Silver blaze puzzle in $1/N_c$ expansions of cold and dense QCD matter

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We consider quantum chromodynamics (QCD) with N_c colors and N_f quark flavors at finite quark chemical potential μ_q or isospin chemical potential μ_I. We specifically address the nature of the "silver blaze" behavior in the framework of $1/N_c$ expansions. Starting with the QCD partition function, we implement Veneziano's N_f/N_c expansion to identify the density onset. We find the baryon mass M_B and the pion mass m_π appearing from different orders of Veneziano's expansion. We argue that the confining properties are responsible for the silver blaze behavior in the region of $m_\pi/2 < \mu_q < M_B/N_c$. We point out, however, that Veneziano's expansion brings about a subtlety along the same line as the baryon problem in finite-density quenched lattice simulations. We emphasize that the large-N_c limit can allow for the physical ordering of M_B and m_π thanks to the similarity between the large-N_c limit and the quenched approximation, while unphysical ghost quarks contaminate the baryon sector if N_c is finite. We also discuss the "orientifold" large-N_c limit that does not quench quark loops.

I. INTRODUCTION

Understanding the phase diagram of quantum chromodynamics (QCD) is one of the most pressing problems. It is theoretically expected that the phase diagram has rich structures as a function of the temperature T, the baryon or quark chemical potential $\mu_B = N_c\mu_q$, the isospin chemical potential μ_I, the external electromagnetic fields, and so on. Hot and/or dense QCD phases such as a quark-gluon plasma, color superconductivity, and the hadronic phase where we live are supposed to have been realized in the early Universe, inside of compact stellar objects, and in the relativistic heavy-ion collisions (see [1] for recent reviews). The theoretical understanding of a limited portion of the phase diagram in the high-T and low-μ_q region has been obtained by means of several techniques including lattice-QCD simulations, effective model studies, and resummed perturbation theories. In contrast, for the dense system at small T, the understanding is far from satisfactory except for normal nuclear matter where the traditional nuclear theory can be applied, and for such high density that perturbative QCD works once all gluons get screened by the Meissner effect in color superconductivity [2,3]. Unlike the case with $\mu_q \ll T$, lattice-QCD simulations based on the Monte Carlo algorithm have a serious difficulty called the sign problem due to a fluctuating phase of the fermion (Dirac) determinant at $\mu_q \neq 0$ [4]. There are several approaches to evade the sign problem: the Taylor expansion [5], the reweighing method [6], the imaginary chemical potential [7], and the complex Langevin equation [8],

but they are not yet well developed to go beyond the small-μ_q/T regime.

In the $T = 0$ limit, as long as μ_B is smaller than the baryon mass minus the nuclear binding energy (i.e. $M_B - B \simeq 923$ MeV), we can make a firm conclusion even without tackling the sign problem. Trivially, no physical excitation is allowed for $\mu_B < M_B - B$ and none of physical properties should depend on μ_B or μ_q then. It is also the case for a system with a finite isospin chemical potential μ_I if it is smaller than $m_\pi/2$. Even though physics is transparent on the intuitive level, the microscopic origin of μ_q independence or μ_I independence has a puzzling character and still deserves theoretical investigations, as first pointed out in Ref. [9]. Even with sufficiently small μ_q or μ_I, since the Dirac operator and thus its eigenvalues have explicit dependence on chemical potentials, one would naively expect that the partition function depends on such μ_q or μ_I, but physically it should not. Interestingly, thus, that nothing happens trivially is a hint of nontrivial physics inherent in the sign problem. This problem is often called the "silver blaze" named after a famous detective story of Sherlock Holmes [9].

In the case of μ_I, it has been convincingly shown already in Ref. [9,10] that the Dirac determinant can be μ_I independent due to a gap in the energy spectrum and this gap is given precisely by $m_\pi/2$. The same argument can hold for the μ_q independence as long as $\mu_q \leq m_\pi/2$. Thus, the silver blaze problem remains profound for the specific window, $m_\pi/2 < \mu_q < (M_B - B)/N_c$. It is argued that the phase fluctuation should be responsible for the μ_q independence

in this region. In fact, there is a demonstration that the average over the phase fluctuations of the Dirac determinant may cancel the μ_q dependence [11]. Such a physical mechanism with fluctuating phase ought to be related to quark confinement [12]; indeed, quark excitation is averaged out by the Z_{N_c}-symmetric phase distribution in the confined phase that is to be identified as the disordered state [13].

The main purpose of this paper is to address the silver blaze behavior within the framework of Veneziano's N_f/N_c expansion and also 't Hooft's $1/N_c$ expansion. To this end we exploit the worldline formalism (or the canonical ensemble representation) in order to expand the Dirac determinant in powers of N_f/N_c [14] (see also arguments in Ref. [12]). The leading order $\mathcal{O}(N_c^2)$ contribution to the free energy is purely gluonic. We will focus on the subleading $\mathcal{O}(N_c N_f)$ and the sub-subleading $\mathcal{O}(N_f^2)$ contributions. At $\mathcal{O}(N_c N_f)$ order, two theories, one with $\mu_I \neq 0$ and $\mu_q = 0$ and the other with $\mu_q \neq 0$ and $\mu_I = 0$, are equivalent to each other in the $\mathcal{C}$-even sector [14–16], as long as μ_I and μ_q are small enough not to induce any finite baryon or isospin density. When the next $\mathcal{O}(N_f^2)$ contribution is added, the equivalence between μ_I and μ_q no longer holds [15]. The virtue of this N_f/N_c expansion lies in the clear separation of the baryon and pion sectors that belong to the $\mathcal{O}(N_c N_f)$ and the $\mathcal{O}(N_f^2)$ contributions, respectively.

Although our formulation provides us with a useful organization of different physics origins, Veneziano's expansion at finite density is a subtle expansion and we should be cautious about the results. In fact, the N_f/N_c expansion amounts to a series of N_f/N_c corrections around the state at $N_f/N_c \to 0$ that is nothing but the quenched limit. It is known that the quenched simulation at finite density[1] sometimes leads to unphysical results; especially the density onset seems to be set not by M_B but by m_π, which would cause additional subtlety in the argument on the silver blaze puzzle in Veneziano's expansion. We argue that in the 't Hooft limit of $N_c \to \infty$ the formulation could be put in a rather clean environment and our argument is validated. With finite N_c or in the orientifold large-N_c limit as we discuss later, it seems that the silver blaze puzzle still remains quite nontrivial.

Our paper is organized as follows: in Sec. II, we review the decomposition of the QCD partition function in the N_f/N_c expansion with a chemical potential. In Sec. III, with $N_f = 2$ fixed, we analyze the expression for the free energy at $\mathcal{O}(N_c N_f)$ and discuss the dependence of the free energy on μ_I and μ_q. Next, in Sec. IV, we turn to the next-order free energy at $\mathcal{O}(N_f^2)$. We briefly discuss a possible

situation in the orientifold large-N_c limit in Sec. V. Finally, Sec. VI is devoted to discussions and conclusions.

II. VENEZIANO EXPANSION OF THE FREE ENERGY

Let us consider QCD generalized for N_c colors and degenerate N_f quark flavors with identical mass m_q. The partition function of this theory on $R^3 \times S^1$ with the antiperiodic boundary condition for quarks is expressed in a form of the functional integration with respect to the gauge fields A_μ. The integrand consists of the weight factors from the Yang-Mills action [denoted by $\exp[-S_{\mathrm{YM}}(A)]$] and the fermionic Dirac determinant [denoted by $\exp[\Gamma(\mu_f)]$]. The latter can be decomposed generally in terms of the winding number ω as

$$\Gamma(\mu_f) = \ln \, \det(D + m_q + \mu_f \gamma^0)$$
$$= \sum_f \sum_{\omega=-\infty}^{\infty} \Gamma_f^{(\omega)}(\mu_f) = \sum_f \sum_{\omega=-\infty}^{\infty} \Gamma_f^{(\omega)}(0) e^{\mu_f \omega/T}. \quad (1)$$

Here, the index f runs over different quark flavors. The explicit expression of $\Gamma_f^{(\omega)}$ as a function of A_μ can be found in the literature, e.g. with use of the worldline formalism [14]. We note that this decomposition with respect to ω can translate into the canonical ensemble with ω identified as the quark number [17,18]. Figure 1 shows an example of the $\omega = 2$ case. We note that the configuration along S^1 may be wandering, and a special straight configuration is nothing but the Polyakov loop.

An important property of Γ is that the connected k-point function of Γ is diagrammatically suppressed by N_f/N_c as [14]

$$\langle \underbrace{\Gamma\Gamma\cdots\Gamma}_{k\,\mathrm{times}} \rangle_{\mathrm{c;YM}} \sim N_c^2 \left(\frac{N_f}{N_c}\right)^k. \quad (2)$$

This N_c^{-k} suppression appears from the gluon interaction $g^2 \sim \mathcal{O}(N_c^{-1})$ that connects Γ's. The expansion of $\langle \exp(\Gamma) \rangle$ in terms of Γ (after taking the flavor sum) leads to an expansion in powers of N_f/N_c, namely, the expansion in the Veneziano limit. Here, we take the average in the vacuum at $N_f/N_c = 0$, i.e. the pure Yang-Mills theory whose weight factor is $\exp(-S_{\mathrm{YM}})$. The subscript "c" in the expectation
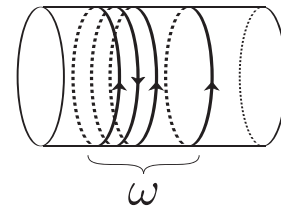

FIG. 1. Schematic picture of the decomposition of the Dirac determinant according to the winding number ω that corresponds to how many times the configuration wraps around S^1. This example shows a case of $\omega = 3 - 1 = 2$.

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[1]This may sound peculiar, as the Yang-Mills theory is density free. Here, finite-density quenched simulations refer to taking the expectation value of finite-density operators, which are typically non-Hermitian, with the vacuum of the Yang-Mills theory.

value denotes the contribution from the "connected" diagrams. Another important property of Γ is that it is exponentially suppressed with ω at least as

$$\Gamma_f^{(\omega)}(0) \sim \exp\left(-\frac{m_q}{T}|\omega|\right) \qquad (3)$$

with the bare quark mass m_q [14]. The suppression could be even faster with m_q replaced with a dynamical M_q when we take the expectation value, $\langle \Gamma_f^{(\omega)}(0) \rangle$, which we will discuss later. This property ensures the convergence of the expansion in terms of ω up to a certain value of μ_f less than the quark mass.

The free energy is then expressed in a form of an expectation value as

$$F = F_{\mathrm{YM}} - T \ln \langle e^{\Gamma(\mu_f)} \rangle_{\mathrm{YM}}$$

$$= F_{\mathrm{YM}} + F^{(1)} + F^{(2)} + \mathcal{O}(N_c^{-1} N_f^3) \qquad (4)$$

with

$$F_{\mathrm{YM}} \sim \mathcal{O}(N_c^2), \qquad (5)$$

$$F^{(1)} = -T \sum_f \sum_{\omega=-\infty}^{\infty} \langle \Gamma_f^{(\omega)}(0) \rangle_{\mathrm{YM}} e^{\mu_f \omega/T} \sim \mathcal{O}(N_c N_f), \qquad (6)$$

$$F^{(2)} = -T \sum_{f,f'} \sum_{\omega,\omega'=-\infty}^{\infty}$$

$$\times \langle \Gamma_f^{(\omega)}(0) \Gamma_{f'}^{(\omega')}(0) \rangle_{\mathrm{c;YM}} e^{(\mu_f \omega + \mu_{f'} \omega')/T} \sim \mathcal{O}(N_f^2), \qquad (7)$$

where F_{YM} represents the pure gluonic energy of $\mathcal{O}(N_c^2)$, the second $F^{(1)}$ with $\langle \Gamma_f^{(\omega)}(0) \rangle_{\mathrm{YM}}$ [in which we omitted "c"

that is irrelevant for the one-point function of $\Gamma_f^{(\omega)}(0)$] is of $\mathcal{O}(N_c N_f)$, and the expansion goes on as the third $F^{(2)}$ with $\langle \Gamma_f^{(\omega)}(0) \Gamma_{f'}^{(\omega')}(0) \rangle_{\mathrm{c;YM}}$ of $\mathcal{O}(N_f^2)$ and so on, according to Eq. (2). It is important to note that this is not yet a consistent ordering of the $1/N_c$ expansion; our identification of $F^{(1)}$ and $F^{(2)}$ is based on the power of N_f/N_c and each of $F^{(1)}$ and $F^{(2)}$ contains subleading (nonplanar) contributions suppressed by higher powers of $1/N_c$.

In this paper we shall work only at sufficiently small temperature, $T \ll \Lambda_{\mathrm{QCD}}$, where the Yang-Mills vacuum should be in the confined phase. (Otherwise, it is rather trivial what is going on in the deconfined phase.) This phase is characterized by the realization of Z_{N_c} (center) symmetry, so that the expectation value of a center non-symmetric operator vanishes, namely,

$$\langle \Gamma^{(\omega)} \rangle_{\mathrm{c;YM}} = \begin{cases} \text{nonzero} & \text{for } \omega = 0 \bmod N_c \\ 0 & \text{otherwise} \end{cases}. \qquad (8)$$

It should be mentioned that the expansion is made around the vacuum of the pure Yang-Mills theory, so that confinement can bear a well-defined meaning and the above expectation value can be strictly vanishing except for $\omega = 0, N_c, 2N_c, \ldots$. Physically speaking, one winding corresponds to a single-quark excitation (see Fig. 1), and each time the winding number reaches N_c, a color singlet is formed out of N_c quarks. We shall thus call such a configuration with $\omega = N_c$ a *baryonic configuration*. We can easily extend the above to more general correlations in the confined phase as

$$\langle \Gamma^{(\omega_1)} \Gamma^{(\omega_2)} \cdots \Gamma^{(\omega_k)} \rangle_{\mathrm{c;YM}} = \begin{cases} \text{nonzero} & \text{for } \omega_1 + \omega_2 + \cdots + \omega_k = 0 \bmod N_c \\ 0 & \text{otherwise} \end{cases}. \qquad (9)$$

Then, in the confined phase at low T, nonvanishing terms out of Eqs. (6) and (7) turn out to be

$$F^{(1)} = -T \sum_f \sum_{\bar{\omega}=-\infty}^{\infty} \langle \Gamma_f^{(\bar{\omega} N_c)}(0) \rangle_{\mathrm{YM}} e^{\bar{\omega} N_c \mu_f/T}, \qquad (10)$$

$$F^{(2)} = -T \sum_{f,f'} \sum_{n=-\infty}^{\infty} \left\{ \langle \Gamma_f^{(n)}(0) \Gamma_{f'}^{(-n)}(0) \rangle_{\mathrm{c;YM}} e^{(\mu_f - \mu_{f'})n/T} + \langle \Gamma_f^{(n+N_c)}(0) \Gamma_{f'}^{(-n)}(0) \rangle_{\mathrm{c;YM}} e^{[\mu_f(n+N_c) - \mu_{f'}n]/T} + \cdots \right\}, \qquad (11)$$

where the ellipsis represents other contributions such as $(\omega = n + 2N_c, \omega' = -n)$, $(\omega = n + 3N_c, \omega' = -n)$, and so on.

In the hadron language, intuitively, $F^{(1)}$ above corresponds to a "multibaryon contribution" with $\bar{\omega}$ baryons. The first term in the next contribution, $F^{(2)}$, corresponds to a "mesonic contribution" and the second in the parentheses is a mixed correlation of baryons and mesons. As long as μ_f

is small enough as compared to the baryonic scale of $F^{(1)}$, these mixed-type contributions are always more suppressed than the pure mesonic contribution. Thus, we can safely neglect the mixed-type term when we discuss the density region up to the onset as in what follows.

Here, let us emphasize that this procedure to take the Z_{N_c}-symmetric average is a vital step to understand the silver blaze problem for the baryon density onset. As we

mentioned, one must take account of the phase fluctuations of the Dirac determinant in this density region of $m_\pi/2 < \mu_q < M_B/N_c$, and we effectively do this by dropping center nonsymmetric operators. Indeed, as argued in Ref. [18], fractional (not a multiple of N_c) excitations of quarks that break center symmetry explicitly are closely related to the sign problem. Usually, in the thermodynamic (i.e. infinite volume) limit in particular, the canonical ensemble becomes quite singular and it loses the strength to solve the sign problem practically. In our present formulation, however, we combine it with the N_f/N_c expansion, so that unwanted quark excitations diminish and the whole machinery is under theoretical control.

III. LARGE-N_c COUNTING OF $F^{(1)}$

In this section, we consider $F^{(1)}$ in an SU(N_c) theory with $N_f = 2$ fixed, and denote corresponding chemical potentials by μ_1 and μ_2. In this way we can introduce a quark (baryon) chemical potential as $\mu_1 = \mu_2 = \mu_q = \mu_B/N_c$ or an isospin chemical potential as $\mu_1 = -\mu_2 = \mu_I$. Then, up to this order of $N_c N_f$, the free energy reads

$$F^{(1)}/T = -\sum_{\bar{\omega}=-\infty}^{\infty} \langle \Gamma_1^{(\bar{\omega} N_c)}(0) \rangle_{\mathrm{YM}} e^{\bar{\omega} N_c \mu_1/T}$$
$$-\sum_{\bar{\omega}=-\infty}^{\infty} \langle \Gamma_2^{(\bar{\omega} N_c)}(0) \rangle_{\mathrm{YM}} e^{\bar{\omega} N_c \mu_2/T}. \quad (12)$$

It is crucially important to note that no difference arises at this order for $\mu_1 = \mu_2 = \mu_q$ and $\mu_1 = -\mu_2 = \mu_I$. In other words, the system at finite quark chemical potential is equivalent to the system at finite isospin chemical potential at $\mathcal{O}(N_c N_f)$ because there is no correlation function involving different flavor sectors. The two flavor sectors do not talk to each other, so to speak [15,19].

Specifically in this section, we shall use a collective notation μ not distinguishing μ_q and μ_I. As we already discussed in the previous section, in the tree level, $\Gamma^{(\bar{\omega} N_c)}(0) \sim \exp(-|\bar{\omega}|N_c m_q/T)$ implies that the free energy in the limit of $T = 0$ becomes independent of μ if $\mu < m_q$. When we perform the functional integration over the gauge fields, this exponential factor would decrease faster; $|\bar{\omega}|N_c m_q$ should be replaced with the baryonic-dressed mass $\bar{\omega} M_B$ (where we picked up only the contribution from ω baryons but neglected any "composite-baryon" possibility, which should be empirically reasonable). This means that

$$\Gamma^{(\bar{\omega} N_c)}(0) \sim \exp(-|\bar{\omega}|N_c m_q/T)$$
$$\rightarrow \langle \Gamma^{(\bar{\omega} N_c)}(0) \rangle_{\mathrm{YM}} \sim \exp(-\bar{\omega} M_B/T). \quad (13)$$

In other words we can state that this is our *definition* of the baryon mass. In fact, together with Eq. (10), we can see that the expansion takes a form of

$$F^{(1)}/T \sim -\sum_{\bar{\omega}=-\infty}^{\infty} \exp[\bar{\omega}(N_c\mu - M_B)/T] \quad (14)$$

apart from prefactors that are not in our interest to locate the density onset. It is obvious that the free energy should not change with μ until μ hits the onset at $\mu_c = M_B/N_c$ [which should be corrected by the binding energy B of nuclear matter that is incorporated, in principle, in the definition (13) for large ω] because no particle can excite at $T = 0$. Thus, from this point of view of the density onset, our definition makes sense to characterize the baryon mass.

In the context of the silver blaze problem, a more nontrivial question is whether our M_B can behave differently from the pion mass m_π or not. The main concern regarding the silver blaze problem lies in the observation that the lowest excitation energy even in the baryonic sector seems to be governed by m_π. We therefore need to treat our M_B very carefully and should clarify if $M_B \simeq (N_c/2)m_\pi$ or not. If this happened, unfortunately, the results are unphysical and any useful information on the silver blaze puzzle in the most nontrivial region is not available at all. As a matter of fact, in the realistic world with $N_c = 3$, it is often the case that $M_B \simeq (N_c/2)m_\pi$ is concluded.

For the case with $N_c = 3$ we can have a valuable hint from lattice-QCD simulations. Along the line of the lattice-QCD setup, it would be instructive to rewrite our $F^{(1)}$ in a slightly different form using the quark number operator $N(\mu)$. It is easy to confirm the following expression,

$$F^{(1)} = -T \langle \Gamma(\mu) \rangle_{\mathrm{YM}} = -T \int^\mu d\mu' \left\langle \frac{d\Gamma(\mu')}{d\mu'} \right\rangle_{\mathrm{YM}}$$
$$= -N_f \int^\mu d\mu' \langle N(\mu') \rangle_{\mathrm{YM}}, \quad (15)$$

up to a μ-independent constant. In the final form $\langle N(\mu) \rangle_{\mathrm{YM}}$ is the same quantity as the quark number expectation value measured in the quenched simulation. We note that this expectation value contains all gluonic loops, i.e., not only planar diagrams but also higher genus diagrams, but no quark loops. Surprisingly, the results from lattice simulations and also from the random matrix model imply that $\langle N(\mu) \rangle_{\mathrm{YM}}$ becomes nonzero when μ exceeds $m_\pi/2$ [20–22]. Strictly speaking, this m_π is not necessarily the physical pion mass, but the quenched pion mass, m_π^{quench}. It is still possible to distinguish m_π^{quench} from the physical baryon mass by looking at how they behave with decreasing m_q; in the $m_q \rightarrow 0$ limit m_π or m_π^{quench} goes to zero but the physical baryon mass should not. If one finds M_B approaching zero in the chiral limit, M_B should be more like the pion mass rather than the physical baryon mass.

Such a striking observation was established first in the so-called "phase quenched" simulation, in which the fluctuating phase of the Dirac determinant is neglected and its modulus, $|\exp[\Gamma(\mu)]|^2$, is implemented in the

simulation. It is understood today that such an approximation is equivalent to replacing the chemical potential with the isospin one, μ_I, so that the onset is determined not by M_B/N_c but by $m_\pi^{\text{quench}}/2$. Later on, it was recognized that the same conclusion was drawn for the quenched simulation in which the whole Dirac determinant is neglected, which is much more nontrivial to understand.

The onset at $m_\pi^{\text{quench}}/2$ in the quenched simulation is caused by the condensation of an "unphysical bound state" of a quark and a conjugate antiquark called the baryonic pion [20,23,24] as sketched in Fig. 2(a). Let us explain what is happening using the language of the so-called partially quenched chiral perturbation theory [25]. In the quenched limit all quark loops should be removed, and we can formulate this by introducing a ghost field ϕ. To cancel the Dirac determinant exactly, ϕ should be a bosonic quark (but should satisfy the antiperiodic boundary condition along the thermal S^1) that yields an inverse of the Dirac determinant. We must utilize such a formulation with quarks and ghosts to deal correctly with the computation of non-Hermitian expectation value like Eq. (15). Because such bosonic ghosts are abundant at finite density, a quark can easily pick an anti-ϕ up and form a bound state q-$\bar\phi$ or the baryonic pion, the mass of which is denoted here as $m_{q\bar\phi}$. In this setup of the quenched limit, if we have a configuration with ω quarks, as is depicted in Fig. 2(b), they turn into ω baryonic pions rather than physical hadrons. Thus, we trivially have $m_\pi^{\text{quench}} = 2m_{q\bar\phi}$, and for the baryonic configuration with N_c quarks, M_B does not access the genuine baryonic sector but simply $M_B = N_c m_{q\bar\phi}$, which immediately leads to the funny observation, $M_B = (N_c/2)m_\pi^{\text{quench}}$. In this way we can understand the subtle nature of the quenched limit when involving non-Hermitian operators.[2]

From the above argument it is highly conceivable that taking the large-N_c limit may cure the subtle situation. The essential point is that the large-N_c limit already encompasses the quenched limit and no quark loops appear. This means that we do not have to introduce the ghost field ϕ to cancel the Dirac determinant. Then, because there is no ϕ, the theory does not have the unphysical baryonic pion. Of course, one can still keep introducing ϕ, but its excitations are negligible as compared to the gluon excitation that is of $\mathcal{O}(N_c^2)$. In other words, forming an unphysical bound state is regarded as the screening effect or the QCD-string breaking. In the large-N_c limit the QCD string extends, so that the linear potential and thus confinement can persist strictly. Because there is no ϕ-induced screening in the
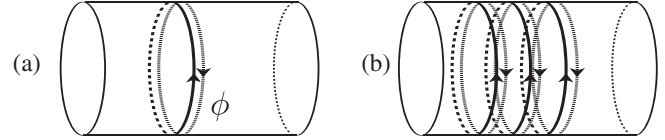

FIG. 2. (a) A bound state of the baryonic pion with a combination of the ghost ϕ. (b) A baryonic configuration screened by ϕ leading to three baryonic pions, the mass of which has nothing to do with the physical baryon mass but is characterized by $3m_{q\bar\phi}$.

large-N_c limit, the baryonic configuration couples to the physical baryon excitation.

To strengthen our argument, let us attempt to confirm explicitly that M_B/N_c is certainly heavier than $m_\pi/2$ within the framework based on the large-N_c limit. As we already mentioned, $F^{(1)}$ contains all the subleading terms in the large-N_c counting, while it is a leading-order contribution in the Veneziano expansion. Here, because we are interested in the behavior of M_B only, it is sufficient for us to focus on the analysis of $\langle \Gamma^{(N_c)}(0) \rangle_{\text{YM}}$. Then, let us pay close attention to the large-N_c expansion of this quantity that consists of N_c quarks propagating in the same direction.

In each quark propagation the self-energy insertion from the interaction with gluons may appear and this is not suppressed by $1/N_c$ as is illustrated in Fig. 3(a). We can also think of higher-order planar diagrams of the self-energy (typically represented by a rainbow-type resummation), which eventually leads to the dynamical quark mass M_q (as is the case in the Dyson-Schwinger studies [26]). This is how the constituent picture of quarks with dynamical mass emerges. For the baryonic configuration, we can also think of different types of diagrams that connect separate quark lines with gluons but the interaction among quarks is always suppressed by $1/N_c$ as explained in Fig. 3(b). Each time one gluon exchange occurs between two quarks (i.e. diquark interaction), it gives a suppression factor by $g^2 \sim 1/N_c$, which is eventually compensated for by a combinatorial enhancement. Thus, we can parametrically write the leading-N_c baryon mass as

$$M_B \simeq N_c M_q - \frac{N_c(N_c - 1)}{2} \cdot \frac{V_{\text{diquark}}}{N_c}, \qquad (16)$$

where V_{diquark} represents the energy gain from the one-gluon exchange interaction in the diquark channel [scaled by N_c so that $V_{\text{diquark}} \sim \mathcal{O}(1)$]. It should be also mentioned that we neglected the kinetic energy in the above expression, which is suppressed in the large-N_c case. The combinatorial factor, $N_c(N_c - 1)/2$, originates from the number of independent diagrams in which two of the N_c quarks are selected out. Before proceeding further, we shall look at the pion sector next.

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[2]We can understand this also from the Dirac eigenvalues; the Banks-Casher type formula for the quark number operator needs an eigenvalue density that can be well defined only for a one-dimensional distribution. The non-Hermiticity makes the eigenvalues spread over the complex plane, and to avoid this, a conjugate sector should be augmented.
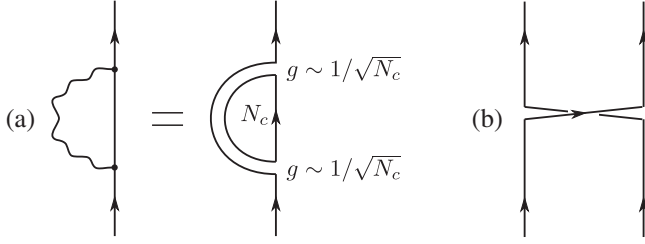
FIG. 3.   (a) Lowest-order diagram of the quark self-energy in the double-line notation. This and all the rainbow types of higher-order diagrams are of the same N_c order as the bare-quark propagation, the sum of which should lead to the dynamical mass M_q. (b) One gluon exchange between two quarks, which is suppressed by $g^2 \sim 1/N_c$. (No N_c factor appears by closing a quark loop.)

IV. LARGE-N_c COUNTING OF $F^{(2)}$

Let us next consider the mesonic contribution of $\mathcal{O}(N_f^2)$ that makes a discrimination between μ_q and μ_I. We first consider the isospin chemical potential, $\mu_1 = -\mu_2 = \mu_I$. Let us remember that, on the one hand, at $\mathcal{O}(N_c N_f)$ the free energies with either an isospin or a quark chemical potential are equivalent to each other, but $F^{(2)}$ at $\mathcal{O}(N_f^2)$, on the other hand, makes a sharp contrast and distinguishes one from the other. The free energy then reads

$$
\begin{aligned}
F^{(2)}(\mu_I)/T = & - \sum_{n=-\infty}^{\infty} \langle \Gamma_1^{(n)}(0)\Gamma_1^{(-n)}(0) \\
& + \Gamma_2^{(n)}(0)\Gamma_2^{(-n)}(0)\rangle_{c;\text{YM}} \\
& - 2\sum_{n=-\infty}^{\infty} \langle \Gamma_1^{(n)}(0)\Gamma_2^{(-n)}(0)\rangle_{c;\text{YM}} e^{2\mu_I n/T}.
\end{aligned}
\tag{17}
$$

In the same way as the analysis in the previous section we can define our pion mass from the following expectation value (under the approximation that we neglect "composite-pion" configurations, which is justified in the large-N_c limit where the meson interaction is turned off):

$$
\langle \Gamma_f^{(n)}(0)\Gamma_{f'}^{(-n)}(0)\rangle_{c;\text{YM}} \sim \exp(-nM_\pi/T).
\tag{18}
$$

Then, using this definition of M_π, apart from unimportant prefactors, we see that the last term of Eq. (17) has the following form of the expansion:

$$
\sim \sum_{n=-\infty}^{\infty} \exp[-n(M_\pi - 2\mu_I)/T].
\tag{19}
$$

Thus, as long as $\mu_I < M_\pi/2$, the expansion is converging and the free energy in the $T = 0$ limit is completely insensitive to μ_I, leading to zero isospin density. In other words the threshold of the pion condensation is given by $M_\pi/2$, and so it is quite reasonable to adopt the above definition (18) of the pion mass.

For the case of the quark chemical potential, $\mu_1 = \mu_2 = \mu_q$, unlike the isospin chemical potential case, we can see that the leading contribution of $F^{(2)}$ is independent of μ_q as

$$
\begin{aligned}
F^{(2)}(\mu_q)/T = & - \sum_{n=-\infty}^{\infty} \langle \Gamma_1^{(n)}(0)\Gamma_1^{(-n)}(0) + \Gamma_2^{(n)}(0)\Gamma_2^{(-n)}(0) \\
& + 2\Gamma_1^{(n)}(0)\Gamma_2^{(-n)}(0)\rangle_{c;\text{YM}}.
\end{aligned}
\tag{20}
$$

Therefore, the onset of the quark number density is solely determined by $F^{(1)}$ and so our baryon mass M_B gives the threshold. It is an interesting and nontrivial observation that the physics of μ_q and that of μ_I belong to different sectors in the power counting of N_f/N_c.

Now let us proceed to a more quantitative aspect of the silver blaze problem; the critical question is how large $M_\pi/2$ is precisely. As we argued in the previous section, the quenched simulation with finite N_c implicitly requires the bosonic ghost fields, while quark loops just decouple in the large-N_c limit. The meson diagrams at large N_c are well known and the ladder resummation as sketched in Fig. 4 gives the meson. We can then parametrize the meson mass as follows:

$$
M_\pi = 2M_q - N_c \cdot \frac{V_{\text{scalar}}}{N_c}.
\tag{21}
$$

It is known [27] that the projection of the one-gluon exchange interaction to the scalar and the diquark channels, respectively, leads to the factor:

$$
V_{\text{diquark}} \propto \frac{N_c + 1}{2N_c}, \qquad V_{\text{scalar}} \propto \frac{N_c^2 - 1}{N_c^2},
\tag{22}
$$

which means that $V_{\text{diquark}} = [N_c/2(N_c - 1)]V_{\text{scalar}}$. For the lightest pseudoscalar (i.e. pion) channel V_{scalar} should be about $2M_q$ at most in order to realize small M_π of the
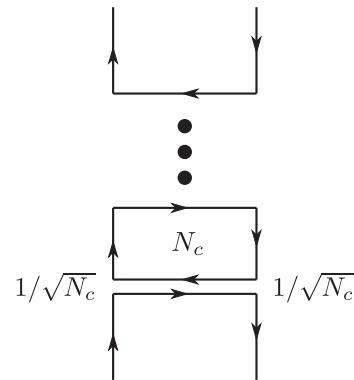

FIG. 4.   Ladder interactions between a quark and an antiquark that form the meson. Lines at the top and the bottom are closed by the antiperiodic boundary condition and the winding number counts how many times the configuration wraps around this circle.

Nambu-Goldstone boson. Using the above coefficients we can make an estimate of the difference between M_B and M_π as

$$\frac{M_B}{N_c} - \frac{M_\pi}{2} \simeq \frac{V_{\text{scalar}}}{4} > 0. \qquad (23)$$

In particular, when the maximally large $V_{\text{scalar}} \simeq 2M_q$ is realized to render the pion mass to vanish, the lower bound of the baryon mass could be $[M_B]_{\text{lowest}} \simeq N_c M_q/2$; that is, the baryon mass cannot be lighter than a half of the sum of the constituent quark mass. Although the quantitative estimate here might be a bit oversimplified, the essential point in this present argument is that it is very likely that M_B is heavier than $(N_c/2)M_\pi$. This hand-waving argument suggests that the diquark interaction is not strong enough to make the baryon as light as the pion in the large-N_c world, which makes a sharp contrast to the finite-N_c quenched world where $M_B \rightarrow 0$ in the chiral limit.

V. "ORIENTIFOLD" LARGE-N_c EXPANSION

Apart from the 't Hooft and the Veneziano large-N_c expansions there is another large-N_c expansion that goes under the name "orientifold expansion" [28]. Consider an $SU(N_c)$ gauge theory coupled to N_f fermions that transform in the two-index antisymmetric representation, denoted by $\psi_{[ij]}$. For $SU(3)$ a Dirac fermion that transforms in the antisymmetric representation is equivalent to a Dirac fermion that transforms in the fundamental representation, since $q^k = \frac{1}{2}\epsilon^{ijk}\psi_{[ij]}$. In the large-$N_c$ limit the fermions that carry two-indices behave like gluons. In particular, fermions are not quenched in the $N_c \rightarrow \infty$ limit. This is the most significant difference between the orientifold expansion and the 't Hooft expansion.

The color singlets of the orientifold theory contain mesons and baryons (in addition to glueballs). The meson, as in ordinary QCD, consists of a pair of a fermion and antifermion. As for the baryons, the issue is more subtle: the most natural candidate consists of N_c fermions, contracted by two epsilon tensors. It turns out, however, that this identification is not correct. It was shown by Bolognesi [29] (see also Ref. [30]) that this simple "baryon" is not stable and, moreover, does not admit the properties of baryons, as anticipated from the Skyrme model. The correct object that should be identified as the baryon consists of $\frac{1}{2}N_c(N_c-1)$ fermions.

Our current discussion of the orientifold theory at finite temperature and density is similar to the previous discussion. We can use the worldline formalism to expand the fermion determinant in powers of N_f. Unlike the 't Hooft expansion case, at present,

$$\langle \underbrace{\Gamma\Gamma\cdots\Gamma}_{k\text{ times}} \rangle_{c;\text{YM}} \sim N_c^2 N_f^k, \qquad (24)$$

which means that the expansion converges if N_f is small enough. It was estimated that at $T = 0$ the worldline

expansion converges for $N_f < 4N_c/(N_c-2)$ [31], when the theory is below the conformal window.

The rest of the discussion is almost identical to the previous discussion. The main issue is that $\mu_B = \frac{1}{2}N_c(N_c-1)\mu$. The free energy $F^{(1)}/T$ at $T = 0$ is μ independent as long as μ is below the onset, according to

$$F^{(1)}/T \sim -\sum_{\bar{\omega}=-\infty}^{\infty} \exp[\bar{\omega}(\mu_B - \tilde{M}_B)/T]. \qquad (25)$$

The question is then how large $\tilde{M}_B$ should be. Because the large-N_c orientifold theory is not quenched, the N_f expansion around the Yang-Mills theory at $N_f = 0$ may be contaminated by the introduction of the corresponding ghost $\tilde{\phi}$ that is required in order to take the average $\langle\cdots\rangle_{\text{YM}}$.

Therefore, if we could perform a quantitative comparison of $F^{(1)}$ calculated with the fundamental fermions and with the antisymmetric fermions, and if N_c is large enough, we could in principle verify our analysis. For the orientifold large-N_c case the scaling between $\tilde{M}_B$ and the corresponding pion mass $\tilde{M}_\pi$ would not be affected, while in the 't Hooft large-N_c case it would show a deviation with increasing N_c, which signals weakened ghosts.

VI. DISCUSSIONS AND CONCLUSIONS

In this paper we discussed QCD with chemical potential in the framework of 't Hooft's $1/N_c$ and Veneziano's N_f/N_c expansions (together with a brief discussion on the "orientifold" expansion). Starting with the QCD partition function, we showed that the free energy at $T = 0$ is μ independent in the regime of small μ where the density of pions or baryons vanishes.

We showed explicitly that at any given order in N_f/N_c the fermionic determinant can be expanded in windings along the temporal (or thermal) direction. This expansion converges only for small values of the chemical potential, the breakdown of which indicates the onset of finite density.

In particular, at $\mathcal{O}(N_f N_c)$ of the free energy, the isospin and the quark chemical potentials are equivalent to each other. The density onset is associated with a baryonic configuration of N_c windings, which leads to a baryonic mass scale M_B. A crucial ingredient in our analysis is the role of the center symmetry, i.e. Z_{N_c}. We used the fact that in the confining vacuum center symmetry is unbroken. Therefore, while the Dirac determinant may depend on the chemical potential, at $T = 0$, the only nonzero contributions come from the zero (modulo N_c) winding sector and the rest vanishes due to Z_{N_c} phase fluctuations.

At $\mathcal{O}(N_f^2)$ of the free energy, on the other hand, the isospin and the quark chemical potentials are no longer equivalent. While no new dependence on the quark chemical potential appears at this order, for the isospin chemical potential we encounter a mesonic configuration of

a quark and an antiquark. In that case the physical picture is as follows: the isospin density onset is characterized by such mesonic configurations, the mass from which is M_π.

An important part of our analysis consists of the estimate of M_B/N_c vs $M_\pi/2$. With Veneziano's expansion we need to evaluate the operator expectation values with the pure Yang-Mills vacuum, and when the finite-density operator is non-Hermitian, the inevitable inclusion of conjugate quarks or ghosts makes the behavior of M_B unphysical. We argue that we can extract the physical information on M_B thanks to 't Hooft's large-N_c limit. Based on diagrammatic analysis, also, we provided an intuitive account for $M_B/N_c > M_\pi/2$ at large N_c. Thus, we conclude that there is definitely a window between the onset of the pion condensation and the finite baryon density in this particular limit.

Apart from the silver blaze puzzle, our expansion scheme is quite unique on its own. We can see an appreciable difference from the standard large-N_c limit if we consider the free energy at finite temperature T: there is an exponentially small but nonzero contribution to the free energy of a form, $F/T \sim \exp[-(M_B - \mu_B)/T]$ in the baryon case and $F/T \sim \exp[-(m_\pi - 2\mu_I)/T]$ in the iso-spin case. Therefore, as long as T is not strictly zero, the free energy is *not* completely T independent, as one might have naively expected from the large-N_c Eguchi-Kawai reduction [32]. This is because we first expanded the Dirac operator to identify the terms of $\mathcal{O}(N_c N_f)$ and $\mathcal{O}(N_f^2)$ and these subleading and sub-subleading terms contain the baryonic and the mesonic excitations, respectively, as "valence" degrees of freedom, though virtual excitations are prohibited in the large-N_c limit. It would be worth revisiting Veneziano's limit to count the physical degrees of freedom not only in the baryon sector [33] but in the meson sector also.

Future applications of our formulation would include concrete implementation in the lattice-QCD simulation; the investigation of the silver blaze behavior in the canonical ensemble can clearly separate two sectors sensitive to the isospin density and to the quark density. Then, one can test whether the onset is really given by the physical pion mass and the physical baryon mass. Once this is confirmed, even without Veneziano's expansion and without potential complication from ghosts, one may be able to justify our understanding of the silver blaze behavior even for the case with finite N_c. On the analytical level which would supplement the numerical efforts, perhaps, the hopping-parameter expansion (which shares a similarity with the large-N_c expansion), together with the resummation of higher-winding configurations [34], will yield a useful exemplification to diagnose the silver blaze puzzle, hopefully in a consistent way with the scenario presented in this work.

ACKNOWLEDGMENTS

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