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Inspecting baby Skyrmions with effective metrics

G. W. Gibbons*

DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, United Kingdom and Laboratoire de Mathématiques et Physique Théorique, Université de Tours, UFR Sciences et Techniques, Parc de Grandmont, 37200 Tours, France

E. Goulart†

CAPES Foundation, Ministry of Education, Brasília/DF, CEP 70040-020 Brasília, DF, Brazil and DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

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In the present paper we investigate the causal structure of the baby Skyrme model using appropriate geometrical tools. We discuss several features of excitations propagating on top of background solutions and show that the evolution of high frequency waves is governed by a curved effective geometry. Examples are given for which the effective metric describes the interaction between waves and solitonic solutions such as kinks, antikinks, and hedgehogs. In particular, it is shown how violent processes involving the collisions of solitons and antisolitons may induce metrics which are not globally hyperbolic. We argue that it might be illuminating to calculate the effective metric as a diagnostic test for pathological regimes in numerical simulations.

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I. INTRODUCTION

The Skyrme model [1] is a nonlinear theory where the fields take values in a Riemannian manifold, typically a Lie group. Originally conceived in the context of high energy physics the model provided a framework where baryons emerged as topologically protected solitons, called Skyrmions. Although the relevant field quantities consisted of just π-meson clouds, the fermionic nucleon was obtained as a specific finite-energy, particlelike, and stable configuration of the bosonic fields (see [2] for details). Later, it was realized that the model appears in the low energy effective field theory of quantum chromodynamics in the limit in which the number of colors is large [3]. Recently, apart from nuclear physics, the model proved to be useful in various condensed matter systems, string theory, and holographic QCD [4,5].

Generically, the associated Euler-Lagrange equations are quasilinear [6]–[8] and subtle issues arise: (i) wave velocities are not given *a priori*, but change as functions of initial data, directions of propagation, and wave polarization; (ii) it is rather usual that the resulting system does not admit solutions which can be continuously extended from the smooth initial configuration: often they tend after a finite time to discontinuous or singular states; (iii) not all initial data are mathematically admissible since for a large class of them the Cauchy problem is ill posed; (iv) there exist the possibility of domains with nonhyperbolic regimes despite the fact that the energy momentum tensor satisfies the dominant energy condition.

Systematic treatment on these topics have proven to be difficult, and efforts to apply finite difference methods in simulations discovered numerical instabilities. Although general results have been obtained for semilinear wave maps, they do not generalize directly to the quasilinear regime [9]–[12]. Quite recently, a local well-posedness result was provided by Wong [13], sharpening some previous conclusions of Crutchfield and Bell [14] and Gibbons [15]. Unfortunately, most of the results concerning the time evolution of Skyrme fields depend on numerics or approximation schemes, and some of them lack satisfactory analytical explanation.

To gain some intuition into the dynamics of the theory it is convenient to obtain an effective metric description of the linearized waves. This is because the effective metrics uniquely determine the causal structure providing at the same time the natural language to treat the well posedness of the initial-boundary value problem. As a consequence, they may help in the identification of pathological solutions when they exist and why instabilities appear in some situations involving collisions/scattering of solitons. Among other developments, the effective metric led to the construction of analog models of gravity, which imitate the kinematical properties of gravitational fields, and to insights into the evolutionary properties of hyperbolic partial differential equations (PDEs) (see [16] for a review).

The point of this article is to investigate the causal structure of the baby Skyrme model using the appropriate geometrical tools. The model is a $(2+1)$-dimensional analogue of the original model with the unit sphere $\mathbb{S}^2$

*G.W.Gibbons@damtp.cam.ac.uk
†egoulart@cbpf.br

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as a target [17]–[19]. Because of the lesser number of dimensions the baby version serves as a toy model for the full theory where some ideas and methods can be tested. They also have physical significance on their own, having applications in some condensed matter systems such as the fractional quantum Hall effect [22]. In particular, we will derive a fourth order Fresnel-like equation for the high frequency waves and show that it factorizes into two quadratic characteristic polynomials. It follows that the causal structure of the theory is governed by a duplicity of effective geometries which depend explicitly on the pulled-back geometry for the map. As a consequence, background solutions of the equations behave as a birefringent medium for the linearized waves. We next show that the model admits signature transitions as well, possibly yielding an elliptic regime for the PDEs. In the last section we evaluate the effective metric for some well known solutions such as kinks and baby Skyrmions and show that a wave interacting with the solitons have drastic modifications on its propagation properties.

II. BABY SKYRME MODELS: GENERAL REMARKS

The model is based on a smooth map $\phi^A : M \to N$ from spacetime M to a target manifold N. Here, (M, η) is equal to $\mathbb{R}^{1+2}$ equipped with a Minkowski metric $\mathrm{diag}(+--)$ and (N, h) is the two-dimensional unit sphere $\mathbb{S}^2$ endowed with a positive definite metric $h_{AB}(\phi)$. The Lagrangian is constructed from the (1,1) pulled-back tensor $\mathbf{L} := h_{AB}(\phi)\partial^a\phi^A\partial_b\phi^B$ and the action is provided by [17]–[19]

$$S = \int \frac{1}{2}\mathrm{Tr}(\mathbf{L}) + \frac{\kappa^2}{4}[\mathrm{Tr}((\mathbf{L})^2) - (\mathrm{Tr}(\mathbf{L}))^2] - V(\phi)d^3x, \tag{1}$$

where κ^2 is a parameter controlling the strength of the nonlinearity.[1] The first term in the Lagrangian reproduces the classical O(3) sigma model while the second is the three-dimensional analogue of the Skyrme term. While in $(3+1)$ the potential $V(\phi)$ is optional, but its presence in the $(2+1)$ model is necessary for the stability of the solitonic solutions.[2]

In what follows we use the conventions: (i) capital latin indices, A, stand for the target space quantities and lower latin indices, a, stand for spacetime tensors; (ii) $(a, b) = ab + ba$ and $[ab] = ab - ba$ for symmetrization and antisymmetrization, respectively; (iii) $h_{ABPQ}(\phi) \equiv h_{A[P}h_{BQ]} = h_{[AP}h_{B]Q}$. Variation with respect to ϕ^A yields a system of second order quasilinear PDEs which can be written in the compact form

$$(H_{AB}\partial^a\phi^B)_{||C}\partial_a\phi^C + V_{,A} = 0, \tag{2}$$

where

$$H_{AB}(\phi, \partial\phi) \equiv h_{AB} - \kappa^2 h_{APBQ}\partial^a\phi^P\partial_a\phi^Q \tag{3}$$

and $||$ represents the covariant derivative with respect to h_{AB}, i.e., $h_{AB||C} = 0$. In terms of the target connection $\Gamma^A{}_{BC}$, Eq. (2) can be written as

$$\partial_a(H_{AB}\partial^a\phi^B) - \Gamma^D{}_{AC}H_{DB}\partial^a\phi^B\partial_a\phi^C + V_{,A} = 0, \tag{4}$$

which reveals that the equation of motion consists of various types of self-interactions arising from the non-standard kinetic terms, the potential, and the target geometry. Nevertheless, quasilinearity implies that the system is linear with respect to higher order derivatives of the dependent field variables. Generically, it is possible to express the equations as

$$M^{ab}{}_{AB}(\phi, \partial\phi)\partial_a\partial_b\phi^B + J_A(\phi, \partial\phi) = 0, \tag{5}$$

where J_A stands for semilinear terms in ϕ^A (lower order derivatives) and the principal symbol is given by

$$M^{ab}{}_{AB} = \left\{\eta^{ab}h_{AB} + \frac{\kappa^2}{2}h_{APBQ}(\partial^a\phi^{(Q}\partial^b\phi^{P)} - 2\eta^{ab}\partial^c\phi^P\partial_c\phi^Q)\right\}. \tag{6}$$

As it is well known, the highest-order terms in derivatives almost completely control the qualitative behavior of solutions of a partial differential equation. We note that, in this case, the principal symbol M is symmetric with respect to ab and AB, i.e., $M^{[ab]}{}_{AB} = M^{ab}{}_{[AB]} = 0$. Also, in the limit of $\kappa^2 \to 0$, Eq. (2) reduces to a semilinear equation

$$\Box\phi^A + \Gamma^A{}_{BC}\partial_a\phi^B\partial^a\phi^C = 0, \tag{7}$$

which reproduces the well-known classical $O(3)$ sigma model in flat spacetime.

III. CAUSAL STRUCTURE

It has been known for some time that the propagation of the excitations of nonlinear field theories in a given background is governed by an effective metric that depends on the background field configuration and on the details of the nonlinear dynamics (see [16] for a review). These propagation features can be analyzed by means of the

[1] Other approaches have been used in the past absorbing the parameter κ into the potential term by rescaling the coordinates. In our discussion it will be convenient to keep it in the calculations for stressing the role played by quasilinearity from the very beginning.

[2] See [20] for supersymmetric extensions of the model supporting Bogomol'nyi-Prasad-Sommerfield solutions and [21] for a model endowed with vector mesons.

eikonal approximation, i.e., the regime of small-amplitude/high-frequency waves propagating on top of a smooth solution $\phi_0^A(x)$. Formally, we consider a one-parameter family of solutions of the form

$$\phi^A(x) = \phi_0^A(x) + \alpha\varphi^A(x)\exp(i\Sigma(x)/\alpha), \tag{8}$$

and let the real parameter $\alpha \to 0$. In this limit, only the higher order derivative terms contribute to the propagation laws for the waves. In other words, we can discard source terms J_A in (5) and consider only the principal part term contributions.[3]

Defining the wave covector $k_a \equiv \partial_a\Sigma$, the equation of motion reduces to the eigenvalue equation

$$[M^{ab}{}_{AB}(\phi_0)k_a k_b]\varphi^B = 0. \tag{9}$$

For a general $k_a \in T_x^*M$ we define the symmetric matrix $M_{AB}(\phi_0, k) \equiv M^{ab}{}_{AB}(\phi_0)k_a k_b$. It follows that (9) can be solved only if k_a satisfy the algebraic conditions

$$F_x(\phi_0, k) \equiv \det(M_{AB}(\phi_0, k)) = 0. \tag{10}$$

As a consequence, at a given spacetime point, the wave normals are characterized by the roots of a multivariate polynomial of fourth order in k_a in the cotangent space. The resulting algebraic variety changes from point to point in a way completely prescribed by the background solution ϕ_0^A and the nonlinearities of the baby Skyrme theory. In general, it will consist at most of two nested sheets, each with the topology of a convex cone.

The general form of F_x is given by a homogeneous polynomial of the form

$$F_x = \det(h_{AB})G^{abcd}(\phi_0)k_a k_b k_c k_d \tag{11}$$

with G^{abcd} a completely symmetric quantity. The latter can be written solely in terms of the spacetime metric and the pulled-back geometry. Thus, the vanishing sets of (11) constitute the baby Skyrmionic analogues of the Fresnel equation encountered in optics (see [23] and references therein for a similar derivation in the context of electrodynamics). They play the role of a fourth order spacetime dispersion relation (at least up to a conformal factor).

Now, the algebraic structure of G^{abcd} reveals that the quartic equation factorizes, yielding the generic birefringence effect; i.e., the characteristic polynomial reduces to a product of two simpler quadratic terms satisfying

$$[\eta^{ab}k_a k_b][(h^{-1})^{cd}k_c k_d] = 0, \tag{12}$$

with h^{-1} a reciprocal quadratic form in the cotangent space given by

$$(h^{-1})^{ab} \equiv (1 - 2\kappa^2\mathcal{L}_s)\eta^{ab} + [\kappa^2(1 - \kappa^2\mathrm{Tr}(\mathbf{L}))L^{ab} + \kappa^4 L^{ac}L_c{}^b], \tag{13}$$

with $\mathcal{L}_s$ denoting the Lagrangian of the model without the potential term.

Thus, the wave fronts are not arbitrarily given but satisfy some relations completely prescribed by the pullback. In general, $\Sigma(x)$ will solve one quadratic polynomial or the other, although it is possible that there exist some directions where the vanishing sets coincide. Consequently, the model admits two different types of waves. One wave travels with the velocity of light while the other travels with a velocity which depends implicitly on the solution.

If the quantity $(h^{-1})^{ab}$ is nondegenerate, it is possible to define its inverse h_{ab} such that $(h^{-1})^{ac}h_{cb} = \delta^a{}_b$. The explicit form of h_{ab} may easily be calculated by applying the Cayley-Hamilton theorem to the matrix $\mathbf{L}$. Because $\det(\mathbf{L}) = 0$, it results in the matrix relation

$$\mathbf{L}^3 = \mathrm{Tr}(\mathbf{L})\mathbf{L}^2 + \frac{1}{2}[\mathrm{Tr}((\mathbf{L})^2) - (\mathrm{Tr}(\mathbf{L}))^2]\mathbf{L}, \tag{14}$$

and we obtain

$$h_{ab} = (1 - 2\kappa^2\mathcal{L}_S)^{-1}[\eta_{ab} - \kappa^2 L_{ab}]. \tag{15}$$

In general, the *effective metric* h_{ab} characterizes a Lorentzian metric on spacetime, the null cones of which are the effective "sound cones" of the theory. The ray vectors associated with the wave fronts are defined as

$$q^a \equiv \eta^{ab}k_b \quad \text{if } \eta^{ab}k_a k_b = 0,$$
$$q^a \equiv (h^{-1})^{ab}k_b \quad \text{if } (h^{-1})^{ab}k_a k_b = 0.$$

It follows from (12) that q^a are the vanishing sets of the dual polynomial G_x, i.e.,

$$G_x(\phi_0, q) \equiv [\eta_{ab}q^a q^b][h_{ab}q^a q^b] = 0. \tag{16}$$

As is well known, these cones completely characterize the causal structure of the theory once a solution is given. Note, however, that (13) and (15) are defined only up to a conformal transformation $\tilde{h}_{ab} \to \Omega^2(x)h_{ab}$. For the sake of simplicity we assume that $\Omega^2 = (1 - 2\kappa^2\mathcal{L}_S)$ and adopt the rescaled effective structure

$$\tilde{h}_{ab} = \eta_{ab} - \kappa^2 L_{ab}, \qquad (\tilde{h}^{-1})^{ab} = \Omega^{-2}h^{ab}. \tag{17}$$

The result that the high-energy perturbations of some nonlinear theories propagate along geodesics that are not null in the background geometry but in an effective spacetime has been obtained several times in the literature [25]. This is also true for the baby Skyrme model. Indeed, defining the effective covariant derivative such that $\tilde{h}_{ab;c} = 0$, it follows that if $\tilde{h}_{ab}q^a q^b = 0$, then

[3] A similar result may be obtained using the Hadmard's method of jump discontinuities (see, for instance, [24]).

$$q^a{}_{;b}q^b = 0, \tag{18}$$

which is the equation of a null geodesic. For an arbitrary smooth solution ϕ_0 the quadratic form $\tilde{h}$ is generally curved. Note, however, that it becomes flat in the limit $\kappa^2 \to 0$. Note also that, although different choices $V(\phi)$ may lead to qualitatively different theories, its particular form does not appear explicitly in the expression of $\tilde{h}_{ab}$.

A. Signature transitions and breakdowns

As $\tilde{h}_{ab}$ is a field dependent quantity, we cannot guarantee that all roots of G_x are automatically real. This will be the case only if the background fields satisfy certain conditions. A direct calculation yields the condition

$$\det(\tilde{h})_{ab}/\det(\eta_{ab}) = (1 - 2\kappa^2\mathcal{L}_S) > 0, \tag{19}$$

which means that, in order to guarantee that the effective metric has the correct signature $(+--)$, the Lagrangian has to satisfy an algebraic constraint. Note that the constraint is the same for all possible potentials.

Following Manton [26] we suppose that L_{ab} can be diagonalized relative to η_{ab}. It is clear that its eigenvalues are necessarily non-negative, so we write them as $L_{ab} = \mathrm{diag}(\lambda_0^2, \lambda_1^2, \lambda_2^2)$. A direct calculation yields the components

$$\tilde{h}_{00} = (1-\kappa^2\lambda_0^2), \quad \tilde{h}_{11} = -(1+\kappa^2\lambda_1^2), \quad \tilde{h}_{22} = -(1+\kappa^2\lambda_2^2). \tag{20}$$

It follows that the 00 component of the effective metric is not always positive, as it should be. For sufficiently small values of λ_0, $\tilde{h}$ defines a metric with a hyperbolic signature. In this regime, all roots of $\tilde{F}$ and $\tilde{G}$ are real, and we have a well-defined causal structure. Nevertheless, for large values of λ_0 it is possible to envisage a situation where the effective metric changes its signature. The new regime is characterized by a spacetime region where the equations are of the elliptic type $(+,+,+)$: *instabilities arise*. The two distinct regions are generally separated by a two-dimensional membrane where the metric is singular.

One concludes that the physical validity of the baby Skyrme model is constrained by the inequality $\lambda_0^2 < \kappa^{-2}$. For static fields, this is always the case since $\lambda_0 = 0$. For time-dependent fields it is important to keep in mind that a signature transition may occur. If this is the case, the system of quasilinear PDEs loses its physical predictability. This danger is particularly important during violent processes involving the collision of solitons and antisolitons [27]. Therefore, it might be illuminating to calculate the effective metric during numerics as a diagnostic tool.

IV. EXAMPLES

We shall illustrate the formalism described above with examples coming from well-known solutions to the baby Skyrme model. To explore these aspects, it is convenient to parametrize the fields with spherical coordinates $\phi^A = (f, \psi)$ in the target and metric $h_{AB} = \mathrm{diag}(1, \sin^2 f)$. From now on we will work with the so-called *old model* potential [28] which explicitly violates the O(3)-rotational iso-invariance of the theory[4]

$$V(f) = \mu^2(1 - \cos f), \tag{21}$$

where μ^2 is a mass parameter which we set equal to 1. This potential is analogous to the pion mass term in the $(3+1)$ dimensional Skyrme model and is generally associated with external fields in the context of magnetic systems.

A. Sine-Gordon-like solutions

As shown in [30], any solution of the sine-Gordon model is also a solution of the baby Skyrme model. Indeed, in the case of a map with constant azimuthal field ($\partial_a\psi = 0$) the equation of motion (4) becomes

$$\partial_a\partial^a f + \sin f = 0 \tag{22}$$

which, of course, is the sine-Gordon equation. Thus, all known solutions of (22) such as kinks, antikinks, and breathers automatically satisfy the baby Skyrme model equations. *Although the simplified model is restrictive from the point of view of admissible solutions, it provides a useful arena for developing insights on how the causal structure behaves if a background solution is given. This simplification can be viewed as a first step toward the analysis of the full nonintegrable model.*[5]

Note that for these types of maps the parameter κ^2 does not appear in the equations. Despite this fact, the effective metric controlling the causal structure is given by

$$\tilde{h}_{ab} = \eta_{ab} - \kappa^2\partial_a f\partial_b f. \tag{23}$$

We analyzed the causal structure associated with a static kink, a collision of kinks, and a collision of kink and antikink. The results are illustrated below.

[4]Other possibilities include the *holomorphic* potential and the so-called *new model* potential [29].

[5]Similar simplifications are often considered in the context of analogue models of gravitation where basic concepts such as emergent horizons can be anticipated without evoking the whole complexity of the system. A typical example is the assumption of inviscid, vorticity-free fluids in the study of artificial black holes [16].

1. Static kink

The kink represents a localized solitary wave, traveling at a velocity $|c| < 1$. It is characterized by a solution of the form

$$f(x,t) = 4\arctan\left(\exp\left(\pm\frac{x - x_0 - ct}{\sqrt{1-c^2}}\right)\right), \tag{24}$$

where the $\pm$ signs correspond to localized structures which are called kink and antikink, respectively. In a reference frame comoving with the solution, we can set $x_0 = c = 0$. A direct calculation yields a diagonal effective geometry of the form

$$\tilde{h}_{00} = 1, \qquad \tilde{h}_{11} = -(1 + 4\mathrm{sech}(x)), \qquad \tilde{h}_{22} = -1. \tag{25}$$

Note that Eqs. (25) are the same for the kink or the antikink solutions, which means that high frequency excitations interact with both solutions in the same way. Obviously, this metric has a Lorentzian signature for all spacetime points.

Figure 1 shows the light cones evaluated with the effective metric in the t-x plane. It is seen that the cones coincide with those of the Minkowskian geometry far from the kink (antikink), and get thinner inside it, signaling that high-energy perturbations propagate subluminally there. We also see that the propagation in the direction y is not affected by the kink (antikink), in accordance with Eqs. (25). The time spent by a given perturbation inside the localized solution depends on the parameters κ and μ, but the solution cannot trap the high-energy perturbations.

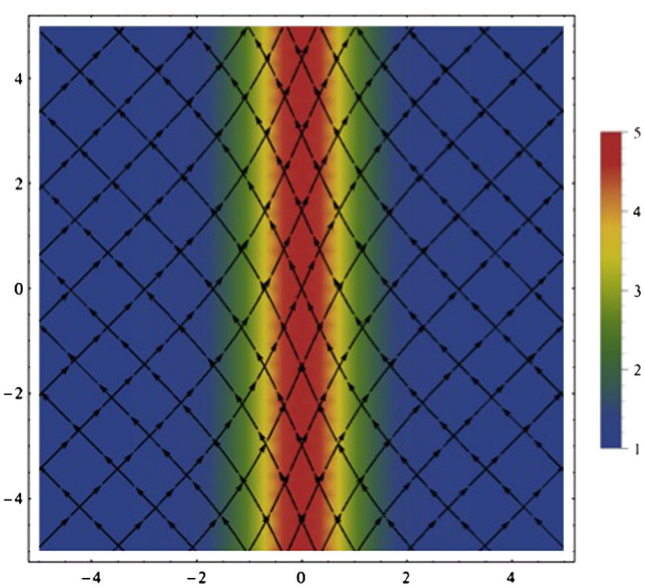

FIG. 1 (color online). Causal structure of the kink (antikink) solution in the t-x plane. The colors represent the behavior of the function $\sqrt{\tilde{h}}$ with parameters $\kappa^2 = 1$ and $\mu = 1$. In this case it is well defined for all possible spacetime events. Note, however, that it only varies significantly near the kink (antikink), yielding the usual cone for distant excitations.

2. Kink-kink collision

Although there are no static solutions describing multisolitons in the sine-Gordon scheme, there are time-dependent solutions, which describe the scattering/collisions of two or more kinks [2]. It is perhaps in the collisions of solitons that the effective geometry may help us in understanding better when instabilities arise and when the solutions are nonphysical. Rather unusually, some of these solutions can be written in analytical form. The kink-kink solution is given by

$$f(x,t) = 4\arctan\left(\frac{c \times \sinh(\frac{x}{\sqrt{1-c^2}})}{\cosh(\frac{ct}{\sqrt{1-c^2}})}\right), \tag{26}$$

and describes the collision between two kinks with respective velocities c and $-c$, approaching the origin from $t \to -\infty$, and moving away from it with velocities $\pm c$ for $t \to +\infty$.

It is well known that there is a repulsive force between two kinks, so they scatter backward after the collision. This behavior is reflected in the causal structure of this solution. Figure 2 illustrates the behavior of $\sqrt{\tilde{h}}$ in the x-t plane for different values of the velocity parameter c. Again, it is possible to show that the signature of the effective metric is Lorentzian for all possible events, implying that the high frequency limit is well defined. Far away from the kinks, the metric becomes nearly flat and $\tilde{h} \approx 1$. Nevertheless, near the collision, this quantity is considerably modified leading to nontrivial interactions between the waves and the solution. Figure 3 shows the behavior of the cones for a particular value of c.

3. Kink-antikink collision

The kink-antikink collision solution is

$$f(x,t) = 4\arctan\left(\frac{\sinh(\frac{ct}{\sqrt{1-c^2}})}{c \times \cosh(\frac{x}{\sqrt{1-c^2}})}\right) \tag{27}$$

with $\pm c$ again describing the velocities of the solitons. As is well known, unlike most topological solitons which are annihilated by antisolitons into radiation, the kink and antikink scatter elastically.

From the physical point of view it seems that there is nothing wrong with the kink-antikink collision. Nevertheless, a direct inspection in the effective metric reveals a hidden pathology in the solution. Indeed, the effective metric is not Lorentzian everywhere and becomes singular in a large bidimensional surface in spacetime. This membrane separates the hyperbolic region from a region where the baby Skyrme equations become elliptic

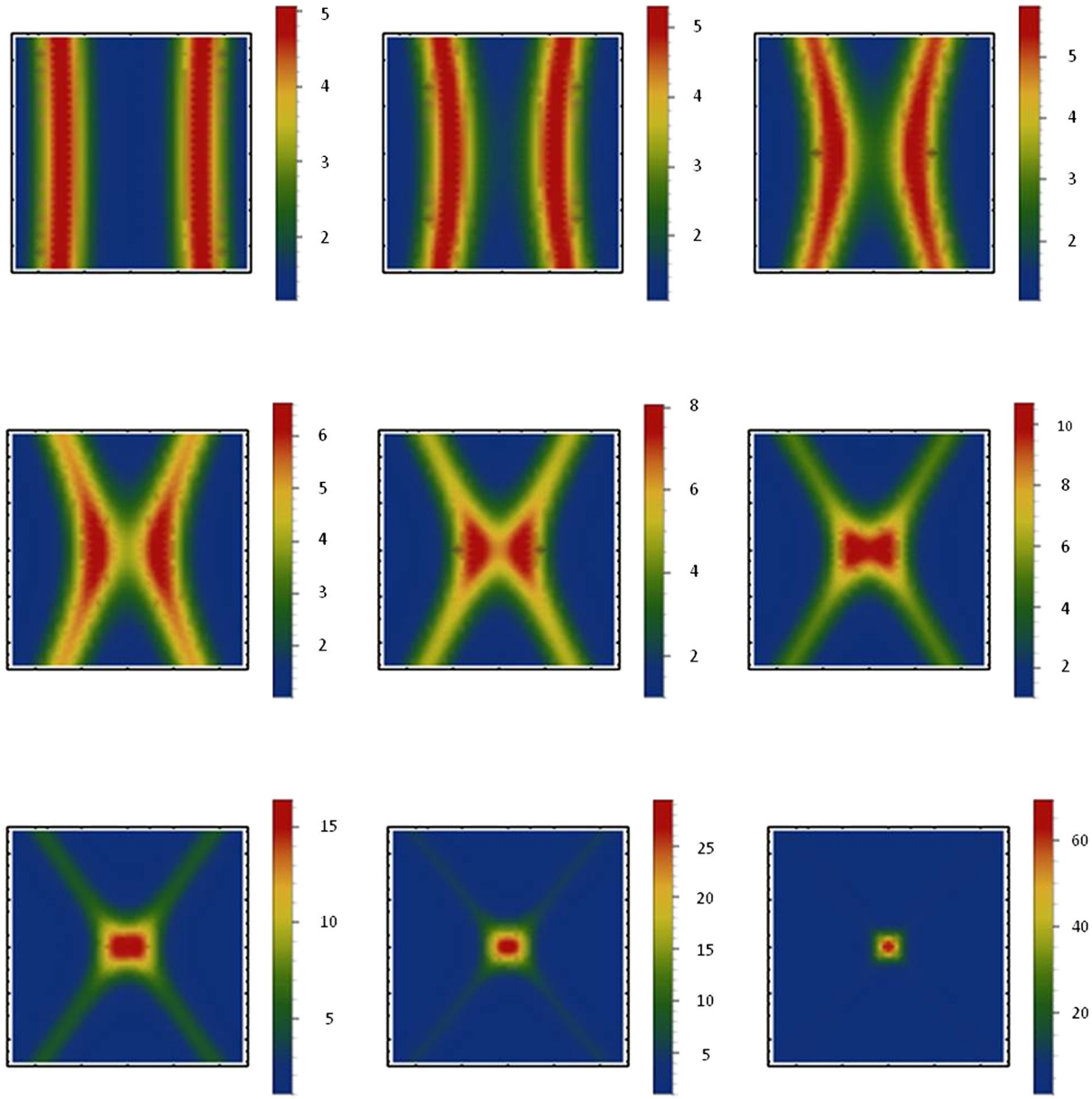

FIG. 2 (color online). Color diagrams representing the function $\sqrt{h}$ for the kink-kink collision in the t-x plane for $0.1 \leq c \leq 0.9$. Note that the function is real for all values of the parameter.

(see Fig. 4). Thus, the PDEs are actually mixed in this regime. There is a good chance that the instabilities reported in numerical simulations involving collisions are associated with this signature transition.

B. Hedgehog solutions

It is well known that the baby-Skyrme model has solitonlike topologically stable static solutions (called baby Skyrmions) and that these solitons can form bound states. To construct these solutions, one must use a radially symmetric ansatz (hedgehog configuration) and reduce the quasilinear PDE (4) to ordinary differential equations. Adopting polar coordinates (r, θ) in the spacetime x-y plane, we consider the class of static solutions of the form

$$f_n = f_n(r), \qquad \psi = n\theta, \tag{28}$$

where $f_n(r)$ is the so-called profile function and $n \in Z$. We assume also the supplementary conditions $f_n(0) = m\pi$ ($m \in Z$) and $\lim_{r\to+\infty} f_n = 0$ to guarantee that the solutions are localized in space and that the total energy is finite. In analogy with the $(3+1)$ case, the domain of this model is compactified to $\mathbb{S}^2$, yielding the topology required for the classification of its field configurations into homotopy classes $\pi_2(\mathbb{S}^2) = \mathbb{Z}$. The above ansatz implies that the

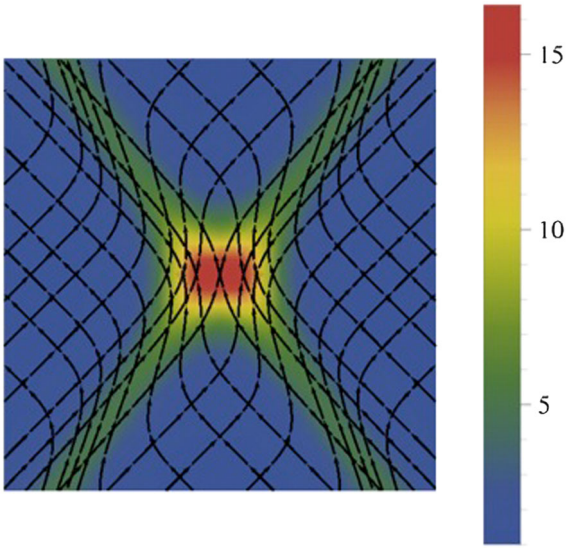

FIG. 3 (color online). Causal structure associated with the kink-kink collision in the t-x plane for $c = 0.7$. Note that the velocities of propagation are drastically modified near the collision.

profile functions f_n satisfy the nonlinear ordinary differential equation

$$\left(r + \frac{n^2 \sin^2 f}{r}\right) f'' + \left(1 - \frac{n^2 \sin^2 f}{r^2} + \frac{n^2 f' \sin f \times \cos f}{r}\right) f' - \frac{n^2 \sin f \times \cos f}{r} - r \times \sin f = 0 \tag{29}$$

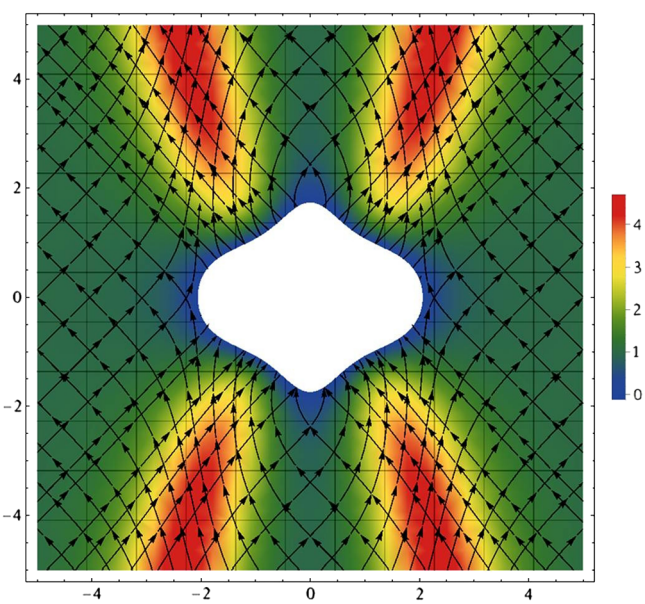

FIG. 4 (color online). Causal structure of the kink-antikink collision in the t-x plane for $c = 0.5$. The colors represent the function $\sqrt{\tilde{h}}$. Note that there exists a region where it vanishes. The white hole in the middle of the figure represents a region with an elliptic signature.

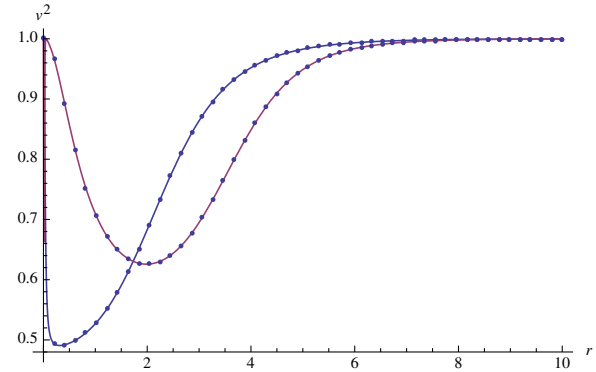

FIG. 5 (color online). Radial velocity in terms of r for the first two baby Skyrmions $n = 1$ (blue curve) and $n = 2$ (red curve). Note that for $r \to \infty$ and $r = 0$ the velocities coincide with the velocity of light.

with $f' = df/dr$. Unfortunately, there are no analytic solutions for this model and the baby Skyrmion solutions have to be computed numerically. Nevertheless, we may readily compute the form of the effective geometry. It follows that the causal structure associated with the baby Skyrmions is given by null intervals $\tilde{h}_{ab} dx^a dx^b = 0$ satisfying

$$dt^2 - (1 + f_n'^2) dr^2 - (r^2 + n^2 \sin^2 f_n) d\theta^2 = 0, \tag{30}$$

where the effective geometry carries with it the topological degree of the map n. In Fig. 5 is shown *the qualitative behavior* of the velocity of propagation $v^2 = dr^2/dt^2$ as a function of the coordinate r for $n = 1$ and $n = 2$.

We note that the rotationally inavariant ansatz (28) is a simplification which does not provide absolute minimums for higher degrees $n \neq 2$. To understand higher order degrees one needs to use a more complicated ansatz. Unfortunately it is not clear at the moment how to treat these solutions using analytical tools. Generically the resulting system is treated numerically using finite difference schemes to evaluate the space derivatives while time evolution integration is made using a Runge-Kutta method. The 3-soliton, for instance, is not rotationally symmetric, and its configuration is similar to a molecule made up of three distorted baby Skyrmions [18]. Typically, for $n > 2$ the hedgehog fields (28) are unstable and the time evolution ends at a less symmetric configurations with lower energy. In principle the effective metric may be computed for these nonsymmetric situations as well. We thus conclude that it would be an interesting task to calculate the effective metrics in simulations involving complicated static solutions as well as in collisions of baby Skyrmions.

V. CONCLUSIONS

The net result of this paper is the derivation of an effective metric description of high frequency waves in the baby Skyrme model. Starting with the geometric optics approximation we derive a fourth order Fresnel-like

equation and show that it factorizes into two quadratic characteristic polynomials. It follows that the causal structure of the theory is governed by two distinct geometrical structures, each playing the role of a second order spacetime dispersion relation (at least up to a conformal factor). As a consequence we show that background solutions of the baby Skyrme equations behave as birefringent medium for the linearized waves. As usual in the context of crystal optics, birefringence is responsible for the phenomenon of double refraction whereby a ray, when incident upon a complex material, is split into two rays taking slightly different paths. We then analyze examples for which the effective metric describes the interaction between waves and solitonic solutions such as kinks, antikinks, and hedgehogs. In particular, it is shown how violent processes involving the collisions of solitons and antisolitons may induce metrics which are not globally hyperbolic. Possible extensions of our work include numerical simulations involving less symmetric static configurations as well as violent processes of collision. It would also be interesting to work with other potentials which may lead to different qualitative behavior. In particular, a similar analysis in the context of the integrable model would be very welcome.

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