Emergent Lorentz signature, fermions, and the standard model

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This article investigates the construction of fermions and the formulation of the Standard Model of particle physics in a theory in which the Lorentz signature emerges from an underlying microscopic purely Euclidean $SO(4)$ theory. Couplings to a clock field are responsible for triggering the change of signature of the effective metric in which the standard fields propagate. We demonstrate that Weyl and Majorana fermions can be constructed in this framework. This construction differs from other studies of Euclidean fermions, as the coupling to the clock field allows us to write down an action which flows to the usual action in Minkowski spacetime. We then show how the Standard Model can be obtained in this theory and consider the constraints on non-Standard-Model operators which can appear in the QED sector due to *CPT* and Lorentz violation.

I. INTRODUCTION

Part of the art of theoretical physics is to find the mathematical structures that allow us to formalize and simplify the laws of nature. These structures include the description of spacetime (dimension, topology, …) and matter and their interactions (fields, symmetries, …). While there is a large amount of freedom in the choice of these mathematical structures, the developments of theoretical physics have taught us that some of them are better suited to describe certain classes of phenomena. However, these choices are only validated by the mathematical consistency of the theory and, in the end, by the agreement of their predictions with experiments.

Among all of these structures, and in the framework of metric theories of gravitation, the signature of the metric is in principle arbitrary. It seems that on the scales that have been probed so far, there is the need for only one time dimension and three spatial dimensions. It is also now universally accepted that the relativistic structure is a central ingredient of the construction of any realistic field theory, in particular as the cleanest way to implement the notion of causality. Spacetime enjoys a locally Minkowski structure and, when gravity is included, the equivalence principle implies (this is not a theoretical requirement, but an experimental fact, required at a given accuracy) that all the fields are universally coupled to the same Lorentzian metric. Thus, we usually take for granted that spacetime is a four-dimensional manifold endowed with a metric of mixed signature, e.g. $(-, +, +, +)$.

While the existence of two time directions may lead to confusion [1,2], several models for the birth of the Universe [3–6] are based on a change of signature via an instanton in which a Riemannian and a Lorentzian manifold are joined across a hypersurface. While there is no time in the Euclidean region, with signature $(+, +, +, +)$, it flips to $(-, +, +, +)$ across this hypersurface, which may be thought of as the origin of time from the Lorentzian point of view. Eddington even suggested [7] that it can flip across some surface to $(-, -, +, +)$, and signature flips also arise in brane or loop quantum cosmology [8–10].

It is legitimate to investigate whether the signature of the metric is only a convenient way to implement causality or whether it is just a property of an effective description of a microscopic theory in which there is no such notion. In Ref. [11], two of us have proposed that at the microscopic level the metric is Riemannian and that the Lorentzian structure, usually thought of as fundamental, is in fact an effective property that emerges in some regions of a four-dimensional space with a positive definite metric. There has been some related work in the past—for instance, the work by Barbero [12] (with more than second-order derivatives in the equations of motion, however), or in Einstein-Aether theory [13] (although without an order parameter connecting the Euclidean and Lorentzian theories) and scalar gravity [14]. We argued that a decent classical field theory for scalars, vectors, and spinors in flat spacetime can be constructed, and that gravity can be included under the form of a covariant Galileon theory instead of general

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relativity. This mechanism of emergent Lorentz signature may also serve as a new way to circumvent the issue of nonunitarity in some higher-derivative quantum gravity theories [15,16].

Among the gaps emphasized in this work, we have pointed out that (1) the construction is restricted to classical field theory, and the spinor sector suffers from a severe fine-tuning to ensure *CPT* invariance (see e.g. Ref. [17] and references therein for recent constraints on *CPT* violation), and that (2) it requires the construction of Majorana and Weyl spinors in order to formulate the Standard Model (SM) and its extensions.

It is well known that Majorana fermions are technically impossible to construct in a 4 D Euclidean theory, but several authors have found alternative constructions [18–22]. However, these techniques are often aimed at a Wick rotation to or from a Lorentzian theory and can involve doubling the fermion degrees of freedom or other aspects which are ill suited to our application. With the aim of developing a theory which flows to the usual actions in Minkowski space (which may look very different in Euclidean space) and the available couplings to the clock field, we arrive at a new formulation for Weyl and Majorana spinors. As our goals and setting are different than in previous studies, we do not need to use the techniques employed there, such as fermion doubling or the *ad hoc* construction of different spinors. The Weyl spinors and coupling to the clock field allow us to directly construct an emergent version of the SM, with its chiral and metric structure inherited from an originally Euclidean theory.

This article is organized as follows: In Sec. II, we briefly review the construction given in Ref. [11]. Following that, in Sec. III, we extend the fermion sector to include Weyl and Majorana fermions, which is quite distinct from the usual considerations in Euclidean space. For fermions, an alternative "derivation" of several of the choices in this construction are detailed in the Appendix. In Sec. IV, we then show how to construct the Standard Model in this framework of an emergent Lorentzian metric. There are additional operators which can arise in this theory and, in Sec. V, we categorize and analyze the constraints on such operators in the QED sector of the SM. Finally, we gather further comments, conclusions, and future directions in Sec. VI.

II. EMERGENT LORENTZ SIGNATURE

In this section, we briefly lay out our conventions and review the construction given in Ref. [11] for a theory with an effective Lorentz signature emerging from a locally Euclidean metric. The Minkowski metric, $\eta_{\mu\nu}$, is mostly positive with the signature $(-,+,+,+)$, while the Euclidean metric has a positive signature and is denoted $\delta_{\mu\nu}$.

A. Basics of the mechanism

From the point of view of the Euclidean theory, at the fundamental level, there is no concept of time (one cannot single out a privileged direction) until the clock field, ϕ, picks out a direction through its derivative having a nonzero vacuum expectation value (VEV). We will always work in some patch $\mathcal{M}_0$ where this VEV can be considered a constant,

$$\partial_\mu \phi = M^2 n_\mu, \tag{1}$$

with M a mass scale for units and n_μ a constant unit vector which now defines a particular direction, related to the notion of time (the direction which will change signature in the effective metric). Thus, we have

$$\mathrm{d}t = n_\mu \mathrm{d}x^\mu \tag{2}$$

and choose

$$t \equiv \frac{\phi}{M^2}. \tag{3}$$

The other three coordinates (with positive signature) are the coordinates of a hypersurface normal to n_μ.

We can now write down actions in the Euclidean theory, which will flow to a Minkowski theory (in the sense that the fields propagate in an effective Minkowski metric) when restricted to $\mathcal{M}_0$ after the gradient of the clock field has a VEV. Here we just summarize the results obtained in Ref. [11], which has further details.

For a scalar field χ with potential $V(\chi)$, we consider a Euclidean action of the form

$$S_\chi = \int \mathrm{d}^4x \left[-\frac{1}{2} \delta^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - V(\chi) + \frac{1}{M^4} (\delta^{\mu\nu} \partial_\mu \phi \partial_\nu \chi)^2 \right]. \tag{4}$$

In $\mathcal{M}_0$, the last term becomes $M^4(\partial_t \chi)^2$, so that the above action leads to the usual action for a scalar field in Minkowski space,

$$S_\chi = \int \mathrm{d}t \mathrm{d}^3x \left[-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - V(\chi) \right]. \tag{5}$$

The case of a vector field, A_μ, with field strength tensor $F^E_{\mu\nu}$ (the E denotes that indices are raised/lowered with the Euclidean metric) is also straightforward. The action

$$S_A = \frac{1}{4} \int \mathrm{d}^4x \left[-F^E_{\mu\nu} F^{\mu\nu}_E + \frac{4}{M^4} F^{\mu\rho}_E F^\nu_{E\rho} \partial_\mu \phi \partial_\nu \phi \right], \tag{6}$$

with the second term equaling $4\delta^{ij} F_{0i} F_{0j}$ in $\mathcal{M}_0$, becomes the standard Maxwell action for a vector field in Minkowski spacetime,

$$S_A = -\frac{1}{4}\int \mathrm{d}t\mathrm{d}^3x\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}F_{\mu\nu}. \tag{7}$$

B. Dirac fermions

We will now consider Dirac fermions, which will require a bit more detail and care, as we have to be careful with the Clifford algebra and the gamma matrices to build a proper action. This will be extended to Weyl and Majorana fermions in the following section, while a more (Clifford) basis agnostic derivation of these conventions can be found in the Appendix.

In general, the gamma matrices γ^μ satisfy[1]

$$\{\gamma^\mu,\gamma^\nu\} = -2g^{\mu\nu} \tag{8}$$

for a metric $g^{\mu\nu}$. These matrices generate the group $[SO(4)$ or $SO(3,1)$ in our case] generators

$$S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu]. \tag{9}$$

In Minkowski space, we will use the common Weyl or chiral representation with the Pauli matrices $\sigma^\mu \equiv (1,\sigma^i), \bar{\sigma}^\mu \equiv (1,-\sigma^i)$, and the gamma matrices

$$\gamma_M^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{10}$$

We define[2] γ_M^5 as

$$\gamma_M^5 \equiv -i\gamma_M^0\gamma_M^1\gamma_M^2\gamma_M^3 = \mathrm{diag}(1,1,-1,-1), \tag{11}$$

which is Hermitian, squares to 1, and anticommutes with all γ_M^μ.

A 4-component Dirac spinor, ψ_M, transforms as

$$\psi_M \to \Lambda_{M,\frac{1}{2}}\psi_M, \qquad \Lambda_{M,\frac{1}{2}} = \exp\left[-\frac{i}{2}\omega_{\mu\nu}S_M^{\mu\nu}\right], \tag{12}$$

with ω an antisymmetric tensor and $\Lambda_{M,\frac{1}{2}}$ not unitary in general. In order to form Lorentz invariants for an action, we define the usual barred spinor:

$$\bar{\psi}_M \equiv \psi_M^\dagger\gamma_M^0, \qquad \bar{\psi}_M \to \bar{\psi}_M\Lambda_{M,\frac{1}{2}}^{-1}. \tag{13}$$

The standard action for the Dirac field in Minkowski space is given by

$$S_\psi^M = \int \mathrm{d}^4x\bar{\psi}_M\left(\frac{i}{2}\gamma_M^\mu\overset{\leftrightarrow}{\partial}_\mu - m\right)\psi_M. \tag{14}$$

In Euclidean space, the gamma matrices are chosen as

$$\gamma_E^0 \equiv i\gamma_M^5, \qquad \gamma_E^i \equiv \gamma_M^i, \tag{15}$$

and γ_E^5 satisfies the same properties, now defined as

$$\gamma_E^5 \equiv \gamma_E^0\gamma_E^1\gamma_E^2\gamma_E^3 = \gamma_M^0. \tag{16}$$

The generators of $SO(4)$, $S_E^{\mu\nu}$, are now Hermitian, and so $\Lambda_{E,\frac{1}{2}}$ is a unitary transformation of the 4-component spinor ψ_E,

$$\psi_E \to \Lambda_{E,\frac{1}{2}}\psi_E, \qquad \Lambda_{E,\frac{1}{2}} = \exp\left[-\frac{i}{2}\omega_{\mu\nu}S_E^{\mu\nu}\right]. \tag{17}$$

Both $\bar{\psi}_E(= \psi_E^\dagger\gamma_M^0 = \psi^\dagger\gamma_E^5)$ and $\psi^\dagger$ transform the same way,

$$\bar{\psi}_E \to \bar{\psi}_E\Lambda_{E,\frac{1}{2}}^{-1}, \qquad \psi_E^\dagger \to \psi_E^\dagger\Lambda_{E,\frac{1}{2}}^{-1}, \tag{18}$$

and can form $SO(4)$ invariants with ψ_E. We will favor the bar notation to make the connection to the Lorentzian theory explicit.

The Euclidean action

$$S_\psi = \int \mathrm{d}^4x\Big\{\bar{\psi}_E\left(\frac{i}{2}\gamma_E^\mu\overset{\leftrightarrow}{\partial}_\mu - m\right)\psi_E$$
$$+ \frac{1}{2M^2}\delta^{\mu\nu}[(i\bar{\psi}\gamma_E^5\overset{\leftrightarrow}{\partial}_\mu\psi) - (i\bar{\psi}\gamma_E^\rho\overset{\leftrightarrow}{\partial}_\mu\psi)\partial_\rho\phi]\partial_\nu\phi\Big\} \tag{19}$$

becomes the Minkowski Dirac action, Eq. (14), after the clock field picks out a direction in $\mathcal{M}_0$.

III. WEYL AND MAJORANA FERMIONS

We now extend the above procedure for Weyl and Majorana fermions. By a Weyl fermion, we mean a 4-component spinor that is an eigenstate of γ^5,

$$\gamma_{E,M}^5\psi_\pm^{E,M} = \pm\psi_\pm^{E,M}, \tag{20}$$

and we recall that in the representation used above, $\gamma_E^5 \neq \gamma_M^5$.

It is important to note that the γ_E^μ representation we have used is *not* the same as the Weyl or chiral representation: it does not make manifest the algebra isomorphism[3]

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[1]The overall sign can be changed by a factor of i in the gamma matrices, changing the Hermiticity of the matrices.

[2]Note that this definition includes a minus sign.

[3]It is important to note that unlike in the Lorentzian case, the representations of these $SU(2)$'s are *not* related by complex conjugation. In other words, a 2-component spinor and its complex conjugate transform under the same $SU(2)$ and can make an $SO(4)$ invariant. For a review of 2-component spinors, see Ref. [23] and references therein, as well as Ref. [24] for Euclidean space.

$SO(4) = SU(2)_- \times SU(2)_+$. In other words, a general $SO(4)$ transformation of a 4-component spinor, ψ^E, in this description does not separate into two 2-component spinors (the top/bottom of the 4-component spinor) transforming in separate $SU(2)$'s. This is also why we have suppressed all spinor indices, as there is not the usual separation into dotted and undotted indices labeling the different $SU(2)$'s.

However, the eigenstates of γ_E^5 take the following form:

$$\psi_\pm^E = \begin{pmatrix} \xi_\pm \\ \pm \xi_\pm \end{pmatrix}, \tag{21}$$

with $\xi_\pm$ transforming as a 2-component spinor under $SU(2)_\pm$. We can also form the usual projection matrices with $(1 \pm \gamma_E^5)/2$. For $\psi_\pm^E$, then, we can make a direct connection with 2-component spinors in this formalism. It should be noted, however, that it is best to work in one form or the other, as the decomposition between 4- and 2-component spinors is completely different in our Euclidean and Lorentzian theories.[4] In the Lorentzian theory, the Weyl spinors are of the form

$$\psi_{L(-)}^M = \begin{pmatrix} \xi_- \\ 0 \end{pmatrix}, \qquad \psi_{R(+)}^M = \begin{pmatrix} 0 \\ \xi_+ \end{pmatrix}. \tag{22}$$

To construct an action in the Euclidean theory which will become the appropriate action in the Lorentzian theory, we might try using the terms

$$i\overline{\psi_\pm^E}\gamma_E^\mu\partial_\mu\psi_\pm^E, \qquad \delta^{\mu\nu}(i\overline{\psi_\pm^E}\gamma_E^\rho\partial_\mu\psi_\pm^E)\partial_\rho\phi\partial_\nu\phi, \tag{23}$$

but unfortunately they vanish identically. Instead, we can construct an appropriate action as

$$S_\pm = \frac{1}{4M^2}\int dt d^3x[\bar{\psi}_\pm^E\gamma_E^5(i\delta^{\mu\nu} - \gamma_E^\nu\gamma_E^\mu)\partial_\mu\psi_\pm^E\partial_\nu\phi + \text{h.c.}]. \tag{24}$$

After the gradient of the clock field has a VEV on $\mathcal{M}_0$, this becomes the standard Lorentzian action for 2-component Weyl spinors,

$$S_\pm = \int dt d^3x \begin{cases} i\xi_{-,L}^\dagger\bar{\sigma}^\mu\partial_\mu\xi_{-,L} \\ i\xi_{+,R}^\dagger\sigma^\mu\partial_\mu\xi_{+,R} \end{cases}, \tag{25}$$

where the subscript indicates the $SU(2)$ representation from the Euclidean ($\mp$) to Lorentzian (L, R). To connect to the 4-component spinors, we recognize that, once the gradient of the clock field has a VEV, we want eigenstates of γ_M^5:

$$\gamma_E^5\psi_\pm^E = \pm\psi_\pm^E \to \gamma_M^5\psi_\pm^M = \pm\psi_\pm^M. \tag{26}$$

This naturally comes out of the action of Eq. (24). By inserting $(-i * i)$ in the second term and using the properties of the gamma matrices, the action becomes

$$S = \int dt d^3x i\bar{\psi}_\pm^M\gamma^\mu\partial_\mu\psi_\pm^M. \tag{27}$$

Finally, we also want to incorporate Majorana spinors (representing fermions which are their own antiparticles). As is well known, we cannot directly have a Majorana spinor in 4D Euclidean space: $\psi_E^C = \psi_E$ is only satisfied for the zero spinor, where $\psi_E^C \equiv C_E\bar{\psi}^T$ is the Euclidean charge conjugate spinor and C_E will be defined below [see Eq. (30)].

However, using the above formulation of Weyl spinors, we can write down a Lagrangian for a single Weyl fermion with a mass term. This captures the physical properties of a Majorana spinor, and in the Lorentzian theory this will correspond to the usual Majorana spinor (a self-conjugate 4-spinor). From our form of Weyl spinors, Eq. (21), we write a Majorana mass term (the right-hand side is exactly the Lorentzian 2-component form, as we have rotated ψ to change the signs) as

$$\frac{1}{4}m(\psi_\pm^E)^T C_E\psi_\pm^E + \text{h.c.} = \frac{1}{2}m\xi_\pm\xi_\pm + \text{h.c.}, \tag{28}$$

with T denoting the transpose, and where we need the matrix C_E to have the term be $SO(4)$ invariant [i.e., to provide the (suppressed) invariant to combine the $\xi_\pm$ spinors as in the usual 2-component formalism]. In other words, we require

$$\Lambda_{E,\frac{1}{2}}^T C_E = C_E\Lambda_{E,\frac{1}{2}}^{-1}, \tag{29}$$

which is satisfied by the matrix[5]

$$C_E = \gamma_E^1\gamma_E^3 \tag{30}$$

with properties

$$C_E^T = C_E^\dagger = C_E^{-1} = -C_E. \tag{31}$$

This is similar to the numerical structure of a charge conjugation matrix,[6] but again, we cannot enforce that a spinor is self-conjugate and nontrivial in the 4D Euclidean theory (as one can see directly given C_E above). When we move to the Lorentzian theory, this matrix becomes

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[4]See Ref. [23] and references therein for details in translating between 2- and 4-component spinors, in 4D Minkowski in particular.

[5]We are not using explicit spinor indices, so we consider this as a numerical identification.

[6]We have not shown how it operates directly on (anti)particles. Also, the matrix satisfies $C_E^{-1}\gamma_E^\mu C_E = (\gamma_E^\mu)^T$ rather than giving $-(\gamma_E^\mu)^T$ as in the usual Minkowski space definition.

$$\tilde{C}_M = \gamma_M^1 \gamma_M^3, \tag{32}$$

which is almost the Lorentzian charge conjugation matrix. If we use a factor[7] of $-\gamma_M^5$ in the mass term from the property of the (now Lorentzian) Weyl spinor, we can now identify (again, as a numerical identity through direct computation of the necessary properties) this with the Lorentzian charge conjugation matrix C_M,

$$C_M = -\gamma_M^5 \gamma_M^1 \gamma_M^3. \tag{33}$$

The structure of this mass term,

$$\frac{1}{2} m \psi_{\pm,M}^T C_M \psi_{\pm,M}, \tag{34}$$

is exactly a Majorana mass term with the identification of the Majorana condition,

$$\psi_M^C \equiv C_M \bar{\psi}_M^T = \psi_M \qquad \text{or} \qquad \bar{\psi}_M = \psi_M^T C_M. \tag{35}$$

Note that the degrees of freedom match, as we have moved from a Weyl spinor in Euclidean space to a Majorana spinor [or equivalently, a single Weyl spinor with a (Majorana) mass term] in Lorentzian space, each with two complex degrees of freedom off shell. In Minkowski space, the Majorana spinors take the following form in terms of 2-component spinors (either a single left- or right-handed spinor):

$$\psi_{M(-)} = \begin{pmatrix} \xi_- \\ \xi_-^\dagger \end{pmatrix}, \qquad \psi_{M(+)} = \begin{pmatrix} \xi_+^\dagger \\ \xi_+ \end{pmatrix}, \tag{36}$$

again with the caveat that one should be careful in mixing the 2- and 4-component languages between the Euclidean and Lorentzian theories.

IV. THE STANDARD MODEL

We have all the ingredients we need to construct the SM in flat spacetime from an originally $SO(4)$ Euclidean theory. The SM contains the gauge field strength terms for each group, kinetic terms for each matter field, and Yukawa terms coupling the Higgs field to the matter fields to give mass terms from the Higgs mechanism. A key structure is that the weak gauge group, $SU(2)_L$, acts only on left-handed fields. It is this chiral structure of the weak force which requires the Yukawa interactions with the Higgs field (or some other mechanism entirely) in order for the fermions to have mass.

We have already seen how to construct kinetic terms (and gauge field strengths) which flow from the Euclidean

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[7]The sign is automatic from the left-handed field, or through a field rotation for the right-handed field (the sign of the Majorana mass term can be changed freely).

theory to the proper terms with a Lorentzian signature for all of the fields we need. Let us now consider the necessary Yukawa interaction terms between the Higgs and fermion matter fields. These terms do not change form as the background metric changes, and we can use the usual terms in the $SO(4)$ theory.

As we must treat left- and right-handed fields differently under the weak force, we rely on the Weyl spinors (or projections) we constructed earlier. A common simplification is to write the SM Lagrangian purely in terms of left-handed fields. In this form, the right-handed fields which do not couple to the weak force are written as antifermions of a new species of left-handed fermions. For instance, for the up and down quarks, the left-handed $SU(2)_L$ doublet is Q, and the right-handed $SU(2)_L$ singlets are $\bar{u}_R$ and $\bar{d}_R$, with the bar purely part of the name. We then use their left-handed antiparticles, $\bar{u}, \bar{d}$, in writing a Lagrangian.

Yukawa terms in the Euclidean theory then look just like in the SM. For example, for the first generation of quarks [with H the Higgs scalar $SU(2)_L$ doublet],

$$Q_E H_E \bar{d}_E + Q_E \epsilon H^\dagger \bar{u}_E + (\text{h.c.}), \tag{37}$$

where Q_E, H_E, $\bar{u}_E$, and $\bar{d}_E$ are all Euclidean Weyl spinors with γ_E^5 eigenvalue -1, and all indices are suppressed [the ϵ tensor combines Q_E and $H_E^\dagger$ antisymmetrically in $SU(2)_L$ indices]. Once we go to the Lorentzian theory, the Euclidean Weyl spinors become the left-handed projections of the SM fields, and we have exactly the SM. The leptons and other generations all follow in the same way.

One thing to note is how the right-handed terms are generated in terms of these left-handed fields. In the Lorentzian theory, the conjugate of a left-handed field is right handed, and vice versa. We do *not* have this group structure in the Euclidean theory. Thus, when we write the Hermitian conjugate terms in the $SO(4)$ theory, they are still fields with γ_E^5 eigenvalue -1. Once we are in the Lorentzian theory, however, Weyl spinors are not self-conjugate, and the Hermitian conjugate terms are right handed. After the Higgs mechanism, the fermions are all (except for the neutrino) paired up into Dirac mass terms, which mix the left- and right-handed components.

V. CONSTRAINTS IN QED

As was remarked in Ref. [11], there is a tuning necessary in the couplings to the clock field to reach the standard Lorentzian theory. In this section, we will restrict ourselves to the QED sector and examine the constraints on these terms by using the work summarized in Refs. [25,26] (see references therein for details on the parameterization of operators and the relevant experimental results). We will work in flat (Minkowski) space with a single fermion flavor (the electron/positron); some constraints may change in more general settings.

We will make a connection from our model to the parameterization of Lorentz-violating operators in the Standard Model Extension (SME) used in Refs. [25,26]. The SME encapsulates the minimal set of dimension-3 and dimension-4 *CPT*- and Lorentz-violating operators and constructs observables which can be constrained by experiment. The general Minkowski space QED Lagrangian (with electromagnetic tensor $F_{\mu\nu}$ and fermion ψ) in the SME is

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\Gamma_\nu\overleftrightarrow{\partial}^\nu\psi - \bar{\psi}M\psi - \frac{1}{4}K_{\mu\nu}F^{\mu\nu}, \qquad (38)$$

with

$$\Gamma_\nu \equiv \gamma_\nu + c_{\mu\nu}\gamma^\mu + d_{\mu\nu}\gamma^5\gamma^\mu + e_\nu$$
$$+ if_\nu\gamma^5 + \frac{1}{2}g_{\lambda\mu\nu}\Sigma^{\lambda\mu}, \qquad (39)$$

$$M \equiv m + a_\mu\gamma^\mu + b_\mu\gamma^5\gamma^\mu + \frac{1}{2}H_{\mu\nu}\Sigma^{\mu\nu}, \qquad (40)$$

$$K_{\mu\nu} \equiv F_{\mu\nu} - 2(k_{AF})^\kappa\epsilon_{\kappa\lambda\mu\nu}A^\lambda + (k_F)_{\kappa\lambda\mu\nu}F^{\kappa\lambda}, \qquad (41)$$

where $\frac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \psi^\nu]$, and all γ^μ are in Minkowski space. The observables are combinations of the free parameters a_μ, b_μ, $c_{\mu\nu}$, $d_{\mu\nu}$, e_ν, f_ν, $g_{\lambda\mu\nu}$, $H_{\mu\nu}$, $(k_{AF})^\kappa$, and $(k_F)_{\kappa\lambda\mu\nu}$ (see Refs. [25,26] for the precise definitions and counting of independent parameters and observable combinations).

Let us start with the photon. We can parameterize any deviation from the interaction term with the clock field, which leads to the Lorentzian theory as

$$\frac{4}{M^4}(1 + \epsilon_A)F_E^{\mu\rho}F_{E\rho}^\nu\partial_\mu\phi\partial_\nu\phi, \qquad (42)$$

with ϵ_A being the deviation from Eq. (6). In the Lorentzian theory, then, we end up with the additional term

$$\epsilon_A\delta_{ij}F^{0i}F^{0j}. \qquad (43)$$

This is a *CPT*-even operator, which violates Lorentz invariance, corresponding to the SME parameter $(k_F)_{\kappa\lambda\mu\nu}$ in Eq. (41): it is constrained to have $|\epsilon_A| < \mathcal{O}(10^{-32})$ (cf. the observables $\tilde{\kappa}$, in particular the component $\tilde{\kappa}_{e+}^{ZZ}$ in Ref. [25]).

In the matter sector, we parametrize a deviation from Eq. (19), which gives the proper Minkowski Lagrangian in $\mathcal{M}_0$, with the parameters $\epsilon_{\psi_{1,2}}$ as

$$\frac{1}{2M^2}\delta^{\mu\nu}[(1 + \epsilon_{\psi_1})(i\bar{\psi}_E\gamma_E^5\overleftrightarrow{\partial}_\mu\psi)$$
$$- (1 + \epsilon_{\psi_2})(i\bar{\psi}\gamma_E^\rho\overleftrightarrow{\partial}_\mu\psi)\partial_\rho\phi]\partial_\nu\phi. \qquad (44)$$

We then have the following Lorentz-violating operators in the theory in Minkowski space, the first of which is *CPT* even, the second *CPT* odd:

$$\frac{i}{2}\epsilon_{\psi_1}(\bar{\psi}\gamma_M^0\overleftrightarrow{\partial}_0\psi) + \frac{1}{2}\epsilon_{\psi_2}(\bar{\psi}\gamma_M^5\overleftrightarrow{\partial}_0\psi). \qquad (45)$$

However, through a field redefinition, this second operator (f_0 in the SME above) can actually be removed at leading order (in ϵ_{ψ_2}) and absorbed into ϵ_{ψ_1} at second order (see the discussion in Refs. [25,26] and references therein). Thus, we do not have *CPT* violation, regardless of the precise tuning, contrary to what was stated originally in Ref. [11]. The *CPT*-even operator must have a coefficient $|\epsilon_{\psi_1}| < \mathcal{O}(10^{-15})$ (corresponding to $\tilde{c}_{TT}/m_e$ in Ref. [25]), and this gives a constraint, through field redefinition, of $|\epsilon_{\psi_2}| < \mathcal{O}(10^{-7})$.

We have seen that there is a precise tuning in the couplings of the SM fields to the clock field needed to avoid Lorentz-violation constraints. Besides the tuning in these coefficients, there are other possible terms which can be dangerous, as noted in Ref. [11]. Of the ten terms which are scalars under $SO(4)$, Hermitian, and include at most one derivative acting on spinors, we have the usual mass and kinetic terms, and the two terms we have already included. There are four additional terms with couplings to the clock field:

$$(\bar{\psi}\gamma_E^\mu\psi)\partial_\mu\phi, \qquad (i\bar{\psi}\gamma_E^5\gamma_E^\mu\psi)\partial_\mu\phi,$$
$$\delta^{\mu\nu}(i\bar{\psi}\overleftrightarrow{\partial}_\mu\psi)\partial_\nu\phi, \qquad \delta^{\mu\nu}(i\bar{\psi}\gamma_E^5\gamma_E^\rho\overleftrightarrow{\partial}_\mu\psi)\partial_\rho\phi\partial_\nu\phi. \qquad (46)$$

The first term corresponds, in the Lorentzian theory, to a γ^5 mass term, which can be removed through a chiral transformation. The third term (corresponding to e_μ in the SME) is *CPT*- and Lorentz-violating, but it can also be removed by transformations and field redefinitions (it can be absorbed into a_μ and is not observable with a single flavor in flat space; see the summary in Ref. [26] and references therein). The second term (b_T in the SME) is *CPT* odd and Lorentz violating, constrained to be less than $\mathcal{O}(10^{-27}$ GeV) (see the combinations $\tilde{b}_T$ and $\tilde{g}_T$ in Ref. [25]). This is problematic, as the generated mass scale is presumably $\sim M$, and there does not appear to be a simple way to forbid such a term. Finally, the fourth term, which is *CPT* even and Lorentz violating, is constrained to be less than $\mathcal{O}(10^{-24})$ (constrained via the tracelessness of $d_{\mu\nu}$; see the observable $\tilde{d}_+$ in Ref. [25]). Again, there is not an obvious way to forbid such a term, and its pure number coefficient is a free parameter.

Finally, we also have two terms which do not involve interactions with the clock field. One is the standard γ_E^5 mass term, $\bar{\psi}\gamma_E^5\psi$, which we will transform away in the Euclidean theory (or it corresponds to the unobservable parameter a_0 in the SME). The second term is $\bar{\psi}\gamma_E^5\gamma_E^\mu\overleftrightarrow{\partial}_\mu\psi$,

which we can write in the Lorentzian theory as

$$\bar{\psi}(i\gamma_M^0\gamma_M^5\overleftrightarrow{\partial}_0 + \gamma_M^0\gamma_M^i\overleftrightarrow{\partial}_i)\psi$$
$$= -i\bar{\psi}(\gamma_M^5\gamma_M^0\overleftrightarrow{\partial}_0 + \Sigma^{0i}\overleftrightarrow{\partial}_i)\psi, \qquad (47)$$

where we have used that $\gamma_M^0\gamma_M^i = \frac{1}{2}[\gamma_M^0, \gamma_M^i] + \frac{1}{2}\{\gamma_M^0, \gamma_M^i\} = \frac{1}{2}[\gamma_M^0, \gamma_M^i]$. The first term in Eq. (47) is the same as the last term discussed in the previous paragraph, and thus it has the same constraint. In the SME, the second term is a component of the trace part of the parameter $g_{\mu\nu\lambda}$ (the coefficient of a *CPT*-odd and Lorentz-violating operator), $g_\mu^{(T)} \equiv g_{\mu\nu}^\nu$ (note that g_{000} does not contribute, since $\Sigma^{00} = 0$). This is not an observable component of g, as it can be removed through a field redefinition (see Ref. [26] and references therein).

VI. DISCUSSION, CONCLUSION, AND OUTLOOK

This article follows the idea that the apparent Lorentzian dynamics of usual field theories is an emergent property and that the underlying field theory is in fact strictly Riemannian. This requires the introduction of the clock field, a scalar field playing the role of the physical time. The microscopic theory is Euclidean, and time evolution is just an effective and emergent property, which holds on some energy scales, and in some regions of the Euclidean space. Through interactions with the clock field, the effective theory flows to the standard Lorentzian picture.

In Ref. [11], we were able to perform a construction in flat spacetime for scalar, vector, and Dirac spinors restricted to classical fields. In order for all the fields to propagate in the same emergent Lorentzian metric, the couplings to the clock field needed to be adjusted with care. This work was a proof of concept in constructing a model with the Lorentzian metric only emerging at energies below the VEV of the gradient of the clock field, with many open and interesting questions. In this work we have addressed several of these questions.

The present analysis has shown that it is possible to construct a Euclidean theory with fermions that reduce, once the gradient of the clock field has a VEV on $\mathcal{M}_0$, to Lorentzian Weyl and Majorana fermions. This completes the basic fields needed in the Standard Model and common extensions. The clock field allows us to avoid the typical difficulties in constructing Euclidean theories of these types of fermions. We then showed that it is possible to construct a Euclidean theory leading to an emergent version of the Standard Model by adding the Standard Model structure to the dynamics necessary for the emergence of a Lorentzian metric.

To finish, we have analyzed with care the fine-tuning required to ensure *CPT* and Lorentz invariance. One crucial point is that the terms necessary in our model do not induce *CPT* violation. Bounds on the deviations from

the adjusted couplings to the clock field, as well as other possible interaction terms in this framework, can be obtained from experimental QED constraints. Forbidding additional operators and ensuring the value of the necessary coupling constants is an open question.

There are still many interesting future directions to pursue in this framework for emergent Lorentz symmetry. One would like to move beyond the classical level and quantize the theory, as well as understand the mechanism which leads to the VEV of the clock field. The possible violation of *CPT* and Lorentz symmetry also needs to be investigated further. Even with these and other open questions, we now have a basic model which can reproduce the Standard Model and its Lorentzian background with time evolution from a purely Riemannian theory with no concept of time.

APPENDIX: GAMMA MATRICES AND FERMIONS

In this appendix, we will try to motivate some of the choices made in our Euclidean formulation of fermions. Our procedure will be to take the action proposed in the Euclidean theory as an ansatz and require that we end with a proper Lorentzian theory. This will then define the relationship between the representations of the gamma matrices (which will not be chosen *a priori*) and identifications between quantities in the two theories.

Let us start with a massless Dirac fermion in the $SO(4)$ theory, coupled to the clock field as in the action of Eq. (19),

$$S_\psi = \int d^4x \left\{ \bar{\psi}_E \left(\frac{i}{2}\gamma_E^\mu\overleftrightarrow{\partial}_\mu - m \right)\psi_E \right.$$
$$\left. + \frac{1}{2M^2}\delta^{\mu\nu}[(i\bar{\psi}_E\gamma_E^5\overleftrightarrow{\partial}_\mu\psi) - (i\bar{\psi}\gamma_E^\rho\overleftrightarrow{\partial}_\mu\psi)\partial_\rho\phi] \right\}, \qquad (A1)$$

but without assuming the form of $\bar{\psi}_E$ or γ_E^μ.

Although we can form an $SO(4)$ invariant with $\psi_E^\dagger\psi_E$, we wish to mirror the usual Lorentzian construction, so we have used $\bar{\psi}$. In order for this to transform as $\psi_E^\dagger$ (i.e., in the opposite way of ψ_E), any matrix we attach to $\psi_E^\dagger$ to form $\bar{\psi}_E$ must commute with the $SO(4)$ generators. Thus, we have

$$\bar\psi_E \equiv \psi_E^\dagger \gamma_E^5. \tag{A2}$$

After the clock field's derivative has a VEV M^2, chosen to define the t direction, the action becomes

$$\mathcal{S}_\psi \to \int \mathrm{d}t\mathrm{d}^3x(\bar\psi i\gamma_E^i\partial_i\psi) + i(\bar\psi\gamma_E^5\partial_0\psi). \tag{A3}$$

Since the clock field has now picked out a direction, morphing $SO(4)$ to $SO(3,1)$, we expect that we should now have a free fermion propagating in Minkowski space. We recover the usual action,

$$\mathcal{S}_M = \int \mathrm{d}t\mathrm{d}^3x i\bar\psi\gamma_M^\mu\partial_\mu\psi = i\bar\psi(\gamma_M^0\partial_0 + \gamma_M^i\partial_i)\psi, \tag{A4}$$

by identifying[8]

$$\bar\psi \equiv \psi^\dagger\gamma_E^5 \to \bar\psi_M \equiv \psi_M^\dagger\beta,$$
$$\gamma_E^i \to \gamma_M^i, \qquad \gamma_E^5 \to \gamma_M^0. \tag{A5}$$

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[8]Note that usually β and γ^0 are used interchangeably because they are numerically the same. However, at least in the chiral representation, the spin structure is different.

From these identifications and the definition of γ_E^5 we know that $\{\gamma_E^5,\gamma_M^i\} = 0$ and $[\gamma_E^5,\gamma_M^0] = 0$. Therefore, $\gamma_E^5\gamma_M^\mu(\gamma_E^5)^{-1} = (\gamma_M^\mu)^\dagger$, with γ_M^0 Hermitian and γ_M^i anti-Hermitian (from the definition of the Clifford algebra as $\{\gamma_M^\mu,\gamma_M^\nu\} = -2\eta^{\mu\nu}$). Combined with γ_E^5 being Hermitian, or by direct computation, we find that it has the right properties (see, e.g., Appendix G of Ref. [23]) to be the matrix β: $\bar\psi_M$ transforms oppositely of ψ_M such that $\bar\psi_M\psi_M$ is a (Hermitian) Lorentz scalar. Furthermore, the Clifford algebra for $SO(4)$ tells us that $(\gamma_E^0)^2 = -1$, and we chose an anti-Hermitian representation, $(\gamma_E^0)^\dagger = -\gamma_E^0$, such that the $SO(4)$ generators we defined were Hermitian. Since γ_E^0 anticommutes with all the γ_M^μ, this implies

$$\gamma_E^0 = i\gamma_M^5. \tag{A6}$$

All the above requirements are then consistent, coming from the proposed $SO(4)$ action. The gamma matrices all match what was given in Sec. II B.

For Weyl spinors (Euclidean spinor eigenstates of γ_E^5) we can follow the same procedure, and we find that we reach the Minkowski Weyl action with the same identification of gamma matrices, including that the spinor is now an eigenstate of γ_M^5.

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