Stable string bit models

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In string bit models, the superstring emerges as a very long chain of "bits," in which s fermionic degrees of freedom contribute positively to the ground state energy in a way to exactly cancel the destabilizing negative contributions of $d = s$ bosonic degrees of freedom. We propose that the physics of string formation be studied nonperturbatively in the class of string bit models in which $s > d$, so that a long chain is stable, in contrast to the marginally stable ($s = d = 8$) superstring chain. We focus on the simplest of these models with $s = 1$ and $d = 0$, in which the string bits live in zero space dimensions. The string bit creation operators are $N \times N$ matrices. We choose a Hamiltonian such that the large N limit produces a string moving in one space dimension, with excitations corresponding to one Grassmann light-cone world sheet field ($s = 1$) and no bosonic world sheet field ($d = 0$). We study this model at finite N to assess the role of the large N limit in the emergence of the spatial dimension. Our results suggest that stringlike states with large bit number M may not exist for $N \leq (M - 1)/2$. If this is correct, one can have finite chains of string bits, but not a continuous string, at finite N. Only for extremely large N can such chains behave approximately like a continuous string, in which case there will also be the (approximate) emergence of a new spatial dimension. In string bit models designed to produce a critical superstring at $N = \infty$, we can then expect only approximate Lorentz invariance at finite N, with violations of order $1/N^2$.

I. INTRODUCTION

String bit models [1] provide one approach to a fundamental formulation of string theory. For another approach, see Ref. [2]. These models are motivated by interpreting the light-cone Hamiltonian for a single string [3],

$$P^- = \frac{1}{2} \int_0^{P^+} d\sigma [\boldsymbol{p}^2 + T_0^2 \boldsymbol{x}'^2], \tag{1}$$

as the large M limit of the Hamiltonian for a harmonic chain of $M = P^+/m$ string bits,[1] where m is the fundamental unit of P^+,

$$H = \frac{1}{2m} \sum_{k=1}^{M} [\boldsymbol{p}_n^2 + T_0^2 (\boldsymbol{x}_{n+1} - \boldsymbol{x}_n)^2]. \tag{2}$$

The idea is to take string bits as the fundamental degrees of freedom of string theory. In this interpretation Lorentz invariance is not built in *a priori*. Moreover, string bits move about in the $d = D - 2$ transverse space dimensions; the spatial coordinate x^- conjugate to P^+ is missing. It will be regained for strings consisting of a very large number

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[1]The compatibility of such a discretization with string interactions is supported by the success of a similar discretization of Mandelstam's interacting light-cone world sheet path integrals [4] carried out in Ref. [5].
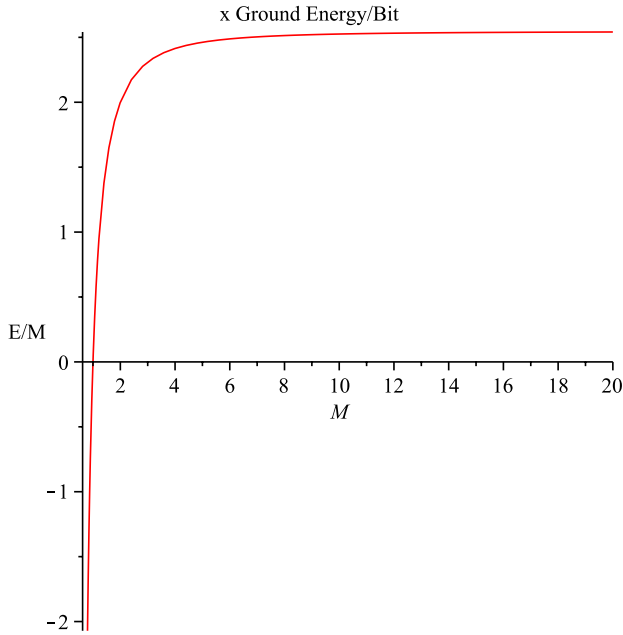
of bits, provided that the excitation spectrum of H scales like $1/M = m/P^+$ in the limit $M \to \infty$. This is one of the earliest implementations of 't Hooft's holography hypothesis [6]. The normal mode frequencies of the closed string bit chain are $\omega_n = (2T_0/m) \sin(n\pi/M)$, with $n = 0, \ldots, (M - 1)$. Thus, all modes with finite n or finite $M - n$ as $M \to \infty$ have the desired scaling behavior.

But the ground state energy of a bosonic closed string bit chain is

$$E_G = \frac{1}{2} \sum_n \omega_n = \frac{dT_0}{m} \sum_{n=1}^{M-1} \sin\frac{n\pi}{M} = \frac{dT_0}{m} \cot\frac{\pi}{2M}$$
$$= \frac{2dT_0 M}{m\pi} - \frac{\pi dT_0}{6Mm} + \mathcal{O}(M^{-3}), \tag{3}$$

where we have taken $M \to \infty$ to regain the continuous string. Since the string interactions conserve bit number, the first term can be dropped [5], and the second term can be identified as $-\pi dT_0/(6P^+)$. For $d = 24$ we obtain the ground state mass squared of the bosonic string $-8\pi T_0$. This result is a version of the zero point energy calculation of Brink and Nielsen [7]. Since the ground state is a tachyon, it signals the instability of bosonic string theory. Moreover, this instability is present at all finite M, which is immediately apparent from the monotonicity of the graph of ground energy per bit number $E_G(M)/M$ as a function of bit number shown in Fig. 1. For a fixed total number of bits, this graph implies that the lowest energy configuration is for the bits to arrange themselves into closed chains of least

FIG. 1 (color online). Energy per bit number vs bit number for bosonic excitations (phonons).

FIG. 2 (color online). Energy per bit number vs bit number for fermionic excitations (statistics waves).

possible bit number. In other words, if the interaction between closed chains is weak, a closed chain of string bits is unstable. Or when a number of bits is brought together, the bits remain individual bits, never joining to form string. In short, a string cannot be expected to emerge from the bosonic string bit model.

This is hardly an unexpected conclusion, since the bosonic string is widely believed to be unstable. This inevitable instability is one motivation for introducing supersymmetry [8]. One imagines that, in addition to the vibrational excitations discussed above, there are also fermionic degrees of freedom [9,10], which are properly thought of as "statistics waves," and which contribute to the ground energy with the opposite sign, cancelling the negative $1/M$ term and leading to massless ground states. Bergman and one of us invented a superstring bit model which accomplishes this cancellation at finite M [11]. The ground state energy of a chain of superstring bits is strictly zero for all finite M.

Employing 't Hooft's large N limit [12] in its Fock space formulation [13], as described in Ref. [1] for the bosonic string, we introduced in Ref. [11] a superstring bit annihilation operator

$$(\phi_{[a_1 \cdots a_n]})_\alpha{}^\beta(\boldsymbol{x}), \qquad n = 0, \ldots, s, \qquad (4)$$

where each a_i is a spinor index running over s values and $\alpha, \beta = 1, \ldots N$ are color indices for the adjoint representation of the color group $U(N)$. Poincaré supersymmetry dictates that $s = d = 8$ for the superstring. The ϕ's are bosonic if n is even and fermionic if n is odd. The square brackets in the subscript remind us that the enclosed indices are completely antisymmetric. Thus, at each transverse
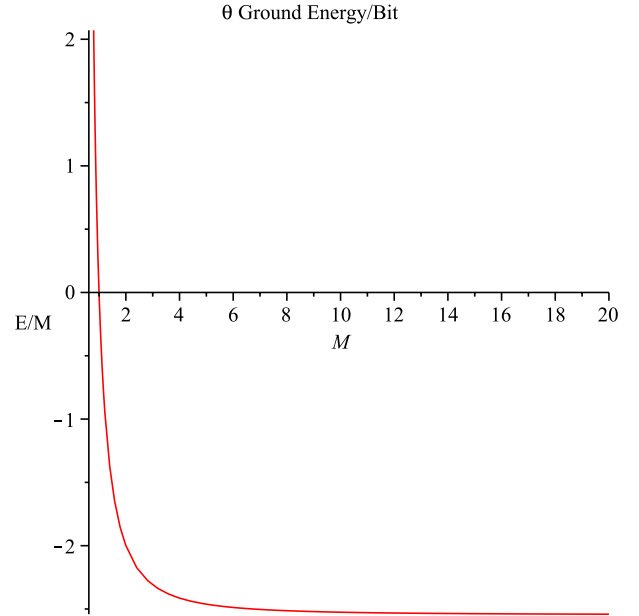
space point $\boldsymbol{x}$, there are 256 degrees of freedom, 128 each of bosonic and fermionic type. We then constructed a Hamiltonian operator which reproduced, in the 't Hooft limit, the mass spectrum of the free superstring when the bit number becomes very large.[2] The problem of devising the correct superstring interactions, which should be consistent with Lorentz invariance, was only partially resolved in Ref. [11]. But temporarily setting aside Lorentz invariance, we stress that our model did induce a (Lorentz noninvariant) interacting superstring theory along with a new dimension of space.

The ground state energy of a closed chain in this superstring bit model is exactly zero for all M, but only because of the cancellation between phonons and statistics waves [14]. This cancellation does not occur if $s \neq d$. In this more general context, we can identify a stable regime ($s > d$) and an unstable regime ($s < d$). The supersymmetric case is at the boundary between these two regimes. Here we are proposing that, since superstring theory emerges from a very special, marginally stable, string bit model, its underlying physics is better understood in terms of the more general class of stable string bit models. Indeed, we can regard holding $d = D - 2 < s$ as a physical infrared cutoff, in the spirit of dimensional regularization.

In these stable models, the energy per bit curve gets flipped and turns into Fig. 2. This figure shows that with fixed M the lowest energy state is a single string, which becomes a string moving in one space dimension when $M \to \infty$. If two closed strings are present, the lowest

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[2]This conclusion rigorously follows only when $N \to \infty$ *before* M becomes large.

energy configuration shares M equally between the two strings. Thus, models in the stable regime provide a sound foundation for the emergence of a string, albeit without Lorentz invariance. In the rest of this article, we analyze the physics of the simplest of these stable models, with $s = 1$ and $d = D - 2 = 0$. Holography is explicitly realized in the 't Hooft limit $N \to \infty$. We will be particularly interested in the physics of these models at finite N and whether finite N can still support chains of an arbitrarily large bit number. Indeed, we find indications, from our study of low-dimension toy string bit models, that a stable string of M bits requires *at least* $N > (M - 1)/2$. If this is so, there will be violations of the Lorentz invariant dispersion law $2P^+P^- - \boldsymbol{p}^2 = m^2$ at finite N because P^+ would necessarily remain discrete at finite N. These violations would be present even if the degrees of freedom and interactions are tuned to satisfy Lorentz invariance order by order in the $1/N$ expansion, which these considerations suggest has zero radius of convergence. One should then be able to translate limits on the accuracy of tests of Lorentz invariance to a lower bound on N, or, if a successful string bit model of all physics could be devised, to an upper bound on Newton's gravitational constant, which is of order $1/N^2$ in these models.

The superstring bit model in zero space dimensions with $s = 1$, studied in the rest of this paper, is a far cry from a successful model of all physics. Nonetheless, it is rich enough to test the soundness of the physical ideas we are advocating. In Sec. II we present the details of the model. Among many possible choices for a Hamiltonian built of single trace operators, we single out the one proposed in Ref. [11] and discuss its $N \to \infty$ limit. In Sec. III, we apply the variational principle to obtain an upper bound on the ground state energy. This bound can be proven only when $N > (M - 1)/2$. Then in Sec. IV, we explore the model at finite N by studying states with a relatively low bit number M. We find all energy eigenstates in the two bit case $M = 2$ and all color singlet and color adjoint states when $M = 3$. We also describe some results for $M = 4, 5$, for which the detailed analysis will appear in a separate article. We close the paper in Sec. V with a discussion of the issues involved with finding string bit models that produce a string in higher dimensions. In the interests of keeping the paper self-contained, we include an Appendix, which calculates the exact energy eigenstates at $N = \infty$ for the model studied here. This is of course a special case of results already obtained in Ref. [11] in a general context. However, it includes two new aspects: (1) the explicit analysis of the consequences of the cyclic symmetry of closed chains of string bits and (2) the calculation of the $N = \infty$ energy eigenvalues of an open (color adjoint) chain, which demonstrates color confinement in this toy string bit model. The energy gap between adjoint and singlet sectors is of order M times the energy scale set by the continuum limit. We also include a second Appendix, which sketches a systematic method to do variational calculations in these models.

II. $s = 1$, $d = 0$ STRING BIT MODEL

The $s = 1$, $d = 0$ superstring bit model contains N^2 bosonic (a_α^β) and N^2 fermionic (b_α^β) string bits. We define conjugate operators $\bar{a}_\alpha^\beta \equiv (a_\beta^\alpha)^\dagger$ and $\bar{b}_\alpha^\beta \equiv (b_\beta^\alpha)^\dagger$, and these operators satisfy the (anti)commutation relations,

$$[a_\alpha^\beta, \bar{a}_\gamma^\delta] = \delta_\alpha^\delta \delta_\gamma^\beta, \qquad \{b_\alpha^\beta, \bar{b}_\gamma^\delta\} = \delta_\alpha^\delta \delta_\gamma^\beta, \qquad (5)$$

all others vanishing. Then we choose a Hamiltonian that is a linear combination of the single trace operators

$$\text{Tr}\bar{a}^2 a^2, \qquad \text{Tr}\bar{b}^2 b^2, \qquad \text{Tr}\bar{b}^2 a^2, \qquad \text{Tr}\bar{a}^2 b^2,$$
$$\text{Tr}\bar{a}\,\bar{b}\,ba, \qquad \text{Tr}\bar{a}\,\bar{b}\,ab, \qquad \text{Tr}\bar{b}\,\bar{a}\,ba, \qquad \text{Tr}\bar{b}\,\bar{a}\,ab, \quad (6)$$

with coefficients scaling as $1/N$. All of these structures conserve bit number and share the feature that the two annihilation operators are consecutive in the trace as are the two creation operators. This feature is necessary for the term to survive the limit $N \to \infty$. Terms without this feature such as $(1/N)\text{Tr}:\bar{a}a\bar{a}a:$ are suppressed at large N, so they can also be included, if desired, without affecting the large N limit. We do not include them here for simplicity only, but in some models they may be necessary for stability reasons.

For definiteness we choose H so that in the large N limit the dynamics reduces to the superstring bit model invented in Ref. [11]. In the large N limit, H maps single trace states to single trace states. To describe these states in this toy model, it is useful to define a supercreation operator,

$$\psi(\theta) = \bar{a} + \bar{b}\theta, \qquad \bar{b} = -\frac{d}{d\theta}\psi, \qquad \bar{a} = \left(1 - \theta\frac{d}{d\theta}\right)\psi, \quad (7)$$

where θ is a Grassmann anticommuting number. Then a basis of single trace states can be taken to be

$$|\theta_1\theta_2\cdots\theta_M\rangle = \text{Tr}[\psi(\theta_1)\psi(\theta_2)\cdots\psi(\theta_M)]|0\rangle. \quad (8)$$

Then it is straightforward to apply each of the candidate terms in the Hamiltonian to such a basis state to get

$$\frac{1}{N}\text{Tr}\bar{a}^2 a^2|\theta_1\theta_2\cdots\theta_M\rangle$$
$$= \sum_{k=1}^{M}\left(1 - \theta_k\frac{d}{d\theta_k}\right)\left(1 - \theta_{k+1}\frac{d}{d\theta_{k+1}}\right)|\theta_1\theta_2\cdots\theta_M\rangle + \mathcal{O}(N^{-1})$$
$$(9)$$

$$\frac{1}{N}\text{Tr}\bar{a}\,\bar{b}\,ba|\theta_1\theta_2\cdots\theta_M\rangle$$

$$=\sum_{k=1}^{M}\left(1-\theta_k\frac{d}{d\theta_k}\right)\theta_{k+1}\frac{d}{d\theta_{k+1}}|\theta_1\theta_2\cdots\theta_M\rangle+\mathcal{O}(N^{-1})$$

(10)

$$\frac{1}{N}\text{Tr}\bar{b}\,\bar{a}\,ab|\theta_1\theta_2\cdots\theta_M\rangle$$

$$=\sum_{k=1}^{M}\theta_k\frac{d}{d\theta_k}\left(1-\theta_{k+1}\frac{d}{d\theta_{k+1}}\right)|\theta_1\theta_2\cdots\theta_M\rangle+\mathcal{O}(N^{-1})$$

(11)

$$\frac{1}{N}\text{Tr}\bar{b}^2b^2|\theta_1\theta_2\cdots\theta_M\rangle$$

$$=\sum_{k=1}^{M}\theta_k\frac{d}{d\theta_k}\theta_{k+1}\frac{d}{d\theta_{k+1}}|\theta_1\theta_2\cdots\theta_M\rangle+\mathcal{O}(N^{-1})$$ (12)

$$\frac{1}{N}\text{Tr}\bar{a}\,\bar{b}\,ab|\theta_1\theta_2\cdots\theta_M\rangle$$

$$=\sum_{k=1}^{M}\theta_k\frac{d}{d\theta_{k+1}}|\theta_1\theta_2\cdots\theta_M\rangle+\mathcal{O}(N^{-1})$$

(13)

$$\frac{1}{N}\text{Tr}\bar{b}\,\bar{a}\,ba|\theta_1\theta_2\cdots\theta_M\rangle=\sum_{k=1}^{M}\theta_{k+1}\frac{d}{d\theta_k}|\theta_1\theta_2\cdots\theta_M\rangle$$

$$+\mathcal{O}(N^{-1})$$ (14)

$$\frac{1}{N}\text{Tr}\bar{b}^2a^2|\theta_1\theta_2\cdots\theta_M\rangle=\sum_{k=1}^{M}\frac{d}{d\theta_k}\frac{d}{d\theta_{k+1}}|\theta_1\theta_2\cdots\theta_M\rangle$$

$$+\mathcal{O}(N^{-1})$$ (15)

$$\frac{1}{N}\text{Tr}\bar{a}^2b^2|\theta_1\theta_2\cdots\theta_M\rangle=\sum_{k=1}^{M}\theta_{k+1}\theta_k|\theta_1\theta_2\cdots\theta_M\rangle$$

$$+\mathcal{O}(N^{-1})$$ (16)

The structure of the $1/N$ terms not shown involves two traces rather than a single trace. The formulas (13), (14), (15), and (16) show a unique correspondence between single trace terms and Grassmann variable operations. The remaining formulas possess some ambiguities in the correspondence. Using Eq. (12) we can remove the quartic term from Eqs. (9), (10) and (11):

$$\frac{1}{N}\text{Tr}[\bar{a}^2a^2-\bar{b}^2b^2]|\theta_1\theta_2\cdots\theta_M\rangle$$

$$=\sum_{k=1}^{M}\left(1-2\theta_k\frac{d}{d\theta_k}\right)|\theta_1\theta_2\cdots\theta_M\rangle+\mathcal{O}(N^{-1})$$ (17)

$$\frac{1}{N}\text{Tr}[\bar{a}\,\bar{b}\,ba+\bar{b}^2b^2]|\theta_1\theta_2\cdots\theta_M\rangle$$

$$=\sum_{k=1}^{M}\theta_k\frac{d}{d\theta_k}|\theta_1\theta_2\cdots\theta_M\rangle+\mathcal{O}(N^{-1})$$ (18)

$$\frac{1}{N}\text{Tr}[\bar{b}\,\bar{a}\,ab+\bar{b}^2b^2]|\theta_1\theta_2\cdots\theta_M\rangle$$

$$=\sum_{k=1}^{M}\theta_k\frac{d}{d\theta_k}|\theta_1\theta_2\cdots\theta_M\rangle+\mathcal{O}(N^{-1}).$$ (19)

In the large N limit, we can make the ansatz

$$|E\rangle=\int d^M\theta|\theta_1\theta_2\cdots\theta_M\rangle\Psi(\theta_1\theta_2\cdots\theta_M)$$ (20)

for the $N=\infty$ energy eigenstate. The function Ψ is the wave function for one of the discretized Grassmann variables of the superstring. Now by construction the states $|\theta_1\theta_2\cdots\theta_M\rangle$ possess cyclic symmetry:

$$|\theta_1\theta_2\cdots\theta_M\rangle=|\theta_M\theta_1\cdots\theta_{M-1}\rangle.$$ (21)

On the other hand, the measure acquires a phase $(-)^{M-1}$ under a one step cyclic transformation. It follows that the wave function can, without loss of generality, be taken to satisfy

$$\Psi(\theta_1\theta_2\cdots\theta_M)=(-)^{M-1}\Psi(\theta_M\theta_1\cdots\theta_{M-1}).$$ (22)

Consulting Eq. (3.17) of Ref. [11], we see that the first quantized Hamiltonian h should be

$$h=\sum_{k=1}^{M}[-iS_kS_{k+1}+i\tilde{S}_k\tilde{S}_{k+1}-iS_k(\tilde{S}_{k+1}+\tilde{S}_{k-1}-2\tilde{S}_k)]$$

$$=\sum_{k=1}^{M}[-iS_kS_{k+1}+i\tilde{S}_k\tilde{S}_{k+1}-iS_k\tilde{S}_{k+1}-iS_{k+1}\tilde{S}_k+2iS_k\tilde{S}_k)].$$

(23)

In these formulas it is understood that $S_{M+1}\equiv S_1$ and $S_0=S_M$. Here the $S,\tilde{S}$ satisfy the Clifford algebras,

$$\{S_k,S_l\}=2\delta_{kl},\quad\{\tilde{S}_k,\tilde{S}_l\}=2\delta_{kl},\quad\{S_k,\tilde{S}_l\}=0,$$ (24)

and have the representations

$$S_k=\theta_k+\frac{d}{d\theta_k},\qquad\tilde{S}_k=i\left(\theta_k-\frac{d}{d\theta_k}\right)$$ (25)

in terms of Grassmann variables. Then we can also write

$$h = \sum_{k=1}^{M} \left[-2i\theta_k\theta_{k+1} - 2i\frac{d}{d\theta_k}\frac{d}{d\theta_{k+1}} - 2\theta_k\frac{d}{d\theta_{k+1}} \right.$$
$$\left. - 2\theta_{k+1}\frac{d}{d\theta_k} - 2 + 4\theta_k\frac{d}{d\theta_k} \right]. \tag{26}$$

Next we apply integration by parts to rewrite

$$\int d^M\theta |\theta_1\theta_2\cdots\theta_M\rangle h\Psi(\theta_1\theta_2\cdots\theta_M)$$
$$= \int d^M\theta\hat{h}|\theta_1\theta_2\cdots\theta_M\rangle\Psi(\theta_1\theta_2\cdots\theta_M), \tag{27}$$

where

$$\hat{h} = \sum_{k=1}^{M} \left[-2i\theta_k\theta_{k+1} - 2i\frac{d}{d\theta_k}\frac{d}{d\theta_{k+1}} + 2\theta_k\frac{d}{d\theta_{k+1}} \right.$$
$$\left. + 2\theta_{k+1}\frac{d}{d\theta_k} + 2 - 4\theta_k\frac{d}{d\theta_k} \right]. \tag{28}$$

Now we compare the terms in $\hat{h}$ to the dictionary, given in Eqs. (9)–(16), to infer the Fock space Hamiltonian:

$$H = \frac{2}{N}\text{Tr}[i\bar{a}^2b^2 - i\bar{b}^2a^2 + \bar{a}\bar{b}ab + \bar{b}\bar{a}ba + \bar{a}^2a^2 - \bar{b}^2b^2]. \tag{29}$$

Without affecting the $N \to \infty$ limit, we are free to add the terms

$$\Delta H = \frac{1}{N}\text{Tr}[2\xi_1\bar{a}\bar{b}ba + 2\xi_2\bar{b}\bar{a}ab$$
$$+ (\xi_1 + \xi_2)(\bar{a}^2a^2 + \bar{b}^2b^2 - M)] \tag{30}$$

to H because this combination of terms only contributes at order N^{-1}. The number of bits in the state is given by the bit number operator $M = \text{Tr}(\bar{a}a + \bar{b}b)$.

In the following we pick, for definiteness, $\xi_2 = -\xi_1 = 1$, so we will be using the Fock space Hamiltonian:

$$H = \frac{2}{N}\text{Tr}[(\bar{a}^2 - i\bar{b}^2)a^2 - (\bar{b}^2 - i\bar{a}^2)b^2 + (\bar{a}\bar{b} + \bar{b}\bar{a})ba$$
$$+ (\bar{a}\bar{b} - \bar{b}\bar{a})ab]. \tag{31}$$

This Hamiltonian is supersymmetric in that it commutes with the Grassmann odd operator $Q = \text{Tr}(\bar{a}be^{i\pi/4} + \bar{b}ae^{-i\pi/4})$. This Hamiltonian will be our paradigm for the rest of the article. A couple of comments are in order. As Appendix A explains, the spectrum of h is symmetric about 0, before imposing the cyclic constraint, which breaks this symmetry. This means that had we chosen $-H$ instead of H, the large N limit would still be described by the Grassmann variables of the superstring. However, in $-H$ the negative coefficient of

$\text{Tr}\bar{a}^2a^2$ would cause a dangerous instability at finite N because a is bosonic. One can add more terms to make $-H$ stable, but we choose $+H$ as our paradigm, to keep the Hamiltonian as simple as possible. The negative coefficient of $\text{Tr}\bar{b}^2b^2$ in H is not a problem because b is fermionic, and the exclusion principle stabilizes the effects of this term.

III. VARIATIONAL ARGUMENT

In the previous section, we have designed a string bit Hamiltonian that reproduces free superstring dynamics at $N = \infty$. In this section we seek finite N information via the variational principle. Inspection of the Hamiltonian (31) suggests that a low energy state should have a large number of fermionic excitations. This encourages us to consider the trial state $|\psi\rangle = \text{Tr}\bar{b}^M|0\rangle$, which is nonzero only for M odd, in the sector with bit number M, and evaluate

$$E(\psi) = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{32}$$

Since there are no $\bar{a}$ excitations, the numerator is easily evaluated

$$\langle\psi|H|\psi\rangle = -\frac{2}{N}\langle 0|\text{Tr}b^M\text{Tr}\bar{b}^2b^2\text{Tr}\bar{b}^M|0\rangle$$
$$= -M\frac{2}{N}\langle 0|\text{Tr}b^M\bar{b}^2b\bar{b}^{M-1}|0\rangle$$
$$= -2M\langle 0|\text{Tr}b^M\text{Tr}\bar{b}^M|0\rangle$$
$$\quad - \frac{2M}{N}\langle 0|\text{Tr}b^M\text{Tr}\bar{b}^2\sum_{k=1}^{M-2}[(-)^k\text{Tr}\bar{b}^k]\bar{b}^{M-k-2}|0\rangle$$
$$= -2M\langle 0|\text{Tr}b^M\text{Tr}\bar{b}^M|0\rangle$$
$$\quad - \frac{2M}{N}\langle 0|\text{Tr}b^M\sum_{k=1}^{M-2}[(-)^k\text{Tr}\bar{b}^k]\text{Tr}\bar{b}^{M-k}|0\rangle. \tag{33}$$

Now $\text{Tr}b^n = 0$ for even n. Thus, only the terms with k odd contribute to the sum on the right. But as already mentioned we must take M odd to get a nonzero trial, so $M - k$ is even implying $\text{Tr}\bar{b}^{M-k} = 0$. Thus, every term in the sum on the right vanishes, and we have

$$\langle\psi|H|\psi\rangle = -2M\langle 0|\text{Tr}b^M\text{Tr}\bar{b}^M|0\rangle = -2M\langle\psi|\psi\rangle. \tag{34}$$

So provided that $\langle\psi|\psi\rangle \neq 0$, we get the variational estimate $E(\psi) = -2M$, so $E_G \leq -2M$.

It remains to calculate the norm of the trial state:

$$\langle\psi|\psi\rangle = \langle 0|\text{Tr}b^M\text{Tr}\bar{b}^M|0\rangle = M\langle 0|\text{Tr}b^{M-1}\bar{b}^{M-1}|0\rangle. \tag{35}$$

To evaluate the right side, we first derive a recursion formula:

$$\langle 0|\text{Tr}b^{M-1}\bar{b}^{M-1}|0\rangle = \langle 0|\text{Tr}b^{M-2}\left[N\bar{b}^{M-2} + \sum_{k=1}^{M-2}(-)^k[\text{Tr}\bar{b}^k]\bar{b}^{M-k-2}\right]|0\rangle$$

$$= \left(N^2 - \sum_{k=\text{odd}}^{M-2}k\right)\langle 0|\text{Tr}b^{M-3}\bar{b}^{M-3}|0\rangle + \langle 0|\text{Tr}b^{M-3}\left[N\sum_{k=1}^{M-3}(-)^k[\text{Tr}\bar{b}^k]\bar{b}^{M-k-3} + \sum_{k=1}^{M-3}[\text{Tr}\bar{b}^k]b\bar{b}^{M-k-2}\right]|0\rangle$$

$$\langle 0|\text{Tr}b^{M-1}\bar{b}^{M-1}|0\rangle = \left(N^2 - \left[\frac{M-1}{2}\right]^2\right)\langle 0|\text{Tr}b^{M-3}\bar{b}^{M-3}|0\rangle$$

$$+ \langle 0|\text{Tr}b^{M-3}\left[-N\sum_{k=\text{odd}}^{M-3}[\text{Tr}\bar{b}^k]\bar{b}^{M-k-3} + N\sum_{k=\text{odd}}^{M-3}[\text{Tr}\bar{b}^k]\bar{b}^{M-k-3}\right]|0\rangle$$

$$- \langle 0|\text{Tr}b^{M-3}\sum_{k=\text{odd}}^{M-3}[\text{Tr}\bar{b}^k]\sum_{l=\text{odd}}^{M-k-3}[\text{Tr}\bar{b}^l]\bar{b}^{M-k-l-3}|0\rangle$$

$$= \left(N^2 - \left[\frac{M-1}{2}\right]^2\right)\langle 0|\text{Tr}b^{M-3}\bar{b}^{M-3}|0\rangle. \qquad (36)$$

The last line is reached by noting the cancellation of the sums in the square brackets and noting that the double sum in the previous line can be rearranged as

$$\sum_{k=\text{odd}}^{M-3}[\text{Tr}\bar{b}^k]\sum_{l=\text{odd}}^{M-k-3}[\text{Tr}\bar{b}^l]\bar{b}^{M-k-l-3}|0\rangle$$

$$= \sum_{n=\text{even},2}^{M-3}\left[\sum_{k=\text{odd},1}^{n-1}\text{Tr}\bar{b}^k\text{Tr}\bar{b}^{n-k}\right]\bar{b}^{M-n-3}|0\rangle, \qquad (37)$$

and the sum within square brackets is zero because the terms cancel in pairs due to the fact that $\text{Tr}\bar{b}^k, \text{Tr}\bar{b}^{n-k}$ are fermionic operators. So we have proved the recursion relation

$$\langle 0|\text{Tr}b^{2n}\bar{b}^{2n}|0\rangle = (N^2 - n^2)\langle 0|\text{Tr}b^{2n-2}\bar{b}^{2n-2}|0\rangle$$

$$= N\prod_{k=1}^{n}(N^2 - k^2). \qquad (38)$$

To apply the variational principle, we should restrict to integer N so the Hamiltonian is truly Hermitian and require $N > n = (M-1)/2$ so that the norm of the state is positive. With these restrictions we can then rigorously conclude that $E_G < -2M$ for M odd and all $N > (M-1)/2$. We gain no information when $N \leq (M-1)/2$. In Appendix B, we explain an extension of this variational argument to more general trial states.

The fact that the upper bound on the ground state energy behaves like $-2M$ is consistent with the ground state having stringy properties similar to those seen at infinite N. The ground energy of the latter has the large M behavior $-8M/\pi$, consistent with the variational bound. It is interesting that the variational result only applies when $N > (M-1)/2$, which suggests, but does not prove, that

high bit number stringy states may require very large N. We shall find further support for these conclusions in the study of low M energy eigenstates that follows.

IV. LOW M EXAMPLES

The eigenvalues and eigenstates of the Hamiltonian (31) in the 't Hooft limit $N \to \infty$ are reviewed in Appendix A. Before studying our model at finite N for specific values of M, it is helpful to note that the case $N = 1$ is trivial to solve. Then there is just a single a and a single b, and H reduces to

$$H_1 = 2[\bar{a}^2a^2 + 2\bar{a}\,\bar{b}\,ba] = 2(M^2 - M), \qquad (39)$$

an explicit function of the bit number operator $M = \bar{a}a + \bar{b}b$. Eigenstates of M are automatically eigenstates of H. Indeed for fixed M there are only two states,

$$|M, b\rangle = \bar{a}^M|0\rangle, \qquad |M, f\rangle = \bar{a}^{M-1}\bar{b}|0\rangle, \qquad (40)$$

the first a boson and the second a fermion. This energy spectrum rules out the emergence of a string and holography for $N = 1$. There are no large bit number states with excitations that scale as $1/M$. Moreover, there are no large bit number states with energies that grow linearly with M. On the other hand, at $N = \infty$, both of these conditions are met. Clearly the energy spectrum must exhibit dramatic qualitative changes, as a function of N. It is likely that, at fixed N and large enough M, there should be states with the qualitative behavior $E \sim M^2$ of the $N = 1$ case, which would definitely not be stringlike. But this does not by itself forbid the lowest energy states from being stringlike.

In the following we shall mostly restrict our analysis to the color singlet sector of state space, for which the

multitrace Fock state basis introduced in Ref. [13] will be exploited. This nonorthogonal basis gives a linearly independent span of the sector with fixed bit number M, provided that N is sufficiently large. Setting up the energy eigenvalue problem in this basis enables a uniform treatment for all continuous values of N, even noninteger ones. In this formulation, the energy eigenvalues of H are contained in the eigenvalues of a matrix $\mathcal{H}(N)$ that is not Hermitian. This will allow us to explore systematically how the energy spectrum changes as N changes. For particular values of N, this basis set is not linearly independent—it is overcomplete. Indeed, for $N = 1$, this overcompleteness is quite dramatic. In these cases the matrix $\mathcal{H}$ is too big and determines more eigenvalues than are physical, some of which can even be complex, because $\mathcal{H}$ is not Hermitian. The overcomplete basis inherits a metric from the original Fock space, which has non-negative eigenvalues for integer N but may have some negative eigenvalues (ghosts) for fractional N. At integer N the eigenstates of $\mathcal{H}$ which do not correspond to true energy eigenstates must have zero norm under the inherited metric. In the following we shall confirm this expectation for some particular low values of M. Interestingly we also find that even for noninteger N all eigenstates with complex energy also have zero norm, although for some values of noninteger N, there are negative norm eigenstates (ghosts). In any case it is clear that to get a complete understanding of the spectrum in this general formalism it is essential to find not only the energy eigenvalues but also the norms of their energy eigenstates.

A. Two bits

The one bit case $M = 1$ is trivial. The only states are $\bar{a}_\alpha^\beta|0\rangle$ and $\bar{b}_\alpha^\beta|0\rangle$, and our selected Hamiltonian applied to them both gives 0. The singlet states at $M = 1$ are simply $\mathrm{Tr}\bar{a}|0\rangle$ and $\mathrm{Tr}\bar{b}|0\rangle$, with zero energy for all N.

The action of H on the two bit Fock space is not quite so trivial:

$$H\bar{a}_\alpha^\beta\bar{a}_\gamma^\delta|0\rangle = \frac{2}{N}[\delta_\gamma^\beta[\bar{a}\,\bar{a}-i\bar{b}\,\bar{b}]_\alpha^\delta + \delta_\alpha^\delta[\bar{a}\,\bar{a}-i\bar{b}\,\bar{b}]_\gamma^\beta]|0\rangle \quad (41)$$

$$H\bar{b}_\alpha^\beta\bar{b}_\gamma^\delta|0\rangle = \frac{2}{N}[\delta_\gamma^\beta[i\bar{a}\,\bar{a}-\bar{b}\,\bar{b}]_\alpha^\delta - \delta_\alpha^\delta[i\bar{a}\,\bar{a}-\bar{b}\,\bar{b}]_\gamma^\beta]|0\rangle \quad (42)$$

$$H\bar{a}_\alpha^\beta\bar{b}_\gamma^\delta|0\rangle = \frac{2}{N}[\delta_\gamma^\beta[\bar{a}\,\bar{b}+\bar{b}\,\bar{a}]_\alpha^\delta - \delta_\alpha^\delta[\bar{a}\,\bar{b}-\bar{b}\,\bar{a}]_\gamma^\beta]|0\rangle. \quad (43)$$

Tracing on $\beta\gamma$ and on $\delta\alpha$ produces two color singlet energy eigenstates:

$$H\mathrm{Tr}\bar{a}\,\bar{a}\,|0\rangle = 4\mathrm{Tr}\bar{a}\,\bar{a}\,|0\rangle, \quad H\mathrm{Tr}\bar{a}\,\bar{b}\,|0\rangle = 4\mathrm{Tr}\bar{a}\,\bar{b}\,|0\rangle. \quad (44)$$

The first is bosonic, and the second is fermionic, giving a supersymmetric degeneracy. The norms of both states are nonzero for all N, and since their energy eigenvalue is

independent of N, it must be equal to the known $N = 1$ eigenvalue: $2(M^2 - M) \to 4$ for $M \to 2$.

Tracing over $\alpha\beta$ and $\gamma\delta$ produces two more singlet states:

$$H\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}|0\rangle = \frac{4}{N}\mathrm{Tr}\bar{a}^2|0\rangle = \frac{1}{N}H\mathrm{Tr}\bar{a}^2|0\rangle \quad (45)$$

$$H\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}|0\rangle = \frac{4}{N}\mathrm{Tr}\bar{a}\,\bar{b}\,|0\rangle = \frac{1}{N}H\mathrm{Tr}\bar{a}\,\bar{b}\,|0\rangle, \quad (46)$$

from which we easily construct the zero energy color singlet eigenstates

$$\left[\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}-\frac{1}{N}\mathrm{Tr}\bar{a}^2\right]|0\rangle, \quad \left[\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}-\frac{1}{N}\mathrm{Tr}\bar{a}\,\bar{b}\right]|0\rangle, \quad E=0. \quad (47)$$

The squared norms of these two states are $2(N^2 - 1)$ and $(N^2 - 1)$, respectively. They have zero norm at $N = 1$, so they will drop out of the spectrum at $N = 1$ as they must.

Tracing only one pair of indices gives the simplified action of H on color adjoint states:

$$H[\bar{a}\,\bar{a}]_\alpha^\delta|0\rangle = 2[\bar{a}\,\bar{a}-i\bar{b}\,\bar{b}]_\alpha^\delta + \frac{2\delta_\alpha^\delta}{N}\mathrm{Tr}[\bar{a}\,\bar{a}] \quad (48)$$

$$H[\bar{b}\,\bar{b}]_\alpha^\delta|0\rangle = 2[i\bar{a}\,\bar{a}-\bar{b}\,\bar{b}]_\alpha^\delta - i\frac{2\delta_\alpha^\delta}{N}\mathrm{Tr}[\bar{a}\,\bar{a}] \quad (49)$$

$$H[\bar{a}\,\bar{b}]_\alpha^\delta|0\rangle = 2[\bar{a}\,\bar{b}+\bar{b}\,\bar{a}]_\alpha^\delta \quad (50)$$

$$H[\bar{b}\,\bar{a}]_\alpha^\delta|0\rangle = 2[\bar{b}\,\bar{a}-\bar{a}\,\bar{b}]_\alpha^\delta + \frac{4\delta_\alpha^\delta}{N}\mathrm{Tr}\bar{a}\,\bar{b}\,. \quad (51)$$

The eigenstates of this adjoint sector are

$$|E,B\rangle_\alpha^\beta = [\bar{a}\,\bar{a}]_\alpha^\delta|0\rangle - i\frac{E-2}{2}[\bar{b}\,\bar{b}]_\alpha^\delta|0\rangle - \frac{\delta_\alpha^\delta}{N}\mathrm{Tr}[\bar{a}\,\bar{a}]|0\rangle,$$
$$E = \pm 2\sqrt{2} \quad (52)$$

$$|E,F\rangle_\alpha^\beta = [\bar{a}\,\bar{b}]_\alpha^\delta|0\rangle + \frac{E-2}{2}[\bar{b}\,\bar{a}]_\alpha^\delta|0\rangle - \frac{E\delta_\alpha^\delta}{2N}\mathrm{Tr}[\bar{b}\,\bar{a}]|0\rangle,$$
$$E = \pm 2\sqrt{2}. \quad (53)$$

The rest of the eigenstates, orthogonal to all these, are at $E = 0$. The total number of states in the two bit Fock space is $2N^4$. All together there are two singlets with $E = 4$, $4(N^2 - 1)$ adjoint states with $E = \pm 2\sqrt{2}$ states, and the rest amount to $2(N^2 - 1)^2$ states with $E = 0$.

Notice that the energy eigenvalues in the two bit sector are actually independent of N. Only the degeneracies and the eigenstates depend on N. This is very special to this sector because of its tiny state space. At $N = \infty$ it is even smaller than the state space of the first quantized Hamiltonian h, due to the cyclic symmetry constraint.

For $M = 2$ we have

$$h_2 = -4\theta_1 \frac{d}{d\theta_2} - 4\theta_2 \frac{d}{d\theta_1} - 4 + 4\theta_1 \frac{d}{d\theta_1} + 4\theta_2 \frac{d}{d\theta_2}$$

$$= 4(\theta_2 - \theta_1)\left(\frac{d}{d\theta_2} - \frac{d}{d\theta_1}\right) - 4. \qquad (54)$$

The eigenfunctions and eigenvalues are

$$1, E = -4; \qquad (\theta_2 + \theta_1), E = -4;$$

$$(\theta_2 - \theta_1), E = +4; \qquad \theta_1\theta_2, E = +4, \qquad (55)$$

but only the last two satisfy the cyclic symmetry constraint $\Psi(\theta_1, \theta_2) = -\Psi(\theta_2, \theta_1)$, which accords exactly with the Fock space analysis.

B. Three bit singlets

The single trace states in the $M = 3$ subspace of states are

$$\mathrm{Tr}\bar{a}^3|0\rangle, \qquad \mathrm{Tr}\bar{a}\bar{b}^2|0\rangle; \qquad \mathrm{Tr}\bar{a}^2\bar{b}|0\rangle, \qquad \mathrm{Tr}\bar{b}^3|0\rangle, \quad (56)$$

and their norms are

$$\langle 0|\mathrm{Tr}a^3\mathrm{Tr}\bar{a}^3|0\rangle = 3\langle 0|\mathrm{Tr}a^2\bar{a}^2|0\rangle = 3N^3 + 3N$$

$$\langle 0|\mathrm{Tr}b^2a\mathrm{Tr}\bar{a}\bar{b}^2|0\rangle = \langle 0|\mathrm{Tr}b^2\bar{b}^2|0\rangle = N^3 - N$$

$$\langle 0|\mathrm{Tr}a^2b\mathrm{Tr}\bar{b}\bar{a}^2|0\rangle = \langle 0|\mathrm{Tr}a^2\bar{a}^2|0\rangle = N^3 + N$$

$$\langle 0|\mathrm{Tr}b^3\mathrm{Tr}\bar{b}^3|0\rangle = 3\langle 0|\mathrm{Tr}b^2\bar{b}^2|0\rangle = 3N^3 - 3N. \quad (57)$$

For starters we work out the eigenstates at $N = \infty$, where it suffices to find the single trace eigenstates, because all remaining color singlet energy eigenstates are tensor products of these at infinite N. In addition, we can work independently in the Bose and Fermi sectors. The action of H on each basis state in the Bose sector is given by

$$H\mathrm{Tr}\bar{a}^3|0\rangle = \frac{6}{N}\mathrm{Tr}(\bar{a}^2 - i\bar{b}^2)a\bar{a}^2|0\rangle$$

$$= 6\mathrm{Tr}(\bar{a}^2 - i\bar{b}^2)\bar{a}|0\rangle + \frac{6}{N}\mathrm{Tr}(\bar{a}^2)\mathrm{Tr}\bar{a}|0\rangle \quad (58)$$

$$H\mathrm{Tr}\bar{a}\bar{b}^2|0\rangle = \frac{2}{N}\mathrm{Tr}[(\bar{a}\,\bar{b} + \bar{b}\,\bar{a})b\bar{b}^2 + (\bar{a}\,\bar{b} - \bar{b}\,\bar{a})a(\bar{b}\,\bar{a} - \bar{a}\,\bar{b})$$

$$- (\bar{b}^2 - i\bar{a}^2)b(\bar{b}\,\bar{a} - \bar{a}\,\bar{b})]|0\rangle$$

$$= 2\mathrm{Tr}[-(3\bar{b}^2 - i\bar{a}^2)\bar{a}]|0\rangle$$

$$+ \frac{2}{N}[-2\mathrm{Tr}(\bar{a}\,\bar{b})\mathrm{Tr}\bar{b} + \mathrm{Tr}(-i\bar{a}^2)\mathrm{Tr}\bar{a}]|0\rangle. \quad (59)$$

Taking $N \to \infty$, the eigenvalue equation becomes

$$E\mathrm{Tr}(c_1\bar{a}^3 + c_2\bar{a}\bar{b}^2)|0\rangle$$

$$= \mathrm{Tr}[6c_1(\bar{a}^2 - i\bar{b}^2)\bar{a} + c_2(2i\bar{a}^3 - 6\bar{b}^2\bar{a})]|0\rangle, \quad (60)$$

which is easily solved:

$$|Eb\rangle_\infty = c_1\mathrm{Tr}\left(\bar{a}^3 + \frac{1}{2i}(E - 6)\bar{a}\bar{b}^2\right)|0\rangle, \qquad E = \pm 4\sqrt{3}. \quad (61)$$

Next we move on to the Fermi sector, for which the action of H gives

$$H\mathrm{Tr}\bar{b}^3|0\rangle = -6\mathrm{Tr}(\bar{b}^2 - i\bar{a}^2)\bar{b}|0\rangle + \frac{6}{N}\mathrm{Tr}(\bar{b}^2 - i\bar{a}^2)\mathrm{Tr}\bar{b}|0\rangle \quad (62)$$

$$H\mathrm{Tr}\bar{a}^2\bar{b}|0\rangle = 2\mathrm{Tr}[(3\bar{a}^2 - i\bar{b}^2)\bar{b}]|0\rangle$$

$$+ \frac{2}{N}[\mathrm{Tr}\bar{a}^2\mathrm{Tr}\bar{b} + 2\mathrm{Tr}(\bar{a}\,\bar{b})\mathrm{Tr}\bar{a}]|0\rangle. \quad (63)$$

We next find the eigenstates at $N = \infty$:

$$|Ef\rangle_\infty = c_1\mathrm{Tr}\left(\bar{b}^3 - \frac{1}{2i}(E + 6)\mathrm{Tr}\bar{a}^2\bar{b}\right)|0\rangle, \qquad E = \pm 4\sqrt{3}. \quad (64)$$

The Fermi–Bose degeneracy here and at all levels is due to the existence of the supercharge $Q = \mathrm{Tr}[e^{i\pi/4}\bar{a}b + e^{-i\pi/4}\bar{b}a]$, which commutes with the Hamiltonian. $[Q, H] = 0$.

To go beyond the $N = \infty$ limit, we require the action of H on the other states in the three bit Fock space. Staying within the color singlet sector, we label the states as

$$|1\rangle = \mathrm{Tr}\bar{a}^3|0\rangle \qquad (65)$$

$$|2\rangle = \mathrm{Tr}\bar{a}^2\mathrm{Tr}\bar{a}|0\rangle \qquad (66)$$

$$|3\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}|0\rangle \qquad (67)$$

$$|4\rangle = \mathrm{Tr}\bar{a}\bar{b}^2|0\rangle \qquad (68)$$

$$|5\rangle = \mathrm{Tr}\bar{a}\,\bar{b}\,\mathrm{Tr}\bar{b}|0\rangle. \qquad (69)$$

Then the result of applying the Hamiltonian on each state can be expressed as

$$H|i\rangle = \sum_j |j\rangle\mathcal{H}_{ji}, \qquad (70)$$

where the matrix $\mathcal{H}$ is

$$\mathcal{H} = \begin{pmatrix} 6 & \frac{8}{N} & 0 & 2i & 0 \\ \frac{6}{N} & 4 & \frac{12}{N} & -\frac{2i}{N} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -6i & -\frac{8i}{N} & 0 & -6 & -\frac{4}{N} \\ 0 & 0 & 0 & -\frac{4}{N} & 4 \end{pmatrix} \neq \mathcal{H}^\dagger. \quad (71)$$

We show the eigenvalues of $\mathcal{H}$ plotted against $1/N$ in Fig. 3. The eigenvalues of $\mathcal{H}$ and their eigenstates will be in one-to-one correspondence with the eigenvalues and eigenstates of H in the three bit sector, provided the basis set $|i\rangle$ is linearly independent. If there are linear dependences among the $|i\rangle$, the eigenstates of H will correspond to a proper subset of those of $\mathcal{H}$. To assess linear independence, we also need to calculate the metric $G_{ij} = \langle i|j\rangle$, which is given by

$$G = \begin{pmatrix} 3+\frac{3}{N^2} & \frac{6}{N} & \frac{6}{N^2} & 0 & 0 \\ \frac{6}{N} & 2+\frac{4}{N^2} & \frac{6}{N} & 0 & 0 \\ \frac{6}{N^2} & \frac{6}{N} & 6 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{1}{N^2} & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{1}{N^2} \end{pmatrix} = G^\dagger. \quad (72)$$

The eigenvalues of G will be real and non-negative for integer values of N because the state space acted on by H is positive definite. But G will have a zero eigenvalue corresponding to each linear dependence in the basis. When N is not an integer, there can be both zero and negative eigenvalues of G because there is no physical state space for such values of N. If we write, for an eigenvector of $\mathcal{H}$,

$|E\rangle = \sum_i |i\rangle v^i$, its norm can be expressed in terms of the matrix G:

$$\langle E|E\rangle = \sum_{ij} v^{i*}\langle i|j\rangle v^j = v^\dagger G v. \quad (73)$$

For integer N this eigenvalue will be in the spectrum of H provided $\langle E|E\rangle \neq 0$. In Fig. 4 we plot this norm for each of the eigenvalues shown in Fig. 3. Whenever the norm is zero, the eigenvalue E is not in the spectrum of H, and in that case it need not even be real because $\mathcal{H}$ is not Hermitian. However, it is not hard to see that $\mathcal{H}^*$ is similar to $\mathcal{H}$, so that complex eigenvalues always occur in complex conjugate pairs. This is what is happening when two eigenvalue curves merge as seen in Fig. 3. To the right of the merger, we plot only the real part of E, and the two eigenvalues have equal and opposite imaginary parts.

At $N = \infty$, the energy eigenvalues are $E = \pm 8\sin(\pi/3) = \pm 4\sqrt{3}$, $E = 4$, $E = 4$, and $E = 0$. The first two eigenvalues come from the single trace states. The next two come from the double trace states, which agree with the two bit results, and the triple trace state $\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}|0\rangle$ has zero energy.

Notice that at $N = 1$ the five curves for the energies, Fig. 3, approach distinct values. On the other hand, Fig. 4 shows that the norms of the lowest four of them are 0 at $N = 1$. These states then disappear from the spectrum at $N = 1$; only the state evolving from $\mathrm{Tr}\bar{a}^3|0\rangle$ at $N = \infty$ survives at this value of N. Also at $N = 1$, $H_1 = 2(M^2 - M) = 12$, which agrees with the top curve of Fig. 3.
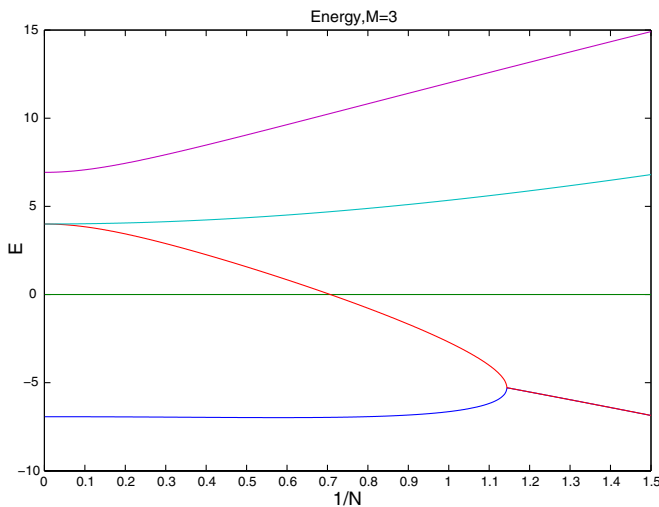

FIG. 3 (color online). The energy eigenvalues of the three bit system as a function of $1/N$. The two energy curves which merge into a single curve, for N a bit less than 1, actually become complex conjugate pairs upon merging. As seen in the next figure, those states have zero norm when they become complex.
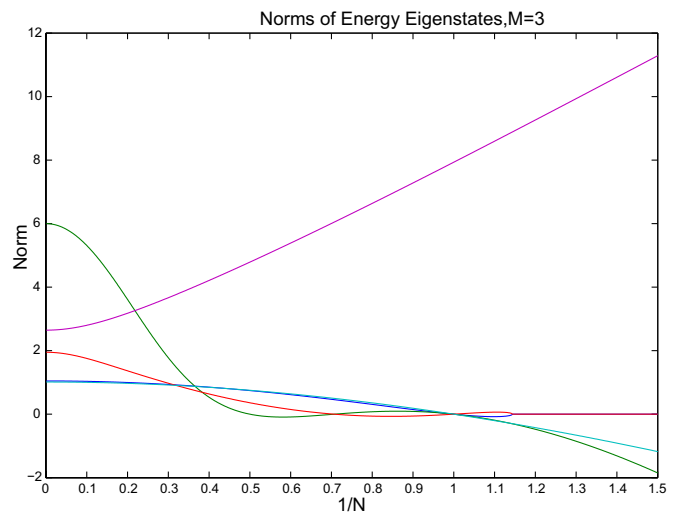

FIG. 4 (color online). The norms of the energy eigenstates of the three bit system as a function of $1/N$. The color of the norm curve is the same as the corresponding energy curve. Note that all states but the one with highest energy have zero norm at $N = 1$. This agrees with our exact treatment of that case. Note the presence of negative norms (ghosts) when $N < 2$.

At $N = 2$ the norm of the state, which evolves from $\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}|0\rangle$, goes to zero, and between $N = 2$ and $N = 1$, its norm goes negative for a while. The presence of such a ghost state, which can happen only for noninteger N, signals a violation of unitarity. Even though at $M = 3$ this disappearing state is a rather trivial one with a zero energy eigenvalue, we have found in higher M cases that the norms of nontrivial states go to zero at integer N. This suggests a pattern that as M goes up, the states start to disappear at the higher integer N.

When $M = 3$ no complex conjugate pair appears in the range $N \geq 1$. But as Fig. 3 shows, two curves do merge in the region $N < 1$. At the merge point, the eigenvalues become complex conjugate pairs. At higher M such complex energies occur at higher N. In our detailed studies at $M = 4$ and $M = 5$, we find that every eigenstate with complex energy has zero norm, indicating that it disappears from the spectrum, whether N is an integer or not. This has to happen at integer N where H is manifestly Hermitian, but not necessarily at unphysical fractional N. The fact that it does indicates that unitarity may be possible for a range of continuous N. In the string interpretation of these models, $1/N$ acts as the string coupling constant, which is not *a priori* quantized. So in this context continuous N might have some physical meaning. Even so our studies do find ranges of noninteger N with negative norm eigenstates (ghosts), so unitarity is by no means assured at fractional N.

We can see that the N dependence of the ground state energy in the three bit sector is very flat from $N = \infty$ to $N = 2$ and only goes up a little bit at $N = 1$. This flatness is also a feature of the $M = 5$ case, suggesting that the stringlike properties of this state may persist, to some extent, as N decreases from ∞. For $M = 4$ (and all even M), this flat ground state is missing because it does not satisfy the cyclic symmetry constraint. The detailed analysis of the singlet spectrum for $M = 4$, $M = 5$ and higher will appear in a separate article.

C. Three bit adjoints

We briefly consider the N dependence of the energy spectrum in the color adjoint sector. In general an adjoint state can be characterized by a monomial of creation operators $\bar{A}_\alpha^\beta$ carrying free color indices. Then the adjoint state would be

$$\bar{A}_\alpha^\beta|0\rangle - \frac{\delta_\alpha^\beta}{N}\mathrm{Tr}\bar{A}|0\rangle. \tag{74}$$

A convenient metric for these states can be taken to be

$$\langle 0|\mathrm{Tr}B\bar{A}|0\rangle - \frac{1}{N}\langle 0|\mathrm{Tr}B\mathrm{Tr}\bar{A}|0\rangle. \tag{75}$$

Specializing to three bits ($M = 3$), we can then form the monomials

$$\bar{A}_1 = [\bar{a}\,\bar{a}\,\bar{a}], \qquad \bar{A}_2 = [\bar{b}\,\bar{b}\,\bar{a}],$$
$$\bar{A}_3 = [\bar{a}\,\bar{b}\,\bar{b}], \qquad \bar{A}_4 = [\bar{b}\,\bar{a}\,\bar{b}] \tag{76}$$

$$\bar{A}_5 = [\bar{a}\,\bar{a}]\mathrm{Tr}[\bar{a}], \qquad \bar{A}_6 = [\bar{a}]\mathrm{Tr}[\bar{a}\,\bar{a}], \qquad \bar{A}_7 = [\bar{b}\,\bar{b}]\mathrm{Tr}[\bar{a}] \tag{77}$$

$$\bar{A}_8 = [\bar{a}\,\bar{b}]\mathrm{Tr}[\bar{b}], \qquad \bar{A}_9 = [\bar{b}\,\bar{a}]\mathrm{Tr}[\bar{b}], \qquad \bar{A}_{10} = [\bar{b}]\mathrm{Tr}[\bar{a}\,\bar{b}], \tag{78}$$

where square brackets surround the monomial carrying the free color indices. Then the analog of the multitrace basis is just

$$|i\rangle_\alpha^\beta = (\bar{A}_i)_\alpha^\beta|0\rangle. \tag{79}$$

The Hamiltonian matrix of the adjoint states in the three bit sector is defined by

$$H|i\rangle = \sum_j |j\rangle \mathcal{H}_{ji}^A, \tag{80}$$

with

$$\mathcal{H}^A = \begin{pmatrix} 4 & 2i & 2i & 0 & \frac{8}{N} & \frac{8}{N} & 0 & 0 & 0 & 0 \\ -2i & -4 & 0 & 2 & -\frac{4i}{N} & -\frac{4i}{N} & 0 & 0 & 0 & -\frac{4}{N} \\ -2i & 0 & 0 & 2 & -\frac{4i}{N} & -\frac{4i}{N} & \frac{4}{N} & -\frac{4}{N} & 0 & \frac{4}{N} \\ 0 & 2 & 2 & 0 & 0 & 0 & \frac{4}{N} & 0 & \frac{4}{N} & -\frac{4}{N} \\ \frac{2}{N} & 0 & 0 & \frac{2i}{N} & 2 & 0 & 2i & 0 & 0 & 0 \\ \frac{4}{N} & -\frac{2i}{N} & -\frac{2i}{N} & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ -\frac{2i}{N} & 0 & 0 & -\frac{2}{N} & -2i & 0 & -2 & 0 & 0 & 0 \\ 0 & \frac{2}{N} & -\frac{2}{N} & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & -\frac{2}{N} & -\frac{2}{N} & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & \frac{4}{N} & 0 & -\frac{4}{N} & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}. \tag{81}$$

The details of the adjoint spectrum will be described in a separate publication. Here we just show the gap between adjoint and singlet states by plotting in Fig. 5 the lowest energy eigenvalues in each of these color representations. It is interesting that at least at $M = 3$ there is no tendency for the gap to close as N decreases from infinity. The cusp seen in the lowest energy adjoint curve is the point at which the next highest adjoint energy merges with the lowest adjoint energy, and thereafter both energies become complex with equal and opposite imaginary parts. To the right of the cusp, the lowest real adjoint energy is yet higher.
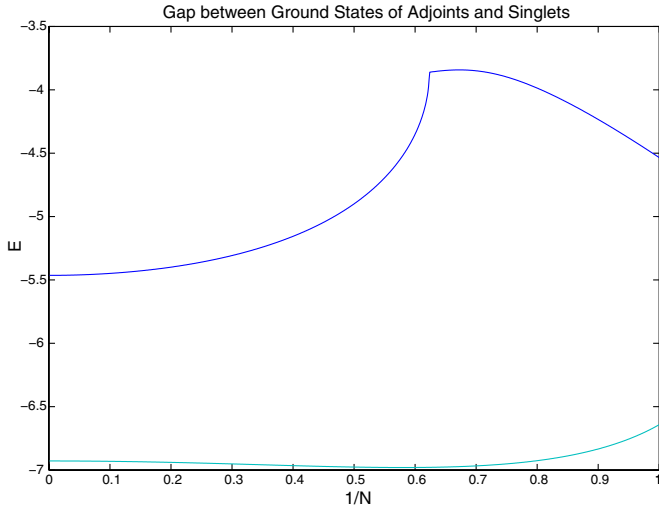
FIG. 5 (color online). The lowest energies of the color adjoint (top curve) and color singlet (bottom curve) states in the three bit sector, showing a gap that persists in the entire range $1 \leq N < \infty$. The cusp is really the point where two real energy curves merge, and the energies to the right of the cusp have equal and opposite imaginary parts.

V. EXTENSIONS AND CONCLUDING REMARKS

In this article we have studied the simplest superstring bit model, which underlies the $s = 1$, $d = 0$ superstring. As mentioned in the introduction, the extension to a superstring bit model underlying the $s > 1$, $d = 0$ superstring is straightforward. One simply enlarges the set of bit creation operators to 128 bosonic and 128 fermionic ones,

$$a_\alpha{}^\beta, b_\alpha{}^\beta \to (\phi_{[a_1,\ldots,a_n]})_\alpha{}^\beta, n = 0, \ldots, s, \qquad (82)$$

where each a_k is a spinor index running over s values. The appropriate Hamiltonian for which the 't Hooft limit gives the $s = 8$, $d = 0$ superstring can be obtained from the one constructed in Ref. [11] by dropping all of the transverse coordinate dependence,

$$H = \frac{1}{N} \sum_{n=0}^{s} \frac{s - 2n}{n!} \mathrm{Tr} \bar{\phi}_{a_1 \cdots a_n} \rho \phi_{a_1 \cdots a_n}$$

$$+ \frac{1}{N} \sum_{n=0}^{s-1} \frac{1}{n!} \mathrm{Tr} \bar{\phi}_{a_1 \cdots a_n} \eta_b \phi_{b a_1 \cdots a_n}$$

$$+ \frac{1}{N} \sum_{n=0}^{s-1} \frac{1}{n!} \mathrm{Tr} \bar{\phi}_{b a_1 \cdots a_n} \bar{\eta}_b \phi_{a_1 \cdots a_n}, \qquad (83)$$

where we have defined

$$\rho = \sum_{k=0}^{s} \frac{1}{k!} \bar{\phi}_{b_1 \cdots b_k} \phi_{b_1 \cdots b_k} \qquad (84)$$

$$\eta_b = \sum_{k=0}^{s-1} \frac{(-)^k}{k!} \bar{\phi}_{b b_1 \cdots b_k} \phi_{b_1 \cdots b_k} + i \sum_{k=0}^{s-1} \frac{(-)^k}{k!} \bar{\phi}_{b_1 \cdots b_k} \phi_{b b_1 \cdots b_k},$$

$$\bar{\eta}_b = -i \eta_b. \qquad (85)$$

Of course when this expression is specialized to $s = 1$, it is just the Hamiltonian we have analyzed in this article. When $s = 8$ its large N limit just describes the spinor sector of the critical superstring in the light-cone gauge. For any $s > 0$ the model can be analyzed in a manner exactly parallel to the analysis in this article. We defer this generalization to a future publication. Only one spatial coordinate, x^-, will be holographically generated in the large N limit, and the excitations of the superstring will be described by s light-cone world sheet Grassmann coordinates θ_a.

The task of constructing a superstring bit model that underlies superstring theory with $d > 0$ calls for some interesting choices. The model constructed in Ref. [11] to describe the $s = 8$, $d = 8$ superstring addressed this problem by simply promoting the quantum mechanical variables $\phi \to \phi(x)$ to fields on the d-dimensional transverse space x. The Hamiltonian for that model included two body terms, quartic in the ϕ's and involving a potential $V(x - y)$. To exactly produce a harmonic superchain, in the 't Hooft limit, required a harmonic oscillator potential $V = T_0(x - y)^2$. We regarded this long range potential as unsatisfactory. After all, the transverse space for the bits was identified with the transverse space after the holographic emergence of x^-, and we felt locality in the dynamics in the emergent $d + 2$-dimensional space-time would be unlikely unless the potential between bits was short range. In subsequent papers [15] we struggled mightily, without complete success, to build satisfactory superstring bit models with a short range potential.

But in retrospect this effort seems philosophically misguided. The principal motivation for string bit models is to replace quantum field theory, with an infinite number of degrees of freedom, with an underlying theory with a finite number of degrees of freedom. Letting the string bits move in transverse space reinserts an infinite number of degrees of freedom in the underlying theory. It is philosophically more coherent to seek a model in which all dimensions of space emerge holographically. We can easily see how this can happen. It has been known for a long time that the Heisenberg chain of spins,

$$H_{hei} = -\frac{1}{2} \sum_k (\sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \Delta \sigma_k^3 \sigma_{k+1}^3), \qquad (86)$$

describes in the continuum limit a spatial coordinate compactified on a circle of radius $R = 2\pi/\sqrt{2T_0(\pi - \mu)}$, where $\Delta = -\cos\mu$ [16].

It is easy to incorporate this idea in string bit models. Append a two valued "flavor index" for each transverse dimension, $\phi_{[a_1 \cdots a_n]} \to \phi_{[a_1 \cdots a_n]}^{f_1 \cdots f_d}$, with $f_i = 1, 2$, and design

the string bit Hamiltonian to produce the Heisenberg Hamiltonian on the long chains that naturally arise in the limit $N \to \infty$. Such a string bit model has $2^s 2^d N^2$ [$= (256N)^2$ for the superstring bit model] degrees of freedom. Pursuit of these ideas is an exciting project for future investigation.

APPENDIX A: DIAGONALIZING H IN THE LARGE N LIMIT

1. Color singlets

We have shown that the action of H on single trace singlet states can be described in the large N limit in terms of the "first-quantized" Hamiltonian h. To find the eigenvalues of h, it is convenient to introduce Fourier transforms

$$\alpha_n = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \theta_k e^{2\pi i k n/M}, \qquad \beta_n = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \frac{d}{d\theta_k} e^{2\pi i k n/M} \tag{A1}$$

$$\theta_k = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} \alpha_n e^{-2\pi i k n/M}, \qquad \frac{d}{d\theta_k} = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} \beta_n e^{-2\pi i k n/M} \tag{A2}$$

$$\{\alpha_n, \beta_m\} = \delta_{m+n,M}. \tag{A3}$$

Then we can express h in terms of these:

$$h = -2M + 2 \sum_{n=1}^{M-1} \left[(\alpha_n \alpha_{M-n} + \beta_n \beta_{M-n}) \sin \frac{2\pi n}{M} \right.$$

$$+ (\alpha_n \beta_{M-n} + \alpha_{M-n} \beta_n) \left(1 - \cos \frac{2\pi n}{M} \right) \right]$$

$$= -2M + 4 \sum_{n=1}^{M-1} \sin \frac{n\pi}{M} \left[(\alpha_n \alpha_{M-n} + \beta_n \beta_{M-n}) \cos \frac{\pi n}{M} \right.$$

$$+ (\alpha_n \beta_{M-n} + \alpha_{M-n} \beta_n) \sin \frac{\pi n}{M} \right]. \tag{A4}$$

For $n < M/2$, we now diagonalize the operator in square brackets, which we call $[]_n$. The functions α_n and α_{M-n} are eigenfunctions of $[]_n$ with eigenvalue $\sin(n\pi/M)$. The remaining two eigenfunctions are of the form $a + b\alpha_n\alpha_{M-n}$:

$$[]_n (a + b\alpha_n \alpha_{M-n}) = a\alpha_n \alpha_{M-n} \cos \frac{\pi n}{M}$$

$$+ b \left[\cos \frac{\pi n}{M} + 2\alpha_n \alpha_{M-n} \sin \frac{\pi n}{M} \right]$$

$$= \epsilon_n (a + b\alpha_n \alpha_{M-n})$$

$$b = \frac{a\epsilon_n}{\cos(n\pi/m)},$$

$$\cos^2 \frac{\pi n}{M} + 2\epsilon_n \sin \frac{\pi n}{M} = \epsilon_n^2$$

$$\epsilon_n = \sin \frac{\pi n}{M} \pm 1. \tag{A5}$$

When M is even, there is a term with $n = M/2$ for which the operator in square brackets is simply

$$[]_{M/2} = 2\alpha_{M/2} \beta_{M/2}, \tag{A6}$$

for which the eigenfunctions are 1 with value $0 = \sin(\pi/2) - 1$ and $\alpha_{M/2}$ with value $2 = \sin(\pi/2) + 1$. If we take the common term $\sin(n\pi/M)$ for each eigenvalue, we see that it contributes

$$4 \sum_{n=1}^{M-1} \sin^2(n\pi/M) = 2M \tag{A7}$$

to the eigenvalue of h. This simply cancels the $-2M$ term in h. Thus, we may write the general eigenvalue of h as

$$E(\{\eta_n\}) = 8 \sum_{n<M/2} \eta_n \sin \frac{n\pi}{M} + 4\eta_{M/2} \tag{A8}$$

$$\eta_n = +1, 0, 0, -1, \quad \text{for } n \neq \frac{M}{2}, \qquad \eta_{M/2} = \pm 1. \tag{A9}$$

Clearly the smallest and largest eigenvalues are

$$E_{\max} = 4 \sum_{n=1}^{M-1} \sin \frac{n\pi}{M} = 4 \cot \frac{\pi}{2M}, \qquad E_{\min} = -E_{\max}. \tag{A10}$$

In the limit of many bits, $M \to \infty$, we have the behavior

$$E_{\max} = \frac{8M}{\pi} - \frac{2\pi}{3M} + \mathcal{O}(M^{-3}) \tag{A11}$$

$$E_{\min} = -\frac{8M}{\pi} + \frac{2\pi}{3M} + \mathcal{O}(M^{-3}). \tag{A12}$$

The term linear in M can be cancelled against a counterterm in the Hamiltonian of the form $(8/\pi)\mathrm{Tr}[\bar{a}a + \bar{b}b]$. Then interpreting M as a discretized $P^+ = Mm$ and identifying $P^- = ET_0/m$, we see that the spectrum is relativistic in the limit $M \to \infty$.

Finally we consider the cyclic symmetry requirements. Under a single step cycle, $\alpha_n \to e^{2in\pi/M}\alpha_n$, so the product $\alpha_n\alpha_{M-n}$ is invariant. When M is odd, the states are supposed to be invariant under such a cycle. This is realized when the values of η_n are restricted to ± 1 or, more generally when a number of the $\eta_k = 0$ for which $\sum_{k,\eta_k=0} k = M$. When M is even, the states must change sign under a single step cycle. This means that the $M/2$ wave function must be $\alpha_{M/2}$, so $\eta_{M/2} = +1$ and all the other $\eta_n = \pm 1$, or for those with $\eta_k = 0$, $\sum_k k = M$. In particular the eigenvalue $E_{\min}$ would be excluded for M even but allowed for M odd. For very large M, so the chains behave as a continuous string, the gap between the even and odd M sectors becomes large compared to the excitation energies of the odd M sector. This means that only the odd M closed chains will participate in the continuum physics. This implies a multiplicatively conserved parity symmetry that forbids an odd number of chains transforming into an even number of chains. In particular a single chain could not decay into two chains.

On the other hand, $E_{\max}$ is allowed for all M. If we want a system for which the ground energy at infinite N is allowed for all M, we could choose its Hamiltonian to be $-h$ rather than h. But this would make the coefficient of the $\mathrm{Tr}\bar{a}^2 a^2$ term negative, which threatens a dangerous instability since this implies attractive interactions between bosons. In Ref. [1] we noted that adding a term $\mathrm{Tr}\bar{a}a\bar{a}a$ with a positive coefficient could stabilize the theory at the expense of complicating the large N analysis. In h the attractive interactions are between fermions which are tamed by the exclusion principle, without such complications.

2. Color adjoints

For completeness, we also consider the color adjoint states in the large N limit. In this case we apply H to states of the form

$$|\theta_1, \cdots \theta_M\rangle_\alpha^\beta = [\psi(\theta_1) \cdots \psi(\theta_M)]_\alpha^\beta |0\rangle \qquad (A13)$$

and determine in the large N limit the first quantized h_A such that

$$H \int d^M\theta |\theta_1, \cdots \theta_M\rangle_\alpha^\beta \Psi(\theta_1, \cdots \theta_M)$$
$$= \int d^M\theta |\theta_1, \cdots \theta_M\rangle_\alpha^\beta h_A \Psi(\theta_1, \cdots \theta_M) + \mathcal{O}(N^{-1}). \qquad (A14)$$

Following similar steps as for the singlets, we find

$$h_A = \sum_{k=1}^{M-1} \left[-2i\theta_k\theta_{k+1} - 2i \frac{d}{d\theta_k} \frac{d}{d\theta_{k+1}} - 2\theta_k \frac{d}{d\theta_{k+1}} \right.$$
$$\left. - 2\theta_{k+1} \frac{d}{d\theta_k} - 2 + 4\theta_k \frac{d}{d\theta_k} \right]. \qquad (A15)$$

We note that, in comparison to h, the only change is the deletion of the term $k = M$. This breaks the closed chain of bits to form an open chain of bits. To diagonalize h_A, we replace periodic boundary conditions with the ones implied by the absence of this term. The net effect of this change of boundary conditions is to change the mode energies from $4\sin(n\pi/M)$ to $4\sin(n\pi/2M)$. Correspondingly, the ground state energy in the adjoint sector at $N = \infty$ is

$$E_A = -4 \sum_{n=1}^{M-1} \sin\frac{n\pi}{2M}$$
$$= -2 \cot\frac{\pi}{4M} + 2$$
$$= -\frac{8M}{\pi} + 2 + \frac{\pi}{6M} + \mathcal{O}(M^{-3}). \qquad (A16)$$

The second term on the right shows that the large N dynamics leads to color confinement The energy gap between the adjoint and singlet sectors at $N = \infty$ is

$$E_A - E_G = 2 - 2\tan\frac{\pi}{4M} = 2 - \frac{\pi}{2M} + \mathcal{O}(M^{-3}). \qquad (A17)$$

In Fig. 6 we plot this energy gap as a function of M. The gap remains finite as $M \to \infty$. But the $1/M$ terms set the scale of energy of the continuum string excitations. Thus, the energy gap between adjoint and singlet sectors becomes infinitely large in comparison to this string energy scale when $M \to \infty$. Note that "perfect" confinement depends on $M \to \infty$. If M is simply extremely large rather than ∞, the mass gap is of order M times the scale set by the $1/M$ excitations.
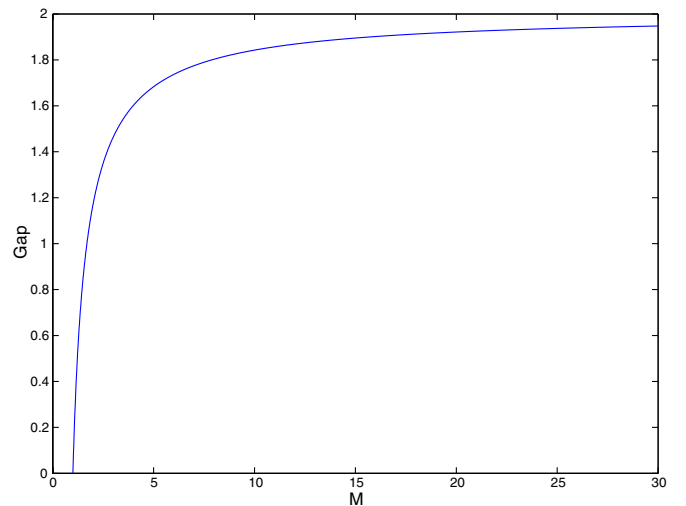

FIG. 6 (color online). The energy gap between color adjoint and color singlet sectors at $N = \infty$.

APPENDIX B: TRUNCATION TO SINGLE TRACE STATES

We mention here an extension of the variational method described in Sec. III. One can take a trial state to be any linear combination of single trace states and vary the energy function with respect to the coefficients in this linear combination. Requiring that the energy function is stationary then implies that the coefficient functions satisfy the eigenvalue equation

$$\sum_j \langle k|H|j\rangle c_j = E\sum_j \langle k|j\rangle c_j, \qquad (B1)$$

where j is summed over the selected states and E is the energy function.

As a simple example, consider the three bit case in the boson sector. Then there are only two single trace states,

$$|1\rangle = \frac{1}{\sqrt{3N(N^2+1)}}\mathrm{Tr}\bar{a}^3|0\rangle,$$

$$|2\rangle = \frac{1}{\sqrt{N(N^2-1)}}\mathrm{Tr}\bar{a}\bar{b}^2|0\rangle, \qquad (B2)$$

which we normalized and happen to be orthogonal. Thus, the factor $\langle k|j\rangle = \delta_{kj}$ and E is then just one of the eigenvalues of the 2×2 matrix:

$$H = \begin{pmatrix} 6\frac{N^2+3}{N^2+1} & -2i\sqrt{3}\sqrt{\frac{N^2-1}{N^2+1}} \\ 2i\sqrt{3}\sqrt{\frac{N^2-1}{N^2+1}} & -6 \end{pmatrix}. \qquad (B3)$$

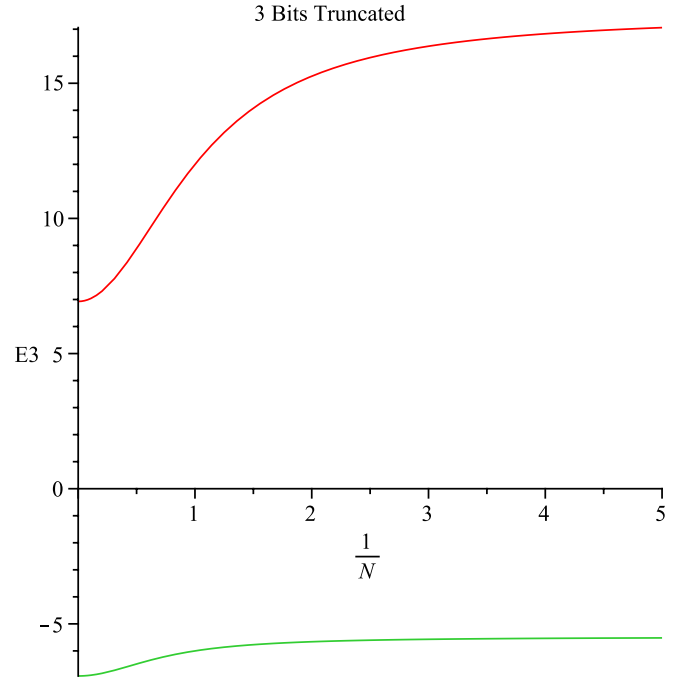The eigenvalues are the roots of a quadratic polynomial:

3 Bits Truncated

FIG. 7 (color online). Variational estimate of two energy eigenvalues of the three bit system.

$$E = \frac{6}{N^2+1} \pm \sqrt{48 + \frac{36}{(N^2+1)^2} + \frac{48}{N^2+1}}. \qquad (B4)$$

We plot these eigenvalues as a function of $1/N$ in Fig. 7. Of course the curves go to the exact eigenvalues at $N = \infty$. It is evident that the lowest of these estimates varies more steeply than the exact eigenvalue as N decreases from ∞.

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