+x+t_2-t_0)e^{z_l x} + \Theta(x)\Theta(x_2-x+t_2-t_0)e^{-z_l x}], \quad (96)$$

where it is worth noting that the resulting integral is the same as the one in kernel $\mathcal{A}$ in Eq. (91).

Then, the result is the same and the kernel can be written as

$$\mathcal{B}(x_1,x_2,t_1,t_2) = (\text{free-field terms}) + (\text{crossed terms}) + (\text{transient terms}) - \frac{\Theta(-x_1)\Theta(-x_2)}{2\beta_\phi}\sum_{j,l}\frac{z_j}{z_l}\frac{R_j R_l}{(z_j+z_l)}$$
$$\times [(\Theta(x_1-x_2+t_1-t_2)\Theta(x_2+t_2-t_0) + \Theta(x_2-x_1+t_2-t_1)\Theta(x_1+t_1-t_0))e^{z_j(x_1+t_1-t_0)}e^{z_l(x_2+t_2-t_0)}$$
$$- \Theta(x_1-x_2+t_1-t_2)\Theta(x_2+t_2-t_0)e^{z_j(x_1-x_2+t_1-t_2)} - \Theta(x_2-x_1+t_2-t_1)\Theta(x_1+t_1-t_0)e^{z_l(x_2-x_1+t_2-t_1)}]. \quad (97)$$

By considering this last equation and Eq. (94), it is now straightforward that the proper field contribution, given by the sum of kernels $\mathcal{A}$ and $\mathcal{B}$, can be written as

$$\mathcal{A}(x_1,x_2,t_1,t_2) + \mathcal{B}(x_1,x_2,t_1,t_2)$$
$$= (\text{free-field terms}) + (\text{crossed terms}) + (\text{transient terms}) + \frac{\Theta(-x_1)\Theta(-x_2)}{2\beta_\phi}\sum_{j,l}\left(1-\frac{z_j}{z_l}\right)\frac{R_j R_l}{(z_j+z_l)}$$
$$\times [(\Theta(x_1-x_2+t_1-t_2)\Theta(x_2+t_2-t_0) + \Theta(x_2-x_1+t_2-t_1)\Theta(x_1+t_1-t_0))e^{z_j(x_1+t_1-t_0)}e^{z_l(x_2+t_2-t_0)}$$
$$- \Theta(x_1-x_2+t_1-t_2)\Theta(x_2+t_2-t_0)e^{z_j(x_1-x_2+t_1-t_2)} - \Theta(x_2-x_1+t_2-t_1)\Theta(x_1+t_1-t_0)e^{z_l(x_2-x_1+t_2-t_1)}]. \quad (98)$$

The last two terms in the brackets contain exponentials whose exponents do not depend on the initial time t_0. Therefore those terms will not vanish in the long-time limit ($t_0 \to -\infty$). This shows that a part of the proper field contribution has not only transient but also steady terms that contribute to the long-time regime. We note that, in fact, these terms in the Hadamard propagator will be constant terms in the expectation values of the energy-momentum tensor components of Eq. (48) after differentiating and calculating the coincidence limit.

Moreover, we can work out these terms and write them in a more familiar way to connect them with previous works. It should be noted that in the first term (associated to $e^{z_j(x_1-x_2+t_1-t_2)}$) the sum over l can be worked out through the residue theorem, while in the second term (associated to $e^{z_l(x_2-x_1+t_2-t_1)}$) the sum over j can be done.

Then, let us first take this last sum over j. Considering that all the poles are simple and $R_j \equiv \mathrm{Res}[\frac{r}{z}, z_j]$, we can write

$$\sum_j \frac{(z_l - z_j)}{(z_j + z_l)} R_j = \sum_j Res\left[\frac{(z_l - z)}{(z + z_l)}\frac{r}{z}, z_j\right]. \tag{99}$$

From Eq. (87) and given that $\mathrm{Re}(z_l) < 0$ for every pole z_l (so $z_l + z_j \neq 0$), we can show that the complex function $\frac{(z_l - z)}{(z+z_l)}\frac{r}{z}$ goes to 0 when $|z| \to +\infty$, independently of the direction in the complex plane, and that its set of poles is given by all the poles z_j, the pole $-z_l$ (which depends on the term in the sum over l that we are considering), and the pole 0 only in the bilinear coupling model. Therefore, through the residue theorem, for a circle $\mathcal{C}_R^+$ of radius R in the complex plane that contains all the poles, when $R \to +\infty$ we can write

$$0 = \int_{\mathcal{C}_\infty} \frac{dz}{2\pi i}\frac{(z_l - z)}{(z + z_l)}\frac{r}{z} = \sum_j Res\left[\frac{(z_l - z)}{(z + z_l)}\frac{r}{z}, z_j\right] - 2r(-z_l) + \alpha - 1, \tag{100}$$

where the last two terms are the result of calculating explicitly the poles at $-z_l$ and at 0.

Therefore, the whole term associated to $e^{z_l(x_2 - x_1 + t_2 - t_1)}$ reads

$$\sum_{j,l}\left(1 - \frac{z_j}{z_l}\right)\frac{R_j R_l}{(z_j + z_l)} e^{z_l(x_2 - x_1 + t_2 - t_1)} = \sum_l \frac{R_l}{z_l}(2r(-z_l) + 1 - \alpha) e^{z_l(x_2 - x_1 + t_2 - t_1)}. \tag{101}$$

Analogously, we can proceed for the other term, associated to $e^{z_j(x_1 - x_2 + t_1 - t_2)}$, by starting with the sum over l. The calculation is the same except that it is over the complex function $\frac{(z - z_j)}{(z+z_j)}\frac{r}{z^2}$ and the pole at $z = 0$ is simple for the current-type model, while it is of second order in the bilinear model. Then we finally have

$$\sum_{j,l}\left(1 - \frac{z_j}{z_l}\right)\frac{R_j R_l}{(z_j + z_l)} e^{z_j(x_1 - x_2 + t_1 - t_2)} = \sum_j R_j\left(2\frac{r(-z_j)}{z_j} + \frac{2\pi\eta\lambda_0^2}{\omega^2}\alpha + (1 - \alpha)\frac{\omega^2}{2\pi\eta\lambda_0^2}\right) \times e^{z_j(x_1 - x_2 + t_1 - t_2)}, \tag{102}$$

where the difference in units between the last two terms in the brackets is due to the fact that the coupling constant λ_0 change its units depending on the coupling model.

The last terms involve differences between both coupling models in Eqs. (101) and (102). It can be shown that they are divergent terms in the coincidence limit, so we discard them by regularizing the expression. This way, we can write the proper field contribution as

$$\mathcal{A}(x_1, x_2, t_1, t_2) + \mathcal{B}(x_1, x_2, t_1, t_2) = (\text{free-field terms}) + (\text{crossed terms}) + (\text{transient terms}) - \frac{\Theta(-x_1)\Theta(-x_2)}{\beta_\phi}\sum_j R_j \frac{r(-z_j)}{z_j}[\Theta(x_1 - x_2 + t_1 - t_2)\Theta(x_2 + t_2 - t_0)e^{z_j(x_1 - x_2 + t_1 - t_2)} + \Theta(x_2 - x_1 + t_2 - t_1)\Theta(x_1 + t_1 - t_0)e^{-z_j(x_1 - x_2 + t_1 - t_2)}]. \tag{103}$$

At this point, we can exploit the residue theorem one more time to obtain a final closed form for these terms. It is straightforward to show that the sum over j (as we did for the others sums and taking into account the convergence requirements) can be written as an integral in the complex plane over a curve $\mathcal{C}^+ = \mathcal{C}_L^+ \mathcal{R}^+$, where $\mathcal{C}_L^+$ is a half-infinite circle enclosing the left half of the complex plane and $\mathcal{R}^+$ is a straight path over the imaginary axis from the bottom to the top. Therefore, since the integrand function $\frac{r(z)r(-z)}{(-z^2)}e^{z(x_1 - x_2 + t_1 - t_2)}$ vanishes for $|z| \to \infty$ with $\mathrm{Re}(z) < 0$, the integral over $\mathcal{C}_L^+$ is null and the integral is directly over the imaginary axis, which can be parametrized as $z = -i\Omega$. Finally, we obtain

$$-\sum_j R_j \frac{r(-z_j)}{z_j} e^{z_j(x_1 - x_2 + t_1 - t_2)} = \int_{\mathcal{C}^+} \frac{dz}{2\pi i}\frac{r(z)r(-z)}{(-z^2)} e^{z(x_1 - x_2 + t_1 - t_2)} = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi}\frac{|r(-i\Omega)|^2}{(\Omega^2)} e^{-i\Omega(x_1 - x_2 + t_1 - t_2)}, \tag{104}$$

where we have used that $r(i\Omega) = r^*(-i\Omega)$ for real Ω.

Finally, the proper field contribution reads

$$\mathcal{A}(x_1,x_2,t_1,t_2)+\mathcal{B}(x_1,x_2,t_1,t_2)=(\text{free-field terms})+(\text{crossed terms})+(\text{transient terms})$$
$$+\frac{\Theta(-x_1)\Theta(-x_2)}{\beta_\phi}\int_{-\infty}^{+\infty}\frac{d\Omega}{2\pi}\frac{|r(-i\Omega)|^2}{\Omega^2}[\Theta(x_1-x_2+t_1-t_2)\Theta(x_2+t_2-t_0)e^{-i\Omega(x_1-x_2+t_1-t_2)}$$
$$+\Theta(x_2-x_1+t_2-t_1)\Theta(x_1+t_1-t_0)e^{i\Omega(x_1-x_2+t_1-t_2)}], \tag{105}$$

where it is remarkable that this expression has the form as the long-time contributions considered without demonstration in Refs. [10,11] for the proper field contribution in a steady canonical quantization scheme in the case of the force between two plates, i.e., the Casimir-Lifshitz problem.

All in all, we have shown that the long-time limit of a Dirac delta plate of real material has contributions from both the bath and the field by itself. The result can also be extended to other configurations in one dimension $(1+1)$, but the calculations are more complicated. The conclusion seems to be general, i.e., we have shown that a situation including boundaries or, analogously, including vacuum regions will present not only the contributions from the baths, but also the proper field contribution in the long-time regime. Therefore, this situation shows a new type of scenario where the long-time regime is steady but it has contributions from two parts of the composite system. Note that the behavior and the steady situation in the case with vacuum regions is radically different from the case of a material over all space. However, the material contribution (considered separately) behaves in the same way, i.e., the contribution associated to the polarization degrees of freedom is transient and vanishes at the steady situation, while the baths' contribution survives and is part of the long-time regime. The great advantage of including boundaries is that it is not the only one that survives. This is due to the fact that while the field tends to vanish inside the material due to dissipation, outside it is fluctuating freely without damping. This means that the fluctuations outside propagate inside the material and finally reach a steady situation in the long-time regime when the material has relaxed and the dynamics are reduced to the steady ones. This allows us to describe the proper contribution effectively by modified vacuum modes, as was done in Refs. [10,11] for the Casimir problem.

Therefore, any quantization procedure in the long-time regime, i.e., any quantization scheme at the steady situation—at least for models in $1+1$—must consider this contribution to obtain the correct results.

Following this train of thought, as a final comment we should note that we have proven this fact in the one-dimensional $(1+1)$ case, but the conclusion for higher dimensions could change. Although we have not preformed the calculation for the $n+1$ case in this scenario including vacuum regions, from the comparison between the reflection and transmission coefficients in Eqs. (84) and (87) and the Laplace transform of the retarded Green functions in Eqs. (85) and (86) for both cases, we can note that for the higher-dimensional problem we have two branch cuts $\sqrt{z^2+k_\parallel}$ involving the parallel momentum $k_\parallel$ instead of a simple z. Therefore, the analytical properties of the Laplace transform are different and the time behavior of the retarded Green function will change critically. It could happen that the proper field contribution vanishes for this higher-dimensional case, but the continuity between the actual case of a real material and the result obtained in the $n+1$ case for a constant dielectric and arbitrary boundaries in Sec. VI C suggest that the result of this section is quite general even for the higher-dimensional case.

VII. FINAL REMARKS

In this article we have extensively used the CTP approach to calculate a general expression for the time evolution of the expectation value of the energy-momentum tensor components, in a completely general nonequilibrium scenario, for a scalar field in the presence of real materials. The interaction is turned on at an initial time t_0, coupling the field to the polarization degrees of freedom of a volume element of the material, which is also linearly coupled to thermal baths at each point of the space. Throughout the work, we studied two coupling models between the field and the polarization degrees of freedom. One is the bilinear model, analogous to the one considered in the QBM theory [7,22,23], and the other is (a more realistic) current-type model, where the polarization degrees of freedom couple to the field's time derivative (as in the EM case interacting with matter).

It is remarkable that the material is free to be inhomogeneous, i.e., its properties (density $\eta_{\mathbf{x}}$, coupling constant to the field $\lambda_{0,\mathbf{x}}$, mass $m_{\mathbf{x}}$, and frequency $\omega_{\mathbf{x}}$ of the polarization degrees of freedom volume elements) can change with the position. The baths' properties (coupling constant to the polarization degrees of freedom $\lambda_{n,\mathbf{x}}$, mass $m_{n,\mathbf{x}}$, and frequency $\omega_{n,\mathbf{x}}$ of each bath oscillator) can also change with position, resulting in effective position-dependent properties over the volume elements of the material, which are represented by the dissipation and noise kernels $D_{\mathrm{QBM},\mathbf{x}}$

and $N_{\mathrm{QBM},\mathbf{x}}$ after the first integration over the baths' degrees of freedom.

On the other hand, thermodynamical nonequilibrium is included by letting both the volume element and the thermal bath have their own temperatures $(\beta_{r_\mathbf{x}}, \beta_{B,\mathbf{x}})$, which is accomplished by choosing the initial density operators of each part to be thermal states. The field also has its own temperature β_ϕ, which analogously comes from the field's initial state, but for simplicity in the calculations we have taken the high-temperature approximation, except for the $0+1$ field case in Sec. VI A, where the calculation can be done for arbitrary temperatures of the field.

It is worth noting that the approach also considers the fact that the material bodies can be of finite extent and have arbitrary shape, i.e., vacuum regions were the field is free are included. All these features are concentrated in the matter distribution function $g(\mathbf{x})$, which takes binary values of 1 or 0 depending on whether there is material at $\mathbf{x}$ or not.

In the field's high-temperature limit, the expectation values of the energy-momentum tensor components are given by Eq. (48), where the Hadamard propagator is defined in Eq. (47). That equation contains the full dynamics of the field correlation, with one contribution clearly associated to the material, which is contained in the third term of Eq. (47); the other is clearly associated to the field by itself, which is contained in the first two terms.

All in all, the third term is directly associated to the material (polarization degrees of freedom plus thermal baths), represented by the field's noise kernel, which is also separated into two contributions, $\mathcal{N}(x,x') = 4\pi\eta_\mathbf{x} g(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')[\mathcal{N}_{r,\mathbf{x}}(\tau,\tau') + \mathcal{N}_{B,\mathbf{x}}(\tau,\tau')]$. One contribution is associated to the polarization degrees of freedom that define the material and has an effective dissipative (damping) dynamics due to the interaction with the baths (QBM), and the other is associated to the baths' fluctuations that [thanks to its (nondamping) dynamics] acts like a source of constant temperatures at each point of space and results in an influence over the field; although they do not interact directly, the polarization degrees act as a bridge and connect them in an indirect interaction. Both contributions define the field's transient dynamics, while the steady situation seems to be determined only by the baths, which have nondamping dynamics. The relaxation dynamics of the polarization degrees of freedom ensures that they will not contribute to the long-time regime.

We have shown that the polarization degrees of freedom never contribute to the steady situation, while the bath always does, independently of the situation considered. However, there is a case (Sec. VI C) where both contributions are absent, namely, the constant dielectric case, where it is trivially expected that there is no contribution because the material dynamics is suppressed.

On the other hand, the first two terms of Eq. (47) are directly associated to the field's initial state. Since we have considered the high-temperature limit, both terms are linear in the field's initial temperature β_ϕ, as is expected. These terms give the transient field evolution from the initial free field over the whole space to the field interacting with the polarization degrees of freedom of the material in certain regions defined by the distribution function $g(\mathbf{x})$. This evolution involves two dynamical aspects. One is related to the fact that the properties of the boundaries are time dependent. This adds extra dynamical features associated to the relaxation process of the material, which enter the field dynamics via the field's dissipation kernel $\partial_{tt}^{2\alpha}\mathcal{D}$ for each coupling model that defines the form of the field's retarded Green function $\mathcal{G}_{\mathrm{Ret}}$ through Eq. (71). In Sec. VI C, this aspect was turned off by suppressing the material's relaxation.

The other dynamical aspect is the adaptation of the free field to the condition of being bounded by the sudden appearance of boundaries. We have shown that this clearly takes place in two scenarios: one was studied in Sec. VI D when the distribution function $g(\mathbf{x})$ has null values for some space point $\mathbf{x}$ (i.e., when there are vacuum regions in the particular problem), while the other is the aforementioned lossless case (Sec. VI C). In both cases, the effect is basically the conversion of the field modes from free field modes to interacting (modified) field modes.

This is completely related to the long-time regime of these terms and it is not so easy to analyze. As we found in Secs. VI A and VI B, the proper field contribution vanishes for both coupling models, so there is no interacting (modified) field modes associated to the field's initial state. Although it is tempting to assume that this result applies to all cases involving material boundaries, in Secs. VI C and VI D we have shown that the proper field contribution does not vanish in the long-time regime. The result is trivially expected for the case of constant dielectric permittivity, as there exists a steady Casimir force between bodies of constant dielectric permittivity (see Refs. [10,28]) involving a $n+1$ field. Therefore, since this case and the real material including losses and absorption are expected to be continuously connected [as we saw in Sec. VI D for the one-dimensional $(1+1)$ case], it is suggested that the result found in the $1+1$ case is quite general and holds for the $n+1$ case with material boundaries (with $n>1$).

Physically, for the case without boundaries, the dissipative (damping) dynamics of the field vanishes at the steady situation, as in the QBM case.

For the case with vacuum regions, the field inside the material tends to behave as a damped field, but in the vacuum regions the field fluctuates without damping. So there is a competition between the behavior in both regions which will make the field evolve to a steady field with modified modes in the long-time regime, beyond the material relaxation, analogously to the case of the steady situation with constant dielectric properties but with frequency-dependent effective properties. The free fluctuations propagate inside the material regions and keep the

field in a continuous situation that survives to the long-time regime. In other words, the transient dynamics of these terms will be different from the constant-dielectric case. The steady situation will also be different due to the frequency dependence of the properties. However, taking into account the considerations made above, the formal expression must be the same as the constant-dielectric case by replacing it with the corresponding actual frequency-dependent permittivity (as it happens for the Casimir force for a $1+1$ field in absorbing media [10]).

Therefore, at least in the $1+1$ case, any quantization scheme for these models in a steady scenario (despite involving thermodynamical nonequilibrium) must take into account the proper field contribution in addition to the expected baths' contribution.

All in all, this is in fact a natural physical conclusion: all the parts of the system that do not have damping dynamics contribute to the long-time regime. Following this train of thought, it is naturally expected that the bath contributes and—since there are regions where the field has no damping dynamics—modified field modes appear in the long-time proper field contribution.

As a future work, and for completeness, it should be possible to extend the calculation to the case of arbitrary field temperatures without many complications. More interesting would be to investigate the implications of this analysis on the vacuum fluctuations in the real Casimir problem, extending the complete study in a $1+1$ scalar field to the $3+1$ EM field. Finally, it would also be interesting to study the heat transfer and other thermodynamical features in situations where the material is thermally inhomogeneous.

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APPENDIX: THE FIELD WIGNER FUNCTIONAL

The Wigner functional for a quantum field can be defined as in Ref. [24],

$$W_\phi[\phi_0(\mathbf{x}),\Pi_0(\mathbf{x}),t_0]=\int \mathcal{D}\varphi(\mathbf{x})e^{-i\int d\mathbf{x}\Pi_0(\mathbf{x})\varphi(\mathbf{x})}\left\langle \phi_0(\mathbf{x})+\frac{1}{2}\varphi(\mathbf{x})\middle|\hat{\rho}_\phi(t_0)\middle|\phi_0(\mathbf{x})-\frac{1}{2}\varphi(\mathbf{x})\right\rangle. \tag{A1}$$

It is worth noting that it sometimes seems easier to compute the Wigner functional in momentum space. However, this is not so easy. Even though the field $\phi(\mathbf{x})$ is real, its Fourier transform $\phi(\mathbf{p})$ is complex and its real and imaginary parts are not independent, because to have a real field $\phi(-\mathbf{p})=\phi^*(\mathbf{p})$. As in Ref. [24], for the Fourier transform we will treat the real and imaginary parts of $\phi(\mathbf{p})$ as independent variables, but we consider $p_i\in(0,+\infty)$ for each momentum component instead of $p_i\in(-\infty,+\infty)$. This way, the Wigner functional in momentum space can be defined as

$$\tilde{W}_\phi[\phi_0(\mathbf{p}),\Pi_0(\mathbf{p}),t_0]=\int \mathcal{D}\varphi(\mathbf{p})e^{-i\int_0^{+\infty}d\mathbf{p}[\Pi_0^*(\mathbf{p})\varphi(\mathbf{p})+\Pi_0(\mathbf{p})\varphi^*(\mathbf{p})]}\left\langle \phi_0(\mathbf{p})+\frac{1}{2}\varphi(\mathbf{p})\middle|\hat{\rho}_\phi(t_0)\middle|\phi_0(\mathbf{p})-\frac{1}{2}\varphi(\mathbf{p})\right\rangle, \tag{A2}$$

with the functional integrations running over real and imaginary components of $\phi(\mathbf{p})$ [24]. Going from Eq. (A1) to (A2) implies a nontrivial Jacobian $\det[\frac{\delta\varphi(\mathbf{x})}{\delta\varphi(\mathbf{p})}]$, which does not depend on the fields because the Fourier transformation is a linear mapping; consequently, it appears merely as a new normalization factor of the Wigner functional.

Now, we consider the scalar field initially in thermodynamical equilibrium. Then, the density matrix operator $\hat{\rho}_\phi(t_0)$ is given by

$$\hat{\rho}_\phi(t_0)=\frac{1}{Z}e^{-\beta_\phi\hat{H}_0}, \tag{A3}$$

where Z is the partition function associated to the initial field's Hamiltonian $\hat{H}_0$, which can be written as

$$\hat{H}_0=\int_0^{+\infty}d\mathbf{p}(\hat{\Pi}^\dagger(\mathbf{p})\hat{\Pi}(\mathbf{p})+p^2\hat{\Phi}^\dagger(\mathbf{p})\hat{\Phi}(\mathbf{p})). \tag{A4}$$

This Hamiltonian is a sum of two harmonic oscillator Hamiltonians for each component at fixed $\mathbf{p}$. Thus, taking $\mathbf{p}$ as a label for each pair of oscillators, we can introduce a complete set of energy eigenstates of the two-dimensional (isotropic) oscillator $|n_1,n_2\rangle$. We can then write Eq. (A2) as

$$\begin{aligned}\tilde{W}_\phi[\phi_0(\mathbf{p}),\Pi_0(\mathbf{p}),t_0]&=\sum_{n_1,n_2}\int \mathcal{D}\varphi(\mathbf{p})e^{-\int_0^{+\infty}d\mathbf{p}[i(\Pi_0^*(\mathbf{p})\varphi(\mathbf{p})+\Pi_0(\mathbf{p})\varphi^*(\mathbf{p}))+\beta_\phi|\mathbf{p}|]}\\&\times\left\langle \phi_0(\mathbf{p})+\frac{1}{2}\varphi(\mathbf{p})|n_1,n_2\rangle\langle n_2,n_1|\phi_0(\mathbf{p})-\frac{1}{2}\varphi(\mathbf{p})\right\rangle.\end{aligned} \tag{A5}$$

The eigenfunctions for the two-dimensional (isotropic) harmonic oscillator are given by

$$\langle \Phi_R, \Phi_I | n_1, n_2 \rangle = \left(\frac{\alpha^2}{\pi 2^{n_1} n_1! 2^{n_2} n_2!} \right)^{1/2} H_{n_1}(\alpha \Phi_R) H_{n_2}(\alpha \Phi_I) e^{-\frac{\alpha^2}{2}(\Phi_R^2 + \Phi_I^2)}, \tag{A6}$$

where $\Phi_{R,I}$ is the real or imaginary part of the field, respectively, H_n are the Hermite polynomials, and $\alpha \equiv (\frac{|\mathbf{p}|}{2})^{1/2}$.

Inserting this into the last expression for the Wigner functional in momentum space and using the identity for the Hermite polynomials,

$$\sum_n^{\infty} \frac{a^n}{n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1-4a^2}} e^{\frac{4axy - 4a^2(x^2+y^2)}{1-4a^2}}, \tag{A7}$$

which holds for $a < 1/2$ (a condition which is satisfied in our case because $a = e^{-\beta_\phi |\mathbf{p}|}/2$), one finds that

$$\begin{aligned} \tilde{W}_\phi[\phi_0(\mathbf{p}), \Pi_0(\mathbf{p}), t_0] &= \int \mathcal{D}\varphi(\mathbf{p}) e^{-i \int_0^{+\infty} d\mathbf{p} (\Pi_0^*(\mathbf{p})\varphi(\mathbf{p}) + \Pi_0(\mathbf{p})\varphi^*(\mathbf{p}) - i\frac{\alpha^2}{4}\varphi^*(\mathbf{p})\varphi(\mathbf{p}) - i\alpha^2 \phi_0^*(\mathbf{p})\phi_0(\mathbf{p}) - i\beta_\phi |\mathbf{p}|)} \\ &\times \prod_{\mathbf{p}} \frac{\alpha^2}{\pi(1 - e^{-2\beta_\phi |\mathbf{p}|})} e^{\frac{\alpha^2 e^{-\beta_\phi |\mathbf{p}|}}{2(1 - e^{-2\beta_\phi |\mathbf{p}|})} [4(1 - e^{-\beta_\phi |\mathbf{p}|}) \phi_0^*(\mathbf{p})\phi_0(\mathbf{p}) - (1 + e^{-\beta_\phi |\mathbf{p}|})\varphi^*(\mathbf{p})\varphi(\mathbf{p})]} \\ &= C e^{\int_0^{+\infty} d\mathbf{p} \alpha^2 \tanh(\frac{\beta_\phi |\mathbf{p}|}{2}) \phi_0^*(\mathbf{p})\phi_0(\mathbf{p})} \int \mathcal{D}\varphi(\mathbf{p}) e^{-i \int_0^{+\infty} d\mathbf{p} (\Pi_0^*(\mathbf{p})\varphi(\mathbf{p}) + \Pi_0(\mathbf{p})\varphi^*(\mathbf{p}))} e^{\int_0^{+\infty} d\mathbf{p} \frac{\alpha^2}{4} \coth(\frac{\beta_\phi |\mathbf{p}|}{2}) \varphi^*(\mathbf{p})\varphi(\mathbf{p})}, \end{aligned} \tag{A8}$$

where in the coefficient C we have included all the terms which are not functionals of the fields and momenta.

Integrating trivially over the real and imaginary components of $\varphi(\mathbf{p})$, we arrive at the three-dimensional generalization of the Wigner functional in momentum space found in Ref. [24] for the one-dimensional case,

$$\begin{aligned} &\tilde{W}_\phi[\phi_0(\mathbf{p}), \Pi_0(\mathbf{p}), t_0] \\ &\quad = C e^{-\frac{\beta_\phi}{2} \int d\mathbf{p} \tilde{\Delta}_{\beta_\phi}(|\mathbf{p}|) [\Pi_0^*(\mathbf{p})\Pi_0(\mathbf{p}) + |\mathbf{p}|^2 \phi_0^*(\mathbf{p})\phi_0(\mathbf{p})]}, \end{aligned} \tag{A9}$$

where the thermal weight factor is given by

$$\tilde{\Delta}_{\beta_\phi}(|\mathbf{p}|) = \frac{2}{\beta_\phi |\mathbf{p}|} \tanh\left(\frac{\beta_\phi |\mathbf{p}|}{2} \right). \tag{A10}$$

This is an even function of the momentum's absolute value, so in Eq. (A9) the integrals over the momentum components were extended to all the real values.

Finally, knowing the Wigner functional in momentum space allows on to easily find the Wigner functional in coordinate space by writing all of the momentum's functions as Fourier transforms of the function in coordinate space. This way, we can write an extension of the result found in Ref. [24],

$$W_\phi[\phi_0(\mathbf{x}), \Pi_0(\mathbf{x}), t_0] = C' e^{-\beta \int d\mathbf{x} \int d\mathbf{x}' \mathcal{H}(\mathbf{x}, \mathbf{x}')}, \tag{A11}$$

where C' is the normalization constant in coordinate space and the integrand $\mathcal{H}$ is given by

$$\mathcal{H}(\mathbf{x}, \mathbf{x}') \equiv \frac{1}{2} \Delta_{\beta_\phi}(\mathbf{x} - \mathbf{x}') [\Pi_0(\mathbf{x})\Pi_0(\mathbf{x}') + \nabla\phi_0(\mathbf{x}) \cdot \nabla\phi_0(\mathbf{x}')], \tag{A12}$$

where the thermal weight factor in coordinate space is given by

$$\Delta_{\beta_\phi}(\mathbf{x} - \mathbf{x}') = \int \frac{d\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')} \tilde{\Delta}_{\beta_\phi}(|\mathbf{p}|). \tag{A13}$$

It is worth noting that, due to the interchange symmetry of the integrand, the thermal weight factor in coordinate space must be symmetric, i.e., $\Delta_{\beta_\phi}(\mathbf{x}' - \mathbf{x}) = \Delta_{\beta_\phi}(\mathbf{x} - \mathbf{x}')$.

It is remarkable that although the expression for the thermal weight factor in momentum space does not change formally with the number of dimensions we are considering, the thermal factor in coordinate space clearly does.

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