Stability of chromomagnetic condensation and mass generation for confinement in SU(2) Yang-Mills theory

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(Received 28 November 2013; published 13 May 2014)

We show that the Nielsen-Olesen instability of the Savvidy vacuum with a homogeneous chromomagnetic condensation disappears in the framework of the functional renormalization group. This result follows from our observations: (i) the vanishing imaginary part of the effective average action is realized for arbitrary infrared cutoff as a novel fixed point solution of the flow equation for the complex-valued effective average action and (ii) an approximate analytical solution for the effective average action is obtained without the pure imaginary part for large infrared cutoff. This result suggests that there exists a physical mechanism for maintaining the stability or staying on the fixed point even for sufficiently small infrared cutoff. We argue that dynamical gluon mass generation (related to two-gluon bound states identified with glueballs) occurs due to the Becchi-Rouet-Stora-Tyutin-invariant vacuum condensate of mass dimension two without causing instability.

I. INTRODUCTION

The dual superconductor picture for the Yang-Mills theory [1] vacuum is an attractive hypothesis for explaining quark confinement. It has been intensively investigated as a promising mechanism for quark confinement up to today since the early proposal in the 1970s by Nambu, Mandelstam, 't Hooft, and Polyakov [2]. In an ordinary (type II) superconductor, electric charges condense into Cooper pairs. As a result, magnetic flux is squeezed into tubes. In the dual superconductor picture of the Yang-Mills theory vacuum, chromomagnetic monopoles are to be condensed into dual Cooper pairs and the chromoelectric flux connecting color charges is to be squeezed into tubes forming the hadron string. Then the nonzero string tension plays the role of the constant of proportionality in the linear potential realizing quark confinement. The key ingredients of this picture are the existence of chromomagnetic monopole condensation and the dual Meissner effect.

There are two methods available for defining the chromomagnetic monopole in the Yang-Mills theory:
(1) Abelian projection (by 't Hooft [3]): partial gauge fixing of the gauge group G to the maximal torus subgroup: $G \to U(1)^r$;
(2) field decomposition (by Cho [4], Duan, and Ge [5], Faddeev and Niemi [6], Shabanov [7], Kondo, Murakami, and Shinohara [8–10]): gauge-invariant decomposition of the gluon field for separating the dominant mode for confinement.
It is very important to answer the question of how to define the gauge-invariant chromomagnetic monopole in the Yang-Mills gauge theory without scalar fields, which should be discriminated from the 't Hooft-Polyakov magnetic

monopole in the Georgi-Glashow model. However, the details of this issue will be discussed elsewhere, since it is not the main issue to be discussed in this paper.

For the dual superconductor picture for the Yang-Mills theory vacuum to be true, the chromomagnetic monopole condensation must give a more stable vacuum than the perturbative one. In view of this, Savvidy [11] has argued based on the general analysis of the renormalization group equation that the dynamical generation of the chromomagnetic field should occur in the Yang-Mills theory, i.e., a non-Abelian gauge theory with asymptotic freedom. Indeed, Savvidy has shown that the vacuum with a non-vanishing homogeneous chromomagnetic field strength, the so-called Savvidy vacuum, has lower energy density than the perturbative vacuum with zero chromomagnetic field. The one-loop effective potential $V(H)$ of the homogeneous chromomagnetic field H obtained in [11] for $SU(2)$ Yang-Mills theory is

$$V_{\text{Savvidy}}(H) = \frac{1}{2}H^2 - \frac{\beta_0 g^2}{16\pi^2}\frac{1}{2}H^2\left(\ln\frac{gH}{\mu^2} + c\right), \quad (1)$$

with $\beta_0 := -\frac{22}{3} < 0$, and a constant c. Then the effective potential $V(H)$ has an absolute minimum at $H = H_0 \neq 0$ away from $H = 0$.

Immediately after his proposal, however, N.K.Nielsen and Olesen [12] have pointed out that the effective potential $V(H)$ of the homogeneous chromomagnetic field H, when calculated carefully at one-loop level in the perturbation theory under the background gauge, develops an imaginary part in the one-loop effective potential:

$$V_{\text{NO}}(H) = \frac{1}{2}H^2 - \frac{\beta_0 g^2}{16\pi^2}\frac{1}{2}H^2\left(\ln\frac{gH}{\mu^2} + c\right) + i\frac{g^2 H^2}{8\pi}, \quad (2)$$

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in addition to the real part which agrees exactly with the prediction of the renormalization group equation, i.e., the Savvidy's result. This is called the Nielsen-Olesen (NO) instability of the Savvidy vacuum. The presence of the pure imaginary part implies that the Savvidy vacuum gets unstable due to gluon–antigluon pair annihilation.

This result is easily understood based on the following observation. In the homogeneous external chromomagnetic field H, the energy eigenvalue E_n of the massless (*off-diagonal*) gluons with the spin $S = 1$ ($S_z = \pm 1$) is given by

$$E_n^\pm = \sqrt{p_\perp^2 + 2gH(n + 1/2) + 2gHS_z}$$
$$(n = 0, 1, 2, \cdots), \qquad (3)$$

where $p_\perp$ denotes the momentum in those space-time directions that are not affected by the magnetic field and the index n is a discrete quantum number that labels the Landau levels. Then the NO instability is understood as originating from the tachyon mode with $n = 0$ and $S_z = -1$ (or the lowest Landau level for the gluon with spin one antiparallel to the external chromomagnetic field), since

$$E_0^- = \sqrt{p_\perp^2 - gH} \qquad (4)$$

becomes pure imaginary when $p_\perp^2 < gH$. In other words, the NO instability of the Savvidy vacuum with homogeneous chromomagnetic condensation is due to the existence of the tachyon mode corresponding to the lowest Landau level which is realized in the homogeneous chromomagnetic field.

It is instructive to compare the Yang-Mills theory with QED to understand the NO instability correctly, since the instability of QED (an Abelian gauge theory without asymptotic freedom) under the applied electric field is well known. In QED, the nonzero magnetic field does not lower the vacuum energy and hence no magnetic condensation occurs, while the electric field causes electron–positron pair creation, destabilizing the QED vacuum. The chromoelectric field destabilizes the vacuum also in Yang-Mills theory. Therefore, the instability of the Yang-Mills vacuum under the chromomagnetic field is quite different from the instability of QED.

Inside each such domain there is a nontrivial chromomagnetic field and the tachyon mode does not appear in the domain supporting $p_\perp^2 > gH$. A way to circumvent the NO instability is to introduce the magnetic domains (domain structure) with a finite extension into the vacuum [13]. The physical vacuum in Yang-Mills theory is split into an infinite number of domains with macroscopic extensions. Inside each such domain there is a nontrivial configuration of the chromomagnetic field and the tachyon mode does not appear in the domain supporting $p_\perp^2 > gH$. This resolution for the NO instability of Yang-Mills theory is called the Copenhagen vacuum or spaghetti vacuum.

What type of vacuum is allowed and preferred in the Yang-Mills theory is an important question related to the physical picture of quark confinement. We can say that the NO instability is an infrared problem in the non-Abelian gauge theory. The domain structure introduces an infrared cutoff that prevents the momenta from taking the smaller values causing the instability. However, it is quite complicated to work out the dynamics of the Yang-Mills theory on the concrete inhomogeneous background. Therefore, there has been a lot of work trying to overcome the NO instability for the homogeneous chromomagnetic field [14–33].

In view of these, we reexamine the NO instability in the $SU(2)$ Yang-Mills theory[1] in the framework of the functional renormalization group (FRG) [35,36] as a realization of the Wilsonian renormalization group [37]. The FRG enables us to examine the effects caused by changing the infrared cutoff in a systematic way.

In this paper we follow the methods developed for FRG in [38–41]. We point out the following results.

(1) The Nielsen-Olesen instability in the effective potential $V(H)$ for the homogeneous chromomagnetic field H, i.e., the imaginary part Im $V(H)$ of $V(H)$, disappears (or is absent from the beginning) in the framework of the FRG. (Therefore, the Nielsen-Olesen instability is an artifact of the one-loop calculation in the perturbation theory and it disappears in the nonperturbative framework beyond the perturbation theory.)

(2) However, this result does not necessarily guarantee the automatic existence of the nontrivial homogeneous chromomagnetic field $H_0 \neq 0$ as the minimum of the effective potential $V(H)$, such that

$$V(H_0) < V(H = 0) = 0. \qquad (5)$$

(Therefore, the absence of the Nielsen-Olesen instability and the existence of the nontrivial minimum for the homogeneous chromomagnetic field in the effective potential are different problems to be considered independently.)[2]

(3) As a physical mechanism for maintaining the stability even for the small infrared cutoff, we propose

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[1] In order to study the true QCD vacuum, we must discuss the $SU(3)$ gauge group. However, the $SU(3)$ case is more difficult from a technical viewpoint than the $SU(2)$ case. In this paper, therefore, we discuss the $SU(2)$ toy model and postpone the physical $SU(3)$ case in a subsequent paper. It should be remarked that the physically interesting case of the $SU(3)$ gauge group cannot be obtained by a simple group-theoretical extension of the $SU(2)$ case and that the different results could be obtained in the case of $SU(3)$, which is suggested from a formal consideration [34].

[2] This question was studied in [41] giving the answer in the affirmative by using the self-dual background, which does not suffer from the instability from the beginning. This work is quite interesting but does not answer other questions raised here for other choices of the background. See Conclusion and Discussion.

the dynamical mass generation for the off-diagonal gluons (and off-diagonal ghosts), which is related to the Becchi-Rouet-Stora-Tyutin (BRST)-invariant vacuum condensation of mass-dimension two [42–45]. This gives a consistent picture compatible with the absence of the instability. (This leads to the Abelian dominance [3,46]: in the string tension extracted from the Wilson loop average [47] and exponential falloff of the off-diagonal gluon propagators [48–51] as well as the magnetic monopole dominance [52] in the Maximal Abelian gauge [53].)

This paper is organized as follows. In Sec. II, we consider the complex-valued flow equation for the effective average action and decompose it into the real and imaginary parts. We show that the flow equation has a solution with vanishing imaginary part of the effective average action for any value of the infrared cutoff Λ, corresponding to the fixed point.

In Sec. III, we derive explicitly the flow equation for the effective average action in the chromomagnetic background. As an infrared regulator, we use the mass type infrared cutoff function to give a closed form for the flow equation. By removing the ultraviolet divergence due to this choice of the infrared function, we obtain a flow equation with the infrared cutoff that is free from the ultraviolet divergence.

In Sec. IV, we show the absence of the NO instability, i.e., vanishing of the imaginary part of the effective average action. This is done based on an approximate solution obtained by solving the flow equation for large values of the infrared cutoff Λ.

In Sec. V, we discuss the mass generation for the off-diagonal gluons and ghosts due to the vacuum condensation of mass-dimension two, which is BRST invariant. Moreover, we argue the relationship between stability and mass generation.

The final section is devoted to conclusions and discussions.

II. COMPLEX-VALUED FLOW EQUATION

The effective average action Γ_Λ with the infrared cutoff Λ is obtained by solving the flow equation [35]

$$\partial_t \Gamma_\Lambda = \frac{1}{2} \mathrm{STr}[(\Gamma_\Lambda^{(2)} + R_\Lambda)^{-1} \cdot \partial_t R_\Lambda], \quad \partial_t := \Lambda \frac{d}{d\Lambda}, \quad (6)$$

where STr denotes the "supertrace" introduced for writing both commuting fields (e.g., gluons) and anticommuting fields (e.g., quarks and the Faddeev-Popov ghosts), R_Λ^Φ is the infrared cutoff function for the field Φ introduced as the infrared regulator term in the form

$$\int \Phi^\dagger R_\Lambda^\Phi \Phi, \quad (7)$$

and $\Gamma_\Lambda^{(2)}$ denotes the second derivatives of Γ_Λ with respect to the field variables Φ,
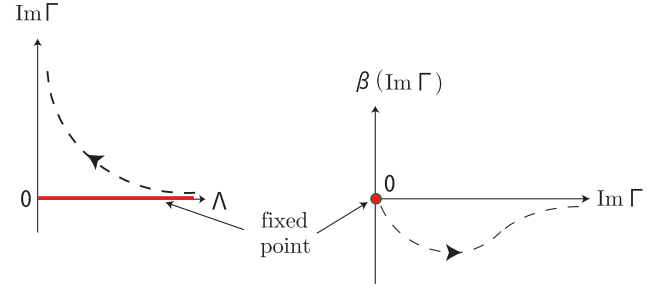

FIG. 1 (color online). Left: imaginary part $\mathrm{Im}\Gamma_\Lambda$ of the average effective action as a function of Λ. Right: β function as a function of $\mathrm{Im}\Gamma_\Lambda$. Here $\mathrm{Im}\Gamma_\Lambda = 0$ corresponds to a fixed point of the renormalization group.

$$(\Gamma_\Lambda^{(2)})_{\Phi^\dagger \Phi} = \frac{\vec{\delta}}{\delta\Phi^\dagger} \Gamma_\Lambda \frac{\overleftarrow{\delta}}{\delta\Phi}, \quad (8)$$

corresponding to the inverse exact propagator at the scale Λ. The ordinary effective action Γ as the generating functional of the one-particle irreducible vertex functions is obtained in the limit $\Lambda \downarrow 0$: $\Gamma = \lim_{\Lambda\downarrow 0}\Gamma_\Lambda$.

We consider the complex-valued effective average action $\Gamma_\Lambda = \Gamma_\Lambda^R + i\Gamma_\Lambda^I$ which is decomposed into the real part $\Gamma_\Lambda^R := \mathrm{Re}\Gamma_\Lambda$ and the imaginary part $\Gamma_\Lambda^I := \mathrm{Im}\Gamma_\Lambda$. Then it is shown (see Appendix A) that the flow equation is decomposed into two parts:

$$\partial_t \Gamma_\Lambda^R = \frac{1}{2} \mathrm{STr}\{[(\Gamma_\Lambda^{R(2)} + R_\Lambda)^2 + (\Gamma_\Lambda^{I(2)})^2]^{-1}$$
$$\times (\Gamma_\Lambda^{R(2)} + R_\Lambda)\partial_t R_\Lambda\}, \quad (9)$$

$$\partial_t \Gamma_\Lambda^I = -\frac{1}{2} \mathrm{STr}\{[(\Gamma_\Lambda^{R(2)} + R_\Lambda)^2 + (\Gamma_\Lambda^{I(2)})^2]^{-1}\Gamma_\Lambda^{I(2)}\partial_t R_\Lambda\}. \quad (10)$$

We find that the flow equation has a remarkable property: the identically vanishing imaginary part $\Gamma_\Lambda^I := \mathrm{Im}\Gamma_\Lambda \equiv 0$ is an exact solution corresponding to a "fixed point":

$$\mathrm{Im}\Gamma_\Lambda \equiv 0 \quad \text{for any value of } \Lambda, \quad (11)$$

in sharp contrast with the real part. See Fig. 1 for the behavior of the "beta" function of Γ_Λ^I defined by

$$\beta(\Gamma_\Lambda^I) := \partial_t \Gamma_\Lambda^I. \quad (12)$$

If $\Gamma_\Lambda^I \neq 0$ for a certain value of Λ, it does not maintain the same value, i.e., $\beta(\Gamma_\Lambda^I) \neq 0$.[3]

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[3]If the right-hand side of the flow equation for Γ_Λ^I was linear in Γ_Λ^I, a stronger statement would be derived: If Γ_Λ^I vanishes for some particular value of Λ, e.g., for $\Lambda = \Lambda_0$, then Γ_Λ^I vanishes automatically for any other value of $\Lambda \leq \Lambda_0$, in particular, for $\Lambda \to 0$, provided that Γ_Λ and Γ_{Λ_0} are related by integrating (or solving) the flow equation:

$$\Gamma_\Lambda^I = 0 \quad \text{at } \Lambda = \Lambda_0 \Rightarrow \Gamma_\Lambda^I = 0 \quad \text{for } \Lambda < \Lambda_0. \quad (13)$$

Thus the problem of showing the absence of the imaginary part $\text{Im}\Gamma = \lim_{\Lambda\downarrow0}\text{Im}\Gamma_\Lambda$ in the effective action $\Gamma = \lim_{\Lambda\downarrow0}\Gamma_\Lambda$ is reduced to proving the vanishing of the imaginary part $\text{Im}\Gamma_\Lambda$ in the effective average action Γ_Λ for a sufficiently large value of Λ:

$$\text{Im}\Gamma_\Lambda = 0 \quad \text{for a certain value of } \Lambda \gg 1. \quad (14)$$

When $\Gamma_\Lambda^{\text{I}} = 0$, the flow equation for $\Gamma_\Lambda^{\text{R}}$ turns into the standard flow equation.

III. FLOW EQUATION IN THE CHROMOMAGNETIC BACKGROUND

We consider the D-dimensional Euclidean Yang-Mills theory. We decompose the $SU(2)$ Yang-Mills field $\mathscr{A}_\mu = \mathscr{A}_\mu^A T^A$ into the background field $\mathscr{V}_\mu = \mathscr{V}_\mu^A T^A$ and the quantum fluctuation field $\mathscr{X}_\mu = \mathscr{X}_\mu^A T^A$ where $T^A = \frac{1}{2}\sigma^A$ with σ^A being the Pauli matrices ($A = 1, 2, 3$):

$$\mathscr{A}_\mu^A = \mathscr{V}_\mu^A + \mathscr{X}_\mu^A (A = 1, 2, 3). \quad (15)$$

We can choose without loss of generality [8] the diagonal field V_μ as the background field,

$$\mathscr{V}_\mu^A(x) = \delta^{A3} V_\mu(x), \quad (16)$$

and the off-diagonal field A_μ^a ($a = 1, 2$) as the quantum fluctuation field,

$$\mathscr{X}_\mu^A(x) = \delta^{Aa} A_\mu^a(x), \quad (a = 1, 2), \quad (17)$$

which means that

$$\mathscr{A}_\mu = V_\mu \frac{\sigma^3}{2} + A_\mu^a \frac{\sigma^a}{2} \quad \text{or} \quad \mathscr{A}_\mu^3 = V_\mu,$$
$$\mathscr{A}_\mu^a = A_\mu^a \quad (a = 1, 2). \quad (18)$$

For the diagonal gauge field V and the off-diagonal gauge field A, the Yang-Mills Lagrangian has the interaction terms of the type: VAA, $VVAA$, and $AAAA$, while the gauge-fixing (GF) and the associated Faddeev-Popov (FP) term for the maximal Abelian (MA) gauge (defined shortly) has the interactions of the type: VAA and $VVAA$. The effective potential $\Gamma(V)$ of V is obtained from the diagrams with the external legs of V by integrating all the internal lines that are connected through the possible interaction vertices. For the one-loop effective potential, accordingly, it is easy to see that only the internal lines of A are allowed, which implies that there is no fluctuating diagonal field A to be integrated out. For large Λ, the deviation from the one loop is not

so significant and the fluctuating diagonal field can be neglected.[4]

In what follows, we prepare the diagonal field $V_\mu(x)$ of the form

$$V_\mu(x) = \frac{1}{2} x_\nu H_{\nu\mu}, \quad (19)$$

so that the x-independent homogeneous background field strength is realized:

$$\mathscr{F}_{\mu\nu}^A[\mathscr{V}](x) := \partial_\mu \mathscr{V}_\nu^A(x) - \partial_\nu \mathscr{V}_\mu^A(x) + \epsilon^{ABC}\mathscr{V}_\mu^B(x)\mathscr{V}_\nu^C(x)$$
$$= \delta^{A3}(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)) = \delta^{A3}H_{\mu\nu}. \quad (20)$$

Then the background field strength realizes the (homogeneous) chromomagnetic field,

$$\boldsymbol{H} = (H_1, H_2, H_3), \quad (21)$$

by choosing the nonvanishing components as

$$H_{23} = -H_{32} := H_1, \qquad H_{31} = -H_{13} := H_2,$$
$$H_{12} = -H_{21} := H_3. \quad (22)$$

The total effective average action Γ_Λ is specified by giving the gauge-invariant part $\Gamma_\Lambda^{\text{inv}}$, the GF part $\Gamma_\Lambda^{\text{GF}}$, and the associated FP ghost part $\Gamma_\Lambda^{\text{FP}}$:

$$\Gamma_\Lambda = \Gamma_\Lambda^{\text{inv}} + \Gamma_\Lambda^{\text{GF}} + \Gamma_\Lambda^{\text{FP}}. \quad (23)$$

We choose the background gauge [56] as the gauge-fixing condition to maintain the gauge invariance for the background field. In the above choice for the background field (16), the background gauge reduces to the maximal Abelian (MA) gauge:

$$F^a := \mathscr{D}_\mu^{ab}[V]A_\mu^b = 0, \quad \mathscr{D}_\mu^{ab}[V] := \partial_\mu \delta^{ab} - g\epsilon^{ab3}V_\mu. \quad (24)$$

Then the gauge-fixing term is given by

$$\Gamma^{\text{GF}} = \int d^D x \frac{1}{2\alpha}(\mathscr{D}_\mu^{ab}[V]A_\mu^b)^2, \quad (25)$$

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[4]Such a distinction between the diagonal and off-diagonal fields can be partially justified and has been used so far based on the results of numerical simulations of $SU(2)$ Yang-Mills theory on a lattice. Beyond one loop, of course, this simplification is not allowed and we must integrate out the diagonal fluctuation field. In fact, in order to show the confinement/deconfinement transition at finite temperature, the diagonal fluctuations play the most important and essential role, as first shown in [54] and confirmed in [55] using the same framework as that of this paper. Such a contribution will be included in the subsequent work where the interplay between the existence/nonexistence of chromomagnetic condensation and confinement/deconfinement at finite temperature will be investigated. In view of these, this paper is the first attempt towards the thorough analysis of the stability of the chromomagnetic condensation in the QCD vacuum.

where α denotes the gauge-fixing parameter. This Γ_{GF} is obtained by integrating out the Nakanishi-Lautrup field N^a from

$$\Gamma^{\text{GF}} = \int d^D x \left\{ N^a (\mathscr{D}_\mu^{ab}[V]A_\mu^b) + \frac{\alpha}{2} N^a N^a \right\}. \quad (26)$$

The FP ghost term is determined according to the standard procedure (see e.g., [57]) as

$$\Gamma^{\text{FP}} = \int d^D x \{ i\bar{C}^a \mathscr{D}_\mu^{ab}[V] \mathscr{D}_\mu^{bc}[V] C^c$$
$$- g^2 \epsilon^{ab3} \epsilon^{cd3} i\bar{C}^a C^d A_\mu^b A_\mu^c + i\bar{C}^a g \epsilon^{ab3} (\mathscr{D}_\mu^{bc}[V]A_\mu^c) C^3 \}. \quad (27)$$

For the gauge-invariant part $\Gamma_\Lambda^{\text{inv}}$, we adopt the ansatz, a function W_Λ of the gauge-invariant term Θ constructed from the field strength $\mathscr{F}_{\mu\nu}^A[\mathscr{A}] := \partial_\mu \mathscr{A}_\nu^A - \partial_\nu \mathscr{A}_\mu^A + \epsilon^{ABC} \mathscr{A}_\mu^B \mathscr{A}_\nu^C$:

$$\Gamma_\Lambda^{\text{inv}} = \int d^D x W_\Lambda(\Theta(x)), \qquad \Theta := \frac{1}{4} (\mathscr{F}_{\mu\nu}^A[\mathscr{A}])^2. \quad (28)$$

Θ is decomposed as [55]

$$\Theta = \frac{1}{4} (\mathscr{F}_{\mu\nu}^A[\mathscr{V}])^2 + \frac{1}{2} A^{\mu a} (Q_{\mu\nu}^{ab} + \mathscr{D}_\mu^{ac}[V] \mathscr{D}_\nu^{cb}[V]) A^{\nu b}$$
$$+ \frac{1}{4} (\epsilon^{3ab} A_\mu^a A_\nu^b)^2, \quad (29)$$

where

$$Q_{\mu\nu}^{ab} := -(\mathscr{D}^2)^{ab} \delta_{\mu\nu} + 2g\epsilon^{ab} H_{\mu\nu},$$
$$(\mathscr{D}^2)^{ab} := \mathscr{D}_\rho^{ac}[V] \mathscr{D}_\rho^{cb}[V]. \quad (30)$$

In the vanishing off-diagonal field limit $A_\mu^a \to 0$, Θ is reduced to

$$\Theta|_{A=0} = \frac{1}{4} (\mathscr{F}_{\mu\nu}^A[\mathscr{V}](x))^2 = \frac{1}{4} (\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x))^2$$
$$= \frac{1}{2} H^2, \quad (31)$$

$$H := \sqrt{\boldsymbol{H}^2} = \sqrt{\frac{1}{2} H_{\alpha\beta} H_{\alpha\beta}} > 0. \quad (32)$$

The off-diagonal gluon fields A_μ^a (and off-diagonal ghost fields C^a, $\bar{C}^a$) should be integrated out in the framework of the FRG following the idea of the Wilsonian renormalization group. For this purpose, we introduce the infrared regulator term ΔS_Λ for the off-diagonal gluon A_μ^a and off-diagonal ghosts C^a, $\bar{C}^a$ by

$$\Delta S_\Lambda = \int_p \left[\frac{1}{2} A_\mu^a(p) R_{\Lambda,\mu\nu}(p^2) \delta^{ab} A_\nu^b(p) \right.$$
$$\left. + \bar{C}^a(p) R_\Lambda(p^2) \delta^{ab} C^b(-p) \right] \quad (a,b=1,2), \quad (33)$$

where

$$\int_p := \int \frac{d^D p}{(2\pi)^D} \quad (34)$$

denotes the integration over the D-dimensional momentum space. We choose the infrared cutoff function with the structure:

$$R_{\Lambda,\mu\nu}(p^2) = \delta_{\mu\nu} R_\Lambda(p^2). \quad (35)$$

We adopt the proper-time form of the flow equation [58]:

$$\partial_t \Gamma_\Lambda = \int_0^\infty d\tau \frac{1}{2} \text{STr}[e^{-\tau(\Gamma_\Lambda^{(2)} + R_\Lambda)} \partial_t R_\Lambda]. \quad (36)$$

After performing the mode decomposition according to the projection method [38–41], the flow equation reads

$$\partial_t \Gamma_\Lambda = \frac{1}{2} \int_0^\infty d\tau \Omega^{-1} \text{Tr}[e^{-\tau(W_\Lambda' Q + R_\Lambda^{\text{gluon}})} \cdot \partial_t R_\Lambda^{\text{gluon}}]$$
$$- \frac{1}{2} \int_0^\infty d\tau \Omega^{-1} \text{Tr}[e^{-\tau(-W_\Lambda' \mathscr{D}^2 + R_\Lambda^{\text{gluon}})} \cdot \partial_t R_\Lambda^{\text{gluon}}]$$
$$+ \frac{1}{2} \int_0^\infty d\tau \Omega^{-1} \text{Tr}[e^{-\tau(-\alpha_\Lambda^{-1} \mathscr{D}^2 + R_\Lambda^{\text{gluon}})} \cdot \partial_t R_\Lambda^{\text{gluon}}]$$
$$- \int_0^\infty d\tau \Omega^{-1} \text{Tr}[e^{-\tau(-\tilde{Z}_\Lambda \mathscr{D}^2 + R_\Lambda^{\text{ghost}})} \cdot \partial_t R_\Lambda^{\text{ghost}}], \quad (37)$$

where we have defined $W_\Lambda'(\Theta) := \frac{d}{d\Theta} W_\Lambda(\Theta)$. Here we have introduced the wave function renormalization constants:

$$Z_\Lambda = Z_\Lambda^{\text{gluon}}, \qquad \bar{Z}_\Lambda = Z_\Lambda^{\text{ghost}}. \quad (38)$$

In this derivation, we have adopted the truncation—neglecting the four-point interactions among the off-diagonal gluons and off-diagonal ghosts $-g^2 \epsilon^{ab3} \epsilon^{cd3} i\bar{C}^a C^d A_\mu^b A_\mu^c$, which do not couple to the background field V_μ.

The spectrum sum is obtained from eigenvalues of the respective operator. The covariant Laplacian $-(\mathscr{D}_\rho[\mathscr{V}])^2$ with the background field $\mathscr{V}$ which gives the (covariant constant) uniform chromomagnetic field H has the spectrum

$$\text{Spect}[-\mathscr{D}_\rho^2[\mathscr{V}]] = p_\perp^2 + (2n+1)gH, \qquad (n = 0, 1, \cdots), \quad (39)$$

where $p_\perp$ denotes the $(D-2)$ dimensional (Fourier) momentum in those space-time directions that are not affected by the magnetic field (say, orthogonal to the 1–2 plane) and the index n is a discrete quantum number that labels the Landau levels. We take into account the fact that the density of states is $\frac{gH}{2\pi}$ for the Landau levels.

Moreover, the operator $Q_{\mu\nu}^{ab}$ with the same background field $\mathcal{V}$ has the spectrum

$$\text{Spect}[Q_{\mu\nu}^{ab}] = \begin{cases} p_\perp^2 + (2n+1)gH & (D-2) \\ p_\perp^2 + (2n+3)gH \text{ multiplicity} & 1 \\ p_\perp^2 + (2n-1)gH & 1 \end{cases}$$
$$(n = 0, 1, \cdots), \tag{40}$$

where the last term contains the Nielsen-Olesen unstable mode for $n = 0$, i.e.,

$$p_\perp^2 - gH, \tag{41}$$

which becomes a tachyonic mode for small momenta $p_\perp^2 < gH$.

The respective trace without the infrared regulator R_Λ is easily obtained [38–40]:

$$\Omega^{-1}\,\text{Tr}[e^{-\tau(W'_\Lambda Q)}] = \frac{NgH}{(4\pi)^{\frac{D}{2}}}(\tau W'_\Lambda)^{1-\frac{D}{2}}\left[\frac{D}{\sinh(\tau W'_\Lambda gH)} + 4\sinh(\tau W'_\Lambda gH)\right]$$
$$= \frac{NgH}{(4\pi)^{\frac{D}{2}}}(\tau W'_\Lambda)^{1-\frac{D}{2}}\frac{2(D-2)e^{-\tau W'_\Lambda gH} + 2e^{-3\tau W'_\Lambda gH} + 2e^{\tau W'_\Lambda gH}}{1 - e^{-2\tau W'_\Lambda gH}},$$

$$\Omega^{-1}\,\text{Tr}[e^{-\tau(-W'_\Lambda \mathscr{D}^2)}] = \frac{NgH}{(4\pi)^{\frac{D}{2}}}(\tau W'_\Lambda)^{1-\frac{D}{2}}\left[\frac{1}{\sinh(\tau W'_\Lambda gH)}\right] = \frac{NgH}{(4\pi)^{\frac{D}{2}}}(\tau W'_\Lambda)^{1-\frac{D}{2}}\left[\frac{2e^{-\tau W'_\Lambda gH}}{1 - e^{-2\tau W'_\Lambda gH}}\right],$$

$$\Omega^{-1}\text{Tr}[e^{-\tau(-\tilde{Z}_\Lambda \mathscr{D}^2)}] = \frac{NgH}{(4\pi)^{\frac{D}{2}}}(\tau\tilde{Z}_\Lambda)^{1-\frac{D}{2}}\left[\frac{1}{\sinh(\tau\tilde{Z}_\Lambda gH)}\right] = \frac{NgH}{(4\pi)^{\frac{D}{2}}}(\tau\tilde{Z}_\Lambda)^{1-\frac{D}{2}}\left[\frac{2e^{-\tau\tilde{Z}_\Lambda gH}}{1 - e^{-2\tau\tilde{Z}_\Lambda gH}}\right], \tag{42}$$

where $N = 2$ for $SU(2)$.

In order to obtain the closed analytical form for the solution and to compare the FRG calculations with the loop calculations, we choose the momentum-independent infrared regular of the mass type

$$R_\Lambda^\Phi = Z_\Lambda^\Phi \Lambda^2, \tag{43}$$

where Z_Λ^Φ denotes the wave function normalization constant for the field Φ. We discuss later (in the end of Sec. IV) whether the result is independent of the choice of the infrared regulator or not.

For the infrared regulator of the mass type, the flow equation thus reads

$$\partial_t \Gamma_\Lambda = \frac{NgH}{(4\pi)^{\frac{D}{2}}}\left\{(W'_\Lambda)^{1-\frac{D}{2}}(2-\eta_\Lambda)Z_\Lambda\Lambda^2 \int_0^\infty d\tau\,\tau^{1-\frac{D}{2}}e^{-\tau Z_\Lambda\Lambda^2}\frac{(D-2)e^{-\tau W'_\Lambda gH} + e^{-3\tau W'_\Lambda gH} + e^{\tau W'_\Lambda gH}}{1 - e^{-2\tau W'_\Lambda gH}}\right.$$
$$-(W'_\Lambda)^{1-\frac{D}{2}}(2-\eta_\Lambda)Z_\Lambda\Lambda^2 \int_0^\infty d\tau\,\tau^{1-\frac{D}{2}}e^{-\tau Z_\Lambda\Lambda^2}\frac{e^{-\tau W'_\Lambda gH}}{1 - e^{-2\tau W'_\Lambda gH}} + \alpha_\Lambda^{\frac{D}{2}-1}(2-\eta_\Lambda)Z_\Lambda\Lambda^2 \int_0^\infty d\tau\,\tau^{1-\frac{D}{2}}e^{-\tau Z_\Lambda\Lambda^2}\frac{e^{-\tau\alpha_\Lambda^{-1}gH}}{1 - e^{-2\tau\alpha_\Lambda^{-1}gH}}$$
$$\left.-(\tilde{Z}_\Lambda)^{1-\frac{D}{2}}(2-\tilde{\eta}_\Lambda)\tilde{Z}_\Lambda\Lambda^2 \int_0^\infty d\tau\,\tau^{1-\frac{D}{2}}e^{-\tau\tilde{Z}_\Lambda\Lambda^2}\frac{2e^{-\tau\tilde{Z}_\Lambda gH}}{1 - e^{-2\tau\tilde{Z}_\Lambda gH}}\right\}, \tag{44}$$

where we have introduced the anomalous dimensions:

$$\eta_\Lambda := -\partial_t \ln Z_\Lambda = -Z_\Lambda^{-1}\partial_t Z_\Lambda,$$
$$\tilde{\eta}_\Lambda := -\partial_t \ln \tilde{Z}_\Lambda = -\tilde{Z}_\Lambda^{-1}\partial_t \tilde{Z}_\Lambda. \tag{45}$$

We find that the integral with respect to τ on the right-hand side of the flow equation is divergent at $D = 4$ in the $\tau = 0$ region, which is an ultraviolet divergence. This divergence is independent of the infrared divergence coming from the $\tau = \infty$ region due to the factor $e^{\tau W'_\Lambda gH}$ for which the Nielsen-Olesen instability is responsible.

This ultraviolet divergence is due to the fact that the momentum-independent infrared cutoff function of the mass type does not suppress the high-momentum modes. This aspect is a shortcoming of the mass-type infrared regulator. Other choices of the infrared regulator are discussed in the end of this section.

Therefore, we first remove the ultraviolet divergence at $\tau = 0$. This is done in the standard way by adopting the minimal subtraction, i.e., $\overline{MS}$ scheme. See Appendix B for the details. Thus we arrive at the flow equation without the ultraviolet divergence:

$$\partial_t \Gamma_\Lambda = \frac{N}{2}\frac{2gH}{(4\pi)^2}\left(-\ln\frac{2gH}{4\pi\mu^2}-\gamma\right)\left\{(W'_\Lambda)^{-1}(2-\eta_\Lambda)Z_\Lambda\Lambda^2\left[\zeta\left(0,\frac{1}{2}+\frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right)+\zeta\left(0,\frac{3}{2}+\frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right)+\zeta\left(0,-\frac{1}{2}+\frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right)\right]\right.$$
$$+\alpha_\Lambda(2-\eta_\Lambda)Z_\Lambda\Lambda^2\zeta\left(0,\frac{1}{2}+\frac{Z_\Lambda\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)-2(2-\tilde\eta_\Lambda)\Lambda^2\zeta\left(0,\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)\Bigg\}$$
$$+\frac{N}{2}\frac{2gH}{(4\pi)^2}\left\{(W'_\Lambda)^{-1}(2-\eta_\Lambda)Z_\Lambda\Lambda^2\left[\zeta^{(1,0)}\left(0,\frac{1}{2}+\frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right)+\zeta^{(1,0)}\left(0,\frac{3}{2}+\frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right)+\zeta^{(1,0)}\left(0,-\frac{1}{2}+\frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right)\right.\right.$$
$$\left.\left.-2\zeta\left(0,\frac{1}{2}+\frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right)\right]+\alpha_\Lambda(2-\eta_\Lambda)Z_\Lambda\Lambda^2\zeta^{(1,0)}\left(0,\frac{1}{2}+\frac{Z_\Lambda\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)-2(2-\tilde\eta_\Lambda)\Lambda^2\zeta^{(1,0)}\left(0,\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)\right\},\tag{46}$$

where $\zeta(z,\lambda)$ is the generalized Riemann ζ function or the Hurwitz ζ function defined by (B3) and its integral representation (B4), and its derivatives $\zeta^{(m,n)}(z,\lambda)$ are defined by (B10).

IV. ABSENCE OF THE INSTABILITY

For large Λ, we can take the approximation

$$\text{(i)}\ \ W_\Lambda(\Theta)=\Theta\Rightarrow W'_\Lambda(\Theta)\equiv 1$$
$$\Leftrightarrow Z_\Lambda\equiv 1\Rightarrow \eta_\Lambda\equiv 0,$$
$$\text{(ii)}\ \ \tilde Z_\Lambda\equiv 1\Rightarrow \tilde\eta_\Lambda\equiv 0,$$
$$\text{(iii)}\ \ \alpha_\Lambda\equiv\alpha_{\Lambda_{\mathrm{UV}}}=\text{const}\geq 0.\tag{47}$$

This is a good approximation for Γ_Λ at sufficiently large Λ. If we choose $\alpha_\Lambda^{-1}\equiv W'_\Lambda$, the second and third terms cancel on the right-hand side of the flow equation (44), which corresponds to the Feynman gauge. If we choose $\alpha_\Lambda\equiv 0$, the third term on the right-hand side of the flow equation (44) vanishes, which corresponds to the Landau gauge.

Then we obtain an approximate flow equation for large Λ:

$$\partial_t\Gamma_\Lambda=\frac{N}{2}\frac{2gH}{(4\pi)^2}\left(-\ln\frac{2gH}{4\pi\mu^2}-\gamma\right)2\Lambda^2\left\{\zeta\left(0,\frac{3}{2}+\frac{\Lambda^2}{2gH}\right)+\zeta\left(0,-\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)-\zeta\left(0,\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)+\alpha_\Lambda\zeta\left(0,\frac{1}{2}+\frac{\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)\right\}$$
$$+\frac{N}{2}\frac{2gH}{(4\pi)^2}2\Lambda^2\left\{\zeta^{(1,0)}\left(0,\frac{3}{2}+\frac{\Lambda^2}{2gH}\right)+\zeta^{(1,0)}\left(0,-\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)-2\zeta\left(0,\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)-\zeta^{(1,0)}\left(0,\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)\right.$$
$$\left.+\alpha_\Lambda\zeta^{(1,0)}\left(0,\frac{1}{2}+\frac{\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)\right\}.\tag{48}$$

Then the flow equation can be cast into the total derivative form:

$$\partial_t\Gamma_\Lambda=\partial_t\left\{\frac{N(2gH)^2}{2\ (4\pi)^2}\left(\ln\frac{2gH}{4\pi\mu^2}+\gamma\right)\left[\zeta\left(-1,\frac{3}{2}+\frac{\Lambda^2}{2gH}\right)+\zeta\left(-1,-\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)-\zeta\left(-1,\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)+\alpha_\Lambda\zeta\left(-1,\frac{1}{2}+\frac{\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)\right]\right.$$
$$-\frac{N(2gH)^2}{2\ (4\pi)^2}\left[\zeta^{(1,0)}\left(-1,\frac{3}{2}+\frac{\Lambda^2}{2gH}\right)+\zeta^{(1,0)}\left(-1,-\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)-2\zeta\left(-1,\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)-\zeta^{(1,0)}\left(-1,\frac{1}{2}+\frac{\Lambda^2}{2gH}\right)\right.$$
$$\left.\left.+\alpha_\Lambda\zeta^{(1,0)}\left(-1,\frac{1}{2}+\frac{\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)\right]\right\},\tag{49}$$

where we have used the relation following from the definition:

$$\zeta^{(m,1)}(z-1,\lambda)=\frac{\partial}{\partial\lambda}\zeta^{(m,0)}(z-1,\lambda)=-\zeta^{(m,0)}(z,\lambda),\tag{50}$$

which yields

$$\partial_t \zeta^{(m,0)}\left(-1, a + \frac{\Lambda^2}{2gH}\right) = \zeta^{(m,1)}\left(-1, a + \frac{\Lambda^2}{2gH}\right)\frac{\Lambda^2}{gH}$$

$$= -\zeta^{(m,0)}\left(0, a + \frac{\Lambda^2}{2gH}\right)\frac{\Lambda^2}{gH}. \quad (51)$$

We take into account the fact that the effective average action Γ_Λ at $\Lambda = \Lambda_{\rm UV} = \infty$ is given by the bare action for the classical chromomagnetic field background:

$$\Gamma_{\Lambda=\infty} = \frac{1}{4}(\mathcal{F}^A_{\mu\nu}[\mathcal{V}])^2 = \frac{1}{2}H^2. \quad (52)$$

Then an approximate solution is obtained by integrating the flow equation from $\Lambda = \Lambda_{\rm UV} = \infty$ to Λ:

$$\Gamma_\Lambda = \frac{1}{2}H^2 + \tilde{V}_\Lambda(H), \quad (53)$$

with

$$\tilde{V}_\Lambda(H) = -\frac{N(2gH)^2}{2(4\pi)^2}\left(\ln\frac{2gH}{4\pi\mu^2} + \gamma\right)\left[\zeta\left(-1, \frac{3}{2} + \frac{\Lambda^2}{2gH}\right)\right.$$
$$+ \zeta\left(-1, -\frac{1}{2} + \frac{\Lambda^2}{2gH}\right) - \zeta\left(-1, \frac{1}{2} + \frac{\Lambda^2}{2gH}\right)$$
$$\left. + \alpha_\Lambda \zeta\left(-1, \frac{1}{2} + \frac{\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)\right]$$
$$+ \frac{N(2gH)^2}{2(4\pi)^2}\left[\zeta^{(1,0)}\left(-1, \frac{3}{2} + \frac{\Lambda^2}{2gH}\right)\right.$$
$$+ \zeta^{(1,0)}\left(-1, -\frac{1}{2} + \frac{\Lambda^2}{2gH}\right) - \zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{\Lambda^2}{2gH}\right)$$
$$\left. - 2\zeta\left(-1, \frac{1}{2} + \frac{\Lambda^2}{2gH}\right) + \alpha_\Lambda \zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)\right], \quad (54)$$

where $\tilde{V}_\Lambda(H) = 0$ at $\Lambda = \infty$.

Using the formula [59–63],

$$\zeta(-1, \lambda) = -\frac{1}{2}\lambda^2 + \frac{1}{2}\lambda - \frac{1}{12}, \quad (\lambda \in \mathbb{R}), \quad (55)$$

we find that the first term proportional to $\ln\frac{gH}{\mu^2}$ is real valued for $\frac{gH}{\mu^2} > 0$, since

$$\zeta\left(-1, \frac{3}{2} + \frac{r}{2}\right) + \zeta\left(-1, -\frac{1}{2} + \frac{r}{2}\right) = -\frac{11}{12} - \frac{1}{4}r^2. \quad (56)$$

On the other hand, the recursion relation [59]

$$\zeta^{(1,0)}(-1, a + 1) = \zeta^{(1,0)}(-1, a) + a\ln a, \quad (57)$$

leads to

$$\zeta^{(1,0)}\left(-1, \frac{3}{2} + \frac{r}{2}\right) + \zeta^{(1,0)}\left(-1, -\frac{1}{2} + \frac{r}{2}\right)$$
$$= 2\zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{r}{2}\right) + \frac{1+r}{2}\ln\frac{1+r}{2} - \frac{-1+r}{2}\ln\frac{-1+r}{2}. \quad (58)$$

Note that $\zeta^{(1,0)}(-1, \lambda)$ is real valued for $\lambda > 0$. See Appendix B. Thus, we arrive at the effective potential for large Λ, e.g., in the case of $\alpha_\Lambda \equiv 1$,

$$V_\Lambda(H) = \frac{1}{2}H^2 + \frac{1}{16\pi^2}\Lambda^2\left[\ln\frac{gH}{\mu^2} + \frac{1}{4} - C\right]$$
$$- \frac{2}{16\pi^2}gH\Lambda^2\ln\frac{\Lambda^2 - gH}{\Lambda^2 + gH} + \frac{1}{16\pi^2}g^2H^2\left[\frac{11}{3}\ln\frac{gH}{\mu^2}\right.$$
$$+ 2\ln\frac{\Lambda^2 + gH}{gH} + 2\ln\frac{\Lambda^2 - gH}{gH} - \frac{11}{3}C - 4\ln 2$$
$$\left. - \frac{1}{3} + 8\zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{\Lambda^2}{2gH}\right)\right], \quad (59)$$

while in the case of $\alpha_\Lambda \equiv 0$,

$$V_\Lambda(H) = \frac{1}{2}H^2 + \frac{1}{16\pi^2}\Lambda^2\left[\frac{1}{2}\ln\frac{gH}{\mu^2} + \frac{1}{4} - \frac{1}{2}C\right]$$
$$- \frac{2}{16\pi^2}gH\Lambda^2\ln\frac{\Lambda^2 - gH}{\Lambda^2 + gH} + \frac{1}{16\pi^2}g^2H^2\left[-\frac{1}{6}\ln\frac{gH}{\mu^2}\right.$$
$$+ 2\ln\frac{\Lambda^2 + gH}{gH} + 2\ln\frac{\Lambda^2 - gH}{gH} - \frac{23}{6}C - 4\ln 2$$
$$\left. - \frac{1}{3} + 4\zeta^{(1,0)}\left(-1, \frac{1}{2} + \frac{\Lambda^2}{2gH}\right)\right]. \quad (60)$$

The same effective potential is obtained by solving the flow equation (44) to obtain the effective potential with the ultraviolet divergence and then removing the ultraviolet divergence by the same method as that above. See Appendix B for more details for the Hurwitz ζ function.

It is instructive to give a comment on the gauge parameter. In the Lorenz gauge there is a privileged choice: $\alpha = 0$ is a fixed point. Whereas there is no special choice for the gauge parameter in the MA gauge: there is no fixed point for α at least in the one-loop level. See Appendix C.

For the large Λ satisfying $\Lambda^2 \geq gH$, $\tilde{V}_\Lambda(H)$ is real valued and $V_\Lambda(H)$ has no imaginary part:

$$\mathrm{Im}V_\Lambda(H) = 0 \quad \text{for } \Lambda^2 \geq gH, \quad (61)$$

and

$$\partial_t \mathrm{Im}V_\Lambda(H) = 0 \quad \text{for } \Lambda^2 \geq gH. \quad (62)$$

Therefore, the Nielsen-Olesen instability disappears for any value of Λ, in particular, even at $\Lambda = 0$ according to the
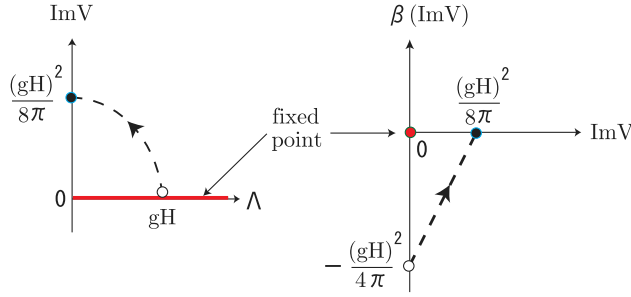
FIG. 2 (color online). Left: imaginary part $\mathrm{Im}V_\Lambda$ of an approximate average effective potential as a function of Λ. Right: β function as a function of $\mathrm{Im}V_\Lambda$. Here $\mathrm{Im}V_\Lambda \equiv 0$ corresponds to a fixed point of the functional renormalization group.

above argument of the fixed point for the pure imaginary part of the flow equation. See Fig. 2.

For the small Λ satisfying $\Lambda^2 < gH$, however, the effective average potential $V_\Lambda(H)$ obtained above has the nonvanishing imaginary part,

$$\mathrm{Im}V_\Lambda(H) = \frac{4}{16\pi^2} g^2 H^2 \frac{\frac{\Lambda^2}{gH}-1}{2} \ln(-1)/i$$

$$= \frac{1}{8\pi} gH(gH - \Lambda^2) \quad \text{for } \Lambda^2 < gH, \quad (63)$$

which yields the nontrivial flow of the imaginary part,

$$\partial_t \mathrm{Im}V_\Lambda(H) = -\frac{1}{4\pi} gH\Lambda^2 < 0 \quad \text{for } \Lambda^2 < gH. \quad (64)$$

The β function for $\mathrm{Im}V_\Lambda$ is obtained as

$$\beta(\mathrm{Im}V_\Lambda) = 2\,\mathrm{Im}V_\Lambda - \frac{1}{4\pi}(gH)^2 < 0 \quad \text{for } \Lambda^2 < gH. \quad (65)$$

This is not a contradiction, since the approximate solution of $V_\Lambda(H)$ obtained above is not considered to be valid in the small Λ region; $\Lambda^2 < gH$. In fact, the derivative $\partial_t \mathrm{Im}V_\Lambda(H)$ has the discontinuity at $\Lambda^2 = gH$. The effective potentials obtained above reproduce the Nielsen-Olesen result by putting $\Lambda = 0$:

$$V_{\mathrm{NO}}(H) = \frac{1}{2}H^2 + \frac{\frac{22}{3}}{16\pi^2}g^2\frac{1}{2}H^2\left(\ln\frac{gH}{\mu^2} + c\right) + i\frac{g^2H^2}{8\pi}. \quad (66)$$

The above argument for the absence of the instability or the vanishing of the imaginary part in the average effective potential $V_\Lambda(H)$ was done for a specific choice of the infrared regulator. However, the result will be true for any other choice of the infrared regulator, since the infrared regulator $R_\Lambda(p^2)$ is constructed in such a way that any infrared cutoff function approaches the asymptotic form of the same form as the mass-type one in the large Λ:

$$R_\Lambda^\Phi(p^2) \to Z_\Lambda^\Phi\Lambda^2 \quad \text{for } \Lambda^2 \to \infty. \quad (67)$$

Indeed, this condition must be imposed to reproduce the "one-loop result" in the large Λ, which is indeed one of the properties required to hold for the infrared regulator [64].

The claim can be explicitly checked for the infrared regulators, e.g., the optimal type [65],

$$R_\Lambda(p^2) = Z_\Lambda(\Lambda^2 - p^2)\theta(\Lambda^2 - p^2), \quad (68)$$

and the step function,

$$R_\Lambda(p^2) = Z_\Lambda\Lambda^2\theta(\Lambda^2 - p^2). \quad (69)$$

This is nontrivial for the exponential type,

$$R_\Lambda(p^2) = \frac{p^2}{e^{\frac{p^2}{Z_\Lambda\Lambda^2}} - 1} = \frac{p^2 e^{-\frac{p^2}{Z_\Lambda\Lambda^2}}}{1 - e^{-\frac{p^2}{Z_\Lambda\Lambda^2}}}, \quad (70)$$

since the momentum integration is difficult to be performed explicitly for this choice.

Moreover, it is important to confirm the statement explicitly for the choice of the infrared regulator $R_\Lambda(\Gamma^{(2)}(p^2))$ with the nontrivial argument $\Gamma^{(2)}(p^2)$ proposed in [38], since it is demonstrated in [40] that such a choice of the argument for the infrared regulator is actually essential to control the physical limit $\Lambda \to 0$.

Here is a good place to review the preceding works on which this work is based. In the paper by Reuter and Wetterich [38], a new nonperturbative flow equation for the average effective action was proposed for Yang-Mills theories. The subsequent works [39–41] are more or less based on this framework. In a subsequent work by them [39], it was applied to the calculation of the gluon condensation and the computation of the effective action for a uniform chromomagnetic field to examine the instability of the Savvidy vacuum. This work improved the earlier error in the evaluation of the flow equation in preceding work, but it did not find a desired gluon condensation in a simple way, since the strong infrared effects were cut off in an *ad hoc* way by introducing effectively an infrared fixed point by hand. Therefore, the resulting flow equation taken at face value shows a Landau-pole-type singularity.

The work by Gies [40] is an improvement of the earlier works by Reuter and Wetterich, which was called "spectrally adjusted" RG flow or spectral adjustment of the RG procedure. Gies has succeeded to estimate the effect of the $\partial_\Lambda\Gamma_\Lambda^{(2)}$ terms coming from the argument of the infrared regulator function $R_\Lambda(\Gamma_\Lambda^{(2)})$ that had been dropped in the preceding work. These terms become essential when the RG flow rapidly changes in the strong coupling domain. In fact, this improvement is necessary to derive the infrared fixed point, namely, the running coupling constant reaching

a finite and nonzero value in the limit $\Lambda \to 0$ without encountering divergence.

In [39] and [40], an ansatz of the power series $W_\Lambda(\Theta) = \sum_{n=0}^{\infty} \frac{1}{n!} w_n(\Lambda)\Theta^n$ is adopted for $W_\Lambda(\Theta)$, not to solve the flow, but to define the running coupling constant from the coefficient $w_1(\Lambda)$ in front of the term $\Theta := \frac{1}{4}F_{\mu\nu}^2$. It should be remarked that different choices for the definition of the running coupling can lead to different results, since the running coupling itself is not a meaningful quantity intrinsically in the sense that it depends on the scheme and the definition.

In the works [39,40], the magnetic field is only used as a technical tool to determine the flow equation. One need not assume that there is a physical magnetic background field. The same results for the running coupling would be obtained with, e.g., a heat-kernel expansion of the traces that are blind to the instability. Therefore, the NO instability is not an issue at all in these works. Still, calculating the flow using the magnetic field as a tool, of course, contains contributions from the Nielsen-Olesen mode, as it is also true for the one-loop calculation.

In the work by Eichhorn, Gies, and Pawlowski [41], on the other hand, the full propagators were used to compute the gluon condensate. The negative eigenvalues of the spin-1 Laplacian can potentially botch the computations. Therefore, they have used the self-dual background in order to avoid these complications from the beginning.

V. GLUON MASS GENERATION AND VACUUM CONDENSATIONS

The above approximate solution (59) eventually has the imaginary part and hence cannot be used in the limit $\Lambda \to 0$. As will be shown in this section, however, the approximate solution obtained in the same type of approximations has the limit $\Lambda \to 0$ without developing the imaginary part, if the effects of mass generation are incorporated into the analysis. Such mass generation is expected to occur, as established in the numerical simulations on the lattice [48–50].[5]

We introduce the mixed composite operators of gluons and ghosts. For $SU(2)$,

$$\mathcal{O} = \frac{1}{2}A_\mu^a A^{\mu a} + \alpha i \bar{C}^a C^a \qquad (a = 1, 2). \quad (71)$$

We then study the mass generation for the off-diagonal gluons (and ghosts), originating from the dimension-two condensation $\langle \mathcal{O} \rangle$. It is shown [43] that the dimension-two

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[5]This means the mass generation for the off-diagonal gluons in the MA gauge. For the diagonal gluon, this is not yet confirmed even for the MA gauge.
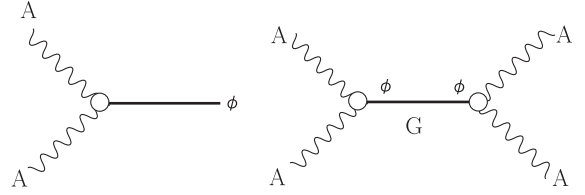

FIG. 3. Left: vertex joining the collective field ϕ to two off-diagonal gluon fields A. Right: an exchange of the collective field ϕ.

condensation $\langle \mathcal{O} \rangle$ is BRST invariant[6] in the modified MA gauge [66] defined by the GF+FP term:[7]

$$\mathscr{L}_{\text{GF+FP}}^{\text{MA}} = N^a F^a + \frac{\alpha}{2}N^a N^a + i\bar{C}^a \mathscr{D}_\mu^{ab}[V]\mathscr{D}_\mu^{bc}[V]C^c$$
$$- g^2 \epsilon^{ab}\epsilon^{cd} i\bar{C}^a C^d A_\mu^c A_\mu^b + gi\bar{C}^a \epsilon^{ab}(\mathscr{D}_\mu^{bc}[V]A_\mu^c)C^3$$
$$+ \alpha g \epsilon^{ab} i\bar{C}^a N^b C^3 + \frac{\alpha}{4}g^2 \epsilon^{ab}\epsilon^{cd}\bar{C}^a \bar{C}^b C^c C^d, \quad (72)$$

which is deduced from the $OSp(D, 2)$-invariant form:

$$\mathscr{L}_{\text{GF+FP}}^{\text{MA}} = i\delta\bar{\delta}\left(\frac{1}{2}A_\mu^a A_\mu^a + \frac{\alpha}{2}i\bar{C}^a C^a\right)$$
$$= -i\delta\left[\bar{C}^a\left(F^a + \frac{\alpha}{2}N^a\right) - \frac{\alpha}{2}gi\bar{C}^a \epsilon^{ab3}\bar{C}^b C^3\right],$$
$$F^a := \mathscr{D}_\mu^{ab}[V]A_\mu^b, \quad (73)$$

where δ and $\bar{\delta}$ are, respectively, the BRST and anti-BRST transformations.

According to [68], we introduce a new field ϕ which is an auxiliary field with no kinetic term represented by the Lagrangian density,

$$\mathscr{L}_\phi = \frac{1}{2}(\phi + G\mathcal{O})^\dagger G^{-1}(\phi + G\mathcal{O})$$
$$= \frac{1}{2}\phi^\dagger G^{-1}\phi + \phi^\dagger \mathcal{O} + \frac{1}{2}\mathcal{O}^\dagger G\mathcal{O}, \quad (74)$$

by inserting the unity,

$$1 = \int D\phi\, e^{-\int d^D x \mathscr{L}_\phi}, \quad (75)$$

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[6]We can construct a gauge-invariant version of the composite operator of mass dimension two; see [8–10].

[7]In the MA gauge, the four-point interaction $AA\bar{C}C$ appears irrespective of the gauge-fixing parameter α and it generates the four-point ghost self-interaction $\bar{C}C\bar{C}C$ by quantum corrections. Therefore, such a four-point ghost self-interaction is indispensable to maintain renormalizability. The naive MA gauge is nonrenormalizable, since it does not include the four-point ghost self-interactions. In the modified MA gauge, the strength of the four-point ghost self-interactions is proportional to the gauge-fixing parameter α. Such a four-point ghost self-interaction follows from the $OSp(D, 2)$ invariance. See [67] for the meaning of the four ghost interactions in the MA gauge.
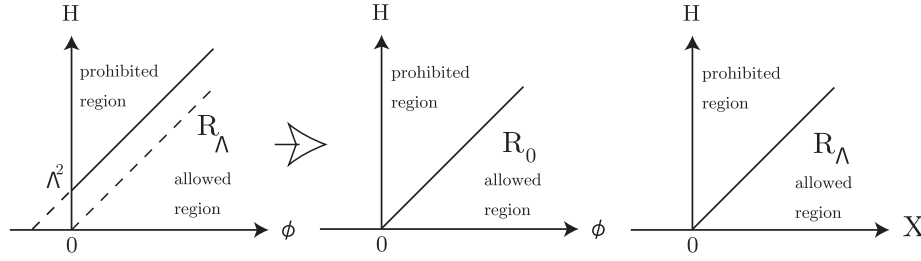
FIG. 4. Left: allowed region $\mathscr{R}_\Lambda$ and the prohibited region in (H, φ) at $\Lambda > 0$ and $\Lambda = 0$. Right: allowed region $\mathscr{R}_\Lambda$ and the prohibited region in (H, X), where X is equal to the shift of φ by $-\Lambda^2$, $X := \varphi + \Lambda^2$.

in the path-integral measure.[8] See Fig. 3. We observe the following:

(i) From the first term $\frac{1}{2}\phi^\dagger G^{-1}\phi$, we observe that G represents the effective propagator of the collective field ϕ, i.e., a two-gluon bound state propagator.

(ii) The second term $\phi^\dagger \mathcal{O}$ yields the cubic interactions ϕAA (and $\phi \bar{C}C$) for the operator $\mathcal{O}$ quadratic in the off-diagonal gluons (and ghosts).

(iii) The third term $\frac{1}{2}\mathcal{O}^\dagger G \mathcal{O}$ involving only the fundamental fields has the form of an exchange of ϕ in the tree approximation.

By including $\mathscr{L}_\phi$, the two-point functions $\Gamma_\Lambda^{(2)}$ are modified as

$$(\Gamma_\Lambda^{(2)})_{A_\mu^a A_\nu^b} = W_\Lambda' Q_{\mu\nu}^{ab} + \varphi \delta_{\mu\nu} \delta^{ab},$$
$$(\Gamma_\Lambda^{(2)})_{\bar{C}^a C^b} = -\tilde{Z}_\Lambda (\mathscr{D}^2)^{ab} + \alpha_\Lambda \varphi \delta^{ab}, \qquad (76)$$

where

$$\varphi = \langle \phi \rangle. \qquad (77)$$

Here we have adopted the truncation— neglecting the four-point interactions among the off-diagonal gluons and off-diagonal ghosts.

We use the infrared regulator of the mass type and the same approximations for W_Λ, $\bar{Z}_\Lambda$, and α_Λ as those adopted in the previous case. Then we obtain the effective average potential $V_\Lambda(H, \varphi)$ describing the chromomagnetic condensation and dynamical mass generation simultaneously. We consider the simplest case of $\alpha_\Lambda \equiv 1$ to clarify the qualitative feature (see [44] for a physical meaning of the dimension-two condensate in this gauge).[9] In this case, the effective potential is given by

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[8]It is shown that the effective field ϕ can be introduced without breaking the BRST symmetry. In fact, it is shown [43] that the operator $\mathcal{O}$ of mass dimension two is BRST invariant up to the total derivative, i.e., $\boldsymbol{\delta}\mathcal{O} = \partial^\mu[A_\mu^a(x)C^a(x)]$ and that the BRST transformation of ϕ is determined from the requirement $\boldsymbol{\delta}(\phi + G\mathcal{O}) = 0$.

[9]The thorough analysis including quantitative features will be given in a subsequent paper.

$$V_\Lambda(H, \varphi) = \frac{1}{2g_\Lambda^2}H^2 + \frac{1}{2G_\Lambda}\varphi^2 + \tilde{V}_\Lambda(H, \varphi), \quad (78)$$

$$\tilde{V}_\Lambda(H, \varphi) = -\frac{1}{4\pi^2}H^2\left(\ln\frac{H}{\mu^2} - C\right)\left[\zeta\left(-1, \frac{3}{2} + \frac{X}{2H}\right)\right.$$
$$\left. + \zeta\left(-1, -\frac{1}{2} + \frac{X}{2H}\right)\right]$$
$$+ \frac{1}{4\pi^2}H^2\left[\zeta^{(1,0)}\left(-1, \frac{3}{2} + \frac{X}{2H}\right)\right.$$
$$\left. + \zeta^{(1,0)}\left(-1, -\frac{1}{2} + \frac{X}{2H}\right) - 2\zeta\left(-1, \frac{1}{2} + \frac{X}{2H}\right)\right],$$
$$(79)$$

where

$$X := \varphi + \Lambda^2. \qquad (80)$$

Here we have rescaled H as $H \to \frac{1}{g}H$ for later convenience so that the quantum part $\tilde{V}_\Lambda$ does not include the g dependence. We find that $\tilde{V}_\Lambda(H, \varphi)$ is obtained from $\tilde{V}_\Lambda(H) = \tilde{V}_\Lambda(H, \varphi=0)$ by shifting the variable $\Lambda^2 \to \Lambda^2 + \varphi$:

$$\tilde{V}_\Lambda(H, \varphi) = \tilde{V}_\Lambda(H, \varphi = 0)|_{\Lambda^2 \to X} = \tilde{V}_\Lambda(H)|_{\Lambda^2 \to X}. \quad (81)$$

The real-valued condition for V_Λ is replaced by

$$X - H > 0, \quad \text{or} \quad H < X := \varphi + \Lambda^2. \qquad (82)$$

In other words, the stability excludes the region

$$H \geq X := \varphi + \Lambda^2. \qquad (83)$$

Therefore, we define the allowed region for stability,

$$\mathscr{R}_\Lambda = \{(H, \varphi); H < X := \varphi + \Lambda^2, H \geq 0, \varphi > 0\}. \quad (84)$$

which is a region below the straight line $H = X$ with the slope 1 and intercept Λ^2. See Fig. 4.

$V_\Lambda(H, \varphi)$ can be made real valued by taking sufficiently large Λ, as in the case of $V_\Lambda(H)$. In the absence of φ, this argument for eliminating the imaginary part does not work in the small Λ region in which the
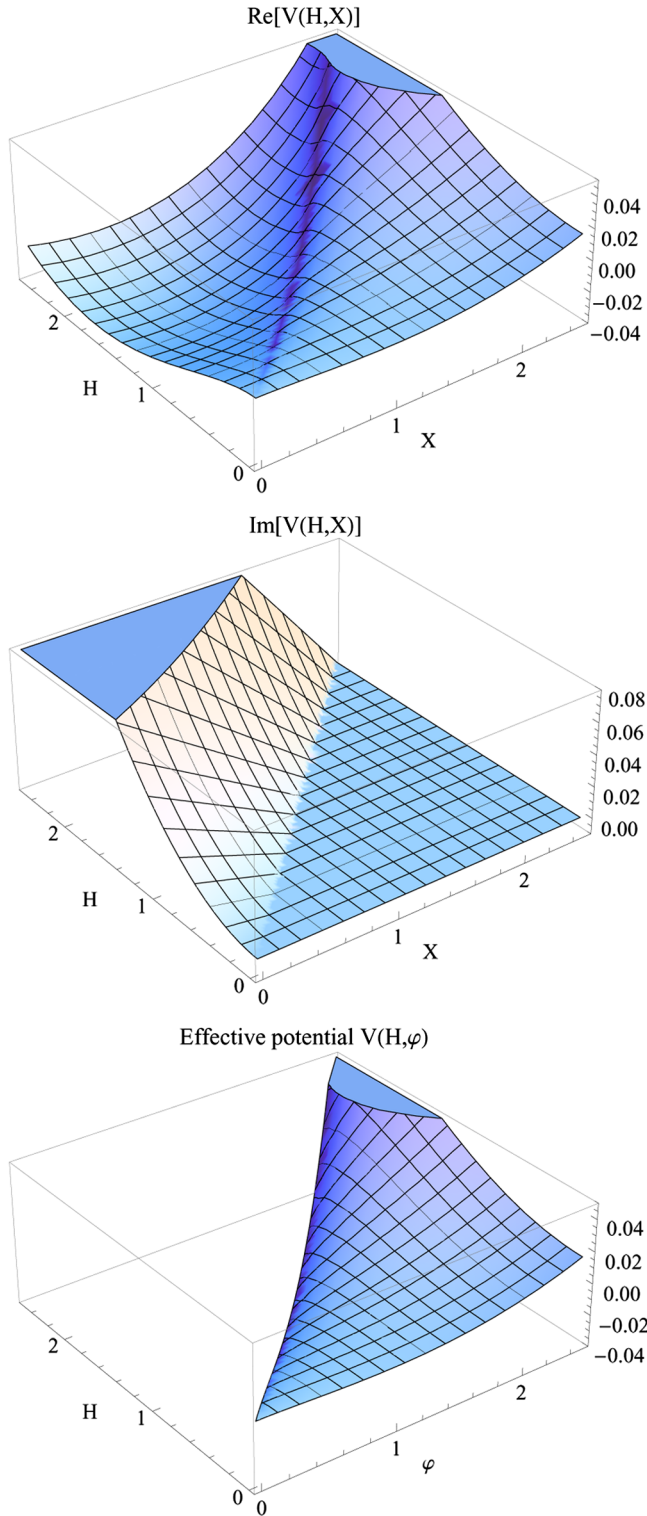
FIG. 5 (color online). Real and imaginary parts of the quantum part of the effective potential $\tilde{V}(H, X)$. The region $0 \leq H \leq X$ is allowed where $\tilde{V}(H, X)$ is real valued, while the region $X < H$ is prohibited where $\tilde{V}(H, X)$ includes the nonzero imaginary part.

inequality $H > \Lambda^2$ is satisfied. This shortcoming is avoided by including φ. In fact, the allowed region for stability $\mathscr{R}_\Lambda$ becomes narrower for a lower value of Λ but survives even in the limit $\Lambda \to 0$. Hence, the H axis or $\varphi = 0$ is excluded in the limit $\Lambda \to 0$.

The running coupling g_Λ is monotonically increasing in decreasing Λ. Therefore, the tree term $\frac{1}{2} g_\Lambda^{-2} H^2$ also becomes negligible for small enough Λ.

We can write down the flow equation for G_Λ. Solving it, we find that G_Λ^{-1} monotonically decreases as Λ decreases. Therefore, the effect of the tree term $\frac{1}{2} G_\Lambda^{-1} \varphi^2$ becomes more and more negligible for smaller Λ. In fact, the increasing of G_Λ in decreasing Λ is reasonable, since the bound state propagator $G_\Lambda(s)$ will approach the structure with a polelike dependence on s for small enough Λ [69–71]. Therefore, the details of the behavior of G_Λ do not change the following result qualitatively.

Thus the existence and location of the minimum can be dominantly determined by the quantum part $\tilde{V}(H, \varphi)$. In view of these, we have looked for the minimum of $\tilde{V}_\Lambda(H, \varphi)$ in the region $\mathscr{R}_\Lambda$. See Fig. 5 for the three-dimensional plot of $\tilde{V}_\Lambda(H, X)$. We find two minima: one minimum at $H \neq 0$ and $\varphi \neq 0$ in the region $H > X$, and another minimum at $H = 0$ and $\varphi \neq 0$ in the region $H < X$. If we trust the above potential, the H axis or $\varphi = 0$ is prohibited in the limit $\Lambda \to 0$, and therefore the former minimum is not allowed in the limit $\Lambda \to 0$, but it might be allowed by finding a more precise improved solution. The latter solution minimum survives in the limit $\Lambda \to 0$, which means that the mass generation occurs with the vanishing chromomagnetic condensation.

The following are details of the potential. In Fig. 6, we have given the plot of the potential $\tilde{V}_\Lambda(X) :=$ $\tilde{V}_\Lambda(H, X)$ at fixed values of H. The region $X > H$ is allowed where $\tilde{V}(X)$ is real valued, while the region $0 < X < H$ is prohibited where $\tilde{V}(X)$ includes the nonzero imaginary part. For relatively small H, a lower (perturbative) minimum for the real part of the effective potential exists for X between zero and H, which is however in the prohibited region. This lower minimum is separated from the higher (nonperturbative) minimum by a little hill with a top at X slightly above H. For higher H, a point is reached where the minimum with smaller X has a lower energy than the one with greater X, around $H/\mu^2 = 0.2$. For H yet higher, the higher minimum disappears altogether and only the lower one remains. The full effective potential has the additional contribution $\frac{1}{2g^2} H^2 + \frac{1}{2G} \varphi^2$, which does not change the above picture but rather strengthens the above tendency. Thus the minimum for the effective potential V can exist for H lower than a yet smaller critical value $H/\mu^2 \sim 0.3$. See Fig. 7 for the collection of all the plots in Fig. 6.
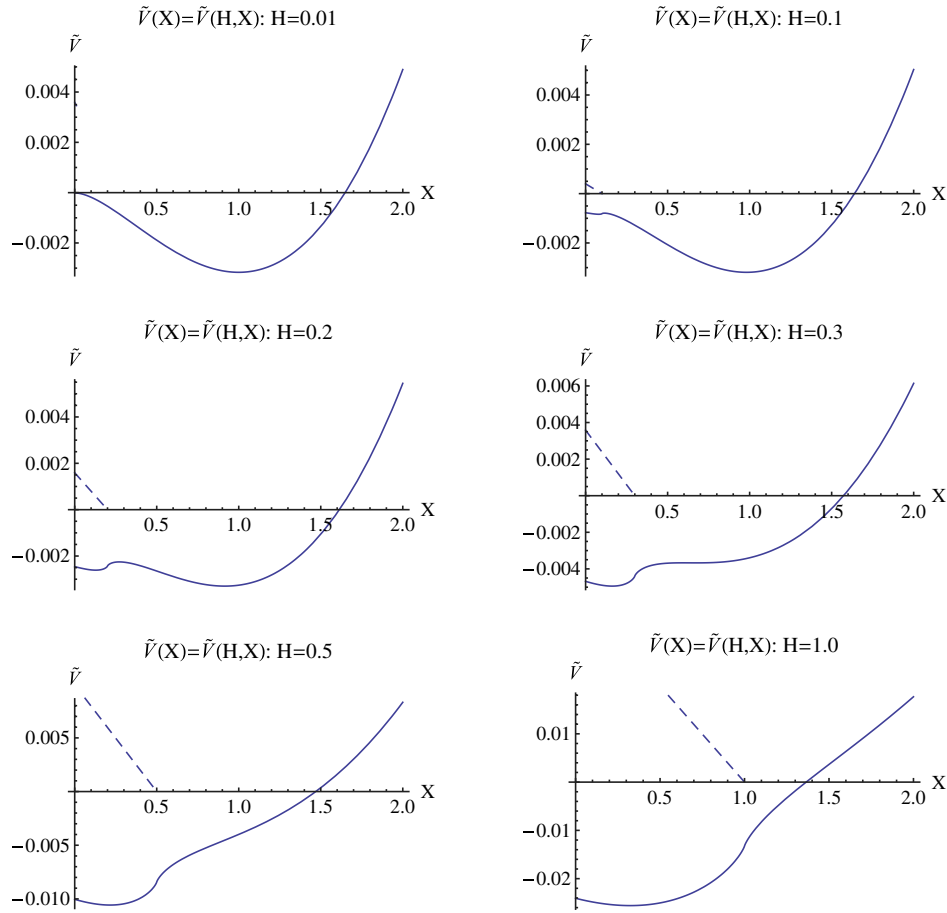
FIG. 6 (color online).   Real part (solid line) and imaginary part (dashed line) of the quantum part of the effective potential $\tilde{V}(X) = \tilde{V}(H,X)$ with various values of H: $H = 0.01$, $H = 0.1$, $H = 0.2$, $H = 0.3$, $H = 0.5$, and $H = 1.0$.

Thus, a nonzero chromomagnetic field decreases the effective gluon mass, and when the chromomagnetic field is sufficiently strong the gluon mass disappears, since the lower minimum is in the prohibited region.

FIG. 7 (color online).   Real part (solid line) and imaginary part (dashed line) of the quantum part of the effective potential $\tilde{V}(X) = \tilde{V}(H,X)$ with various values of H: $H = 0.01$, $H = 0.1$, $H = 0.2$, $H = 0.3$, $H = 0.5$, and $H = 1.0$. The region $H \leq X$ is allowed where $\tilde{V}(X) = \tilde{V}(H,X)$ is real valued, while the region $0 < X < H$ is prohibited where $\tilde{V}(X) = \tilde{V}(H,X)$ includes the nonzero imaginary part.

This conclusion is different from the result of Vercauteren and Verschelde [33] that the mass is lowered to a value slightly lower than gH after a phase transition occurred when the chromomagnetic field is strong enough.
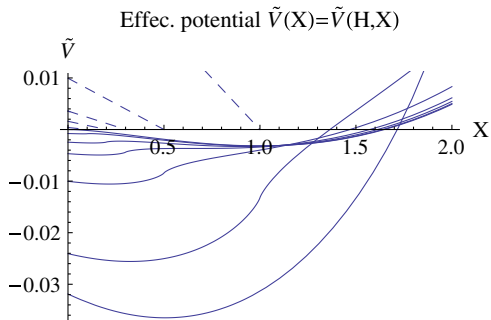
In Fig. 8, we have given the plot of the potential $\tilde{V}_\Lambda(H) := \tilde{V}_\Lambda(H,X)$ at fixed values of X. The region $0 < H < X$ is allowed where $\tilde{V}(H)$ is real valued, while the region $H > X$ is prohibited where $\tilde{V}(H)$ includes the nonzero imaginary part. For relatively small X, a lower (perturbative) minimum for the real part of the effective potential exists for H between zero and X, which is however in the prohibited region. When going to higher values of X, we find that $H = 0$ turns into a local minimum of the potential. For H slightly below X, there is a maximum and for H higher than X there is a higher (nonperturbative) minimum. When increasing X, the higher minimum first deepens out, reaching a lowest value for X, and it then goes up. See Fig. 9 for the collection of all the plots in Fig. 8.

We have considered the effect of the gluon mass on the chromomagnetic field condensation. When the gluon mass
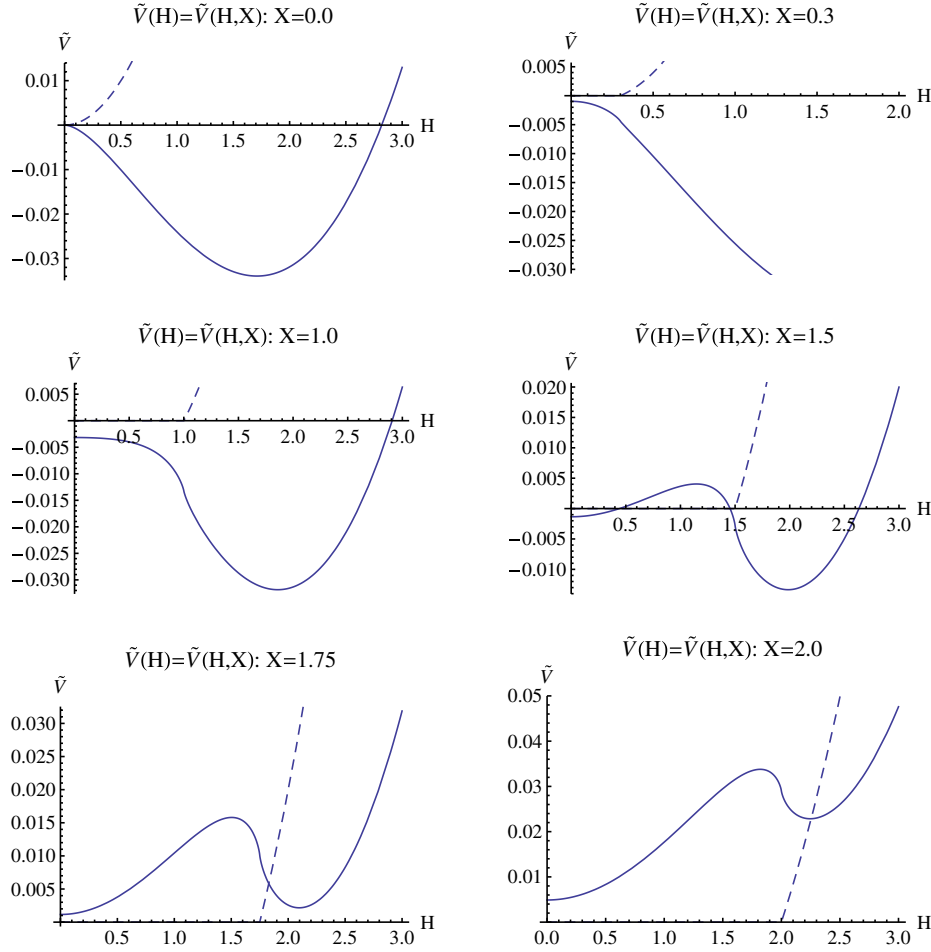
FIG. 8 (color online).   Real part (solid line) and imaginary part (dashed line) of the quantum part of the effective potential $\tilde{V}(H) = \tilde{V}(H, X)$ with various values of X: $X = 0$, $X = 0.3$, $X = 1.0$, $X = 1.5$, $X = 1.75$, and $X = 2.0$.

is sufficiently large, the vacuum is no longer unstable against the formation of a homogeneous chromomagnetic field, and the Nielsen-Olesen instability, caused by the imaginary part in the effective potential, is resolved.
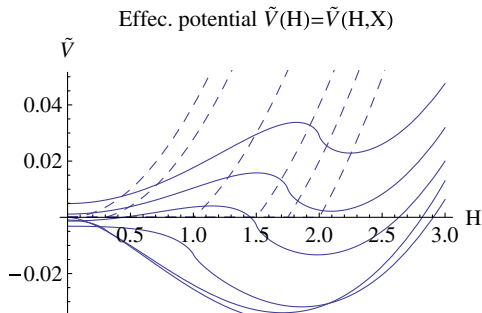

FIG. 9 (color online).   Real part (solid line) and imaginary part (dashed line) of the effective potential $\tilde{V}(H) = \tilde{V}(H, X)$ with various values of X: $X = 0$, $X = 0.3$, $X = 1.0$, $X = 1.5$, $X = 1.75$, and $X = 2.0$. The region $0 \leq H \leq X$ is allowed where $\tilde{V}(H)$ is real valued, while the region $X < H$ is prohibited where $\tilde{V}(H)$ includes the nonzero imaginary part.

VI. CONCLUSION AND DISCUSSION

In this paper, we have shown that the Nielsen-Olesen instability of the Savvidy vacuum with homogeneous chromomagnetic condensation is avoided in the framework of the FRG. Actually, we have shown that the imaginary part of the effective average action vanishes at sufficiently large infrared cutoff Λ, and this property can survive at $\Lambda = 0$. This behavior can be understood as a fixed point solution of the flow equation for the complex-valued effective average action. Therefore, the Nielsen-Olesen instability is an artifact of the loop calculation in the perturbation theory.

First, the most important observation given in Sec. II in this paper is the "fixed point" structure that exists in the imaginary part $\mathrm{Im}\Gamma_\Lambda$ of the complex-valued average effective action Γ_Λ governed by the FRG equation of the Wetterich type.

This fixed point is different from the infrared fixed point of the usual RG. The fixed point of this paper is restricted to the fixed point for all the scales from the ultraviolet down to the infrared, i.e., for any value of the flow parameter Λ, and

the fixed point is considered only for the imaginary part of the complex-valued average effective action. In this sense, the claim of this paper is that the complex valued FRG equation has the fixed point solution, i.e., the identically vanishing imaginary part $\mathrm{Im}\Gamma_\Lambda \equiv 0$ as an exact solution, while the real part $\mathrm{Re}\Gamma_\Lambda$ does not have such a remarkable structure. This novel concept is schematically shown in Fig. 1 using the beta function $\beta(\mathrm{Im}\Gamma_\Lambda)$ defined for the imaginary part $\mathrm{Im}\Gamma_\Lambda$ of the average effective action Γ_Λ.

If the average effective action as the solution of the FRG equation exhibits this fixed point structure, then the stability holds at any scale Λ including $\Lambda = 0$, since the imaginary part is identically vanishing and hence vanishing also at $\Lambda = 0$. This fact is the most important discovery of this paper, which has not been recognized in the preceding works to the best of the author's knowledge. In Fig. 1, two possibilities for the solution are drawn: the fixed-point solution with $\mathrm{Im}V \equiv 0$ and the nonfixed-point solution with $\mathrm{Im}V \not\equiv 0$.

Second, we have proceeded to show that the solution of the FRG equation satisfies the fixed point criterion. This is the content of Sec. IV. Of course, no one knows the exact solution of the FRG equation for the Yang-Mills theory. And we do not know even the explicit analytical form of the approximate solution which is valid for any Λ. In order to examine the stability, however, it is enough to show that all the solutions satisfy the fixed point structure at large but arbitrary value of Λ (for a finite interval of large Λ), since the smooth solutions must remain on the fixed point once they are on the fixed point, i.e., showing $\mathrm{Im}V = 0$ at large Λ. See Fig. 1.

For large Λ, in fact, we can find a good approximate analytical solution due to the asymptotic freedom of the theory, although this is not the case for small Λ due to the strong interactions at the infrared region. Hence, the first check whether or not the solution satisfies the fixed point criterion, i.e., $\mathrm{Im}V \equiv 0$, was performed in the large Λ region in Sec. IV. The result shows that the solution satisfies the criterion, i.e., no imaginary part at large but arbitrary value of Λ (for a finite interval of large Λ). This result is explicitly obtained for some infrared cutoff functions. But it must hold for any infrared cutoff function on general ground, since the infrared cutoff function is required to satisfy the same asymptotic behavior for large Λ. Thus, the stability must be shown without discussing other details of the solution. The approximate solution given in this paper is valid for large Λ at best $\Lambda > gH$. Therefore, Fig. 2 is consistent with Fig. 1.

To show the recovery of stability or the vanishing of the imaginary part just at $\Lambda = 0$ starting from the stability region at large Λ, we need to control the approximate solution along the flow from the large Λ all the way down to $\Lambda = 0$, which is quite a difficult task. Fortunately, we do not need to do so for concluding only the stability.

Next, we tried to find a better approximate solution that is valid for even lower values of Λ for understanding the physics behind the restoration of stability or the elimination of instability. This is the content of Sec. V. We have discussed the physical mechanism for keeping the stability for smaller Λ: the stability is maintained even for small Λ once the mass generation occurs for the off-diagonal gluons (and off-diagonal ghosts).

In fact, we have found two minima of the effective potential as a function of the chromomagnetic field condensate H and dynamical mass generation due to dimension-two vacuum condensation φ:

(i) One minimum at $H \neq 0$ and $\varphi \neq 0$ in the allowed region of stability $\mathscr{R}_\Lambda$ for relatively small Λ: Both the chromomagnetic field condensate and dynamical mass generation due to dimension-two vacuum condensation occur simultaneously in the region of validity for the infrared scale Λ.

(ii) Another minimum at $H = 0$ and $\varphi \neq 0$ in the region $\mathscr{R}_\Lambda$: This minimum survives in the limit $\Lambda \to 0$, which means that the mass generation occurs with the vanishing chromomagnetic condensation.

If we accept our result for solving the flow equation at face value, however, our approximate solution for the effective action is valid only for the infrared cutoff Λ above Λ_0, i.e., $\Lambda \geq \Lambda_0 \approx 0.335$ GeV. In fact, the running Yang-Mills coupling constant $\alpha_\Lambda := g_\Lambda^2/(4\pi)$ ceases running at $\Lambda = \Lambda_0$ where $\alpha_\Lambda = \alpha^0 \approx 1.88\ldots$. This is the same situation as that encountered in the work [39].

In order to obtain the true effective action, we need to solve the flow equation all the way down to $\Lambda \to 0$. In fact, a finite value for the running coupling constant has been obtained even at $\Lambda = 0$ in the framework of the FRG [40,41], although it had been shown for the first time in the framework of the Schwinger-Dyson equation [72].

The comparison of our result for the effective potential with that of [41] suggests that (i) $H \neq 0$ and $\varphi \neq 0$ is realized in the Yang-Mills vacuum. Using these solutions [40,41], moreover, we are able to discuss the possible relationship between the stability and the scaling/ decoupling [72–74] solutions that were recently claimed to be the true infrared solutions in the deep infrared region realizing quark and gluon confinement [75–78]. These issues will be further discussed in future works.

APPENDIX A: DECOMPOSITION OF A COMPLEX-VALUED MATRIX

In order to obtain the inverse matrix $P + iQ$ of the complex matrix $A + iB$, we set

$$(A + iB)(P + iQ) = \mathbf{1} = (P + iQ)(A + iB), \quad \text{(A1)}$$

which yields

$$\begin{cases} AP - BQ = \mathbf{1} = PA - QB, \\ AQ + BP = 0 = PB + QA. \end{cases} \quad \text{(A2)}$$

From the second equation, we obtain

$$Q = -A^{-1}BP = -PBA^{-1}. \quad \text{(A3)}$$

Substituting this relation into the first equation to eliminate Q, we obtain P,

$$AP + BA^{-1}BP = \mathbf{1} = PA + PBA^{-1}B$$
$$\Rightarrow (A + BA^{-1}B)P = \mathbf{1} = P(A + BA^{-1}B)$$
$$\Rightarrow P = (A + BA^{-1}B)^{-1}$$
$$= A(AA + BA^{-1}BA)^{-1} = (AA + ABA^{-1}B)^{-1}A, \quad \text{(A4)}$$

and hence

$$Q = -(AB^{-1}A + B)^{-1}$$
$$= -A^{-1}BA(AA + BA^{-1}BA)^{-1}$$
$$= -(AA + ABA^{-1}B)^{-1}ABA^{-1}. \quad \text{(A5)}$$

If $[A, B] = 0$, i.e., $AB = BA$, then $B^{-1}A^{-1} = A^{-1}B^{-1}$, which leads to $A^{-1}B = BA^{-1}$ and $AB^{-1} = B^{-1}A$. Therefore, we obtain

$$P = (A^2 + B^2)^{-1}A, \qquad Q = -(A^2 + B^2)^{-1}B. \quad \text{(A6)}$$

Note that if $B \to 0$, then $P \to A^{-1}$ and $Q \to 0$.

APPENDIX B: REMOVING THE ULTRAVIOLET DIVERGENCE

The ultraviolet divergence of (44) at $\tau = 0$ is removed as follows. (i) We introduce the parameter,

$$\epsilon := 2 - \frac{D}{2} = \frac{4 - D}{2} \quad (D = 4 - 2\epsilon), \quad \text{(B1)}$$

and replace D by $D = 4 - 2\epsilon$. (ii) Expand the right-hand side into the Laurent series in powers of ϵ, and (iii) extract the terms of order ϵ^0 (ϵ-independent terms).

By using the rescaling of τ, the flow equation (44) reads

$$\partial_t \Gamma_\Lambda = \frac{N}{2} \frac{(2gH)^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \left\{ (W'_\Lambda)^{-1}(2 - \eta_\Lambda)Z_\Lambda\Lambda^2 \int_0^\infty ds\, s^{1-\frac{D}{2}} e^{-s\frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}} \frac{(D-3)e^{-\frac{1}{2}s} + e^{-\frac{3}{2}s} + e^{\frac{1}{2}s}}{1 - e^{-s}} + \alpha_\Lambda(2 - \eta_\Lambda)Z_\Lambda\Lambda^2 \right.$$

$$\left. \times \int_0^\infty ds\, s^{1-\frac{D}{2}} e^{-s\frac{Z_\Lambda\Lambda^2}{2\alpha_\Lambda^{-1}gH}} \frac{e^{-\frac{1}{2}s}}{1 - e^{-s}} - (\tilde{Z}_\Lambda)^{-1}(2 - \tilde{\eta}_\Lambda)\tilde{Z}_\Lambda\Lambda^2 \int_0^\infty ds\, s^{1-\frac{D}{2}} e^{-s\frac{\tilde{Z}_\Lambda\Lambda^2}{2gH}} \frac{2e^{-\frac{1}{2}s}}{1 - e^{-s}} \right\}. \quad \text{(B2)}$$

We introduce the generalized Riemann ζ function or the Hurwitz ζ function $\zeta(z, \lambda)$ defined by

$$\zeta(z, \lambda) := \sum_{n=0}^\infty \frac{1}{(n + \lambda)^z}, \quad \text{(B3)}$$

which has its integral representation [59]:

$$\zeta(z, \lambda) = \frac{1}{\Gamma(z)} \int_0^\infty ds\, s^{z-1} \frac{e^{-\lambda s}}{1 - e^{-s}} \quad (\text{Re } z > 1, \text{Re}\lambda > 0). \quad \text{(B4)}$$

Although the Hurwitz ζ function $\zeta(z, \lambda)$ is originally defined for Re $z > 1$, Re $\lambda > 0$, it can be analytically continued to other regions in the complex z plane as an analytic function. Then the flow equation is rewritten as

$$\partial_t \Gamma_\Lambda = \frac{N}{2} \frac{(2gH)^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \Gamma\left(2 - \frac{D}{2}\right) \left\{ (W'_\Lambda)^{-1}(2 - \eta_\Lambda)Z_\Lambda\Lambda^2 \right.$$

$$\times \left[(D-3)\zeta\left(2 - \frac{D}{2}, \frac{1}{2} + \frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right) \right.$$

$$+ \zeta\left(2 - \frac{D}{2}, \frac{3}{2} + \frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right) + \zeta\left(2 - \frac{D}{2}, -\frac{1}{2} + \frac{Z_\Lambda\Lambda^2}{2W'_\Lambda gH}\right) \bigg]$$

$$+ \alpha_\Lambda(2 - \eta_\Lambda)Z_\Lambda\Lambda^2\zeta\left(2 - \frac{D}{2}, \frac{1}{2} + \frac{Z_\Lambda\Lambda^2}{2\alpha_\Lambda^{-1}gH}\right)$$

$$\left. - 2(\tilde{Z}_\Lambda)^{-1}(2 - \tilde{\eta}_\Lambda)\tilde{Z}_\Lambda\Lambda^2\zeta\left(2 - \frac{D}{2}, \frac{1}{2} + \frac{\Lambda^2}{2gH}\right) \right\}. \quad \text{(B5)}$$

This expression has the divergence at $D = 4$, since $\Gamma(0) = \infty$, although $\zeta(0, \lambda) < \infty$ for $\lambda < \infty$.

Therefore, we rewrite the flow equation into

$$
\partial_t \Gamma_\Lambda = \frac{N}{2} \frac{(2gH)^{1-\epsilon}}{(4\pi)^{2-\epsilon}} \Gamma(\epsilon) \Big\{ (W'_\Lambda)^{-1} (2 - \eta_\Lambda) Z_\Lambda \Lambda^2
$$

$$
\times \Big[(1 - 2\epsilon) \zeta \Big(\epsilon, \frac{1}{2} + \frac{Z_\Lambda \Lambda^2}{2 W'_\Lambda gH} \Big)
$$

$$
+ \zeta \Big(\epsilon, \frac{3}{2} + \frac{Z_\Lambda \Lambda^2}{2 W'_\Lambda gH} \Big) + \zeta \Big(\epsilon, -\frac{1}{2} + \frac{Z_\Lambda \Lambda^2}{2 W'_\Lambda gH} \Big) \Big]
$$

$$
+ \alpha_\Lambda (2 - \eta_\Lambda) Z_\Lambda \Lambda^2 \zeta \Big(\epsilon, \frac{1}{2} + \frac{Z_\Lambda \Lambda^2}{2 \alpha_\Lambda^{-1} gH} \Big)
$$

$$
- 2 (\tilde{Z}_\Lambda)^{-1} (2 - \tilde{\eta}_\Lambda) \tilde{Z}_\Lambda \Lambda^2 \zeta \Big(\epsilon, \frac{1}{2} + \frac{\Lambda^2}{2gH} \Big) \Big\}. \quad \text{(B6)}
$$

For $\epsilon \ll 1$, we can use the expansions around $\epsilon = 0$,

$$
\Gamma(\epsilon) = \epsilon^{-1} - \gamma + O(\epsilon), \quad \text{(B7)}
$$

$$
\mu^{2\epsilon} \Big(\frac{2gH}{4\pi} \Big)^{-\epsilon} = \exp \Big[-\epsilon \ln \Big(\frac{2gH}{4\pi\mu^2} \Big) \Big]
$$

$$
= 1 - \epsilon \ln \frac{2gH}{4\pi\mu^2} + O(\epsilon^2), \quad \text{(B8)}
$$

and

$$
\zeta(\epsilon, \lambda) = \zeta(0, \lambda) + \epsilon \zeta^{(1,0)}(0, \lambda) + O(\epsilon^2), \quad \text{(B9)}
$$

where we have defined

$$
\zeta^{(m,n)}(z, \lambda) := \frac{\partial^m}{\partial z^m} \frac{\partial^n}{\partial \lambda^n} \zeta(z, \lambda). \quad \text{(B10)}
$$

APPENDIX C: GENERALIZED RIEMANN ζ FUNCTION

The generalized Riemann ζ-function $\zeta^{(0,0)}(1 - n, \lambda)$ can be represented as

$$
\zeta^{(0,0)}(1 - n, \lambda) = -\frac{1}{n} B_n(\lambda), \quad \text{(C1)}
$$

where $B_n(\lambda)$ is the Bernoulli polynomial of degree n.
For $n = 1$,

$$
\zeta^{(0,0)}(0, \lambda) = -B_1(\lambda) = -\lambda + \frac{1}{2}. \quad \text{(C2)}
$$

For $n = 2$,

$$
\zeta^{(0,0)}(-1, \lambda) = -\frac{1}{2} B_2(\lambda) = -\frac{1}{2} \Big(\lambda^2 - \lambda + \frac{1}{6} \Big). \quad \text{(C3)}
$$

The expansion of the derivative of the generalized Riemann ζ function $\zeta^{(1,0)}(1 - n, \lambda)$ for large λ is given by [59]

$$
\zeta^{(1,0)}(1 - n, \lambda) = \frac{1}{n} \Big(\ln \lambda - \frac{1}{n} \Big) B_n(\lambda) - \frac{1}{2n} \lambda^{n-1}
$$

$$
- \frac{1}{n} \sum_{k=2}^{n} B_k \sum_{j=0}^{k-1} {}_n C_j \frac{(-1)^j}{k - j} \lambda^{n-k}
$$

$$
+ (-1)^{n-1} (n - 1)!
$$

$$
\times \sum_{k=n+1}^{\infty} \frac{B_k}{k(k-1)\ldots(k-n)} \lambda^{n-k}, \quad \text{(C4)}
$$

where B_k is the Bernoulli numbers.
For $n = 1$,

$$
\zeta^{(1,0)}(0, \lambda) = (\ln \lambda - 1) B_1(\lambda) - \frac{1}{2} + \frac{1}{2} B_2 \lambda^{-1} + O(\lambda^{-2})
$$

$$
= (\ln \lambda - 1) \Big(\lambda - \frac{1}{2} \Big) - \frac{1}{2} + O(\lambda^{-1}). \quad \text{(C5)}
$$

For $n = 2$,

$$
\zeta^{(1,0)}(-1, \lambda) = \frac{1}{2} \Big(\ln \lambda - \frac{1}{2} \Big) B_2(\lambda) - \frac{1}{4} \lambda + \frac{1}{2} B_2 \lambda^{-1}
$$

$$
+ O(\lambda^{-2})
$$

$$
= \frac{1}{2} \Big(\ln \lambda - \frac{1}{2} \Big) \Big(\lambda^2 - \lambda + \frac{1}{6} \Big) - \frac{1}{4} \lambda + O(\lambda^{-1}).
$$

$$
\text{(C6)}
$$

The following recursion relation holds [59]:

$$
\zeta^{(1,0)}(-1, a + n) = \zeta^{(1,0)}(-1, a) + \sum_{n=0}^{n-1} (k + a) \ln(k + a).
$$

$$
\text{(C7)}
$$

In particular,

$$
\zeta^{(1,0)}(-1, a + 1) = \zeta^{(1,0)}(-1, a) + a \ln a. \quad \text{(C8)}
$$

APPENDIX D: FLOW OF GAUGE PARAMETERS IN MA GAUGE

It was shown [79–82] that the gauge-fixing parameter β of the diagonal part in the Lorentz gauge obeys the RG equation to the one-loop calculation:

$$
\mu \frac{\partial}{\partial \mu} \beta_R = \frac{44}{3} \beta_R \frac{g_R^2}{(4\pi)^2}, \quad \text{(D1)}
$$

and that the gauge-fixing parameter α of the off-diagonal part in the modified maximal Abelian gauge obeys the RG equation:

$$
\mu \frac{\partial}{\partial \mu} \alpha_R = \Big[-2\alpha_R^2 + \frac{8}{3} \alpha_R - 6 \Big] \frac{g_R^2}{(4\pi)^2}. \quad \text{(D2)}
$$

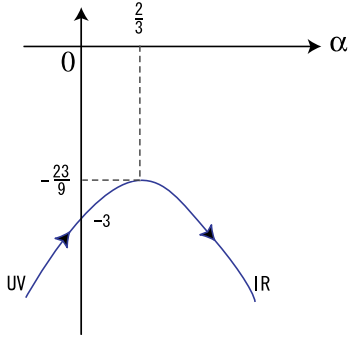It is well known that the running of the gauge coupling constant is governed by the differential equation:

FIG. 10 (color online). Flow of the gauge parameter α in the modified MA gauge for the off-diagonal gauge field. The arrow is directed to the IR region, i.e., decreasing the renormalization scale μ.

$$\beta(g_R^2) := \mu \frac{\partial g_R^2}{\partial \mu} = -\frac{22}{3} \frac{C_2(G)}{(4\pi)^2} g_R^4. \quad \text{(D3)}$$

Equation (D3) is a closed equation for g_R, which is solved exactly as a function of μ:

$$g_R^2(\mu) = \frac{g_R^2(\mu_0)}{1 + \frac{22}{3} \frac{C_2(G)}{(4\pi)^2} g_R^2(\mu_0) \ln \frac{\mu}{\mu_0}} = \frac{1}{\frac{22}{3} \frac{C_2(G)}{(4\pi)^2} \ln \frac{\mu}{\Lambda_{\text{QCD}}}}, \quad \text{(D4)}$$

where we have used the boundary condition $g_R(\mu_0) = \infty$ at $\mu_0 = \Lambda_{\text{QCD}}$. Using the solution (D4), the derivative $\frac{1}{g_R^2} \mu \frac{\partial}{\partial \mu}$ in (D1) and (D2) is rewritten as

$$\frac{1}{g_R^2} \mu \frac{\partial}{\partial \mu} = \frac{22}{3} \frac{C_2(G)}{(4\pi)^2} \frac{\partial}{\partial \ln \ln \frac{\mu}{\Lambda_{\text{QCD}}}}. \quad \text{(D5)}$$

We apply (D5) to rewrite the differential equation (D1) into the form that does not explicitly depend on g^2:

$$\frac{22}{3} 2 \frac{\partial}{\partial \ln \ln \frac{\mu}{\Lambda_{\text{QCD}}}} \beta_R = \frac{44}{3} \beta_R, \quad \text{(D6)}$$

which is easily solved. The integration,

$$\int_\beta^{\bar{\beta}} \frac{d\beta}{\beta} = \int_{\mu_0}^\mu d \ln \ln \frac{\mu}{\Lambda_{\text{QCD}}}, \quad \text{(D7)}$$

yields

$$\bar{\beta} = \beta \frac{\ln(\mu/\Lambda_{\text{QCD}})}{\ln(\mu_0/\Lambda_{\text{QCD}})} = \beta \frac{g^2}{\bar{g}^2}. \quad \text{(D8)}$$

In what follows, we use β to denote the initial value, $\bar{\beta} := \beta_R$ the running parameter and β_* the fixed point of RG. As $\mu \to \infty$ or $\bar{g} \to 0$, $\bar{\beta} \to +\infty$ for $\beta > 0$ and $\bar{\beta} \to -\infty$ for $\beta < 0$, while $\bar{\beta} \equiv 0$ for $\beta = 0$. As $\mu \to 0$ or $\bar{g} \to \infty$, $\bar{\beta} \to 0$. Hence, $\beta_* = 0$ is the IR fixed point for β.

In a similar way, (D2) is cast into

$$\frac{44}{3} \frac{\partial}{\partial \ln \ln \frac{\mu}{\Lambda_{\text{QCD}}}} \alpha_R = -2\alpha_R^2 + \frac{8}{3} \alpha_R - 6 < 0. \quad \text{(D9)}$$

Before solving this equation, we can observe that $\bar{\alpha}$ is monotonically increasing (decreasing) in decreasing (increasing) μ towards the IR (UV) direction, and that there is no fixed point for α, in sharp contrast to the Lorentz gauge. See Fig. 10.

Equation (D9) is solved by the integration

$$\int_\alpha^{\bar{\alpha}} \frac{d\alpha_R}{-2\alpha_R^2 + \frac{8}{3}\alpha_R - 6} = \frac{3}{44} \int_{\mu_0}^\mu d \ln \ln \frac{\mu}{\Lambda_{\text{QCD}}}. \quad \text{(D10)}$$

First, we consider sufficiently small α ($|\alpha| \ll 1$), neglecting the order α^2 term,

$$\int_\alpha^{\bar{\alpha}} \frac{d\alpha_R}{\alpha_R - \frac{9}{4}} = \frac{2}{11} \int_{\mu_0}^\mu d \ln \ln \frac{\mu}{\Lambda_{\text{QCD}}}, \quad \text{(D11)}$$

which yields

$$\bar{\alpha}(\mu) = \frac{9}{4} + \left(\alpha - \frac{9}{4}\right) \left[\frac{\ln(\mu/\Lambda_{\text{QCD}})}{\ln(\mu_0/\Lambda_{\text{QCD}})}\right]^{\frac{2}{11}}$$

$$= \frac{9}{4} + \left(\alpha - \frac{9}{4}\right) \left(\frac{g^2}{\bar{g}^2}\right)^{\frac{2}{11}} \to -\infty \quad (\mu \to \infty, \bar{g} \to 0).$$

(D12)

Next, we take into account the $O(\alpha^2)$ term too. Applying the formula

$$\int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}} \quad (b^2 < 4ac),$$

(D13)

to (D10), we obtain

$$\arctan \frac{-3\bar{\alpha} + 2}{\sqrt{23}} = \arctan \frac{-3\alpha + 2}{\sqrt{23}}$$

$$+ \frac{\sqrt{23}}{22} \ln \left[\frac{\ln(\mu/\Lambda_{\text{QCD}})}{\ln(\mu_0/\Lambda_{\text{QCD}})}\right]. \quad \text{(D14)}$$

Thus the running gauge parameter obeys

$$\bar{\alpha}(\mu) = \frac{2}{3} - \frac{\sqrt{23}}{3} \tan \left\{ \arctan \frac{-3\alpha + 2}{\sqrt{23}} + \frac{\sqrt{23}}{22} \ln \left[\frac{g^2}{\bar{g}^2}\right] \right\}. \quad \text{(D15)}$$

This shows that $\bar{\alpha} \to -\infty$ as $\mu \to \infty$ ($\bar{g} \to 0$) irrespective of the value of α. Note that $\arctan x$ is multivalued, unless $-\pi/2 < \arctan x < \pi/2$.

For higher loop calculations in the MA gauge, see [83].

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