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Gauged Hopfions

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We discuss the $U(1)$ gauged version of the $3+1$ dimensional Faddeev-Skyrme model supplemented by the Maxwell term. We show that there exist axially symmetric static solutions coupled to the noninteger toroidal flux of magnetic field, which revert to the usual Hopfions $\mathcal{A}_{m,n}$ of lower degrees $Q = mn$ in the limit of the gauge coupling constant vanishing. The masses of the static gauged Hopfions are found to be less than the corresponding masses of the usual ungauged solitons $\mathcal{A}_{1,1}$ and $\mathcal{A}_{2,1}$, respectively; they become lighter as the gauge coupling increases. The dependence of the solutions on the gauge coupling is investigated. We find that in the strong coupling regime the gauged Hopfion carries two magnetic fluxes, which are quantized in units of 2π, carrying n and m quanta, respectively. The first flux encircles the position curve and the second one is directed along the symmetry axis. Effective quantization of the field in the gauge sector may allow us to reconsider the usual arguments concerning the lower topological bound in the Faddeev-Skyrme-Maxwell model.

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I. INTRODUCTION

Spatially localized finite energy particlelike soliton solutions play a prominent role in a wide variety of nonlinear physical systems, from modern cosmology and quantum field theory to condensed matter physics (for a general review see, e.g., [1]).

Interesting examples of the topological solitons appear in the scalar models of the Skyrme family. In $d = 3+1$ dimensions it includes the original Skyrme model [2] and the Faddeev-Skyrme model [3], in $d = 2+1$ there is a simplified baby Skyrme model [4,5] which resembles the basic properties of the genuine Skyrme model in many aspects. The structure of these models looks similar, the corresponding Lagrangian includes the usual sigma model term, the Skyrme term, which is quartic in derivatives of the field, and the potential term which does not contain the derivatives.[1] However the topological properties of the corresponding solitons are different.

A peculiar feature of the soliton solutions, both Skyrmions and Hopfions, is that they do not saturate the topological lower bound. In order to attain it and get a relation[2] between the masses of the solitons and their topological charges Q, one has to modify the model, for example eliminate the quadratic in derivatives term [8], which yields exact Bogomol'nyi solutions in the Skyrme-type submodel [9]. Similarly, the topological bound is saturated in the truncated Faddeev-Skyrme model [10]. Alternatively, one can extend the model by coupling the Skyrmions to an infinite tower of vector mesons [11].

Another peculiarity of the soliton solutions of the Skyrme family is that they can be constructed only numerically, one has to apply rather complicated numerical methods which need a serious amount of computation power. This task becomes particularly complicated in the case of the Hopfions in the Faddeev-Skyrme model which are stringlike configurations classified by the linking number, the first Hopf map $S^3 \to S^2$.

A physically natural extension of the Skyrme model is related with the possibility of gauging of the global symmetry group. Originally, the Abelian gauged Skyrme model was proposed to model the monopole catalysis of the proton decay [12], the axially-symmetric gauged Skyrmions were considered in [13]. Similar analysis of the gauged baby Skyrmions [14] reveals a very interesting feature of the corresponding solitons: they carry a magnetic flux which is not topologically quantized. Gauge versions of some systems, which resemble a modified Faddeev-Skyrme model, were also considered in the papers [15–18]. However, to the best of our knowledge, the analysis of the fully coupled Faddeev-Skyrme-Maxwell system has not yet been done.

The Hopfions have been intensively studied over recent years [19–22]. These solitons have a number of physical applications, for example, in the study of Bose-Einstein condensates [23], nonlinear optics [24], and nonconventional superconductivity [25].

It was shown that whereas the minimal energy solitons of the Faddeev-Skyrme model of degree $Q = 1, 2$ are axially symmetric, the higher degree solutions should be not just closed flux-tubes of the fields but knotted field configurations. Note that the solitons of the model possess both

[1]Note that the potential term is optional in $d = 3+1$ dimensions, on the other hand it is obligatory to stabilize the solitons in the low-dimensional baby-Skyrme model [6].

[2]This relation is linear for Skyrmions, however for the Hopfions the corresponding Vakulenko-Kapitanski bound is $E = cQ^{3/4}$ where $c = (3/16)^{3/8}$ [7].

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rotational and internal rotational (or isorotational) degrees of freedom, such a rotation might seriously affect the structure of the Hopfions [26,27].

The soliton solutions of the Faddeev-Skyrme model are invariant with respect to the global $SO(2)$ symmetry. Furthermore, for axially symmetric configurations the rotations about the third axis in space and in isospace are identical. Therefore, by analogy with the similar situation in the planar baby Skyrme model [14], we can couple the usual Maxwell electrodynamics to the Faddeev-Skyrme model by gauging this symmetry. Note that in such a theory, unlike the low-dimensional baby Skyrme model [28], the Maxwell term alone probably cannot be used as a substitute for the Skyrme term in order to stabilize the Hopfions.

In this paper we discuss the topologically stable static soliton solutions of the full coupled gauged Faddeev-Skyrme-Maxwell system which carry two magnetic fluxes. We study numerically the dependency of shapes of the gauged Hopfions, their masses and magnetic fluxes on the gauge coupling constant, both in the perturbative limit and in the strong coupling limit. Since the consistent consideration of the solitons with higher Hopf charges is related with the complicated task of full numerical simulations in 3d [21], we restrict our consideration to the case of the static axially symmetric unlinked Hopfions $\mathcal{A}_{1,1}$ and $\mathcal{A}_{2,1}$ of charges $Q = 1, 2$, respectively. We find that the gauged Hopfion carries a toroidal magnetic flux which, in the strong coupling regime is effectively reduced to two magnetic fluxes, one of which encircles the position curve of the Hopfion and the second one is directed along the symmetry axis. Further, we demonstrated that both fluxes are quantized in units of 2π, carrying n and m quanta, respectively. We observe that the mass of the gauged Hopfions is decreasing as the gauge constant grows.

More systematic investigation of the gauged Hopfions for larger values of Q and with nonvanishing electric field will be presented elsewhere.

II. THE MODEL

A gauged version of the Faddeev-Skyrme model can be constructed if we take into account global $SO(2)$ invariance of the $3+1$ dimensional Lagrangian

$$\mathcal{L}_{\rm FS} = \frac{1}{32\pi^2\sqrt{2}}\left(\partial_\mu\phi^a\partial^\mu\phi^a - \frac{\kappa}{2}(\varepsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c)^2 - \mu^2[1-(\phi^3)^2]\right), \tag{1}$$

where κ is the dimensional coupling constant and a triplet of scalar real fields $\phi^a = (\phi^1, \phi^2, \phi^3)$ satisfy the constraint $\phi^a\cdot\phi^a = 1$. An additional potential term $V = \mu^2[1-(\phi^3)^2]$ breaks the global $SO(3)$ symmetry of the model. Note that if the model (1) becomes restricted to the xy plane, it corresponds to the double vacuum baby Skyrme model [29].

Under the scaling transformations of the domain space $x \to \lambda x$ the sigma model term scales as λ, the Skyrme term scales as λ^{-1} and the potential term scales as λ^3. Hence, even if the latter term is absent, the existence of static solitons of the model (1) is allowed by the Derrick theorem.

Topological restriction on the field ϕ^a is that it approaches its vacuum value at spacial boundary, i.e., $\phi^a_\infty = (0,0,1)$. This allows a one-point compactification of the domain space $\mathbb{R}^3$ to S^3 and the field of the finite energy solutions of the model, the Hopfions, is a map $\phi^a\colon \mathbb{R}^3 \to S^2$ which belongs to an equivalence class characterized by the homotopy group $\pi_3(S^2) = \mathbb{Z}$. Explicitly, the Hopf invariant is defined nonlocally as

$$Q = \frac{1}{16\pi^2}\int_{\mathbb{R}^3}\varepsilon_{ijk}\mathcal{F}_{ij}\mathcal{A}_k, \tag{2}$$

where $\mathcal{F}_{ij} = \varepsilon_{abc}\phi^a\partial_i\phi^b\partial_j\phi^c$ and one-form $\mathcal{A} = \mathcal{A}_k dx^k$ is defined via $\mathcal{F} = d\mathcal{A}$, i.e., the two-form $\mathcal{F}$ is closed, $d\mathcal{F} = 0$.

For the lowest values of the corresponding Hopf charge $Q = 1, 2$ the simplest soliton solutions can be constructed using the axially symmetric ansatz [19] written in terms of two functions $f = f(r,\theta)$ and $g = g(r,\theta)$ which depend on the radial variable r and the polar angle θ:

$$\begin{aligned}\phi^1 &= \sin f(r,\theta)\cos(m\varphi - ng(r,\theta)),\\ \phi^2 &= \sin f(r,\theta)\sin(m\varphi - ng(r,\theta)),\\ \phi^3 &= \cos f(r,\theta),\end{aligned} \tag{3}$$

where $n, m \in \mathbb{Z}$. An axially-symmetric configuration of this type is commonly referred to as $\mathcal{A}_{m,n}$, where the first subscript corresponds to the number of twists along the loop and the second label is the usual $O(3)$ sigma model winding number associated with the map $S^2 \to S^2$. The Hopf invariant of this configuration is $Q = mn$.

The unbroken global symmetry of the configurations with respect to the rotations around the third axis allows us to rotate the components of the axially-symmetric configuration as $(\phi_1 + i\phi_2) \mapsto (\phi_1 + i\phi_2)e^{i\alpha}$, where α is the angle of rotation. Thus, we can gauge this subgroup by a $U(1)$ gauge field A_μ defining the covariant derivative as (cf. [19,28,30])

$$D_\mu\phi^a = \partial_\mu\phi^a + gA_\mu\varepsilon_{abc}\phi^b\phi^c_\infty, \tag{4}$$

where g is the gauge coupling constant.

Note that the field configuration has finite energy if $D_\mu\phi^a \to 0$ as $r \to \infty$. Hence, on the spacial asymptotic, the field of the gauged Hopfion must lie in an orbit of the gauge group, unless the global symmetry is explicitly broken by the potential term. In other words, this condition

generically does not imply that the field ϕ^a necessarily tends to a constant on the spacial asymptotic.

The total Lagrangian of the gauged Faddeev-Skyrme-Maxwell model can be written as

$$\mathcal{L} = \frac{1}{32\pi^2\sqrt{2}}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^a D^\mu\phi^a - \frac{\kappa}{2}(\varepsilon_{abc}\phi^a D_\mu\phi^b D_\nu\phi^c)^2 - \mu^2[1-(\phi^3)^2]\right) \quad (5)$$

where we introduced the usual Maxwell term and the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here we suppose that the topological charge of the configuration is defined as usual by (2).

Note that the integrated Maxwell term transforms as λ^{-1} under the scaling transformations $x \to \lambda x$, i.e., it has the same scaling properties as the Skyrme term. Setting $\lambda = \sqrt{\kappa}$ allows us to rescale the Skyrme coupling constant to $\kappa = 1$.

The question about the possible existence of soliton solutions of the gauged model (5) was briefly discussed recently in review [31]. It was pointed out that, in some sense, the Maxwell term looks similar to the Skyrme term, thus one can consider an additional linking between the genuine $U(1)$ gauge field A_μ and the "potential" $\mathcal{A}_\mu$ which appears in the definition of the Hopf charge (2) [17,31]. However it may result in instability of the configuration since it seems to be that there are no topological restrictions on the Maxwell field and there is no lower energy bound in the generalized Vakulenko-Kapitanski relation [17]. However, as we will see below, there still is an effective quantization of the field fluxes in the gauge sector which may allow us to evade a possible collapse of the configuration.

Hereafter, we restrict the consideration to the original model without the potential term, so we set $\mu = 0$. Then in normalized units of energy, in which $E \to E/(32\pi^2)\sqrt{2}$, the static energy of the gauged Hopfion is defined by the functional

$$E = \int_{\mathbb{R}^3}\left[(D_i\phi^a)^2 + \frac{1}{2}(\varepsilon_{abc}\phi^a D_i\phi^b D_j\phi^c)^2\right] \quad (6)$$

and the electromagnetic part of the total energy functional is the usual sum of magnetic and electric components:

$$E_{em} = \frac{1}{2}\int_{\mathbb{R}^3}[B_k^2 + E_k^2]. \quad (7)$$

Let us consider a purely magnetic field generated by the axially symmetric Maxwell potential

$$A_0 = A_r = 0; \quad A_\theta = A_1(r,\theta); \quad A_\phi = A_2(r,\theta)\sin\theta \quad (8)$$

represented in terms of the two functions $A_i(r,\theta)$ $i = 1, 2$, here the gauge fixing condition is used to exclude the radial component of the vector-potential. Note that the "trigonometric" parametrization (3) is not very convenient from the point of view of numerical calculations [32,33], here we used it to produce an initial configurations in the given topological sector. However the original triplet of the scalar fields ϕ^a was considered as dynamical variables in the corresponding system of the Euler-Lagrange equations.

Certainly, the $U(1)$ gauge potential obeys the usual Maxwell equation

$$\partial_\mu F^{\mu\nu} = j^\nu \quad (9)$$

with the current

$$j_\mu = 2g\varepsilon_{abc}\phi^a D_\mu\phi^b(\phi^c_\infty - D_\nu\phi^c\partial^\nu(\phi^d\phi^d_\infty)). \quad (10)$$

The complete set of the field equations, which follow from the variation of the action of the Faddeev-Skyrme-Maxwell model (5), can be solved when we impose the boundary conditions. As usual, they follow from the regularity on the symmetry axis and symmetry requirements as well as the condition of finiteness of the energy and the topology. In particular, we have to take into account that the magnetic field is vanishing on the spacial asymptotic. Explicitly, we impose

$$\phi^1|_{r\to\infty} \to 0, \quad \phi^2|_{r\to\infty} \to 0, \quad \phi^3|_{r\to\infty} \to 1,$$
$$\partial_r A_1|_{r\to\infty} \to 0, \quad A_2|_{r\to\infty} \to 0, \quad (11)$$

at infinity and

$$\phi^1|_{r\to 0} \to 0, \quad \phi^2|_{r\to 0} \to 0, \quad \phi^3|_{r\to 0} \to 1,$$
$$A_1|_{r\to 0} \to 0, \quad A_2|_{r\to 0} \to 0, \quad (12)$$

at the origin. The condition of regularity of the fields on the symmetry axis yields

$$\phi^1|_{\theta\to 0,\pi} \to 0, \quad \phi^2|_{\theta\to 0,\pi} \to 0, \quad \phi^3|_{\theta\to 0,\pi} \to 1,$$
$$A_1|_{\theta\to 0,\pi} \to 0, \quad A_2|_{\theta\to 0,\pi} \to 0. \quad (13)$$

III. NUMERICAL RESULTS

The numerical calculations are mainly performed on an equidistant grid in spherical coordinates r and θ, employing the compact radial coordinate $x = r/(1+r) \in [0:1]$ and $\theta \in [0, \pi]$. To find solutions of the Euler-Lagrange equations which follow from the Lagrangian (5) and depend parametrically on the coupling constant g, we implement a simple forward differencing scheme on a rectangular lattice with lattice spacing $\Delta x = 0.01$. Typical grids used have sizes 120×70. The relative errors of the solutions are of order of 10^{-4} or smaller. We also introduce an additional

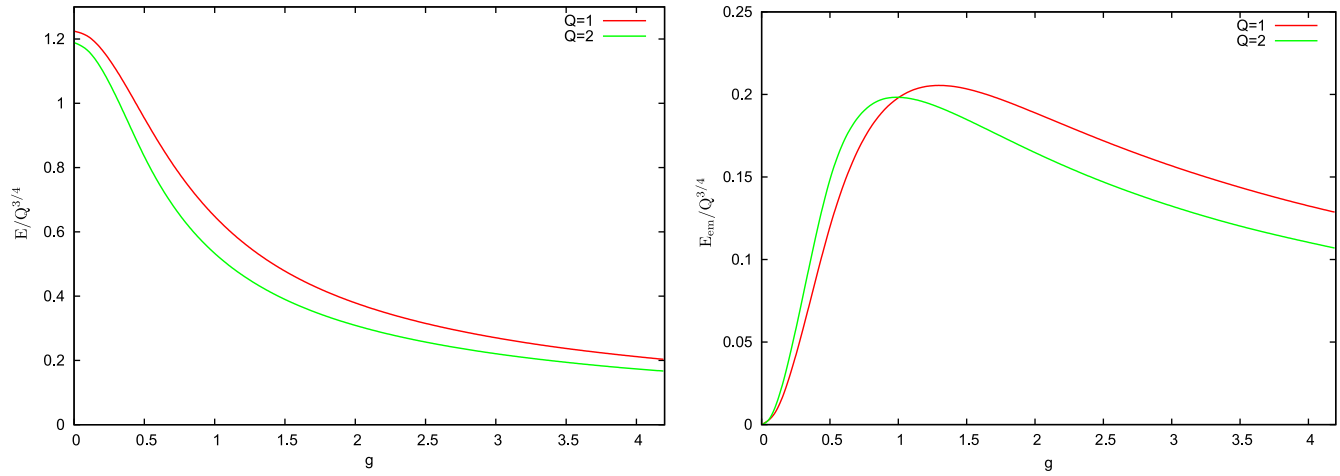

FIG. 1 (color online). Left: The normalized energy E of the $\mathcal{A}_{1,1}$ and $\mathcal{A}_{2,1}$ gauged Hopfions; Right: The corresponding magnetic energy as a function of the coupling constant g at $\mu = 0$.

Lagrangian multiplier to constrain the field to the surface of unit sphere.

Each of our simulations began at $g = 0$ at fixed value of μ, then we proceed by making small increments in g.

In Fig. 1 we have plotted the graphs of energy of gauged Hopfion defined by the functional (6), and magnetic energy as function of the gauge coupling. Here we used the normalized units of energy and took into account the Ward's conjecture [34] concerning the lower energy bound, in these units it becomes $E \geq Q^{3/4}$.

As the gauge coupling increases from zero, the energy of the gauged Hopfion decreases since the toroidal magnetic flux is formed. The size of the configuration is decreasing, it becomes deformed as shown in Fig. 2 where we exhibited the energy density isosurfaces of the gauged $\mathcal{A}_{1,1}$ and $\mathcal{A}_{2,1}$ Hopfions at $g = 0$ and $g = 2$, respectively. This pattern is in agreement with behavior of the solitons in the $SO(3)$ gauged Skyrme model [35].

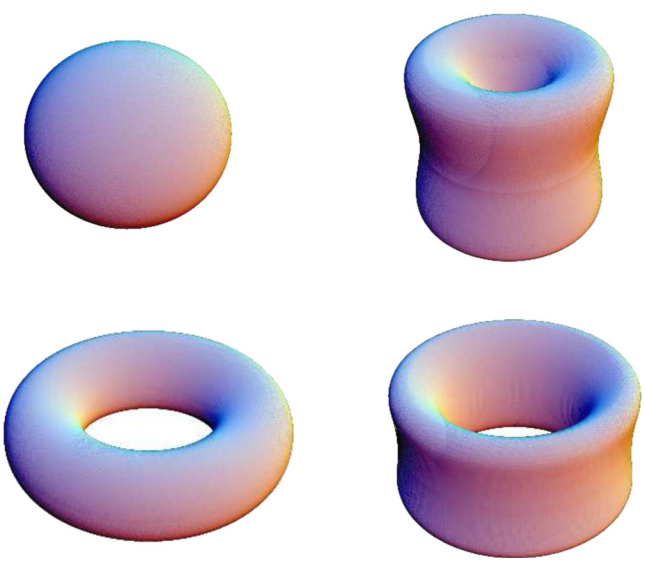

FIG. 2 (color online). Top row: energy density isosurfaces for gauged solitons $\mathcal{A}_{1,1}$ at $g = 0$ (left plot) and $g = 2$ (right plot). Bottom row: energy density isosurfaces for gauged solitons $\mathcal{A}_{2,1}$ at $g = 0$ (left plot) and $g = 2$ (right plot).

Note that as the coupling remains smaller than one, the electromagnetic energy E_{em} defined by (7) is increasing, however in the strong coupling limit its contribution begins to decrease as g continues to grow, see Fig. 1, right plot. We can understand this effect if we note that the conventional rescaling of the potential $A_\mu \to gA_\mu$ leads to $F_{\mu\nu}^2 \to \frac{1}{g^2} F_{\mu\nu}^2$. Thus, the very large gauge coupling effectively removes the Maxwell term leaving the limiting configuration of gauged Hopfion coupled to a circular magnetic vortex of constant flux. Apparently, in such a limit the strong coupling with a vortex yields an effective "mass term" $g^2(A_1^2 + A_2^2)[1 - (\phi^3)^2]/r^2$. Unlike the usual symmetry breaking term in (1), it affects both the gauge potential and the field components ϕ^1 and ϕ^2 which became massive due to the coupling to the gauge sector.

On the other hand, the gauge field remains massless on the symmetry axis since $\phi^1 = \phi^2 = 0$ at $\theta = 0, \pi$, because of the boundary condition (13) we imposed there. Furthermore, it is massless along the position curve of the Hopfion where $\phi^3 = -1$, thus a magnetic flux should also appear there.

Indeed, in Fig. 3 we display the results of our numerical calculations of the magnetic field of the gauged Hopfion in the xz plane and in the xy plane. Evidently, in the weak coupling regime, this is a toroidal field which encircles the position curve. The flux of such a field is not quantized since there is no topological reason for that. As the gauge coupling increases, the vortex is getting smaller and the magnitude of the magnetic field increases significantly. Effectively, using the Maxwell equation (9), one can set this magnetic field into correspondence with a circular electric current $\vec{j}$ [36]. Note that there is an interesting similarity

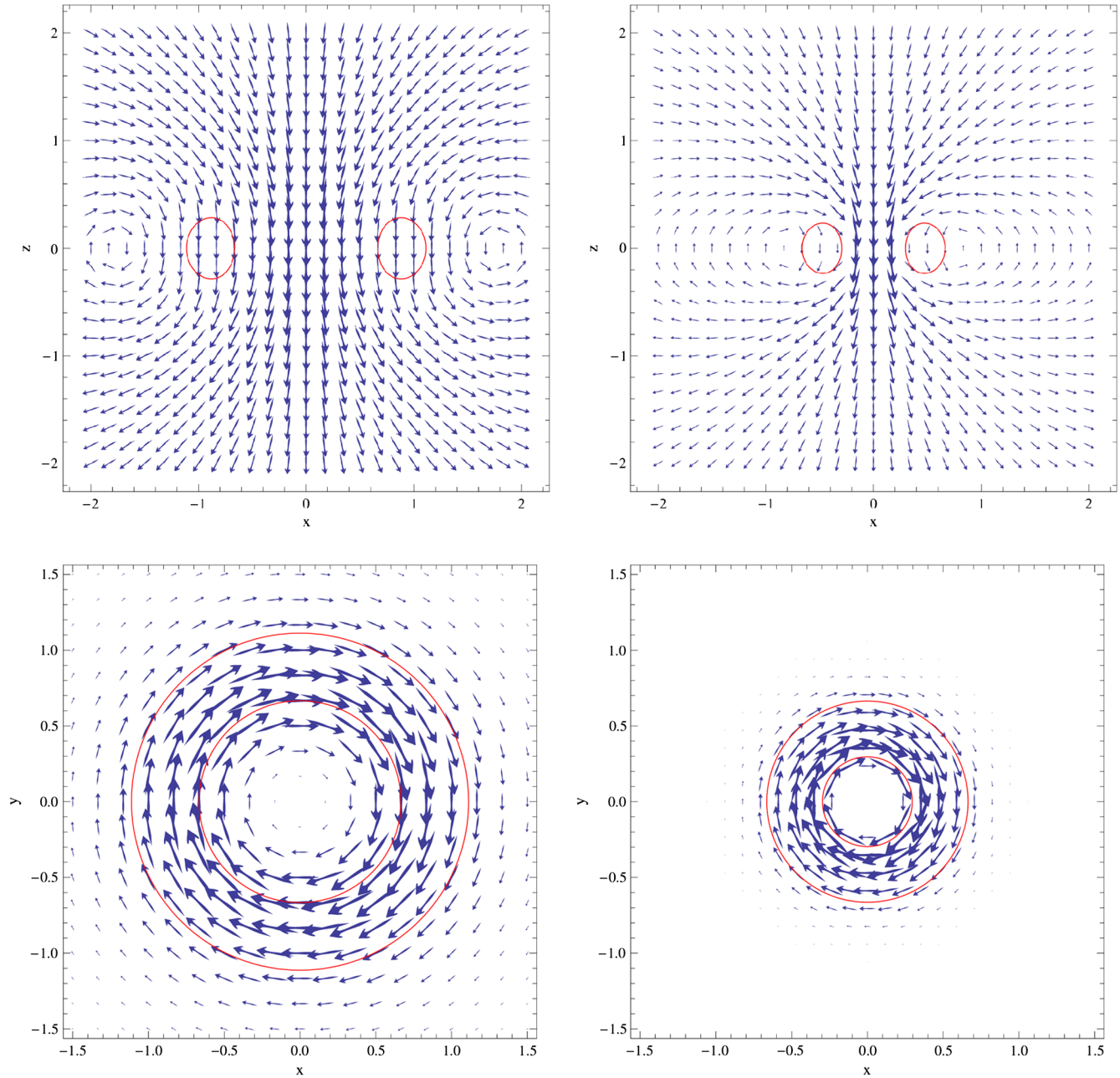

FIG. 3 (color online). Magnetic field orientation of the gauged $\mathcal{A}_{1,1}$ Hopfion. Top row: The magnetic flux in the xz plane at $g = 0.1$ (left) and $g = 2$ (right). The red profile indicates the position curve of the gauged Hopfion given by the isosurface of the form $\phi^3 = -0.80$. Bottom row: The magnetic flux in the xy plane at $g = 0.1$ (left) and $g = 2$ (right). The position curve is indicated by the solid closed curves.

between the magnetic flux around the gauged Hopfion and the magnetic field generated by the vortexlike configurations in the $SU(2)$ Yang-Mill-Higgs system [37–39]

It is instructive to compare our results with the pattern of evolution of the gauged baby Skyrmions [14]. In the latter case the solitons also carry magnetic flux $\Phi = g\int_{\mathbb{R}^2} B$ which is in general, nonquantized. The flux of the gauged baby Skyrmions is associated with the position of the solitons; it is orthogonal to the xy plane. An interesting observation is that as the gauge coupling grows, the magnetic flux of the degree n baby Skyrmions varies from 0 to $-2\pi n$, i.e., in the strong coupling regime the magnetic flux is quantized though there are no topological reasons for it.

Since the axially symmetric Hopfions can be thought of as planar Skyrmions placed along a twisted closed string [19,40], this picture is certainly consistent with our results. Indeed, as shown in Fig. 3, the circular magnetic flux is

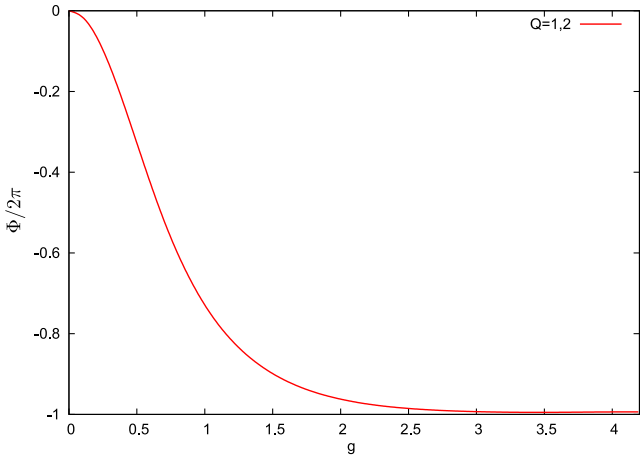

FIG. 4 (color online). The magnetic flux in units of 2π through the $z\rho$-plane as a function of the coupling constant g for the solutions of degree $Q = 1, 2$.

orthogonal to the xz plane, although its radius is slightly larger than the radius of the position curve of the Hopfion. Certainly, the total flux through the xz plane is zero, in order to evaluate the magnitude of the flux we have to consider the xz half-plane or, equivalently, the $z\rho$ plane. We found that in the strong coupling limit the magnetic flux of the gauged Hopfion through the $z\rho$ plane becomes quantized in units of 2π, see Fig. 4.

A particularly interesting observation is that the total circular flux of the configurations of degrees $Q = 1, 2$ through the $z\rho$ plane shows the same dependence on the coupling constant, independently of the Hopf degree of the soliton. We can understand this pattern when we recall that, as it was mentioned above, the axially symmetric configuration of the type $\mathcal{A}_{m,n}$ can be thought of as composed from the planar baby Skyrmion of charge n twisted m times along the circle. Thus, the circular magnetic flux is associated with the planar charge $n = 1$, which is the same in both cases we are considering.

The situation changes in the strong coupling limit $g \geq g_{cr} \sim 3$. Then the contribution of the Maxwell term becomes negligible and the condition of regularity of the energy functional (6) is satisfied if $D_k\phi^a = 0$ as $r \to \infty$. Furthermore, our simulations show that at the large coupling the component $A_\varphi(r, \theta)$ develops a sharp plateau $gA_\varphi = -m$ in the vicinity of the position curve. Here the integer m, as it is defined in (3), corresponds to the number of twists along the position curve. The plateau further extends as the gauge coupling grows. On the other hand, in the strong coupling limit the position curve itself expands from a circle S^1 to some region. In some sense it resembles the Meissner effect—at critical value of the gauge coupling $g_{cr} \sim 3$ the magnetic field is expelled from the Hopfion.

Indeed, in this region the covariant derivative in azimuthal direction $D_\varphi\phi^a = 0$ and the component A_φ is a pure gauge, i.e., $gA_\varphi = iU\partial_\varphi U^{-1}$, where $U = e^{im\varphi}$. Clearly this corresponds to the linear string of magnetic flux through the center of the Hopfion, which is quantized in units of 2π and carries m quanta.

We illustrated this observation in Fig. 5, where both the profiles of the rescaled component of the potential A_φ in units of g/m and the component ϕ^3 in the xy plane are shown in the weak and strong coupling regimes.

Hence, the corresponding integrated flux through the Hopfion along the symmetry axis is $-2\pi m$. Similar to the flux through the xz plane, which encircles the Hopfion, it becomes quantized in the strong coupling regime. We conclude that the gauged Hopfion carries *two* magnetic fluxes, the first circular flux encircles the position curve while the second one is directed along the third axis. In the strong coupling regime they both are quantized in units of 2π and carry n and m quanta, respectively.

Evidently, the appearance of the effective quantization of the magnetic field matches the underlying topology of the Hopfion configuration. This observation may be used to identify related topological invariants in the Maxwell sector of the gauged model. This mechanism could allow us to reconsider the usual arguments concerning implementation of the Protogenov-Verbus topological bound [18]. Physically, appearance of the quantized magnetic flux

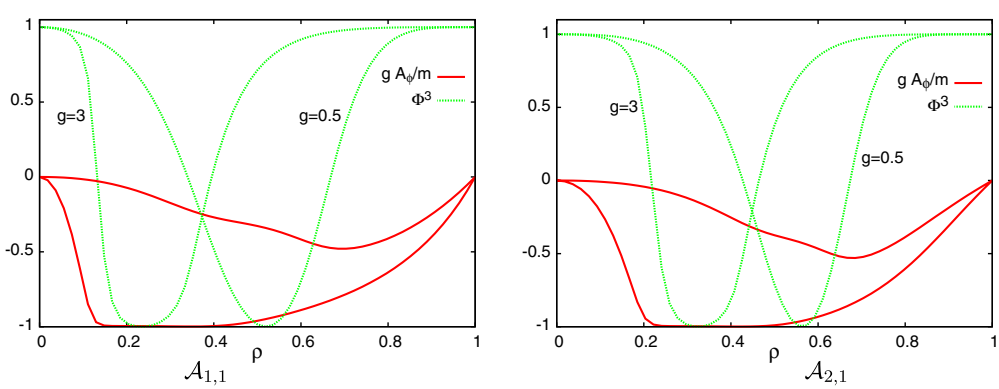

FIG. 5 (color online). The magnetic flux $gA_\phi(\tilde{x}, 0)/m$ (solid line) and the scalar field component $\phi^3(\tilde{x}, 0)$ (dashed line) of the gauged Hopfions of degree $Q = 1, 2$ in the xy plane; the radial variable ρ is compactified onto the unit interval, $\tilde{x} = \rho/(1 + \rho) \in [0, 1]$.

through the center of the Hopfion may yield a constraint which could affect the usual scaling arguments [18].

IV. CONCLUSIONS

The main purpose of this work was to present a new type of gauged solitons in the Faddeev-Skyrme-Maxwell theory. Our consideration is restricted to the simple axially-symmetric Hopfions $\mathcal{A}_{m,n}$ of lower degree $Q = mn$. Similar to the corresponding solutions in the Skyrme model they are topologically stable, in the weak coupling regime they carry noninteger toroidal magnetic flux. In the strong coupling regime the configuration is associated with two magnetic fluxes, one of which represents a circular vortex, and the second one is orthogonal to the position curve. In this limit we observe an effective quantization of both fluxes, the first flux is quantized in units of the winding number n and the second flux is quantized in units of m, respectively.

Certainly, this is a first step toward complete investigation of the gauged Hopfions. Clearly, this study should be extended to the Hopfions of higher degrees and different geometry. One can expect by analogy with isorotations of the solitons of the Skyrme systems (see [26,27] and [41,42]) that the coupling of the Hopfions to the electromagnetic field may drastically affect their structure. Since for a given degree Q there are several different soliton solutions of rather similar energy and the number of solutions seems to grow with Q [21], various bifurcations may occur as the gauge coupling varies. Further, we cannot exclude possible instabilities of gauged Hopfions at some critical coupling.

We do not investigate here possible effects of the potential term in (1) on the properties of the gauged Hopfions. We can expect, by analogy with the case of the gauged baby Skyrme model [14] that the results may strongly depend on the ratio μ^2/g. On the other hand, the limiting truncated Faddeev-Skyrme-Maxwell system which appears in the strong coupling limit should be considered in depth, a possible interplay between the mass generating terms and the potential may drastically affect the stability of the configuration.

An important feature of the Faddeev-Skyrme model is that the energy of the Hopfions is related with the topological degree via the Vakulenko-Kapitanski bound $E \geq Q^{3/4}$ [7]. We expect that the coupling to the Maxwell field will affect this relation, in particular, the solutions may approach the topological bound. This problem will be investigated elsewhere.

As a direction for future work, it would be interesting to study spinning gauged Hopfions, these electrically charge solutions with an intrinsic angular momentum can be constructed by analogy with similar configuration in the gauged Skyrme model [18].

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