Uniform WKB, multi-instantons, and resurgent trans-series

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We illustrate the physical significance and mathematical origin of resurgent trans-series expansions for energy eigenvalues in quantum mechanical problems with degenerate harmonic minima, by using the uniform WKB approach. We provide evidence that the perturbative expansion, combined with a global eigenvalue condition, contains all information needed to generate all orders of the nonperturbative multi-instanton expansion. This provides a dramatic realization of the concept of resurgence, whose structure is naturally encoded in the resurgence triangle. We explain the relation between the uniform WKB approach, multi-instantons, and resurgence theory. The essential idea applies to any perturbative expansion, and so is also relevant for quantum field theories with degenerate minima which can be continuously connected to quantum mechanical systems.

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I. INTRODUCTION

A. Why are resurgent trans-series important?

In a large variety of quantum theoretical settings, it is well known that perturbative (P) and nonperturbative (NP) physics are closely related [1–4]. In quantum mechanical systems with degenerate (harmonic) minima, perturbation theory leads to divergent and nonalternating series [5–8].[1] This leads to two interrelated fundamental problems:

(i) analysis of these divergent series (for example, by Borel summation) leads to imaginary contributions to observables (such as energy) that must be real;

(ii) this Borel summation procedure is ambiguous, with the ambiguity manifest in the sign of the imaginary nonperturbative contributions [5–7,10–14].

Resurgent trans-series analysis resolves these two problems, producing an expression for the observable (such as energy) that is real and unambiguous. This approach unifies the perturbative (P) series with a sum over all nonperturbative (NP) contributions, forming a so-called "trans-series" expression, and the various terms in this trans-series are connected by an infinite ladder of intricate interrelations which encode the cancellation of all imaginary and ambiguous terms [15–17]. We refer to this generalized notion of summability as *Borel-Écalle summability.* For example, the leading ambiguous imaginary term arising from a Borel analysis of the divergent perturbative series is of order $\pm i\pi e^{-2S_I/g^2}$. This is exactly canceled by an identical term in the instanton–anti-instanton amplitude, $[\mathcal{I}\bar{\mathcal{I}}]_\pm \sim e^{-2S_I/g^2} \pm i\pi e^{-2S_I/g^2}$, whose imaginary part is also ambiguous, and which lives in the nonperturbative part of the trans-series. We refer to this cancellation mechanism in quantum mechanics as the Bogomolny-Zinn-Justin (BZJ) mechanism [6,10]. A very important aspect of the theory of "resurgence" is the statement that these cancellations occur to all NP orders, including P fluctuations around NP saddles. Thus the full trans-series is real, unique and unambiguous [14].

This beautiful BZJ mechanism of cancellation of ambiguities between non-Borel-summable perturbation theory and the nonperturbative multi-instanton sector has been explored in some detail for quantum mechanics (QM) problems with degenerate minima [5–7,10–12,18–21], but in fact this resurgent structure is a general property of perturbation theory that is also relevant for quantum field theory (QFT), in particular when there are degenerate classical vacua. For example, in asymptotically free quantum field theories such as 4D $SU(N)$ gauge theory or 2D $\mathbb{CP}^{N-1}$ theories there are infrared renormalons that lead to non-Borel summability of perturbation theory. This is a serious problem, because it means that perturbation theory on its own is ill defined, just as is the case for the QM problems with degenerate minima. Until recently it was not known how to cancel the resulting ambiguous imaginary parts against nonperturbative amplitudes, because for both 4D $SU(N)$ gauge theory and 2D $\mathbb{CP}^{N-1}$, the IR renormalons lead to nonperturbative effects with exponential factors having exponents depending parametrically on N as $2S_I/N$, and such nonperturbative factors do not appear in these theories defined on $\mathbb{R}^4$ or $\mathbb{R}^2$, respectively [22,23]. However, a resolution of this problem has recently been

[1] In this paper, we are concerned with quantum mechanical systems which only admit real instantons. In more generic cases where there are both real and complex saddles, the connection between perturbation theory and nonperturbative saddles is more involved. An example of this type of more general problem is discussed in [9].

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proposed [24,25], motivated by another problem in the nonperturbative sector, which is that the instanton gas analysis (which works well for QM) is inconsistent for these QFTs defined on $\mathbb{R}^4$ or $\mathbb{R}^2$. The dilute instanton gas approximation assumes that the interinstanton separation is much larger than the size of the instanton, while classical scale invariance implies that instantons of arbitrary size come with the same action (leading to uncontrolled infrared divergences); hence the assumption is invalid. A regularization of the QFT by spatial compactification (either twisting the boundary conditions or center-stabilizing deformation) at a weak-coupling semiclassical regime produces fractionalized instantons, "molecules" of which are associated with nonperturbative factors of the form $e^{-2S_I/(g^2N)}$. This is appropriate for canceling the ambiguities from the semiclassical realization of IR renormalon singularities. For $\mathbb{CP}^{N-1}$ models, the N dependence matches precisely the N dependence coming from the IR renormalons [25], while for 4D gauge theory the dependence is parametrically correct [24].

Since this is a new type of QFT argument, using resurgent analysis to relate the IR renormalon problem of perturbation theory in asymptotically free theories with the IR divergence of the nonperturbative instanton gas, and trans-series expansions are still somewhat unfamiliar in much of the physics community, this paper is designed to be a simple pedagogical introduction to the physical origin of trans-series expansions. Our presentation is mainly in terms of two important quantum mechanical examples, the double-well and Sine-Gordon potentials, since these contain already much of the physics relevant for the discussion of nonperturbative effects due to degenerate minima in gauge theories and $\mathbb{CP}^{N-1}$ models. In fact, these field theories can be continuously connected to the quantum mechanical systems with periodic potentials. However, beyond our pedagogical presentation, we also make a new observation. For these theories (and others listed below), we show in explicit detail that

(i) The perturbative series contains *all* information about the nonperturbative sector, to all nonperturbative orders.

(ii) Perturbation theory around the perturbative vacuum and fluctuations about all nonperturbative saddles (multi-instantons) are interrelated in a precise manner: high orders of fluctuations about one saddle are determined by low orders about "nearby" saddles (in the sense of action).

These are extremely nontrivial facts, providing clear and direct illustrations of the surprising power of resurgent analysis. The first point was observed previously in the double-well system [26], but here we show that the result is more general [27].

There is some body of work concerning trans-series expansions for wave functions, special functions and solutions to Schrödinger-like equations, as well as nonlinear differential equations [16,17,28,29]. Since we are motivated by attempts to compute QFT quantities such as a mass gap, to be very concrete we focus on energy eigenvalues, rather than on wave functions, but these approaches are obviously closely related. There is also an important set of ideas concerning exact quantization conditions [30–32], although these have mostly been investigated for QM potentials without degenerate vacua. We also stress that the basic idea of resurgent trans-series analysis is much more general, applying to both linear and nonlinear problems, and therefore should be applicable to functional problems like QFT, matrix models and string theory [24,25,33–35].

B. Where do the trans-series come from?

In this paper we concentrate on trans-series expressions for energy eigenvalues in certain QM problems, with a coupling constant g^2. Our notation is chosen to match the coupling parameter g^2 in certain QFTs such as Yang-Mills or $\mathbb{CP}^{N-1}$ models. The general perturbative expansion of an energy level has the form

$$E^{(N)}_{\text{pert.theory}}(g^2) = \sum_{k=0}^{\infty} g^{2k} E_k^{(N)} \tag{1}$$

where N is an integer labeling the energy level, and the perturbative coefficients $E_k^{(N)}$ can be computed by straightforward iterative procedures. For the cases we study here, potentials with degenerate harmonic vacua, this perturbative expansion is not Borel summable, which means that *on its own* it is incomplete, and indeed inconsistent.

This situation can be remedied by recognizing that the full expansion of the energy at small coupling is in fact of the "trans-series" form [4,15–20]:

$$E^{(N)}(g^2) = E^{(N)}_{\text{pert.theory}}(g^2) + \sum_{\pm}\sum_{k=1}^{\infty}\sum_{l=1}^{k-1}\sum_{p=0}^{\infty} \underbrace{\left(\frac{1}{g^{2N+1}}\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k-instanton}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zeromode}} \underbrace{c^{\pm}_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \tag{2}$$

In (2) we have artificially separated the perturbative expansion in the zero-instanton sector. The second part of the trans-series involves a sum over powers of nonperturbative factors $\exp[-c/g^2]$, multiplied by prefactors that are themselves series in g^2 and in $\ln(\pm 1/g^2)$. The basic building blocks of the trans-series, g^2, $\exp[-c/g^2]$ and $\ln(-1/g^2)$, are called "trans-monomials," and are familiar from QM and QFT. In physical terms, the trans-series is a

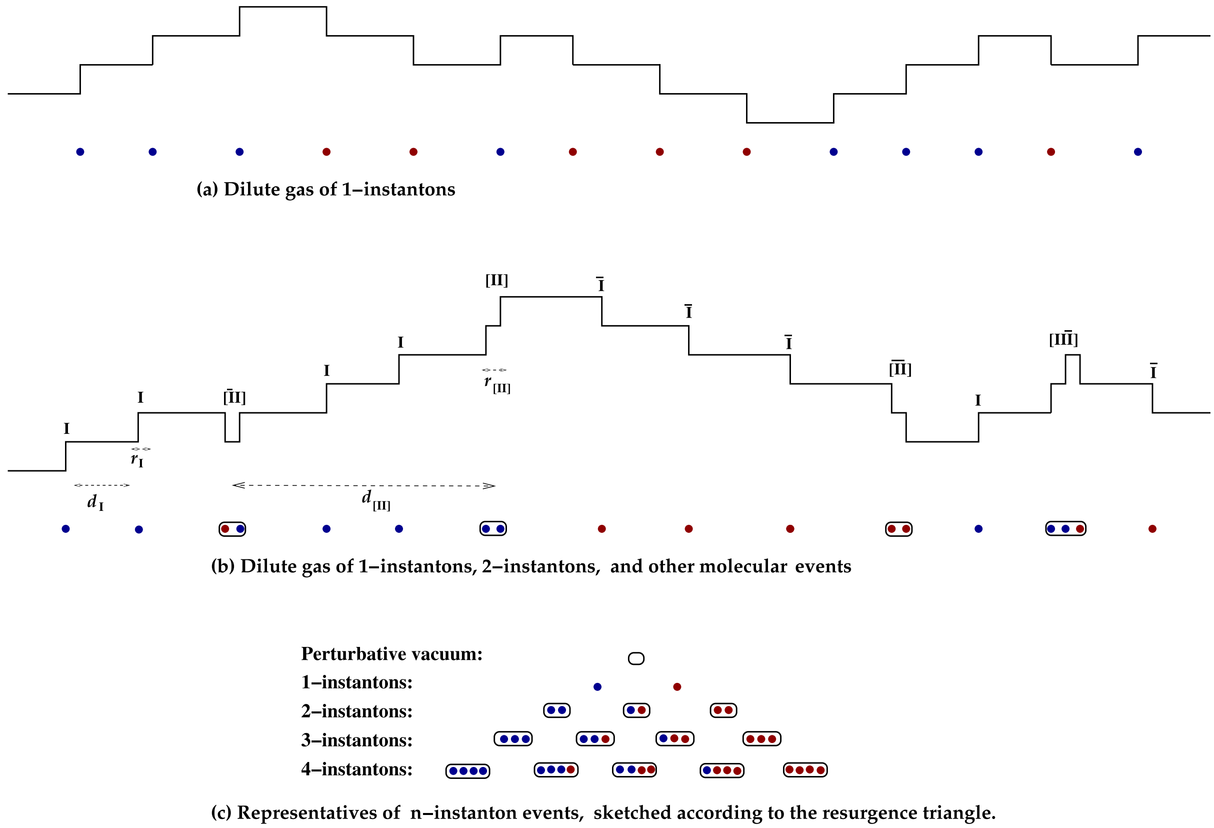

FIG. 1 (color online). (a) Dilute gas of 1-instantons for a periodic potential (as given in typical textbook treatment). (b) Dilute gas of 1-instantons, 2-instantons, 3-instantons, etc. 2-instanton events (topological molecules) such as $[\mathcal{I}\mathcal{I}]$, $[\bar{\mathcal{I}}\bar{\mathcal{I}}]$, $[\mathcal{I}\bar{\mathcal{I}}]$ are rarer, but present. The amplitude associated with neutral 2-instantons or any other k-instanton with a neutral 2-instanton subcomponent is multifold ambiguous. This ambiguity cures the ambiguity of perturbation theory around the perturbative vacuum. (c) n-instanton events classified according to homotopy (columns) and resurgence (refined structure in each column). This picture is the result of uniform WKB and a multi-instanton approach.

sum over instanton contributions, with the perturbative fluctuations about each instanton, and logarithmic terms coming from quasizero modes (QZMs). A trans-series therefore combines perturbation theory with a dilute gas of 1-instantons, 2-instantons, 3-instantons, etc.[2] Note that in a typical textbook treatment, only the proliferation of 1-instanton events is accounted for. However, in order to make sense of (i.e. to define consistently) the semiclassical expansion, one needs to take into account a dilute gas of both 1-instantons and k-instantons, where $k \geq 2$. See Fig. 1 for a snapshot of the Euclidean vacuum of the theory for the case of periodic potential. The subfigure shows examples of k-instantons (molecular events).

At second order in the instanton expansion, quasizero-mode logarithms are first generated. Remarkably, the expansion coefficients $c_{k,l,p}$ of the trans-series are intertwined amongst themselves, and also with the coefficients of the perturbative expansion, in such a way that the total trans-series is real and unambiguous. This intertwining can be represented graphically by the "resurgence triangle" introduced in [25], shown in Fig. 1(c), and discussed in detail below for both the double-well and Sine-Gordon potentials. For example, a Borel analysis of the perturbative series requires an analytic continuation in g^2, producing nonperturbative imaginary parts, but these are precisely canceled by the imaginary parts associated with the $\ln(-1/g^2)$ factors in the nonperturbative portion of the trans-series. This applies not just at leading order, but to all subsequent orders arising from Borel summation of the divergent fluctuation expansion around any instanton sector. Ambiguities only arise if one looks at just one isolated portion of the trans-series expansion, for example just the perturbative part, or just some particular multi-instanton sector. When viewed as a whole, the trans-series expression is unique and exact. We call this generalized summability of a non-Borel summable series Borel- Écalle summability [15].

We have three main goals in this paper:

(1) Explain in a simple manner how such a trans-series expansion (2) arises, and also in what sense it is generic.

[2] n-instanton is a correlated n-event, and should be distinguished from uncorrelated n 1-instanton events.

(2) Explain the origin of the interrelations within the trans-series, and their physical consequences.
(3) In its strongest form, "resurgence" claims that complete knowledge of the perturbative series is sufficient to generate the remainder of the trans-series, including all orders of the nonperturbative expansion. We show here in simple and explicit detail how this works in QM systems with degenerate harmonic vacua.

We comment that there is not yet universal agreement in the mathematical literature concerning the rigorous proof of the generality of trans-series expansions for resurgent functions. References [19] contain proofs, but in a recent talk Kontsevich has raised questions about the rigor of mathematical results concerning resurgent functions [36]. Nevertheless, each of the trans-series monomials has a clear physical meaning, and here we show using relatively elementary techniques (uniform WKB) that the energy eigenvalues have precisely this trans-series structure in QM systems with degenerate harmonic vacua. Moreover, these trans-monomial elements also have clear physical meaning in quantum field theory.

II. UNIFORM WKB FOR POTENTIALS WITH DEGENERATE MINIMA

A. The spectral problem

Consider the spectral problem

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x). \tag{3}$$

We are interested in cases where the potential $V(x)$ has degenerate minima, which are locally harmonic: $V(x)\approx x^2 + \cdots$. The two paradigmatic cases we study in detail are the double-well (DW) and Sine-Gordon (SG) potentials:

$$V_{\mathrm{DW}}(x) = x^2(1+gx)^2 = x^2 + 2gx^3 + g^2x^4 \tag{4}$$

$$V_{\mathrm{SG}}(x) = \frac{1}{g^2}\sin^2(gx) = x^2 - \frac{1}{3}g^2x^4 + \cdots \tag{5}$$

The Sine-Gordon case can be directly related to the Mathieu equation by simple changes of variables. This permits detailed comparison with known results for Mathieu functions [37,38].

It is convenient to rescale the coordinate variable to $y = gx$:

$$-g^4\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2E\psi(y) \tag{6}$$

where

$$V_{\mathrm{DW}}(y) = y^2(1+y)^2 \tag{7}$$

$$V_{\mathrm{SG}}(y) = \sin^2(y). \tag{8}$$

It is well known that in both these cases the perturbative energy levels are split by nonperturbative instanton effects. This level splitting is (at leading order) a single-instanton effect, and is textbook material [4,39–41]. From (6) we see that g^4 plays the role of $\hbar^2$, and so we expect these nonperturbative effects to be characterized by exponential factors of the form

$$\exp\left(-\frac{c}{g^2}\right) \tag{9}$$

for some constant $c > 0$.

More interestingly, the perturbative series for these spectral problems is non-Borel-summable, and in the Borel-Écalle approach is defined by the analytic continuation $g^2 \to g^2 \pm i\epsilon$, which induces a nonperturbative imaginary part, even though both potentials are completely stable and the energy should be purely real. As mentioned in the Introduction, the resolution of this puzzle is the Bogomolny-Zinn-Justin mechanism: the nonperturbative imaginary part is in fact at the two-instanton order, and is canceled by a corresponding nonperturbative imaginary contribution coming from the instanton/anti-instanton amplitude [4,6,10,11]. The resurgent trans-series expression (2) for the energy eigenvalue encodes the fact that there is an infinite tower of such cancellations, thereby relating properties of the perturbative sector and the nonperturbative sector. The BZJ cancellation is the first of this tower. A new observation we make here (see Sec. V) is that we do not need to compute separately the perturbative and nonperturbative sectors: in fact, the perturbative series encodes *all* information about the nonperturbative sector, to all nonperturbative orders.

B. Strategy of the uniform WKB approach

Before getting into details, we first state our strategy, and the basic result, which explains already why the expression for the energy eigenvalues has the trans-series form in (2). Since the potentials we consider have degenerate harmonic vacua, in the $g^2 \to 0$ limit each classical vacuum has the form of a harmonic oscillator well. Therefore it is natural to use a parabolic *uniform WKB* ansatz for the wave function [26,42–45]:

$$\psi(y) = \frac{D_\nu(\frac{1}{g}u(y))}{\sqrt{u'(y)}}. \tag{10}$$

Here D_ν is a parabolic cylinder function [37] (the solution to the harmonic problem), and ν is an ansatz parameter that is to be determined. When $g^2 = 0$ we would have an isolated harmonic well, and ν would be an integer N. For $g^2 > 0$, we find that ν is close to an integer [see (13) below].

Substituting this uniform WKB ansatz form (10) of the wave function into the Schrödinger equation (6) produces a nonlinear equation for the argument function $u(y)$, and this equation can be solved perturbatively. Purely local analysis in the immediate vicinity of the potential minimum, where the potential is harmonic, leads to a perturbative expansion of the energy (explained in Sec. II C below):

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu). \tag{11}$$

The coefficient $E_k(\nu)$ depends on the as-yet-undetermined ansatz parameter ν. In fact, $E_k(\nu)$ is a polynomial in ν, of degree $(k+1)$. In the $g^2 \to 0$ limit, the ansatz parameter ν tends to an integer N, labeling the unperturbed harmonic oscillator energy level. Indeed, when $\nu = N$, the expansion (11) coincides precisely with standard Rayleigh-Schrödinger perturbation theory:

$$E(\nu = N, g^2) \equiv E^{(N)}_{\text{pert.theory}}(g^2). \tag{12}$$

This perturbative series expression is incomplete, and indeed ill defined, because the series is not Borel summable. The fact that it is incomplete should not be too surprising because so far the analysis has been purely *local*, making no reference to the existence of neighboring degenerate classical vacua. To fully determine the energy we must impose a *global* boundary condition that relates one classical vacuum to another. When we do this we learn that ν is only exponentially close to the integer N, with a small correction $\delta\nu$ that is a function of both N and g^2:

$$\nu_{\text{global}}(N, g^2) = N + \delta\nu(N, g^2). \tag{13}$$

The explicit form of the correction term $\delta\nu(N, g^2)$ is derived and discussed below in Sec. III. For now we state that generically it has a trans-series form:

$$\delta\nu(N, g^2) = \sum_{\pm} \sum_{k=1}^{\infty} \sum_{l=1}^{k-1} \sum_{p=0}^{\infty} d^{(\pm)}_{k,l,p} \left(\frac{1}{g^{2N+1}} \exp\left[-\frac{c}{g^2} \right] \right)^k \times \left(\ln\left[\mp \frac{1}{g^2} \right] \right)^l g^{2p}. \tag{14}$$

We show in Sec. III that this form follows directly from properties of the parabolic cylinder functions, and so it is generic to problems having degenerate vacua that are harmonic.[3]

Having solved the global boundary condition to determine the parameter ν as a function of N and g^2, as in (13) and (14), to obtain the corresponding energy eigenvalue we insert this value $\nu_{\text{global}}(N, g^2)$ back into the perturbative expansion (11) for the energy, leading to the final exact expression for the energy eigenvalue:

$$E^{(N)}(g^2) = E(N + \delta\nu(N, g^2), g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(N + \delta\nu(N, g^2)). \tag{15}$$

Reexpanding the polynomial coefficients $E_k(N+\delta\nu(N,g^2))$ for small coupling g^2, we obtain the trans-series expression (2) for the Nth energy level, $E^{(N)}(g^2)$:

$$E_N(g) = E(N, g) + (\delta\nu) \left[\frac{\partial E}{\partial \nu} \right]_N + \frac{(\delta\nu)^2}{2} \left[\frac{\partial^2 E}{\partial \nu^2} \right]_N + \cdots \tag{16}$$

We stress that this uniform WKB approach makes it very clear why the trans-series form of the energy is generic for problems with degenerate harmonic classical vacua: all properties of the $g^2 \to 0$ limit reduce to properties of the parabolic cylinder functions, which lead directly to the trans-series form for $\delta\nu(N, g^2)$ in (14). In particular, all analytic continuations needed to analyze questions of resurgence and cancellation of ambiguities can be expressed in terms of the known analytic continuation properties of the parabolic cylinder functions [37].

C. Perturbative expansion of the uniform WKB ansatz

Recalling that the parabolic cylinder function $D_\nu(z)$ satisfies the differential equation [37]

[3]For a curious counterexample to the oft-held belief that non-Borel-summable expansions occur for any potential with degenerate vacua, consider the *nonharmonic* case of two square wells, separated by a distance $1/g$, and with a central barrier of height $1/g^2$. This g dependence is chosen to mimic that of the double-well potential. This is an elementary problem, soluble in terms of hyperbolic trigonometric functions, and an expansion of the eigenvalue condition for small g produces a trans-series expansion, but without any $\ln(-1/g^2)$ terms. Moreover, one finds that the “perturbative” small g^2 expansion is in fact summable. Thus, the trans-series structure is quite different in this nonharmonic case. One could argue that this case is ill defined because the bottom of each well is flat, so there is no real vacuum location, but the same conclusion can be obtained by replacing the square wells by triangular wells, which is also a soluble problem, in terms of Airy functions. Periodic versions of these cases also produce interesting trans-series. Thus, the harmonic nature of the classical vacua is a significant feature of the argument.

$$\frac{d^2}{dz^2}D_\nu(z)+\left(\nu+\frac{1}{2}-\frac{z^2}{4}\right)D_\nu(z)=0 \tag{17}$$

we see that the uniform WKB ansatz (10) converts the Schrödinger equation (6) to the following nonlinear equation for the argument function $u(y)$ appearing in (10):

$$V(y)-\frac{1}{4}u^2(u')^2-g^2E+g^2\left(\nu+\frac{1}{2}\right)(u')^2 +\frac{g^4}{2}\sqrt{u'}\left(\frac{u''}{(u')^{3/2}}\right)'=0. \tag{18}$$

Here u' means du/dy. At first sight, it looks like (18) is more difficult to solve than the original Schrödinger equation (6), but we will see that the perturbative solution of (18) has some advantages over the perturbative solution of (6). We solve (18) for $u(y)$ and E by making simultaneous perturbative expansions:

$$E=E_0+g^2E_1+g^4E_2+\cdots \tag{19}$$

$$u(y)=u_0(y)+g^2u_1(y)+g^4u_2(y)+\cdots \tag{20}$$

Note that the expansion coefficients E_k and $u_k(y)$ also depend on the as-yet-undetermined parameter ν that appears in the ansatz (10), and consequently in the equation (18). This parameter ν is not determined by the local perturbative expansions in (19) and (20); the parameter ν requires global nonperturbative information describing how one perturbative vacuum potential well connects to another. This is discussed below in Sec. III.

1. Leading order: Origin of the usual exponential WKB factor

At zeroth order in g^2 the equation (18) implies

$$u_0^2(u_0')^2=4V\Rightarrow u_0^2(y)=4\int_0^y dy\sqrt{V} \tag{21}$$

where the lower limit is chosen to satisfy the small y limiting behavior of the nonlinear equation (18). In particular, since each well is locally harmonic, $V(y)\approx y^2$, we learn that

$$u_0(y)\approx\sqrt{2}y+\cdots,\qquad y\to 0. \tag{22}$$

Correspondingly, the $O(g^2)$ term in (18) then tells us that the perturbative expansion for the energy begins as

$$E=(2\nu+1)+\cdots \tag{23}$$

The results (22)–(23) are simply reflections of the locally harmonic nature of the $g^2\to 0$ limit.

For the DW and SG potentials, (21) yields

$$\text{DW:}\ u_0(y)=\sqrt{2}y\sqrt{1+\frac{2y}{3}};\qquad \text{SG:}\ u_0(y)=2\sqrt{2}\sin\left(\frac{y}{2}\right). \tag{24}$$

From the asymptotic behavior of the parabolic cylinder function [37], in the $g^2\to 0$ limit we find the expected exponential WKB factor:

$$D_\nu(z)\sim z^\nu e^{-z^2/4},\qquad (z\to+\infty)\Rightarrow\psi(y)\sim\exp\left[-\frac{u_0^2}{4g^2}\right]\sim\exp\left[-\frac{1}{g^2}\int_0^y\sqrt{V}\right]. \tag{25}$$

We discuss the prefactors below in Sec. IV.[4]

2. Higher orders

The higher-order perturbative solution is straightforward but tedious. Imposing the boundary condition of finiteness of $u^2(y)$ at $y=0$, one finds that the energy $E(\nu,g^2)$ has an expansion of the form

$$E(B,g^2)=2B-\sum_{k=1}^{\infty}g^{2k}p_{k+1}(B),\qquad B\equiv\nu+\frac{1}{2} \tag{26}$$

where it proves convenient to express the coefficients in terms of the parameter $B\equiv\nu+\frac{1}{2}$. The leading term is universal [recall (23)], and the coefficients, $p_{k+1}(B)$, of this expansion are polynomials of degree $(k+1)$ in B. Moreover, they have definite parity: $p_k(-B)=(-1)^k p_k(B)$. For example, in the two explicit cases of the double-well and Sine-Gordon potentials:

$$\begin{aligned}E_{\rm DW}(B,g^2)=2B&-2g^2\left(3B^2+\frac{1}{4}\right)-2g^4\left(17B^3+\frac{19}{4}B\right)\\&-2g^6\left(\frac{375}{2}B^4+\frac{459}{4}B^2+\frac{131}{32}\right)\\&-2g^8\left(\frac{10689}{4}B^5+\frac{23405}{8}B^3+\frac{22709}{64}B\right)\\&-\cdots\end{aligned} \tag{27}$$

[4]The relation between the uniform WKB wave function and instanton amplitude is the following: in our normalization of Hamiltonian (3), $m=\frac{1}{2}$. Thus, the BPS bound for the instanton action is $S[y]=\frac{1}{g^2}\int dt(\frac{1}{4}\dot{y}^2+V(y))\geq\frac{1}{g^2}\int_0^{2y_{\text{mid-point}}}dy\sqrt{V}=\frac{S_I}{g^2}$. Thus, according to (25), the leading uniform WKB wave function at $y_{\text{mid-point}}$ is $\psi(y_{\text{mid-point}})\sim e^{-S_I/(2g^2)}$, and is exponentially small; i.e, the value of the uniform WKB wave function at the midpoint between the two harmonic minima (see Figs. 2 or 3) is the square root of instanton fugacity. Also see the discussion around (43).

$$E_{SG}(B,g^2) = 2B - \frac{g^2}{2}\left(B^2+\frac{1}{4}\right) - \frac{g^4}{8}\left(B^3+\frac{3}{4}B\right) - \frac{g^6}{32}\left(\frac{5}{2}B^4+\frac{17}{4}B^2+\frac{9}{32}\right) - \frac{g^8}{128}\left(\frac{33}{4}B^5+\frac{205}{8}B^3+\frac{405}{64}B\right) - \cdots \tag{28}$$

An important observation is that if we replace ν by an integer quantum number N, so that $B = N + \frac{1}{2}$, then the expansions (26)–(28) coincide precisely with the corresponding expansion obtained from standard Rayleigh-Schrödinger perturbation theory about the Nth harmonic oscillator level:

$$E\left(B = N+\frac{1}{2}, g^2\right) = E^{(N)}_{\text{pert.theory}}(g^2). \tag{29}$$

In particular, note that for each $B > 0$, the perturbative expansion in g^2, as in (26)–(28), is a divergent nonalternating series, which is not Borel summable. This fact will be crucial below when we come to discuss the global boundary conditions that connect one perturbative vacuum to another; see Sec. III.

The corresponding perturbative expansion for $u(y)$ [the function that appears in the argument of the parabolic cylinder function in the uniform WKB ansatz (10)] is of the form

$$u(y) = u(y,B,g^2) = \sum_{k=0}^{\infty} g^{2k} u_k(y,B). \tag{30}$$

With respect to its dependence on B, the coefficient function $u_k(y,B)$ is a polynomial of degree k in B, with definite parity: $u_k(y,-B) = (-1)^k u_k(y,B)$. For the DW and SG cases,

$$u_{DW}(y) = \sqrt{2}y\sqrt{1+\frac{2y}{3}} + g^2 B \frac{\ln[(1+\frac{2y}{3})(1+y)^2]}{\sqrt{2}y\sqrt{1+\frac{2y}{3}}} + \cdots \tag{31}$$

$$u_{SG}(y) = 2\sqrt{2}\sin\frac{y}{2} + g^2 B\frac{\ln\cos\frac{y}{2}}{\sqrt{2}\sin\frac{y}{2}} + \cdots \tag{32}$$

Higher-order terms are straightforward to generate but cumbersome to write.

While the perturbative expansion (19) of the energy yields, with the identification $\nu \to N$, exactly the same perturbative series for the energy eigenvalue as Rayleigh-Schrödinger perturbation theory [see (29)], the situation is quite different for the wave-function expansion in (20) and (30). To recover the Rayleigh-Schrödinger perturbation theory wave function for the Nth level, we use the uniform WKB ansatz (10), identify $\nu \to N$, rewrite $y = gx$, and expand in g^2:

$$\psi^{(N)}(x) = \frac{D_N(\frac{1}{g}[u_0(gx)+g^2u_1(gx)+g^4u_2(gx)+\cdots])}{\sqrt{(d/dx)[u_0(gx)+g^2u_1(gx)+g^4u_2(gx)+\cdots]/g}} \equiv \frac{D_N(\sqrt{2}x)}{\sqrt{2}} + g^2\psi_1^{(N)}(x) + g^4\psi_2^{(N)}(x) + \cdots \tag{33}$$

The leading term is the familiar harmonic oscillator wave function for the unperturbed Nth level. Interestingly, if we truncate the perturbative expansion of $u(gx)$ at some order g^{2k}, and use this *inside* the uniform WKB expression (10), we obtain a much better approximation to the wave function than the truncation of the Rayleigh-Schrödinger perturbation theory wave function at the same order g^{2k}. The uniform WKB approximation effectively gives a resummation of many orders of Rayleigh-Schrödinger perturbation theory.

III. GLOBAL BOUNDARY CONDITIONS

A. Relating one minima to another

So far the entire discussion has been local, in the neighborhood of the minimum of one of the classical vacua. To proceed, we need to specify how one classical vacuum relates to another. Here the details of the double-well and Sine-Gordon cases differ slightly, but in each case we impose a global boundary condition at the midpoint of the barrier between two neighboring classical vacua (we restrict ourselves here to symmetric barriers). The result illustrates the physics of level splitting (DW) and band spectra (SG), respectively.

Consider first the DW potential. Each level labeled by the index N splits into two levels due to tunneling between the two classical vacua. To see how this arises, consider $N = 0$ and note that the ground state wave function is a nodeless function, which is therefore an even function about the midpoint between the two wells ($y_{\text{midpoint}} = -\frac{1}{2}$), while the first excited state wave-function (which also has $N = 0$) has one node and is therefore an odd function about this midpoint; see Fig. 2. Thus, the global boundary condition to be imposed at y_{midpoint} is

$$\text{ground state:}\ \psi'_{DW}\left(-\frac{1}{2}\right) = 0 \tag{34}$$

$$\text{first excited state:}\ \psi_{DW}\left(-\frac{1}{2}\right) = 0. \tag{35}$$

Because of the reflection symmetry of the DW potential about the midpoint, in effect we only need to solve the DW

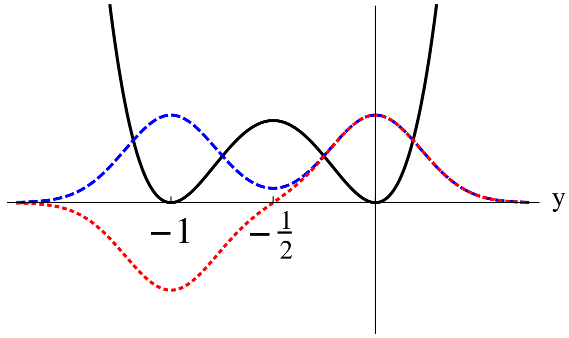

FIG. 2 (color online). The global boundary condition for the lowest two states in the double-well potential $V(y) = y^2(1+y)^2$. The lower state wave function is nodeless and has vanishing derivative at the midpoint of the barrier. The upper state wave function has one node at the midpoint of the barrier.

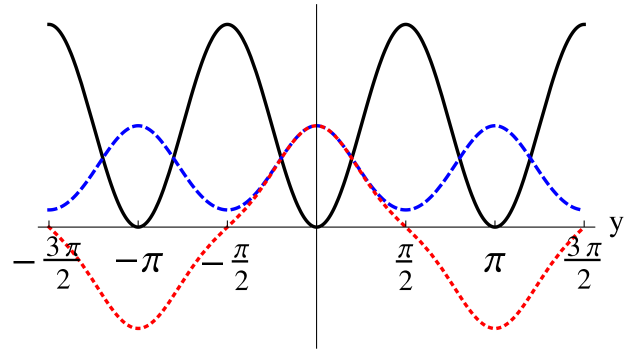

FIG. 3 (color online). The global boundary condition for the band-edge states of the lowest band for the Sine-Gordon potential $V(y) = \sin^2 y$. The lower band-edge wave function is nodeless and has vanishing derivative at the midpoint of each barrier. The upper band-edge wave function has one node at the midpoint of each barrier.

problem in the right-hand half-space, $-\frac{1}{2} \leq y < \infty$, with either a Neumann (ground state) or Dirichlet (first excited state) boundary condition at $y = -\frac{1}{2}$, and in both cases with a Dirichlet boundary condition at $y = +\infty$. For higher energy levels (i.e., for higher values of N), we interchange the Neumann or Dirichlet boundary conditions at $y = -\frac{1}{2}$, according to whether N is odd or even.

For the SG potential, each perturbative level labeled by the index N splits into a continuous band of states. This phenomenon arises from the Bloch condition: $\psi(y+\pi) = e^{i\theta}\psi(y)$, where θ is a real angular parameter $\theta \in [0, \pi]$ that labels states in a given band of the spectrum. This Bloch boundary condition is efficiently expressed in terms of the discriminant, using the standard Floquet analysis [37,41]. Define two independent solutions $w_I(y)$ and $w_{II}(y)$, normalized as follows at some arbitrary chosen point (which we take here to be at $y = -\frac{\pi}{2}$, the center of a barrier between two classical vacua, as in the DW case):

$$\begin{pmatrix} w_I(-\frac{\pi}{2}) & w_I'(-\frac{\pi}{2}) \\ w_{II}(-\frac{\pi}{2}) & w_{II}'(-\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{36}$$

The Bloch condition is expressed in terms of the discriminant, which is itself expressed in terms of the functions w_I and w_{II}' evaluated at a location shifted by one period, for example at $y = +\frac{\pi}{2}$:

$$\cos\theta = \frac{1}{2}\left(w_I\left(\frac{\pi}{2}\right) + w_{II}'\left(\frac{\pi}{2}\right)\right) \tag{37}$$

$$= w_I\left(\frac{\pi}{2}\right). \tag{38}$$

In the last step we have used the symmetry of the SG potential which implies that $w_{II}'(\frac{\pi}{2}) = w_I(\frac{\pi}{2})$, in order to write the Bloch condition in the compact form (38). The band edge wave functions are either periodic or antiperiodic functions of y, with period π, depending on whether N is even or odd. For example, for the $N = 0$ perturbative level, the wave function for the lower edge of the resulting band is a periodic function, while for the upper edge it is an antiperiodic function. See Fig. 3.

B. Global boundary conditions in the uniform WKB approach

Since the uniform WKB approximation (10) to the wave function is expressed in terms of parabolic cylinder functions, the implementation of the global boundary condition in this approach is intimately related to the properties of the parabolic cylinder functions. Moreover, since the argument of the parabolic cylinder function in the uniform ansatz (10) goes like $u_0(y)/g$ in the $g^2 \to 0$ limit, and $u_0(y_{\text{midpoint}})$ is finite, we see that the global boundary condition in the $g^2 \to 0$ limit is directly related to the asymptotic behavior of the parabolic cylinder functions at large values of their argument. It is at this stage that we must confront the fact that the perturbative expansion of the energy in (26), and also the perturbative expansion of the function $u(y)$ in (30), is in fact a non-Borel-summable divergent series in g^2. This is because $g^2 > 0$ is a Stokes line, and we encounter the familiar problems of trying to make a perturbative expansion on a Stoke line [28,29,31,32,46,47]. The theory of Borel-Écalle resurgent summation provides a well-defined approach to this problem:

(1) Analytically continue in g^2 off the positive real axis. Then all the divergent series become Borel summable. This is often expressed [4,6,10,11] as continuing all the way to $g^2 \to -g^2$, in which case the nonalternating non-Borel-summable series become alternating and Borel summable. In fact, it is enough to go slightly off the positive real g^2 axis: $g^2 \to g^2 \pm i\epsilon$, which avoids the Borel poles and/or branch cuts.

(2) Having obtained the Borel summed expressions, analytically continue in g^2 back to the positive real axis.
(3) This procedure produces nonperturbative imaginary contributions as the Borel summed series are continued back to the positive real g^2 axis; moreover, the overall sign of such a term is ambiguous, depending on whether one approaches the positive real g^2 axis from above or below. The remarkable fact is that if one makes all analytic continuations consistently in the global boundary condition, then in the trans-series expansion all ambiguities in the perturbative expansions are strictly correlated with corresponding ambiguities in the nonperturbative sectors, in such a way that all ambiguities cancel, producing an exact and unambiguous trans-series expression for the energy eigenvalue.

C. Global boundary condition for the double-well system

To derive the explicit form of the global boundary condition, recall that the global boundary conditions (34)–(35) are imposed at the barrier midpoint $y_{\text{midpoint}} = -\frac{1}{2}$. When we analytically continue g^2 off the positive real axis, this renders the g^2 expansion (30) of the argument $\frac{1}{g}u(-\frac{1}{2})$ of the parabolic cylinder function D_ν appearing in the uniform WKB ansatz (10) Borel summable. But now this argument $\frac{1}{g}u(-\frac{1}{2})$ is also a complex number, off the real positive axis. Thus in the limit where the modulus of g^2 approaches zero, the appropriate asymptotic behavior of the parabolic cylinder function is not just given by $D_\nu(z) \sim z^\nu e^{-z^2/4}, (z \to +\infty)$, as used in (25). We now need to use the (resurgent) asymptotic behavior of the parabolic cylinder functions throughout the relevant region of the complex plane, given by [37]:

$$D_\nu(z) \sim z^\nu e^{-z^2/4} F_1(z^2) + e^{\pm i\pi\nu} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} z^{-1-\nu} e^{z^2/4} F_2(z^2), \quad \frac{\pi}{2} < \pm \arg(z) < \pi \tag{39}$$

where

$$F_1(z^2) = \sum_{k=0}^{\infty} \frac{\Gamma(k-\frac{\nu}{2})\Gamma(k+\frac{1}{2}-\frac{\nu}{2})}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{1}{2}-\frac{\nu}{2})} \frac{1}{k!} \left(\frac{-2}{z^2}\right)^k \tag{40}$$

$$F_2(z^2) = \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2}+\frac{\nu}{2})\Gamma(k+1+\frac{\nu}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})} \frac{1}{k!} \left(\frac{2}{z^2}\right)^k. \tag{41}$$

Notice that there are two different exponential terms $e^{\pm z^2/4}$ in (39). Normally one or other is dominant or subdominant, but for certain rays of z^2 in the complex plane they may be equally important. This is a manifestation of the Stokes phenomenon [28,29,32,46,47].

Consider first the global boundary condition with Dirichlet boundary condition at the midpoint, as in (35). Using the full analytic expression (39), the global boundary condition (35) can be written as

$$\psi_{\text{DW}}\left(-\frac{1}{2}\right) = 0 \Rightarrow D_\nu\left(\frac{u(-\frac{1}{2})}{g}\right) = 0$$

for arbitrary $\arg(g^2)$. Hence

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\nu} = -\xi H_0(\nu, g^2); \quad \text{(upper level)} \tag{42}$$

where the "instanton factor" is related to the zeroth-order uniform WKB wave function at the midpoint,

$$\xi \equiv \sqrt{\frac{1}{\pi g^2}} \exp\left[-\frac{u_0^2(-\frac{1}{2})}{2g^2}\right] = \frac{1}{\sqrt{\pi g^2}} \exp\left[-\frac{1}{6g^2}\right] = \frac{1}{\sqrt{\pi g^2}} \exp\left[-\frac{S_I}{g^2}\right], \tag{43}$$

and the perturbative "fluctuations around the instanton" are given by the function

$$H_0(\nu, g^2) \equiv \left(\frac{u^2(-\frac{1}{2})}{2}\right)^{\nu+\frac{1}{2}} \frac{F_1\left(\frac{u^2(-\frac{1}{2})}{g^2}\right)}{F_2\left(\frac{u^2(-\frac{1}{2})}{g^2}\right)} \times \exp\left[-\frac{1}{2g^2}\left(u^2\left(-\frac{1}{2}\right) - u_0^2\left(-\frac{1}{2}\right)\right)\right]. \tag{44}$$

The form of (42) follows directly from the global boundary condition (35) and the asymptotic properties (39) of the parabolic cylinder functions.

The expression (42) is an implicit relation for ν as a function of the coupling g^2. As $g^2 \to 0$, it is clear that ν is close to a non-negative integer N. We solve by expanding $\nu = N + \delta\nu$, noting that

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\nu} = -N!\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-N}\left\{\delta\nu - \left[\gamma + \ln\left(\frac{e^{\pm i\pi}2}{g^2}\right) - h_N\right] \times (\delta\nu)^2 + \cdots\right\} \tag{45}$$

where h_N is the Nth harmonic number [37] and γ is the Euler-Mascheroni constant. This implies that (for the odd state)

$$\nu = N + \frac{(\frac{2}{g^2})^N H_0(N,g^2)}{N!}\xi - \frac{(\frac{2}{g^2})^{2N}}{(N!)^2} \times \left[H_0\frac{\partial H_0}{\partial N} + \left(\ln\left(\frac{e^{\pm i\pi}2}{g^2}\right) - \psi(N+1)\right)H_0^2\right]\xi^2 + O(\xi^3). \tag{46}$$

This expansion is the trans-series form of the parameter ν mentioned already in (13)–(14) in Sec. II. This discussion makes it clear that the "instanton" exponential factor ξ, the logarithmic factors, and the powers of g^2 all come from the expansion of the gamma function and the exponential factor in (42), which ultimately originate in the asymptotic form of the parabolic cylinder function (39). This explains why the trans-series form for the energy eigenvalue is generic for problems with degenerate harmonic vacua.

Notice that the leading imaginary part in (46) occurs at $O(\xi^2)$, showing that it is generically a two-instanton effect, and moreover it is directly related to the square of the real part at $O(\xi)$:

$$\mathrm{Im}[\nu - N] = \pm\pi(\mathrm{Re}[\nu - N])^2 + O(\xi^3). \tag{47}$$

This is the first of a set of *confluence equations* [25], as discussed below in Sec. IV. The $\pm$ sign here comes from the ambiguity in the analytic continuation of g^2; it will be shown to be correlated with the ambiguity in the Borel summation of the perturbative series, in such a way that the ambiguous imaginary parts cancel.

If we repeat this argument using the Neumann boundary condition at the midpoint, then after some computation we find that the only change is a change in sign on the rhs of (42), which leads to a change of sign of the odd powers of ξ in (46). Thus, to leading order in the exponentially small instanton factor ξ, using (16) and (46), the splitting of the levels is symmetric[5]:

$$E_{\mathrm{DW}}^{(N)} = E_{\mathrm{DW}}(N,g^2) \pm \left(\frac{2}{g^2}\right)^N \frac{H_0(N,g)[\frac{\partial E}{\partial \nu}]_N}{N!}\xi + O(\xi^2). \tag{48}$$

1. Resurgent expansion for DW vs instantons

We can expand the left-hand side of (42) up to kth order, and at the same time, the right-hand side up to $(k-1)$-th order in $\delta\nu$. This suffices to systematically extract the trans-series up to kth order in the instanton expansion. Let us do this exercise for the ground state energy ($N = 0$): let $\nu = 0 + \delta\nu$, and define

$$\sigma_\pm = \ln\left(\frac{e^{\pm i\pi}2}{g^2}\right) = \sigma \pm i\pi, \qquad \sigma = \ln\left(\frac{2}{g^2}\right). \tag{49}$$

[5]At higher orders in ξ the splitting is no longer symmetric.

Then, expanding both sides of the global boundary condition we find

$$\frac{1}{\Gamma(-\delta\nu)}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\delta\nu} = \{-\delta\nu Q_0 + (\delta\nu)^2 Q_1^\pm + (\delta\nu)^3 Q_2^\pm + \cdots\} = -\xi H_0(\nu,g^2) = -\xi\left[H_0 + H_0'(\delta\nu) + \frac{1}{2}H_0''(\delta\nu)^2 + \cdots\right] \tag{50}$$

where $H_0 = H_0(0,g^2)$, $H_0' = [\frac{\partial H_0(\nu,g^2)}{\partial\nu}]_{\nu=0}$, etc., and $Q_n(\sigma)$ is an nth-order polynomial, encoding the quasizero-mode integrations in the instanton picture as described below, the first few of which are given by

$$\begin{aligned} Q_0 &\equiv Q_0(\sigma_\pm) = 1, \\ Q_1^\pm &\equiv Q_1(\sigma_\pm) = \gamma + \sigma_\pm \\ Q_2^\pm &\equiv Q_2(\sigma_\pm) = -\frac{1}{2}(\gamma+\sigma_\pm)^2 + \frac{\pi^2}{12} \\ Q_3^\pm &\equiv Q_3(\sigma_\pm) = \frac{1}{6}(\gamma+\sigma_\pm)^3 - \frac{\pi^2}{12}(\gamma+\sigma_\pm) - \psi^{(2)}(1). \end{aligned} \tag{51}$$

The subscript n counts, in the instanton picture, the number of quasizero modes associated with $(n+1)$-instanton events. Solving for $\delta\nu$ iteratively in the instanton fugacity ξ, we write $\delta\nu = \sum a_n\xi^n$. Then, it is easy to show that

$$\delta\nu = \xi H_0 + \xi^2[H_0H_0' + H_0^2 Q_1(\sigma_\pm)] + \xi^3\left[H_0(H_0')^2 + \frac{1}{2}H_0^2H_0'' + 3H_0^2H_0'Q_1(\sigma_\pm) - 3H_0^3Q_2(\sigma_\pm) + \frac{\pi^2}{3}H_0^3\right] + \cdots \tag{52}$$

2. Remarks and connection to instanton picture

We can interpret various terms in the trans-series expansion due to topological defects with action nS_I. The origin of terms proportional to ξ, ξ^2, ξ^3 are, respectively, 1-, 2-, and 3-defects; see Fig. 4. Although there is no strict topological charge, one can still assign a topology to instanton and anti-instanton events by their asymptotics. Doing so will help us to disentangle the contributions to the physical observable and clarify the cancellations taking place in the (truncated) resurgence triangle. For $\mathcal{I}$ and $\bar{\mathcal{I}}$, we assign "topological charges," ±1 as follows:

$$\begin{aligned} \mathcal{I}&: \ Q_T = y(\infty) - y(-\infty) = 0 - (-1) = +1 \\ \bar{\mathcal{I}}&: \ Q_T = y(\infty) - y(-\infty) = -1 - (0) = -1. \end{aligned} \tag{53}$$

Consequently, the topological excitations in the double-well problem and their topological charges are given by

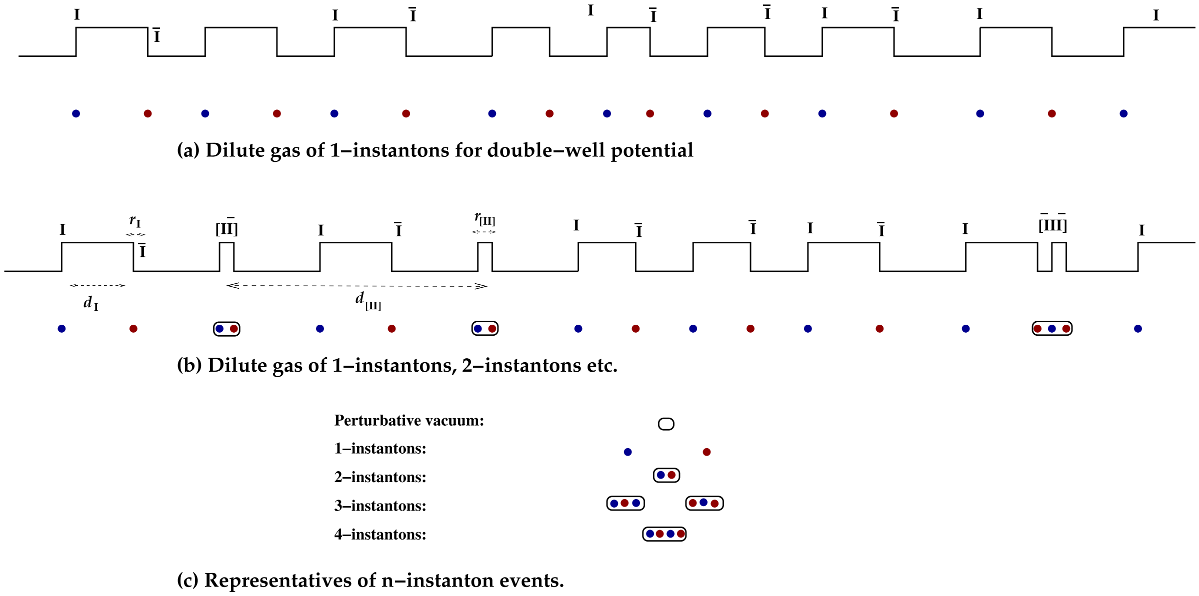

FIG. 4 (color online). Same as Fig. 1, for the double-well potential.

$$
\begin{aligned}
&\text{1-defects}: \mathcal{I} \sim (\ldots)\xi, \quad Q_T = +1, \quad \bar{\mathcal{I}} \sim (\ldots)\xi, \quad Q_T = -1,\\
&\text{2-defects}: [\mathcal{I}\bar{\mathcal{I}}]_\pm = [\bar{\mathcal{I}}\mathcal{I}]_\pm \sim (\ldots)\xi^2 \mp i(\ldots)\xi^2 \quad Q_T = 0\\
&\text{3-defects}: [\mathcal{I}\bar{\mathcal{I}}\mathcal{I}]_\pm \sim (\ldots)\xi^3 \pm i(\ldots)\xi^3, \quad Q_T = +1,\\
&\qquad\qquad [\bar{\mathcal{I}}\mathcal{I}\bar{\mathcal{I}}]_\pm \sim (\ldots)\xi^3 \pm i(\ldots)\xi^3, \quad Q_T = -1. \qquad (54)
\end{aligned}
$$

Note that there are no $[\mathcal{I}\mathcal{I}]$, $[\mathcal{I}\mathcal{I}\mathcal{I}]$, … type events; i.e, $Q_T \geq 2$ are not present, despite the fact that the action of the n-event is just nS_I. This is so because we are dealing with a double-well potential. The situation is different for the periodic SG potential; see Sec. III D 1. The resurgence triangle [25] provides a simple graphical representation of the trans-series structure. Each cell is labeled by (n, m). The rows are sectors with fixed action (nS_I), $n = 0, 1, 2, \ldots$, and columns are sectors with fixed topological charge $Q_T = m = +1$, 0, -1(compare with periodic potential, for which $|m| \leq n$). The truncated resurgence triangle for the DW system is

$$
\begin{array}{ccc}
\underline{m=+1} & \underline{m=0} & \underline{m=-1}\\
 & f_{(0,0)} & \\
e^{-\frac{1}{6g^2}} f_{(1,1)} & & e^{-\frac{1}{6g^2}} f_{(1,-1)}\\
 & e^{-\frac{2}{6g^2}} f_{(2,0)} & \\
e^{-\frac{3}{6g^2}} f_{(3,1)} & & e^{-\frac{3}{6g^2}} f_{(3,-1)}\\
 & e^{-\frac{4}{6g^2}} f_{(4,0)} & \\
\vdots & \vdots & \vdots
\end{array} \qquad (55)
$$

Various comments are in order in connection with the trans-series (52), instanton and multi-instanton amplitudes (54), and the truncated resurgence triangle (55).

(i) In the instanton picture, the interpretation of $Q_n(\sigma_\pm)$ is the following. For an $(n+1)$-instanton event, there are $(n+1)$ low lying modes. In the noninteracting instanton gas picture, these are $(n+1)$-position moduli of these defects. Including interaction, n of these become quasizero modes that need to be integrated exactly, and one is the "center of action" of the $(n+1)$-defect. In this way, one obtains the amplitude of the correlated $(n+1)$-instanton event, and $Q_n(\sigma_\pm)$ or $Q_n(\sigma)$ is its prefactor. (See the next item.) We call Q_n the *quasizero-mode polynomial*; the degree of polynomial n counts the number of quasizero modes that are integrated over.[6]

(ii) The polynomials are of two types: those which are two-fold ambiguous $Q_n^\pm = Q_n(\sigma^\pm)$ and those which are not $Q_n = Q_n(\sigma)$. For the $(n+1)$-instanton configuration with only instantons, unambiguous polynomials $Q_n(\sigma)$ arise. (This does not happen in the double-well system, because an instanton is always followed by an anti-instanton. But it does happen for the periodic potential as we discuss later.) Whenever there are both correlated instanton and anti-instanton pairs in an $(n+1)$-instanton event, polynomials with ambiguities arise. Such is always the case in the double-well potential.

[6]The correspondence with the notation of Zinn-Justin [10] is the following. $Q_1(\sigma) = P_2(\sigma)$, $2(Q_1)^2 + Q_2 = P_3(\sigma)$, etc. ZJ uses subscript $(n+1)$ for a polynomial of degree n, because the polynomial multiplies an $(n+1)$-event amplitude. Since the degree n of the polynomial is equal to the number of integrated quasizero modes, and the number of QZM is one less than the number of the constituents of a correlated event, we call this polynomial $Q_n(\sigma)$.

(iii) These polynomials are universal. They will appear in any QM mechanics problems with degenerate minima.[7] In any given theory, we have to consider both instantons as well as correlated/molecular $(n+1)$-events. These polynomials are an integral part of the $(n+1)$-correlated instanton events.

(iv) The ambiguities in $Q_n(\sigma_\pm)$ cancel the ambiguities associated with the non-Borel summability of the perturbation theory according to the rules of the resurgence triangle. For example, the ambiguity in $Q_1^\pm$ and $Q_2^\pm$ cancel the ambiguities associated with non-Borel summability of the perturbation theory around the perturbative vacuum and one-instanton sector, respectively.

3. The truncated resurgence triangle and (graded) partition functions

The structure associated with the truncated resurgence triangle can also be seen by studying partition functions graded by parity symmetry. Parity in our DW potential is defined as

$$P:\ y \to -1 - y \quad \left(\text{reflection with respect to } y = -\frac{1}{2}\right) \tag{56}$$

and commutes with the Hamiltonian, $[P, H] = 0$.

We can define two types of partition functions. One is regular, and one is twisted by the insertion of the parity operator:

$$Z(\beta, g^2) = \mathrm{tr} e^{-\beta H} \longrightarrow \int_{y(t+\beta)=y(t)} Dy(t) e^{-S[y]}$$
$$\tilde{Z}(\beta, g^2) = \mathrm{tr} P e^{-\beta H} \longrightarrow \int_{y(t+\beta)=P[y(t)]=-y(t)-1} Dy(t) e^{-S[y]}. \tag{57}$$

The boundary conditions associated with $Z(\beta, g^2)$ forbid the contribution of a single instanton effect as well as any topological configuration which has the same asymptotic behavior as the single instanton. On the flip side, the boundary conditions associated with $\tilde{Z}(\beta, g^2)$ forbid the contribution of the perturbative vacuum saddle as well as any topological configuration which has the same asymptotic behavior as the perturbative vacuum. For example, $Z(\beta, g^2)$ receives a contribution from the $m = 0$ column, while $\tilde{Z}(\beta, g^2)$ receives a contribution from the ± 1 columns in the resurgence triangle. In the periodic SG potential, the columns are characterized by a winding number associated with their θ-angle dependence; see Sec. III D 1.

[7]They also have natural generalization to QFT, which is not explored here.

D. Global boundary condition for the Sine-Gordon system

A similar analysis applies to the SG system. We first take the appropriate linear combinations of the two linearly independent parabolic cylinder functions to match the normalization conditions for the functions $w_I(y)$ and $w_{II}(y)$ in (36). Define even and odd functions on the interval $y \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$:

$$f_1(y) = \frac{1}{\sqrt{u'(y)}}\left(D_\nu\left(\frac{u(y)}{g}\right) + D_\nu\left(-\frac{u(y)}{g}\right)\right) \quad \text{(even)} \tag{58}$$

$$f_2(y) = \frac{1}{\sqrt{u'(y)}}\left(D_\nu\left(\frac{u(y)}{g}\right) - D_\nu\left(-\frac{u(y)}{g}\right)\right) \quad \text{(odd)} \tag{59}$$

where we note that $u(y)$ is odd, and $u'(y)$ is even and positive on this interval. The Wronskian is

$$\mathcal{W} \equiv f_1(y) f_2'(y) - f_1'(y) f_2(y) = -\frac{4}{g}\sqrt{\frac{\pi}{2}}\frac{1}{\Gamma(-\nu)} \tag{60}$$

which is independent of y, and is nonzero except when ν is a non-negative integer. Then, the appropriately normalized basis solutions (36) can be written as

$$w_I(y) = \frac{1}{\mathcal{W}}\left(f_2'\left(-\frac{\pi}{2}\right) f_1(y) - f_1'\left(-\frac{\pi}{2}\right) f_2(y)\right) \tag{61}$$

$$w_{II}(y) = \frac{1}{\mathcal{W}}\left(-f_2\left(-\frac{\pi}{2}\right) f_1(y) + f_1\left(-\frac{\pi}{2}\right) f_2(y)\right). \tag{62}$$

Using the parity properties of f_1 and f_2 we can therefore write the Bloch condition (38) in various equivalent ways:

$$\cos\theta = \frac{1}{\mathcal{W}}\left(f_2'\left(\frac{\pi}{2}\right) f_1\left(\frac{\pi}{2}\right) + f_1'\left(\frac{\pi}{2}\right) f_2\left(\frac{\pi}{2}\right)\right) \tag{63}$$

$$= 1 + \frac{2}{\mathcal{W}} f_1'\left(\frac{\pi}{2}\right) f_2\left(\frac{\pi}{2}\right) \tag{64}$$

$$= -1 + \frac{2}{\mathcal{W}} f_2'\left(\frac{\pi}{2}\right) f_1\left(\frac{\pi}{2}\right). \tag{65}$$

Thus, as in the double-well case, the global boundary condition is imposed at the midpoint between two neighboring perturbative vacua: $y_{\text{midpoint}} = \frac{\pi}{2}$. Moreover, the global condition is expressed in terms of parabolic cylinder functions evaluated at y_{midpoint}. This Bloch condition results in the perturbative energy level splitting into a continuous band, with states within the band labeled by the angular parameter θ. The bottom of the lowest band has $\theta = 0$ and its wave function is an even function, while the top of the lowest band $\theta = \pi$ and its wave function is an odd function. For these lowest band-edge states the Bloch condition takes a simpler form reminiscent of the DW case (34), (35):

$$(\text{lower, even state}; \theta = 0):\ f_1'(y_{\text{midpoint}}) = 0 \quad (66)$$

$$(\text{upper, odd state}; \theta = \pi):\ f_1(y_{\text{midpoint}}) = 0. \quad (67)$$

The Bloch condition determines the ansatz parameter ν as a function of the coupling g^2, for each value of the Bloch angle θ. As before, $u(\frac{\pi}{2})$ is a non-Borel-summable divergent series in g^2, so we need to analytically continue in g^2 off the Stokes line ($g^2 > 0$) in order to properly define the Bloch condition. This requires again the full analytic continuation behavior (39) of the parabolic cylinder functions that enter into the definition of the functions f_1 and f_2. Proceeding in a manner similar to the DW potential, we can write the boundary conditions (63)–(65) as an equation determining ν in terms of g^2:

$$\frac{1}{\Gamma(-\nu)}\left(\frac{2}{g^2}\right)^{-\nu} \pm i\frac{\pi}{2}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{+\nu}\frac{\xi^2[H_0(\nu,g^2)]^2}{\Gamma(1+\nu)} = -\xi H_0(\nu,g^2)\cos\theta. \quad (68)$$

This is the analog of (42) for DW, now applied to the SG potential.

At leading nonperturbative order, we find that the parameter ν is exponentially close to an integer. For example, for the lowest band we write $\nu = 0 + \delta\nu + \cdots$, and the Bloch condition (63) becomes in the small g^2 limit:

$$\cos\theta = -\frac{g}{4}\sqrt{\frac{2}{\pi}}\Gamma(-\nu)\left(f_2'\left(\frac{\pi}{2}\right)f_1\left(\frac{\pi}{2}\right) + f_1'\left(\frac{\pi}{2}\right)f_2\left(\frac{\pi}{2}\right)\right) \quad (69)$$

$$\sim \frac{g}{4}\sqrt{\frac{2}{\pi}}(\delta\nu)\frac{\pi}{2u(\pi/2)}\exp\left[\frac{(u(\pi/2))^2}{2g^2}\right]. \quad (70)$$

Using from (32) the fact that

$$u\left(\frac{\pi}{2}\right) \sim 2 - \frac{g^2}{4}(2\nu+1)\ln 2 + \cdots \quad (71)$$

we find that

$$\delta\nu \sim -\frac{4}{\sqrt{\pi}g}\cos\theta e^{-\frac{2}{g^2}} \quad (72)$$

which gives the familiar instanton result for the splitting of the lowest band. Incorporating fluctuation terms we find

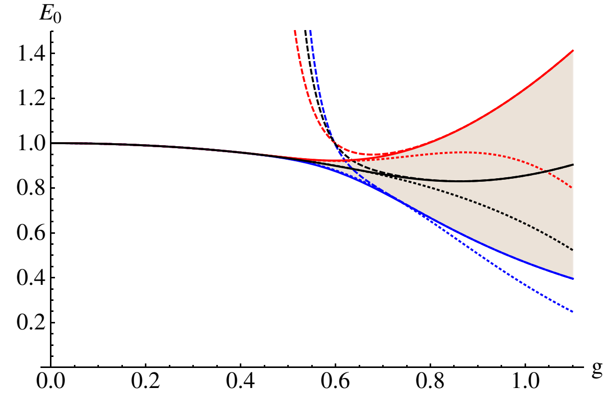

FIG. 5 (color online). A comparison of the exact band edges, and the center of the band, for the lowest band of the Sine-Gordon potential (solid lines), with the weak-coupling trans-series expansion (dotted lines), and the strong-coupling results (dashed lines). The exact results are generated using the Mathematica functions MathieuCharacteristicA and Mathieu-CharacteristicB, which compute Mathieu band edges numerically. The weak-coupling expansions have been plotted here using the expression in (74), and the strong-coupling expansions have been plotted using (75). Note the excellent numerical agreement.

$$E_\theta^{(\text{lowest band})} \sim \left[1 - \frac{g^2}{4} - \frac{g^4}{16} - \cdots\right] - \cos\theta\frac{8}{\sqrt{\pi}g}e^{-\frac{2}{g^2}} \times \left[1 - \frac{7g^2}{16} - \frac{59g^4}{512} - \cdots\right] + O(e^{-\frac{4}{g^2}}). \quad (73)$$

This is in agreement with the Mathieu equation results [7,37,38]. In Fig. 5 we show that this expansion gives an excellent approximation to the lowest band for the Sine-Gordon potential. The edges of the lowest band are given by $\theta = 0$, π in the small coupling limit, $g^2 \to 0$. We can also compute the strong-coupling limit, $g^2 \to \infty$, by treating the potential as a perturbation of the free periodic or antiperiodic solution on the single-period interval. We obtain the following weak- and strong-coupling expressions:

$$[E_0^\pm]_{(\text{weak-coupling})} = \left[1 - \frac{g^2}{4} - \frac{g^4}{16} - \frac{3g^6}{64} - \cdots\right] \pm \frac{8}{\sqrt{\pi}g}e^{-2/g^2} \times \left[1 - \frac{7g^2}{16} - \frac{59g^4}{512} - \cdots\right] + \cdots \quad (74)$$

$$[E_0^\pm]_{(\text{strong-coupling})} = \begin{cases} g^2 + \frac{1}{4g^2} - \frac{1}{128g^6} + \cdots \\ \frac{1}{2g^2} - \frac{1}{32g^6} + \frac{7}{2^{15}g^{14}} - \cdots \end{cases} \quad (75)$$

Note that the strong-coupling expansion is convergent (it is not unusual for functions to have convergent expansions for large/small argument, but asymptotic expansions for small/large argument). We can also plot the *exact* expressions for the band edges, by writing the global boundary conditions (66)–(67) directly in terms of Mathieu functions, which can be plotted. Figure 5 shows excellent agreement of the exact result with the asymptotic limits.

If we include more exponentially small terms (along with the perturbative fluctuations around them) in the weak-coupling trans-series expansion, the trans-series will approach the exact result even for larger values of the coupling. Analogously, if we include more terms in the strong-coupling expansion, the series will actually approach the exact result even for smaller values of the coupling.

1. Resurgent expansion for SG vs instantons

Similar to Sec. III C 1, we can expand the first (second) term in the left-hand side of (68) up to kth [$(k-2)$-th] order, and at the same time, the right-hand side up to $(k-1)$-th order in $\delta\nu$. This suffices to systematically extract the trans-series up to kth order in the instanton expansion. Doing so also helps us to visualize the differences and similarities between the resurgent expansions in periodic and double-well potentials.

For the ground state, we let $\nu = N + \delta\nu = 0 + \delta\nu$. Then,

$$\begin{aligned}&\{-\delta\nu Q_0 + (\delta\nu)^2 Q_1 + (\delta\nu)^3 Q_2\}\\ &\quad\pm\frac{i\pi\xi^2}{2}[H_0^2 + (\delta\nu)(2H_0H_0{}' + Q_1^{\pm})]\\ &= \xi\left[H_0 + (\delta\nu)H_0' + \frac{1}{2}(\delta\nu)^2 H_0''\right]\cos\theta \qquad (76)\end{aligned}$$

where Q_n and $Q_n^{\pm}$ are the quasizero-mode polynomials given in (51). Solving for $\delta\nu$ iteratively in instanton fugacity ξ, we find

$$\begin{aligned}\delta\nu &= -\xi H_0\cos\theta + \xi^2\left([H_0H_0' + H_0^2Q_1]\cos^2\theta \mp \frac{i\pi}{2}H_0^2\right)\\ &\quad + \xi^3\left(\left[-H_0(H_0')^2 - 3H_0'H_0^2Q_1 - H_0^3(2(Q_1)^2 + Q_2) - \frac{1}{2}H_0^2H_0''\right]\cos^3\theta\right.\\ &\quad \left. + \left[\pm i\pi Q_1H_0^3 \pm \frac{3}{2}i\pi H_0^2H_0' \pm \frac{1}{2}i\pi H_0^3Q_1^{\pm}\right]\cos\theta\right) + \cdots\\ &= -\frac{1}{2}\xi H_0 e^{i\theta} - \frac{1}{2}\xi H_0 e^{-i\theta} + \frac{1}{4}\xi^2[H_0H_0' + H_0^2Q_1]e^{2i\theta} + \frac{1}{2}\xi^2[H_0H_0' + H_0^2Q_1^{\pm}] + \frac{1}{4}\xi^2[H_0H_0' + H_0^2Q_1]e^{-2i\theta} + \cdots \qquad (77)\end{aligned}$$

2. Remarks and connection to instanton picture

We can interpret the various terms in the trans-series expansion due to topological defects with action nS_I and winding number $m \le n$. For example, the origin of terms proportional to ξ, ξ^2, ξ^3 are, respectively, 1-, 2-, and 3-defects. Let us classify the n-defects contributing to the trans-series expansion at order n:

$$\begin{aligned}&\text{1-defects: } \mathcal{I} \sim (\ldots)\xi e^{i\theta}, \qquad \bar{\mathcal{I}} \sim (\ldots)\xi e^{-i\theta},\\ &\text{2-defects: } [\mathcal{I}\mathcal{I}] \sim (\ldots)\xi^2 e^{2i\theta}, \qquad [\bar{\mathcal{I}}\bar{\mathcal{I}}] \sim (\ldots)\xi^2 e^{-2i\theta} \qquad [\mathcal{I}\bar{\mathcal{I}}]_{\pm} = [\bar{\mathcal{I}}\mathcal{I}]_{\pm} \sim (\ldots)\xi^2 \mp i(\ldots)\xi^2\\ &\text{3-defects: } [\mathcal{I}\mathcal{I}\mathcal{I}] \sim (\ldots)\xi^3 e^{3i\theta}, \qquad [\bar{\mathcal{I}}\bar{\mathcal{I}}\bar{\mathcal{I}}] \sim (\ldots)\xi^3 e^{-3i\theta}, [\bar{\mathcal{I}}\mathcal{I}\mathcal{I}]_{\pm} = [\mathcal{I}\bar{\mathcal{I}}\mathcal{I}]_{\pm} = [\mathcal{I}\mathcal{I}\bar{\mathcal{I}}]_{\pm} \sim (\ldots)\xi^3 e^{i\theta} \pm i(\ldots)\xi^3 e^{i\theta},\\ &\qquad [\bar{\mathcal{I}}\bar{\mathcal{I}}\mathcal{I}]_{\pm} = [\mathcal{I}\bar{\mathcal{I}}\bar{\mathcal{I}}]_{\pm} = [\bar{\mathcal{I}}\mathcal{I}\bar{\mathcal{I}}]_{\pm} \sim (\ldots)\xi^3 e^{-i\theta} \pm i(\ldots)\xi^3 e^{-i\theta}. \qquad (78)\end{aligned}$$

Note the multiplicity of the n-defects. At action level n, the events with topological charge $m = n - 2k$, $k = 0, 1, \ldots, n$ have multiplicity $\binom{n}{k}$. For example, 2- and 3-defects have multiplicities 1, 2, 1 and 1, 3, 3, 1, respectively, and combine to give $\cos^2\theta$, $\cos^3\theta$ terms in the trans-series (77). In general, we have instanton events of the form $[\mathcal{I}^{n-k}\bar{\mathcal{I}}^k]$ with n units of action and θ dependence $e^{i(n-2k)\theta}$. The multiplicities of these correlated events are $\binom{n}{k}$. Hence,

$$\sum_{k=0}^{n}\binom{n}{k} e^{i(n-2k)\theta} = (e^{i\theta} + e^{-i\theta})^n = 2^n\cos^n\theta \qquad (79)$$

giving the result obtained above, e.g., in (77).

A way to organize trans-series is through the structure of the resurgence triangle, where each cell is labeled by (n, m), $|m| \le n$. The rows are sectors with fixed action (nS_I), $n = 0, 1, 2, \ldots$, and the columns are sectors with fixed topological charge $|m| \le n$. The resurgence triangle for the periodic potential is

$$\begin{array}{c} f_{(0,0)} \\ e^{-\frac{2}{g^2}+i\theta} f_{(1,1)} \quad e^{-\frac{2}{g^2}-i\theta} f_{(1,-1)} \\ e^{-\frac{4}{g^2}+2i\theta} f_{(2,2)} \quad e^{-\frac{4}{g^2}} f_{(2,0)} \quad e^{-\frac{4}{g^2}-2i\theta} f_{(2,-2)} \\ e^{-\frac{6}{g^2}+3i\theta} f_{(3,3)} \quad e^{-\frac{6}{g^2}+i\theta} f_{(3,1)} \quad e^{-\frac{6}{g^2}-i\theta} f_{(3,-1)} \quad e^{-\frac{6}{g^2}-3i\theta} f_{(3,-3)} \\ e^{-\frac{8}{g^2}+4i\theta} f_{(4,4)} \quad e^{-\frac{8}{g^2}+2i\theta} f_{(4,2)} \quad e^{-\frac{8}{g^2}} f_{(4,0)} \quad e^{-\frac{8}{g^2}-2i\theta} f_{(4,-2)} \quad e^{-\frac{8}{g^2}-4i\theta} f_{(4,-4)} \\ \iddots \qquad \vdots \qquad \ddots \end{array} \tag{80}$$

Various comments are in order in connection with the trans-series (77), instanton and multi-instanton amplitudes (78), and the resurgence triangle (80).

(i) The n-instanton, e.g., $\mathcal{I}$, $[\mathcal{I}\mathcal{I}]$, $[\mathcal{I}\mathcal{I}\ldots\mathcal{I}]$ (similarly for n-anti-instanton) amplitudes, associated with the edges of the triangle $m = \pm n$ are the leading semiclassical configuration in the corresponding homotopy class, and they are unambiguous. This is because instanton-instanton interactions are repulsive and defining the n-instanton amplitude does not require the BZJ prescription. The quasizero-mode integrations produce ambiguity-free $Q_{n-1}(\sigma)$ polynomials. However, the perturbative expansion around the n-instanton is non-Borel-summable and still ambiguous.

(ii) Since an n-instanton amplitude has θ-dependence $e^{in\theta}$, it cannot mix with perturbation theory around the perturbative vacuum, which is clearly insensitive to θ. Since the basis $\{e^{in\theta}, n \in \mathbb{Z}\}$ forms an orthogonal complete set for periodic functions with period 2π, we can define superselection sectors in the resurgence triangle; i.e., columns with different θ angle dependence are associated with different homotopy classes and do not mix with each other in the cancellation of their ambiguities.

(iii) The ambiguous part in the $O(\xi^2)$ term does not depend on the Bloch angle θ. The contribution of $[\bar{\mathcal{I}}\mathcal{I}]_\pm$ produces ambiguous quasizero-mode polynomials $Q_1^\pm = Q_1(\sigma_\pm)$. It must be this way if this imaginary term is to cancel against an imaginary term arising from the non-Borel-summable perturbative series, because the perturbative series is independent of θ. This pattern continues throughout the entire trans-series, and the resurgence triangle [25], in which the resurgent cancellations are characterized by their θ dependence.

3. The resurgence triangle and graded partition functions

The structure associated with the resurgence triangle can also be seen by studying graded partition functions. Consider the Fourier expansion of the partition function in the orthonormal basis $\{e^{in\theta}, n \in \mathbb{Z}\}$:

$$Z(\beta, g, \theta) = \sum_{m=-\infty}^{+\infty} e^{im\theta} Z_m(\beta, g). \tag{81}$$

Then, it is not hard to realize that $Z_m(\beta, g)$ has a resurgent expansion associated with the mth column in (80). In the operator formalism, this data can be extracted by studying the twisted partition function $Z_m = \mathrm{tr} T^m e^{-\beta H}$ with the insertion of the translation operator T. In path integrals, this corresponds to restricting the boundaries of the path integration, namely,

$$Z_m = \mathrm{tr} T^m e^{-\beta H} \longrightarrow \int_{x(t+\beta)=x(t)+m\frac{\pi}{g}} Dx(t) e^{-S[x]}. \tag{82}$$

For example, the boundary conditions associated with the $Z_{\pm 1}$ *forbid* the contribution of the perturbative vacuum sector, as well as any topological configuration which has the same asymptotic behavior as the perturbative vacuum. The leading saddle contributing to $Z_{\pm 1}$ is a one-instanton event, from which one can extract the bandwidth at leading order.

IV. EXPLICIT RESURGENCE RELATIONS

A. Comparison with Zinn-Justin and Jentschura

In order to discuss the explicit resurgent relations encoded in the trans-series expressions for the energy eigenvalues, it is convenient at this stage to comment on the similarities and differences between the uniform WKB approach, discussed in this paper, and the approach of Zinn-Justin and Jentschura (ZJJ), who have presented extensive results for the resurgent relations [20].

1. Double-well potential

For the DW problem, ZJJ express their exact quantization condition as [20]

$$\frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2} - B(E, g^2)\right) \left(\frac{2}{g^2}\right)^{B(E,g^2)} e^{-A(E,g^2)/2} = \pm i \tag{83}$$

where the $\pm$ sign in (83) refers to the splitting of a given perturbative level into two separate levels, and the perturbative function $B(E, g^2)$ and the nonperturbative function $A(E, g^2)$ were computed to be (converting the results of [20] to our notation: $E_{ZJ} \to \frac{E}{2}$, and $g_{ZJ} \to g^2$)

$$B_{\rm DW}(E, g^2) = \frac{E}{2} + g^2\left(\frac{3}{4}E^2 + \frac{1}{4}\right) + g^4\left(\frac{35}{8}E^3 + \frac{25}{8}E\right) + g^6\left(\frac{1155}{32}E^4 + \frac{735}{16}E^2 + \frac{175}{32}\right) + g^8\left(\frac{45045}{128}E^5 + \frac{45045}{64}E^3 + \frac{31185}{128}E\right) + \cdots \qquad (84)$$

$$A_{\rm DW}(E, g^2) = \frac{1}{3g^2} + g^2\left(\frac{17}{4}E^2 + \frac{19}{12}\right) + g^4\left(\frac{227}{8}E^3 + \frac{187}{8}E\right) + g^6\left(\frac{47431}{192}E^4 + \frac{34121}{96}E^2 + \frac{28829}{576}\right) + g^8\left(\frac{317629}{128}E^5 + \frac{264725}{48}E^3 + \frac{842909}{384}E\right) + \cdots \qquad (85)$$

Notice that our global boundary condition (42) has the form

$$\frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{2} - B\right)\left(\frac{2}{g^2}\right)^B e^{-A(B,g^2)/2} = \pm i \qquad (86)$$

where $B \equiv \nu + \frac{1}{2}$, and $A = A(B, g^2)$ is a known function of B and g, given in (42)–(44).

To understand the precise relation between ZJJ's result (83)–(85) and our expression (86), observe that if we invert the expression (84) for $B = B(E, g^2)$ to write it as $E = E(B, g^2)$ we obtain

$$E_{\rm DW}(B, g^2) = 2B - 2g^2\left(3B^2 + \frac{1}{4}\right) - 2g^4\left(17B^3 + \frac{19}{4}B\right) - 2g^6\left(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\right) - 2g^8\left(\frac{10689}{4}B^5 + \frac{23405}{8}B^3 + \frac{22709}{64}B\right) - \cdots \qquad (87)$$

which agrees precisely with the perturbative expansion (27) for $E(B, g^2)$ that was found in the perturbative expansion of the uniform WKB approach. Recall that this is exactly the usual perturbative expansion for the energy of the Nth level, when we identify $B = N + \frac{1}{2}$. Moreover, if we now insert this expression for $E = E_{\rm DW}(B, g^2)$ as a function of B into ZJJ's expression (85) for $A = A_{\rm DW}(E, g^2)$, we obtain the expansion of $A_{\rm DW}(B, g^2)$ in powers of the coupling:

$$A_{\rm DW}(B, g^2) = \frac{1}{3g^2} + g^2\left(17B^2 + \frac{19}{12}\right) + g^4\left(125B^3 + \frac{153}{4}B\right) + g^6\left(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\right) + g^8\left(\frac{87549}{4}B^5 + \frac{50715}{2}B^3 + \frac{217663}{64}B\right) + \cdots \qquad (88)$$

This matches precisely the function $A_{\rm DW}(B, g^2)$ obtained from our global condition (42)–(44).

Thus, the conditions (83) and (86) are equivalent. However, the philosophy is subtly different. In [20], the expression (83) is regarded as an equation for the energy E as a function of g^2, provided both functions $B(E, g^2)$ and $A(E, g^2)$ are known. On the other hand, we regard (86) as an equation for B (equivalently for $\nu \equiv B - 1/2$) as a function of g^2, provided the function $A(B, g^2)$ is known, and we then insert the resulting $B(g^2)$ into the perturbative expansion (26) in order to obtain the resurgent trans-series expression for the energy eigenvalue. We will see in Sec. V that there is a surprising advantage to the latter, uniform WKB, perspective.

2. Sine-Gordon potential

A similar correspondence applies to the SG potential. ZJJ express their exact (Bloch) quantization condition as

$$\left(\frac{2}{g^2}\right)^{-B(E,g^2)}\frac{e^{A(E,g^2)/2}}{\Gamma(\frac{1}{2} - B(E,g^2))} + \left(-\frac{2}{g^2}\right)^{B(E,g^2)} \times \frac{e^{-A(E,g^2)/2}}{\Gamma(\frac{1}{2} + B(E,g^2))} = \frac{2\cos\theta}{\sqrt{2\pi}} \qquad (89)$$

where θ is the Bloch angle, and the perturbative function $B(E, g^2)$ and the nonperturbative function $A(E, g^2)$ were computed to be (converting the results of [20] to our notation: $E_{ZJ} \to \frac{E}{2}$, and $g_{ZJ} \to \frac{g^2}{4}$)

$$B_{\rm SG}(E, g^2) = \frac{1}{2}E + \frac{g^2}{16}(1 + E^2) + \frac{g^4}{128}(5E + 3E^3) + \frac{g^6}{64}\left(\frac{17}{32} + \frac{35}{16}E^2 + \frac{25}{32}E^4\right) + \frac{g^8}{256}\left(\frac{721}{128}E + \frac{525}{64}E^3 + \frac{245}{128}E^5\right) + \cdots \qquad (90)$$

$$A_{\rm SG}(E,g^2)=\frac{4}{g^2}+\frac{3g^2}{16}(1+E^2)+\frac{g^4}{16}\left(\frac{23}{4}E+\frac{11}{8}E^3\right)+\frac{g^6}{64}\left(\frac{215}{64}+\frac{341}{32}E^2+\frac{199}{64}E^4\right)+\frac{g^8}{256}\left(\frac{4487}{128}E+\frac{326}{8}E^3+\frac{1021}{128}E^5\right)+\cdots \quad (91)$$

[Note there is a small typo in (6.32) of [20]. The term $-\frac{199}{4}$ should be $+\frac{199}{4}$.]

We invert the first expression (90) to obtain

$$E_{\rm SG}(B,g^2)=2B-\frac{g^2}{2}\left(B^2+\frac{1}{4}\right)-\frac{g^4}{8}\left(B^3+\frac{3}{4}B\right)-\frac{g^6}{32}\left(\frac{5}{2}B^4+\frac{17}{4}B^2+\frac{9}{32}\right)-\frac{g^8}{128}\left(\frac{33}{4}B^5+\frac{205}{8}B^3+\frac{405}{64}B\right)-\cdots \quad (92)$$

which agrees precisely with the perturbative expression (28) found in the uniform WKB approach. Substituting $E_{\rm SG}(B,g^2)$ for E in order to reexpress A as $A=A(B,g^2)$, we find

$$A_{\rm SG}(B,g^2)=\frac{4}{g^2}+\frac{g^2}{4}\left(3B^2+\frac{3}{4}\right)+\frac{g^4}{16}\left(5B^3+\frac{17}{4}B\right)+\frac{5g^6}{4096}(176B^4+328B^2+27)+\frac{9g^8}{16384}(336B^5+1120B^3+327B)+\cdots \quad (93)$$

In the ZJJ approach [20], the expression (89) determines the energy E as a function of g^2, provided both functions $B(E,g^2)$ and $A(E,g^2)$ are known. On the other hand, in the uniform WKB approach, this same condition is viewed as determining B as a function of g^2, given the function $A(B,g^2)$, and this is then inserted into the perturbative expansion $E(B,g^2)$ to determine the energy.

B. Cancellation of ambiguities (beyond Bogomolny–Zinn-Justin)

We first demonstrate the cancellation between the ambiguous imaginary terms arising from the non-Borel-summability of the perturbative series and the ambiguous imaginary terms arising from the analytic continuation in g^2 in the global boundary condition including perturbative fluctuations around the nonperturbative factors. This cancellation of ambiguities at two-instanton order is known as the Bogomolny–Zinn-Justin mechanism. Below, we provide evidence that this is also true if one includes perturbative fluctuations around the nonperturbative saddle $[\mathcal{I}\bar{\mathcal{I}}]$.

1. Double-well potential

The energy trans-series for the level N can be written as

$$E^{(N)}(g^2)=E\left(B=N+\frac{1}{2},g^2\right)+\delta\nu\left[\frac{\partial E(B,g^2)}{\partial B}\right]_{B=N+\frac{1}{2}}+\frac{1}{2}(\delta\nu)^2\left[\frac{\partial^2 E(B,g^2)}{\partial B^2}\right]_{B=N+\frac{1}{2}}+\cdots \quad (94)$$

The first term is the perturbative series, which is non-Borel-summable. The resummation results in the imaginary ambiguous term of order two-instantons, e^{-2S_I/g^2}. For the reality of the resurgent trans-series for real coupling, this must be canceled by an imaginary part coming from the higher nonperturbative terms in the trans-series.

From (46) we see that the first imaginary term arises in the $O(\xi^2)$ term in $\delta\nu$, which is the two-instanton sector. To this order it has the form

$$\mathrm{Im}(\delta\nu)=\pm\pi\left(\frac{(-\frac{2}{g^2})^N}{N!}\times\exp\left[-\frac{1}{2}\left(A_{\rm DW}\left(N+\frac{1}{2}\right)-\frac{1}{3g^2}\right)\right]\right)^2\xi^2+\cdots \quad (95)$$

$$=\pm\pi\left(\frac{(-\frac{2}{g^2})^N}{N!}\right)^2\left[1-g^2q_2\left(N+\frac{1}{2}\right)+g^4\left(\frac{1}{2}q_2^2\left(N+\frac{1}{2}\right)-q_3\left(N+\frac{1}{2}\right)\right)+\cdots\right]\xi^2+\cdots \quad (96)$$

where the polynomials $q_k(B)$ are defined in (88):

$$A_{\rm DW}(B,g)-\frac{1}{3g^2}\equiv\sum_{k=1}^{\infty}g^{2k}q_{k+1}(B). \quad (97)$$

Note that the prefactor of ξ^2 is a perturbative series in g^2.

The leading imaginary part of the energy coming from the two-instanton sector, including the perturbative fluctuations around it, can be found by calculating $\mathrm{Im}([\delta\nu\frac{\partial E}{\partial B}]_{B=\frac{1}{2}})$. Using (87) we find

$$\frac{\partial E_{\rm DW}}{\partial B}=2\left(1-6Bg^2-\left(51B^2+\frac{19}{4}\right)g^4+\cdots\right). \quad (98)$$

For example, for the $N=0$ level we get (recall, $\xi^2\sim e^{-2S_I/g^2}$)

$$\mathrm{Im}\left(\left[\delta\nu\frac{\partial E}{\partial B}\right]_{B=\frac{1}{2}}\right)=\pm 2\pi\left(1-\frac{35g^2}{6}-\frac{1277g^4}{72}-\cdots\right)\times\left(1-3g^2-\frac{35g^4}{2}-\cdots\right)\xi^2 \quad (99)$$

$$=\pm 2\pi\left(1-\frac{53}{6}g^2-\frac{1277}{72}g^4-\cdots\right)\xi^2. \quad (100)$$

Compare this with the large-order behavior of perturbation theory quoted in Eq. (8.7) of [20] (converted to our notation)

$$E_k^{(0)}\sim-\frac{2}{\pi}3^{k+1}k!\left[1-\frac{53}{6}\frac{1}{(3k)}-\frac{1277}{72}\frac{1}{(3k)^2}-\cdots\right]. \quad (101)$$

Given the subleading corrections to large-order terms (101), we can obtain the imaginary part by standard dispersion relation arguments [4]. Remarkably, not only does the leading term cancel, but also the subleading terms are canceled once we include the prefactor. This precise correspondence between the coefficients of the behavior of high orders of perturbation theory about the vacuum and the coefficients of the low orders of fluctuations about the 2-instanton sector is an explicit example of resurgence. The behavior near one saddle (P saddle) "resurges" in the behavior near another saddle ($[\mathcal{I}\bar{\mathcal{I}}]$, a NP saddle) [15].

2. Sine-Gordon potential

For the SG potential, we note the important distinction that the imaginary part in the $O(\xi^2)$ term does not depend on the Bloch angle θ. It must be this way if this term is to cancel against an imaginary term arising from the non-Borel-summable perturbative series, because the perturbative series is clearly independent of θ. For example, for the $N=0$ level we get

$$\mathrm{Im}\left(\left[\delta\nu\frac{\partial E_{\mathrm{SG}}}{\partial B}\right]_{B=\frac{1}{2}}\right)=\pm 2\pi\left(1-\frac{3g^2}{8}-\frac{13g^4}{128}-\cdots\right)\times\left(1-\frac{g^2}{4}-\frac{3g^4}{32}-\cdots\right)\xi^2 \quad (102)$$

$$=\pm 2\pi\left(1-\frac{5}{2}\frac{g^2}{4}-\frac{13}{8}\left(\frac{g^2}{4}\right)^2-\cdots\right)\xi^2. \quad (103)$$

Compare this with the large-order behavior of perturbation theory quoted in Appendix A of [7] (converted to our notation)

$$E_k^{(0)}\sim-\frac{2}{\pi}k!\left[1-\frac{5}{2k}-\frac{13}{8}\frac{1}{k^2}-\cdots\right] \quad (104)$$

from which we obtain the imaginary part by standard dispersion relation arguments [4].

To recap, for the DW and SG problems, the instanton actions are given by $S_I^{\mathrm{DW}}=\frac{1}{6}$ and $S_I^{\mathrm{SG}}=\frac{1}{2}$, respectively. The instanton factor is $\xi=\sqrt{\frac{2S_I}{\pi g^2}}e^{-S_I/g^2}$, while the imaginary part associated with Borel resummation of vacuum energy is $\pm 2\pi\xi^2=\pm 2\times\frac{2S_I}{g^2}e^{-2S_I/g^2}$. Including fluctuations around the $[\mathcal{I}\bar{\mathcal{I}}]$ in the trans-series, we can write

$$\mathrm{Im}\left(2\times[\mathcal{I}\bar{\mathcal{I}}]_\pm\sum_{k=0}^{\infty}a_k^{[\mathcal{I}\bar{\mathcal{I}}]}g^{2k}\right)=\pm\frac{4S_I}{g^2}e^{-2S_I/g^2}(a_0^{[\mathcal{I}\bar{\mathcal{I}}]}+a_1^{[\mathcal{I}\bar{\mathcal{I}}]}g^2+a_2^{[\mathcal{I}\bar{\mathcal{I}}]}g^4+\cdots)+O(e^{-4S_I/g^2}). \quad (105)$$

This implies that, using the dispersion relations,

$$E_k^0=\frac{1}{\pi}\int_0^\infty \mathrm{Im}E_0(g^2)\frac{d(g^2)}{(g^2)^{k+1}}, \quad (106)$$

the large-order behavior of the perturbation theory (including the subleading $1/k$ suppressed terms) is given by

$$E_k^{(0)}\sim-\frac{2}{\pi}\frac{k!}{(2S_I)^k}\times\left[a_0^{[\mathcal{I}\bar{\mathcal{I}}]}+a_1^{[\mathcal{I}\bar{\mathcal{I}}]}\left(\frac{2S_I}{k}\right)+a_2^{[\mathcal{I}\bar{\mathcal{I}}]}\left(\frac{2S_I}{k}\right)^2+\cdots\right] \quad (107)$$

which can be checked against the result obtained via Bender-Wu recursion relations, an independent method to calculate (107).

This implies that in both DW and SG cases, not only does the leading term cancel, but also the subleading terms are canceled once we include the prefactor. Once again, there is a precise correspondence between the coefficients of the behavior of high orders of perturbation theory about the vacuum and the coefficients of the low orders of fluctuations about the 2-instanton sector: this is resurgence at work.

V. GENERATING NP PHYSICS FROM P PHYSICS

At first sight (and naively), there is no real difference between the ZJJ and uniform WKB approaches. However, the latter approach reveals a simple and elegant relation between perturbative and nonperturbative physics that is not obvious in the former.

In ZJJ, one computes the perturbative function $B(E,g^2)$ and the nonperturbative function $A(E,g^2)$, and imposes an exact quantization condition. Although calculation of $B(E,g^2)$ is straightforward, the evaluation of $A(E,g^2)$ is more challenging.

In the uniform WKB approach, one computes the perturbative function $E(B,g^2)$ and the prefactor function $A(B,g^2)$, and imposes a global boundary condition. This reveals an extremely simple (but nonobvious) relation between the two functions $E(B,g^2)$ and $A(B,g^2)$:

$$\frac{\partial E}{\partial B} = -\frac{g^2}{S}\left(2B + g^2\frac{\partial A}{\partial g^2}\right) \tag{108}$$

where S is the numerical coefficient of the instanton action in $\xi \equiv e^{-S/g^2}/\sqrt{\pi g^2}$. This relation was not observed in [20], because the relation is not apparent when looking at the expansions of the functions $B(E,g^2)$ and $A(E,g^2)$. (Note that $E_{ZJ} \to \frac{E}{2}$, and $g_{ZJ} \to g^2$. We used ZJJ in [27].)

Equation (108) has a magical implication: that all the information in the nonperturbative expression $A(B,g^2)$ is completely determined by the perturbative expression $E(B,g^2)$. Thus, the nonperturbative computation of $A(B,g^2)$ is actually unnecessary. The overall factor S appearing in the formula can also be deduced from the large-order growth of $E(B,g^2)$, or can be computed trivially by usual instanton methods, but the crucial thing is that it is also already encoded in late non-alternating terms of, for example, the ground state perturbative expansion $E(B=0+\frac{1}{2},g^2) \sim \sum_n a_n g^{2n}$ where $a_n \sim n!/(2S)^n$.

This is astonishing, especially in light of the extremely complicated nonperturbative multi-instanton analysis required to compute $A(E,g^2)$ in [20]. All features of the nonperturbative sector are encoded in the perturbative sector, provided we know the perturbative expansion $E(B,g^2)$ as a function of both the coupling g^2 and the level number parameter B.

This is an explicit realization of Écalle's statement that all information about the trans-series is encoded in the perturbative sector. For this double-well potential, this fact was noticed previously, in a slightly different form, in a beautiful paper by Álvarez [26]. Below we show that it is more general [27].

A. Double-well potential

For the sake of comparison, we recall the DW potential expressions:

$$\begin{aligned} E_{\mathrm{DW}}(B,g^2) = 2B &- 2g^2\left(3B^2+\frac{1}{4}\right) - 2g^4\left(17B^3+\frac{19}{4}B\right) \\ &- 2g^6\left(\frac{375}{2}B^4+\frac{459}{4}B^2+\frac{131}{32}\right) \\ &- 2g^8\left(\frac{10689}{4}B^5+\frac{23405}{8}B^3+\frac{22709}{64}B\right) \\ &- \cdots \end{aligned} \tag{109}$$

$$\begin{aligned} A_{\mathrm{DW}}(B,g^2) &= \frac{1}{3g^2} + g^2\left(17B^2+\frac{19}{12}\right) + g^4\left(125B^3+\frac{153}{4}B\right) \\ &+ g^6\left(\frac{17815}{12}B^4+\frac{23405}{24}B^2+\frac{22709}{576}\right) \\ &+ g^8\left(\frac{87549}{4}B^5+\frac{50715}{2}B^3+\frac{217663}{64}B\right) - \cdots \end{aligned} \tag{110}$$

Notice the similarities between terms in the expansion of $A_{\mathrm{DW}}(B,g^2)$ and $E_{\mathrm{DW}}(B,g^2)$. To make this completely explicit, compute

$$\begin{aligned} \frac{\partial E_{\mathrm{DW}}(B,g^2)}{\partial B} &= 2 - 12Bg^2 - 2g^4\left(51B^2+\frac{19}{4}\right) \\ &- 2g^6\left(750B^3+\frac{459B}{2}\right) \\ &- 2g^8\left(\frac{53445B^4}{4}+\frac{70215B^2}{8}+\frac{22709}{64}\right) - \cdots \end{aligned} \tag{111}$$

And, for comparison, compute

$$\begin{aligned} -6g^4\frac{\partial A_{\mathrm{SG}}(B,g^2)}{\partial g^2} &= 2 - 6g^4\left(17B^2+\frac{19}{12}\right) - 18g^6\left(125B^3+\frac{153}{4}B\right) \\ &- 18g^8\left(\frac{17815}{12}B^4+\frac{23405}{24}B^2+\frac{22709}{576}\right) - \cdots \end{aligned} \tag{112}$$

We deduce the remarkably simple relation between the perturbative expression $E_{\mathrm{DW}}(B,g^2)$ and $A_{\mathrm{DW}}(B,g^2)$:

$$\frac{\partial E_{\mathrm{DW}}}{\partial B} = -12Bg^2 - 6g^4\frac{\partial A_{\mathrm{DW}}}{\partial g^2} \tag{113}$$

which is nothing but (108) with $S = 1/6$, the instanton action. This means that the nonperturbative expression $A_{\mathrm{DW}}(B,g^2)$ is completely determined by the perturbative expression $E_{\mathrm{DW}}(B,g^2)$.

B. Sine-Gordon potential

Remarkably, exactly the same thing happens for the SG potential. Again, for the sake of comparison, we recall the expressions:

$$E_{\rm SG}(B,g^2) = B - \frac{g^2}{2}\left(B^2+\frac{1}{4}\right) - \frac{g^4}{8}\left(B^3+\frac{3B}{4}\right) - \frac{g^6}{32}\left(\frac{5B^4}{2}+\frac{17B^2}{4}+\frac{9}{32}\right) - \frac{g^8}{128}\left(\frac{33B^5}{4}+\frac{205B^3}{8}+\frac{405B}{64}\right)+\cdots \quad (114)$$

$$A_{\rm SG}(B,g^2) = \frac{4}{g^2} + \frac{g^2}{4}\left(3B^2+\frac{3}{4}\right) + \frac{g^4}{16}\left(5B^3+\frac{17B}{4}\right) + \frac{5g^6}{4096}(176B^4+328B^2+27) + \frac{9g^8}{16384}(336B^5+1120B^3+327B)+\cdots \quad (115)$$

Notice again the similarities between terms in the expansion of $A_{\rm SG}(B,g^2)$ and $E_{\rm SG}(B,g^2)$. To make this completely explicit, compute

$$\frac{\partial E_{\rm SG}(B,g^2)}{\partial B} = 2 - Bg^2 - \frac{g^4}{8}\left(3B^2+\frac{3}{4}\right) - \frac{g^6}{32}\left(10B^3+\frac{17B}{2}\right) - \frac{g^8}{128}\left(\frac{165B^4}{4}+\frac{615B^2}{8}+\frac{405}{64}\right)-\cdots \quad (116)$$

And, for comparison, compute

$$-\frac{1}{2}g^4\frac{\partial A_{\rm SG}(B,g^2)}{\partial g^2} = 2 - \frac{g^4}{8}\left(3B^2+\frac{3}{4}\right) - \frac{g^6}{32}\left(10B^3+\frac{17B}{2}\right) - \frac{g^8}{128}\left(\frac{165B^4}{4}+\frac{615B^2}{8}+\frac{405}{64}\right)-\cdots \quad (117)$$

We deduce the remarkably simple relation:

$$\frac{\partial E_{\rm SG}}{\partial B} = -Bg^2 - \frac{1}{2}g^4\frac{\partial A_{\rm SG}}{\partial g^2} = -\frac{2(g^2/4)B}{S} - \frac{(g^2/4)^2}{S}\frac{\partial A}{\partial(g^2/4)}. \quad (118)$$

In the second equality, we observe that instanton action $S=1/2$ and expansion parameter $g^2 \to g^2/4$, and hence, this is again the same as (108) [27].

C. Fokker-Planck potential

In [20], ZJJ present expressions for $B(E,g)$ and $A(E,g)$ for the Fokker-Planck potential (in this section we use their conventions for the coupling and normalizations):

$$V_{\rm FP}(y) = \frac{1}{2}y^2(1-y)^2 + g\left(y-\frac{1}{2}\right). \quad (119)$$

This is essentially a double-well potential, with a linear symmetry breaking term. It can be thought of as the SUSY QM version of the double-well problem. ZJJ give the results:

$$B_{\rm FP}(E,g) = E + 3gE^2 + g^2\left(35E^3+\frac{5}{2}E\right) + g^3\left(\frac{1155}{2}E^4+105E^2\right)+\cdots \quad (120)$$

The nonperturbative function $A_{\rm FP}(E,g)$ is [20]

$$A_{\rm FP}(E,g) = \frac{1}{3g} + g\left(17E^2+\frac{5}{6}\right) + g^2\left(227E^3+\frac{55}{2}E\right) + g^3\left(\frac{47431}{12}E^4+\frac{11485}{12}E^2+\frac{1105}{72}\right)+\cdots \quad (121)$$

Inverting, to write E as a function of B, we find

$$E_{\rm FP}(B,g) = B - 3gB^2 - g^2\left(17B^3+\frac{5}{2}B\right) - g^3\left(\frac{375}{2}B^4+\frac{165}{2}B^2\right)+\cdots \quad (122)$$

Inserting the expression for $E=E_{\rm FP}(B,g)$ we obtain

$$A_{\rm FP}(B,g) = \frac{1}{3g} + g\left(17B^2+\frac{5}{6}\right) + g^2\left(125B^3+\frac{55}{2}B\right) + \frac{5}{72}g^3(21378B^4+11370B^2+221)+\cdots \quad (123)$$

Thus, we observe the relation

$$\frac{\partial E_{\rm FP}(B,g)}{\partial B} = -6Bg - 3g^2\frac{\partial A_{\rm FP}(B,g)}{\partial g}. \quad (124)$$

So, again, the nonperturbative function $A_{\rm FP}(B,g)$ is determined by the perturbative function $E_{\rm FP}(B,g)$.

D. Symmetric anharmonic oscillator Potential

Another example studied by ZJJ is the $O(d)$ symmetric anharmonic oscillator (AHO), with potential (in this section we use their conventions for the coupling and normalizations):

$$V_{\mathrm{AHO}}(\vec{x}) = \frac{1}{2}\vec{x}^2 + g(\vec{x}^2)^2. \tag{125}$$

The radial problem with angular momentum l leads to a spectral problem characterized by a parameter $j = l + d/2 - 1$. ZJJ find the following expressions, as a function of j:

$$\begin{aligned} B_{\mathrm{AHO}}(E,g) = E &+ \frac{g}{2}(j^2 - 3E^2 - 1) \\ &+ \frac{g^2}{4}(-15j^2E + 35E^3 + 25E) \\ &+ \frac{g^3}{16}(-35j^4 + 630j^2E^2 + 210j^2 - 1155E^4 \\ &- 1470E^2 - 175) + \cdots \end{aligned} \tag{126}$$

The function $A_{\mathrm{AHO}}(E,g)$ is [20] (note that we adopt the sign convention from [20])

$$\begin{aligned} A_{\mathrm{AHO}}(E,g) = -\frac{1}{3g} &+ g\left(\frac{3j^2}{4} - \frac{17E^2}{4} - \frac{19}{12}\right) \\ &+ g^2\left(-\frac{77j^2E}{8} + \frac{227E^3}{8} + \frac{187E}{8}\right) \\ &+ g^3\left(-\frac{341j^4}{64} + \frac{3717j^2E^2}{32} + \frac{1281j^2}{32} \right. \\ &\left. - \frac{47431E^4}{192} - \frac{34121E^2}{96} - \frac{28829}{576}\right) + \cdots \end{aligned} \tag{127}$$

Inverting, to write E as a function of B, we find

$$\begin{aligned} E_{\mathrm{AHO}}(B,g) = B &+ \frac{1}{2}g(3B^2 - j^2 + 1) \\ &+ \frac{1}{4}g^2(-17B^3 + 9Bj^2 - 19B) \\ &+ \frac{1}{16}g^3(375B^4 - 258B^2j^2 + 918B^2 \\ &+ 11j^4 - 142j^2 + 131) + \cdots \end{aligned} \tag{128}$$

Converting A to a function of B, we find

$$\begin{aligned} A_{\mathrm{AHO}}(B,g) = -\frac{1}{3g} &+ \frac{1}{12}g(-51B^2 + 9j^2 - 19) \\ &+ \frac{1}{8}g^2(125B^3 - 43Bj^2 + 153B) \\ &+ \frac{1}{576}g^3(-53445B^4 + 26730B^2j^2 \\ &- 140430B^2 - 909j^4 + 14778j^2 - 22709) \\ &+ \cdots \end{aligned} \tag{129}$$

Thus, we see that for all j, we have the relation

$$\frac{\partial E_{\mathrm{AHO}}(B,g)}{\partial B} = 3Bg + 3g^2\frac{\partial A_{\mathrm{AHO}}(B,g)}{\partial g}. \tag{130}$$

So, again, the nonperturbative function $A_{\mathrm{AHO}}(B,g)$ is determined by the perturbative function $E_{\mathrm{AHO}}(B,g)$.

VI. CONCLUSIONS

In this paper we have given an elementary derivation, using a uniform WKB expansion, of the appearance of trans-series expressions of the form (2) for energy eigenvalues in quantum problems with degenerate harmonic minima. We have shown that this trans-series form is generic for such problems because it can be related, in the small g^2 limit, to basic analyticity properties of the parabolic cylinder functions that underly harmonic vacuum problems. The global boundary conditions that relate neighboring vacua for the double-well potential and the periodic Sine-Gordon potential problems lead naturally to resurgent relations connecting different parts of the trans-series expansion, again due to analyticity properties of the parabolic cylinder functions.

The trans-series expansion unifies the perturbative and nonperturbative sectors, in such a way that ambiguities are canceled between sectors, yielding real and unambiguous results. The global boundary conditions are expressed in terms of two functions, the perturbative energy $E = E(B, g^2)$, and a nonperturbative function $A = A(B, g^2)$ that contains the single-instanton factor and fluctuations around it. Here $B = N + \frac{1}{2}$, where N is an integer labeling the energy level or band. Given these two functions, the global boundary condition generates the entire trans-series expansion, incorporating *all* multi-instanton effects to all orders both perturbatively and nonperturbatively.

Finally, we have shown that there is a remarkably simple relation between the functions $E(B, g^2)$ and $A(B, g^2)$, which means that $A(B, g^2)$ is completely determined by knowledge of $E(B, g^2)$. Thus, the entire trans-series, including all perturbative, nonperturbative and quasizero-mode terms, is encoded in the perturbative expansion [27]. In other words, the fluctuations around the vacuum saddle point contain information about all other nonperturbative saddles, including their nonperturbative actions as well as perturbative fluctuations around them. This is a physical manifestation of the mathematical concept of resurgence. A more complete understanding of this remarkable phenomenon in the language of path integrals would facilitate further application of the ideas and methods of resurgence to quantum field theory and string theory [24,25,33–35,48–51].

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