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Post-linear Schwarzschild solution in harmonic coordinates: Elimination of structure-dependent terms

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This paper deals with a special kind of problems that appear in solutions of Einstein's field equations for extended bodies: many structure-dependent terms appear in intermediate calculations that cancel exactly in virtue of the local equations of motion or can be eliminated by appropriate gauge transformations. For a single body at rest, these problems are well understood for both the post-Newtonian and the post-Minkowskian cases. However, the situation is still unclear for approximations of higher orders. This paper discusses this problem for a "body" of spherical symmetry to post-linear order. We explicitly demonstrate how the usual Schwarzschild field can be derived directly from the field equations in the post-linear approximation in the harmonic gauge and for an arbitrary spherically symmetric matter distribution. Both external and internal solutions are considered. The case of static incompressible fluid is then compared to the well-known results from the literature. The results of this paper can be applied to generalize the well-known post-Newtonian and post-Minkowskian multipole expansions of the metric in the post-linear approximation.

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I. INTRODUCTION

There might be several reasons for an interest in the post-linear Schwarzschild problem. Our main interest for that comes from the problem of high-accuracy astrometry in the framework of general relativity. Recently, a series of igh-accuracy astrometric space missions were proposed such as Gaia [1], with accuracies of a few microarcseconds (μas) or the Nearby Earth Astrometric Telescope (NEAT) proposed to ESA [2], for which accuracies around 50 nanoarcseconds (nas) are under discussion. For all of these missions, the light propagation has to be calculated at a very high level of accuracy that lies beyond the level of 1 μas in observed directions. Already for a mission like Gaia, the influence of the oblateness (quadrupole moment) of the bodies as well as their barycentric motion cannot be neglected. Largest post-post-Newtonian effects in the light propagation also have to be taken into account [3]. Astrometric missions with angular accuracies beyond 1 μas will certainly come in the near future, and also the day will come when the subtle effects of higher post-Newtonian level will be required. For those reasons, it is of great importance to have a metric tensor for a system of N gravitationally interacting arbitrarily shaped and composed, deformable, and rotating bodies to second post-Newtonian or second post-Minkowskian order (keeping all terms in the velocities but only linear and quadratic terms in the gravitational constant). Such a metric will form the basis for the modeling of light trajectories. Some first steps toward such a metric have been done [4,5], but the problem is far from being solved. Clearly, further work is needed.

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Tremendous work in general relativity has been done with the harmonic gauge that was found to be a useful and simplifying gauge for many kinds of applications. It is logical to continue using the harmonic gauge for further refinements of the theory needed for the high-accuracy astrometry and celestial mechanics. The harmonic condition is defined by the following equation [$g = \det(g_{\alpha\beta})$ is the determinant of the metric tensor $g_{\alpha\beta}$]:

$$\frac{\partial}{\partial x^\alpha}((-g)^{1/2} g^{\alpha\beta}) = 0. \tag{1}$$

Several equivalent forms of the harmonic conditions can be found, e.g., in Sec. 7.4 of Ref. [6].

For some "body" (which in principle can be composed of a whole set of individual bodies) at rest, the external metric in the harmonic gauge that is fully specified by two families of multipole moments, mass, and spin moments (M_L and S_L) is known for both the post-Newtonian [7] and the post-Minkowskian cases [8]. For a system of pointlike masses, the whole post-Minkowskian problem, the metric in harmonic coordinates, and the light-ray trajectories was solved in Ref. [9]. This work was extended by including the spin monopoles of the bodies by Kopeikin and Mashhoon [10]. Kopeikin *et al.* [11] found an analytical post-Minkowskian solution for the light propagation in the field of an extended body *at rest*; here, the full multipole structure was taken into account.

Problems arise that are related with the internal structure of the bodies. For a single body at rest, these problems are well understood for both the post-Newtonian and the post-Minkowskian cases [7,8], in which many structure-dependent terms appear in intermediate calculations that cancel exactly in virtue of the local equations of motion or can be eliminated by corresponding gauge transformations. However, for the post-linear case, the situation is still unclear. In the course of our studies for the general problem mentioned above, we found that, even for the spherically symmetric case of a single body, the complete derivation of the external metric (the Schwarzschild metric) is interesting.

We use fairly standard notations: G is the Newtonian constant of gravitation, and c is the vacuum speed of light. We use the signature $(-+++)$ throughout this paper. Lowercase Latin indices i, j, … take values 1, 2, 3. Lowercase Greek indices μ, ν, … take values 0, 1, 2, 3. Repeated indices imply the Einstein's summation irrespective of their positions (e.g., $a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3$ and $a^\alpha b^\alpha = a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$). We use two special objects: $\delta^{ij} = \mathrm{diag}(1,1,1)$ is the Kronecker delta, and ε_{ijk} is the fully antisymmetric Levi-Civitá symbol ($\varepsilon_{123} = +1$). The three-dimensional coordinate quantities ("3-vectors") referred to the spatial axes of the corresponding reference system are set in boldface: $\mathbf{a} = a^i$. The scalar product of any two 3-vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to the Euclidean metric δ_{ij} is denoted by $\mathbf{a}\cdot\mathbf{b}$ and can be computed as $\mathbf{a}\cdot\mathbf{b} = \delta_{ij} a^i b^j = a^i b^i$. A comma before an index designates the partial derivative with respect to the corresponding coordinates: $A_{,\mu} = \partial A(t,\mathbf{x})/\partial x^\mu$, $A_{,i} = \partial A(t,\mathbf{x})/\partial x^i$. For partial derivatives with respect to the coordinate times t, we use $A_{,t} = \partial A(t,\mathbf{x})/\partial t$. A dot over any quantity designates the total derivative with respect to the coordinate time of the corresponding reference system, e.g., $\dot{A} = dA/dt$. Parentheses surrounding a group of indices denote symmetrization, e.g., $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$. Angle brackets surrounding a group of indices or, alternatively, a caret on top of a tensor symbol denote the symmetric trace-free (STF) part of the corresponding object, e.g., $\hat{A}_{ij} \equiv A_{\langle ij\rangle} \equiv STF_{ij} A_{ij} = A_{(ij)} - \frac{1}{3}\delta^{ij} A_{kk}$. For sequences of spatial indices, we shall use multi-indices; a spatial multi-index containing l indices is denoted by the same Latin character in the upper case L (K for k indices, etc.), $L = i_1 \ldots i_l$, where each Cartesian index takes values 1, 2, 3. We use also $L-1 = i_1 \ldots i_{l-1}$, etc. A multisummation is understood for repeated multi-indices: $A_L B_L \equiv \sum_{i_1 \ldots i_l} A_{i_1 \ldots i_l} B_{i_1 \ldots i_l}$. For a spatial vector v^i, we denote $v^L \equiv v^{i_1} v^{i_2} \ldots v^{i_l}$. For an L-order partial derivative, we denote $\partial_L \equiv \partial_{i_1} \ldots \partial_{i_l}$. For true tensorial quantities like the energy-momentum tensor $T^{\mu\nu}$ or the metric tensor $g_{\mu\nu}$, the position of each index (spatial or not) is of great importance. For certain other quantities, like, e.g., w^i, σ^{ij}, or q^{ij} introduced below, the position of indices (upper or lower) is irrelevant (e.g., $w_i = w^i$).

In Sec. II, we deal with the most generic case of an arbitrary spherically symmetric mass distribution. The special case of a static spherically symmetric incompressible matter distribution will be treated in Sec. III, and the results will be compared to those known from the literature in Sec. IV. Conclusions are formulated in Sec. V.

II. GENERAL SPHERICALLY SYMMETRIC CASE

In this section, we deal with the most general case. Our goal is to derive the external metric in the post-linear approximation for a general spherically symmetric compact matter distribution. From Birkhoff's theorem, it is clear that this external metric will be the usual Schwarzschild metric that is determined by a single parameter, the mass of the central body. The central point of this paper is to demonstrate how other terms related with the structure of the body (e.g., its radius R) that appear in intermediate calculations cancel exactly or can be removed by a suitable gauge transformation. Other aspects of the problem related with the usage of harmonic coordinates are also of general interest.

A. Metric tensor and field equations

The post-linear metric tensor in harmonic coordinates will be written in the form

$$g_{00} = -1 + \frac{2}{c^2} w - \frac{2}{c^4} w^2 + \mathcal{O}(c^{-6}), \tag{2}$$

$$g_{0i} = -\frac{4}{c^3} w^i + \mathcal{O}(c^{-5}), \tag{3}$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2} w + \frac{2}{c^4} w^2\right) + \frac{4}{c^4} q_{ij} + \mathcal{O}(c^{-5}). \tag{4}$$

Here, the metric potentials w, w^i, and q_{ij} obey the equations (see, e.g., Ref. [5])

$$\Delta w - \frac{1}{c^2} w_{,tt} = -4\pi G\sigma + \mathcal{O}(c^{-4}), \tag{5}$$

$$\Delta w^i = -4\pi G\sigma^i + \mathcal{O}(c^{-2}), \tag{6}$$

$$\Delta q_{ij} = -w_{,i} w_{,j} - 4\pi G\sigma^{ij} + \mathcal{O}(c^{-1}), \tag{7}$$

where

$$\sigma = \frac{T^{00} + T^{ss}}{c^2}, \qquad \sigma^i = \frac{T^{0i}}{c}, \qquad \sigma^{ij} = T^{ij} - \delta_{ij} T^{ss} \tag{8}$$

and $T^{\mu\nu}$ are the components of the energy-momentum tensor. The metric potentials w and w^i in Eqs. (2)–(4) are needed to orders $\mathcal{O}(c^{-2})$ and $\mathcal{O}(c^0)$, respectively.

B. Formal solution of the field equations

We consider an isolated compact matter distribution and, as usual, require space-time to be asymptotically flat and covered by one single global coordinate system $x^\mu = (ct, x^i)$ with

$$\lim_{\substack{|\mathbf{x}|\to\infty \\ t=\mathrm{const}}} g_{\mu\nu} = \eta_{\mu\nu}, \tag{9}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ is the flat metric tensor of Minkowski space-time. For this reason, the field equations should be solved with the boundary conditions

$$\lim_{\substack{|\mathbf{x}|\to\infty \\ t=\mathrm{const}}} w(t,\mathbf{x}) = 0, \qquad \lim_{\substack{|\mathbf{x}|\to\infty \\ t=\mathrm{const}}} w^i(t,\mathbf{x}) = 0, \qquad \lim_{\substack{|\mathbf{x}|\to\infty \\ t=\mathrm{const}}} q_{ij}(t,\mathbf{x}) = 0. \tag{10}$$

The solution of Eqs. (5)–(7) satisfying these boundary conditions that will be used in the following reads

$$w(t,\mathbf{x}) = G\int_V \frac{\sigma(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \frac{1}{2c^2}G\frac{\partial^2}{\partial t^2}\int_V \sigma(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x' + \mathcal{O}(c^{-4}), \tag{11}$$

$$w^i(t,\mathbf{x}) = G\int_V \frac{\sigma^i(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \mathcal{O}(c^{-2}), \tag{12}$$

$$q_{ij}(t,\mathbf{x}) = \frac{1}{4\pi}\int_V \frac{w_{,i}(t,\mathbf{x}')w_{,j}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + G\int_V \frac{\sigma^{ij}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \mathcal{O}(c^{-1}). \tag{13}$$

Here, V is the support of the matter distribution.

C. Spherically symmetric compact matter distribution

In the following, we consider a matter distribution for which $T^{\mu\nu}$ has compact support; that is, in our reference system $(t,\mathbf{x})$, there exists a quantity $R > 0$ so that for $r \equiv |\mathbf{x}| > R$ the energy-momentum tensor vanishes, $T^{\mu\nu}(t,\mathbf{x}) = 0$. In the following, the matter located within the area $|\mathbf{x}| \le R$ will be often called the body. Moreover, we will consider a spherically symmetric matter distribution for which at an arbitrary point one has

$$\sigma = \frac{T^{00}+T^{ss}}{c^2} = \sigma(t,r) \tag{14}$$

$$\sigma^i = \frac{1}{c}T^{0i} = B(t,r)n^i, \qquad B(t,r) = \frac{1}{c}T^{0i}n^i, \tag{15}$$

$$T^{ij} = A(t,r)\hat{n}^{ij} + \delta^{ij}C(t,r), \qquad A(t,r) = \frac{3}{2}T^{ij}\hat{n}^{ij}, \qquad C(t,r) = \frac{1}{3}T^{kk}, \tag{16}$$

$$\sigma^{ij} = A(t,r)\hat{n}^{ij} - 2\delta^{ij}C(t,r). \tag{17}$$

This form of the energy-momentum tensor is in agreement with the most general form of the spherically symmetric metric tensor (see, e.g., Sec. 13.5 of Ref. [6]) and the corresponding field equations. Thus, matter is fully characterized by four independent scalar functions of time t and radial coordinate $r = |\mathbf{x}|$: $\sigma(t,r)$, $A(t,r)$, $B(t,r)$, and $C(t,r)$. No further assumptions on these four functions are made. The body might be nonstatic; it can oscillate or collapse, etc. In the calculations below, the time t plays a role as an additional parameter, and we will often omit the explicit dependence of these functions on time.

D. Computation of the gravitational potentials w and w^i

The gravitational potentials w and w^i in the required approximation have been extensively discussed in the literature. Here, we summarize the results needed for our further work. We specialize Eqs. (11) and (12) for the case of the spherically symmetric matter distribution from Sec. II C:

$$w(t,\mathbf{x}) = Gr^2\int_0^{2\pi}d\lambda'\int_0^{\pi}d\theta'\sin\theta' \times \int_0^{R/r}dz\frac{z^2\sigma(t,zr)}{\sqrt{1+z^2-2z\mathbf{n}'\cdot\mathbf{n}}} + \frac{1}{2c^2}Gr^4\frac{\partial^2}{\partial t^2}\int_0^{2\pi}d\lambda'\int_0^{\pi}d\theta'\sin\theta' \times \int_0^{R/r}dz z^2\sigma(t,zr)\sqrt{1+z^2-2z\mathbf{n}'\cdot\mathbf{n}} + \mathcal{O}(c^{-4}), \tag{18}$$

$$w^i(t,\mathbf{x}) = Gr^2\int_0^{2\pi}d\lambda'\int_0^{\pi}d\theta'\sin\theta' n'^i \times \int_0^{R/r}dz\frac{z^2B(t,zr)}{\sqrt{1+z^2-2z\mathbf{n}'\cdot\mathbf{n}}} + \mathcal{O}(c^{-2}). \tag{19}$$

The computations of these and similar integrals discussed below are straightforward and can be performed by using

$$\frac{1}{\sqrt{1+z^2-2zx}} = \begin{cases} \sum_{n=0}^{\infty} P_n(x)z^n, & |z| < 1, \\ \sum_{n=0}^{\infty} P_n(x)z^{-n-1}, & |z| > 1, \end{cases} \tag{20}$$

$$\sqrt{1+z^2-2zx} = \begin{cases} \sum_{n=0}^{\infty} C_n^{(-1/2)}(x)z^n, & |z| < 1, \\ \sum_{n=0}^{\infty} C_n^{(-1/2)}(x)z^{-n+1}, & |z| > 1, \end{cases} \tag{21}$$

where $P_n(x)$ are Legendre polynomials, $C_n^{(\alpha)}(x)$ are Gegenbauer polynomials, and

$$\int_0^{2\pi} d\lambda' \int_0^{\pi} d\theta' \sin\theta' \hat{n}'^L P_s(\mathbf{n}'\cdot\mathbf{n}) = \frac{4\pi}{2l+1}\hat{n}^L\delta^{ls}, \qquad l\geq 0, \tag{22}$$

$$\int_0^{2\pi} d\lambda' \int_0^{\pi} d\theta' \sin\theta' C_s^{(-1/2)}(\mathbf{n}'\cdot\mathbf{n}) = 4\pi\left(\delta^{0s}+\frac{1}{3}\delta^{2s}\right), \qquad s\geq 0. \tag{23}$$

Equation (22) can be demonstrated in different ways, e.g., by using the representation of $\hat{n}^L$ in terms of spherical functions and the STF basis tensors and noting that $P_s(\mathbf{n}'\cdot\mathbf{n})$ can be represented as a sum of associated Legendre polynomials depending on the spherical coordinates of $\mathbf{n}$ and $\mathbf{n}'$. The orthogonality of the associated Legendre functions can then be used. Equation (23) follows, e.g., from the explicit formulas for the Gegenbauer polynomials $C_s^{(-1/2)}(x)$.

1. Internal part

For an internal point $(t,\mathbf{x})$ with $r=|\mathbf{x}|\leq R$, the formal solution of Eqs. (5) and (6) reads

$$\begin{aligned} w(t,\mathbf{x}) = {}& \frac{4\pi G}{r}\int_0^r dy y^2\sigma(t,y) + 4\pi G\int_r^R dy y\sigma(t,y) \\ & + \frac{2\pi G}{c^2}\frac{\partial^2}{\partial t^2}\left(r\int_0^r dy y^2\sigma(t,y) + \frac{1}{3r}\int_0^r dy y^4\sigma(t,y)\right. \\ & \left. + \int_r^R dy y^3\sigma(t,y) + \frac{1}{3}r^2\int_r^R dy y\sigma(t,y)\right) \\ & + \mathcal{O}(c^{-4}), \end{aligned} \tag{24}$$

$$w^i(t,\mathbf{x}) = \frac{4\pi G}{3}\left(\frac{x^i}{r^3}\int_0^r dy y^3 B(t,y) + x^i\int_r^R dy B(t,y)\right) + \mathcal{O}(c^{-2}). \tag{25}$$

2. External part

For an external point $(t,\mathbf{x})$ with $r=|\mathbf{x}|\geq R$, the formal solution of Eqs. (5) and (6) can be simplified so that

$$\begin{aligned} w(t,\mathbf{x}) = {}& \frac{4\pi G}{r}\int_0^R dy y^2\sigma(t,y) \\ & + \frac{2\pi G}{c^2}\frac{\partial^2}{\partial t^2}\left(r\int_0^R dy y^2\sigma(t,y) + \frac{1}{3r}\int_0^R dy y^4\sigma(t,y)\right) \\ & + \mathcal{O}(c^{-4}), \end{aligned} \tag{26}$$

$$w^i(t,\mathbf{x}) = \frac{4\pi G}{3}\frac{x^i}{r^3}\int_0^R dy y^3 B(t,y) + \mathcal{O}(c^{-2}). \tag{27}$$

3. General multiple expansions for the external part

As is well known, the solution of Eqs. (5) and (6) outside an arbitrary compact matter distribution admits an expansion in terms of multipole moments (e.g., Ref. [7]). Such an expansion takes the form

$$w = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left[M_L\partial_L\frac{1}{r} + \frac{1}{2c^2}\ddot{M}_L\partial_L r\right] + \frac{4}{c^2}\Lambda_{,t} + \mathcal{O}(c^{-4}), \tag{28}$$

$$w^i = -G\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\left[\dot{M}_{iL-1}\partial_{L-1}\frac{1}{r} + \frac{l}{l+1}\varepsilon_{ijk}S_{kL-1}\partial_{jL-1}\frac{1}{r}\right] - \Lambda_{,i} + \mathcal{O}(c^{-2}), \tag{29}$$

where

$$\Lambda = G\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+1)!}\frac{2l+1}{2l+3}\mathcal{P}_L\partial_L\frac{1}{r}. \tag{30}$$

The Blanchet–Damour moments, M_L and S_L, are given by

$$M_L = \int_V \sigma\hat{x}^L d^3x + \frac{1}{2(2l+3)}\frac{1}{c^2}\frac{d^2}{dt^2}\int_V \sigma\hat{x}^L x^2 d^3x - \frac{4(2l+1)}{(l+1)(2l+3)}\frac{1}{c^2}\frac{d}{dt}\int_V \sigma^i\hat{x}^{iL}d^3x, \qquad l\geq 0, \tag{31}$$

$$S_L = \int_V \varepsilon^{ij\langle a_l}\hat{x}^{L-1\rangle i}\sigma^j d^3s, \qquad l\geq 1. \tag{32}$$

The additional moments $\mathcal{P}_L$ are defined by

$$\mathcal{P}_L = \int_V \sigma^i\hat{x}^{iL}d^3x, \qquad l\geq 0. \tag{33}$$

Here, V again denotes the support of the matter distribution.

Since we consider an isolated matter distribution of compact support, it is well known that, according to the local equations of motion, the lower multipole moments satisfy the equations [12,13]

$$\dot{M} = \mathcal{O}(c^{-4}), \qquad \ddot{M}_i = \mathcal{O}(c^{-4}), \qquad \dot{S}_i = \mathcal{O}(c^{-2}). \tag{34}$$

It is also clear that M_i can always be chosen to be identically zero by the choice of the origin of the reference system as the post-Newtonian center of mass.

4. General skeletonized harmonic gauge

The terms containing Λ can be eliminated from Eqs. (2) and (3) by a transformation of the time coordinate

$$t' = t - \frac{4}{c^4}\Lambda, \qquad \mathbf{x}' = \mathbf{x}. \tag{35}$$

This coordinate transformation obviously retains the harmonics gauge. This transformation changes the metric tensor as

$$g'_{00} = g_{00} - \frac{8}{c^4}\Lambda_{,t} + \mathcal{O}(c^{-5}), \tag{36}$$

$$g'_{0i} = g_{0i} - \frac{4}{c^3}\Lambda_{,i} + \mathcal{O}(c^{-5}), \tag{37}$$

$$g'_{ij} = g_{ij} + \mathcal{O}(c^{-5}). \tag{38}$$

This gauge is called a skeletonized harmonic gauge [12], in which Λ terms do not appear in the post-Newtonian metric, neither in Eq. (2) and (3) nor in the terms $\mathcal{O}(c^{-2})$ of Eq. (4). In this approximation, the metric is "skeletonized" by the Blanchet–Damour moments M_L and S_L. However, it is important to understand that the transformation (35) does not change g_{ij} and therefore terms depending on Λ are still present in the terms $\mathcal{O}(c^{-4})$ in g_{ij}.

5. Multipole moments for a spherically symmetric matter distribution

For the spherically symmetric matter distribution (14)–(17), one can easily show that

$$\begin{aligned} M &= \int_V \sigma d^3x - \frac{1}{2c^2}\frac{d^2}{dt^2}N \\ &= 4\pi\int_0^R dy y^2\sigma(t,y) - \frac{2\pi}{c^2}\frac{d^2}{dt^2}\int_0^R dy y^4\sigma(t,y), \end{aligned} \tag{39}$$

$$N = \int_V \sigma r^2 d^3x = 4\pi\int_0^R dy y^4\sigma(t,y), \tag{40}$$

$$\begin{aligned} \mathcal{P} &= \int_V \sigma^i x^i d^3x = \frac{1}{2}\dot{N} + \mathcal{O}(c^{-2}) \\ &= 4\pi\int_0^R dy y^3 B(t,y) + \mathcal{O}(c^{-2}), \end{aligned} \tag{41}$$

$$M_L = 0, \qquad l \geq 1, \tag{42}$$

$$S_L = 0, \qquad l \geq 1, \tag{43}$$

$$\mathcal{P}_L = 0, \qquad l \geq 1. \tag{44}$$

In this case, the Blanchet–Damour mass M coincides with the Tolman mass [13] and thus coincides with the mass parameter of the Schwarzschild metric as discussed, e.g., in Ref. [6]. The relation $\mathcal{P} = \frac{1}{2}\dot{N} + \mathcal{O}(c^{-2})$ holds for an arbitrary matter distribution and follows from the Newtonian equation of continuity [see Eq. (67) below] and the Ostrogradsky–Gauss theorem.

It is easy to see that w and w^i from Eqs. (26) and (27) admit multipole expansions (28)–(30) with multipole moments given by Eqs. (39)–(44).

In the following, we work only with the skeletonized harmonic gauge and drop the primes over the coordinates. Thus, the metric tensor in this gauge at the external point $(t,\mathbf{x})$ with $|\mathbf{x}| \geq R$ takes the form

$$g_{00} = -1 + \frac{2}{c^2}\frac{GM}{r} - \frac{2}{c^4}\frac{G^2M^2}{r^2} + \mathcal{O}(c^{-5}), \tag{45}$$

$$g_{0i} = \mathcal{O}(c^{-5}), \tag{46}$$

$$\begin{aligned} g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}\frac{GM}{r} + \frac{2}{c^4}\frac{G^2M^2}{r^2}\right) + \frac{4}{c^4}\left(q_{ij} + \delta_{ij}\frac{G\ddot{N}}{3r}\right) \\ + \mathcal{O}(c^{-5}). \end{aligned} \tag{47}$$

E. Computation of q_{ij}

We now come to the computation of q_{ij}, as a solution of Eq. (7), which can be split according to

$$q_{ij} = q^w_{ij} + q^\sigma_{ij}, \tag{48}$$

where

$$\Delta q^w_{ij} = -w_{,i}w_{,j} + \mathcal{O}(c^{-1}), \tag{49}$$

$$\Delta q^\sigma_{ij} = -4\pi G\sigma^{ij} + \mathcal{O}(c^{-1}). \tag{50}$$

1. Computation of q^w_{ij}

The gravitational potential w in the Newtonian approximation is determined by Eq. (24), where the terms $\mathcal{O}(c^{-2})$ are omitted. Since $w = w(t,r)$, we get

$$\frac{\partial w}{\partial x^i} = \frac{x^i}{r}\frac{\partial w}{\partial r}, \qquad \frac{\partial w}{\partial r} = \begin{bmatrix} -\dfrac{GM_r}{r^2}, & r \leq R \\ -\dfrac{GM}{r^2}, & r \geq R \end{bmatrix}, \tag{51}$$

where M_r is the mass contained in a sphere of radius r,

$$M_r = \int_{|\mathbf{x}|\leq r}\sigma d^3x = 4\pi\int_0^r \sigma(t,y)y^2 dy, \tag{52}$$

and $M \equiv M_R$ is the total mass of the body. Note, that M_r is some unknown function of r, while M does not depend on r. Therefore, one gets

$$\Delta q^w_{ij} = -w_{,i}w_{,j} = -G^2M^2f^2(r)\frac{x^ix^j}{r^6}, \tag{53}$$

where

$$f(r) \equiv M_r/M, \tag{54}$$

so that $f(0) = 0$ and $f(r) = 1$ for $r \geq R$. The solution for q^w_{ij} can be written as

$$q^w_{ij} = G^2M^2(I_{ij} + E_{ij}), \tag{55}$$

where $I_{ij} = I_{ij}(t, \mathbf{x})$ is the potential with the source defined by the gravitational potential $w(t, r)$ inside the body (for $r \leq R$),

$$\begin{aligned} I_{ij} &= \frac{1}{4\pi}\int_{|\mathbf{x}'|\leq R} f^2(r')\frac{x'^ix'^j}{r'^6}\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= \frac{1}{4\pi}\frac{1}{r^2}\int_0^{2\pi}d\lambda'\int_0^\pi d\theta' \sin\theta' n'^i n'^j \\ &\quad\times \int_0^{R/r} dz \frac{f^2(zr)}{z^2\sqrt{1+z^2-2z\mathbf{n}'\cdot\mathbf{n}}}, \end{aligned} \tag{56}$$

and $E_{ij} = E_{ij}(t, \mathbf{x})$ is the potential with the source defined by the gravitational potential $w(t, r)$ outside the body (for $r \geq R$),

$$\begin{aligned} E_{ij} &= \frac{1}{4\pi}\int_{|\mathbf{x}'|\geq R}\frac{x'^ix'^j}{r'^6}\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= \frac{1}{4\pi}\frac{1}{r^2}\int_0^{2\pi}d\lambda'\int_0^\pi d\theta' \sin\theta' n'^i n'^j \\ &\quad\times \int_{R/r}^\infty dz \frac{1}{z^2\sqrt{1+z^2-2z\mathbf{n}'\cdot\mathbf{n}}} \\ &= \frac{1}{4\pi}\frac{1}{r^2}\int_0^{2\pi}d\lambda'\int_0^\pi d\theta' \sin\theta' n'^i n'^j \\ &\quad\times \int_0^{r/R} dz \frac{z}{\sqrt{1+z^2-2z\mathbf{n}'\cdot\mathbf{n}}}, \end{aligned} \tag{57}$$

where $r' = |\mathbf{x}'|$, $r = |\mathbf{x}|$, $\mathbf{n}' = \mathbf{x}'/r'$, $\mathbf{n} = \mathbf{x}/r$. Both potentials I_{ij} and E_{ij} are nonzero both inside and outside of the matter distribution. For each of these two integrals, $I_{ij}(t, \mathbf{x})$ and $E_{ij}(t, \mathbf{x})$, two cases should be considered: the external case with $|\mathbf{x}| = r \geq R$ (labeled by a superscript $+$) and the internal case for $|\mathbf{x}| = r \leq R$ (labeled by a superscript $-$). Straightforward calculations show that

$$\begin{aligned} I^-_{ij} &= \frac{1}{3r^2}\delta^{ij}\left(r\int_0^r dy\frac{f^2(y)}{y^2} + r^2\int_r^R dy\frac{f^2(y)}{y^3}\right) \\ &\quad+ \frac{1}{5r^2}\hat{n}^{ij}\left(\frac{1}{r}\int_0^r dy f^2(y) + r^4\int_r^R dy\frac{f^2(y)}{y^5}\right), \end{aligned} \tag{58}$$

$$I^+_{ij} = \frac{1}{3r}\delta^{ij}\int_0^R dy\frac{f^2(y)}{y^2} + \frac{1}{5r^3}\hat{n}^{ij}\int_0^R dy f^2(y). \tag{59}$$

$$E^-_{ij} = \frac{1}{6}\frac{1}{R^2}\delta^{ij} + \frac{1}{20}\frac{r^2}{R^4}\hat{n}^{ij}, \tag{60}$$

$$E^+_{ij} = \frac{1}{3}\frac{1}{Rr}\delta^{ij} - \frac{1}{6}\frac{1}{r^2}\delta^{ij} + \frac{1}{4}\frac{1}{r^2}\hat{n}^{ij} - \frac{1}{5}\frac{R}{r^3}\hat{n}^{ij}. \tag{61}$$

Note, that the integrals in Eq. (59) do not depend on r. Therefore, the dependence of I^+_{ij} on r is explicitly found.

2. Computation of q^σ_{ij}

We now turn to the computation of q^σ_{ij} determined by Eq. (50). Using Eq. (17), we have

$$\begin{aligned} q^\sigma_{ij} &= G\int_{|\mathbf{x}'|\leq R}\frac{\sigma^{ij}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= Gr^2\int_0^{2\pi}d\lambda'\int_0^\pi d\theta' \sin\theta' \hat{n}'^{ij} \\ &\quad\times\int_0^{R/r}dz\frac{z^2A(zr)}{\sqrt{1+z^2-2z\mathbf{n}'\cdot\mathbf{n}}} \\ &\quad- 2Gr^2\delta^{ij}\int_0^{2\pi}d\lambda'\int_0^\pi d\theta' \sin\theta' \\ &\quad\times\int_0^{R/r}dz\frac{z^2C(zr)}{\sqrt{1+z^2-2z\mathbf{n}'\cdot\mathbf{n}}}. \end{aligned} \tag{62}$$

Here, we do not specify explicitly that A and C may also depend on time t. Again, two cases $r \geq R$ and $r \leq R$ should be considered using Eqs. (20)–(22). For an internal point with $r \leq R$, the integral expression for q^σ_{ij} reads

$$\begin{aligned} q^{\sigma,-}_{ij} &= \frac{4\pi G}{5}\frac{\hat{n}^{ij}}{r^3}\int_0^r dy y^4 A(y) + \frac{4\pi G}{5}r^2\hat{n}^{ij}\int_r^R dy\frac{A(y)}{y} \\ &\quad- \frac{8\pi G}{r}\delta^{ij}\int_0^r dy y^2 C(y) - 8\pi G\delta^{ij}\int_r^R dy y C(y). \end{aligned} \tag{63}$$

For an external point with $r \geq R$, the integral expression for q^σ_{ij} can be simplified to

$$q^{\sigma,+}_{ij} = \frac{4\pi G}{5}\frac{\hat{n}^{ij}}{r^3}\int_0^R dy y^4 A(y) - \frac{8\pi G}{r}\delta^{ij}\int_0^R dy y^2 C(y). \tag{64}$$

Again, the integrals on the last line of Eq. (64) do not depend on r, and the dependence of $q^{\sigma,+}_{ij}$ on r is explicitly given by Eq. (64).

For the general spherically symmetric case considered here, $f = f(r)$, $A(r)$ and $C(r)$ are arbitrary functions, and no further simplification of the internal potentials I^-_{ij} and $q^{\sigma,-}_{ij}$ can be done. Clearly, the internal potentials I^-_{ij}, E^-_{ij}, and $q_{ij}^{\sigma,-}$ are not needed for the derivation of the external metric. They will be used below for comparisons in the special case of a body composed of an incompressible fluid.

F. External metric

Gathering all the partial results, we can now write the following expression for the potential $q_{ij}(t,\mathbf{x})$ at an external point with $|\mathbf{x}| \geq R$:

$$\begin{aligned} q_{ij}^{+} &= G^2M^2(I_{ij}^{+} + E_{ij}^{+}) + q_{ij}^{\sigma,+} \\ &= \frac{G^2M^2}{3r}\delta^{ij}\int_0^R dy\frac{f^2(y)}{y^2} + \frac{G^2M^2}{5r^3}\hat{n}^{ij}\int_0^R dyf^2(y) \\ &\quad + \frac{G^2M^2}{3Rr}\delta^{ij} - \frac{G^2M^2}{6r^2}\delta^{ij} + \frac{G^2M^2}{4r^2}\hat{n}^{ij} - \frac{G^2M^2R}{5r^3}\hat{n}^{ij} \\ &\quad + \frac{4\pi G}{5}\frac{\hat{n}^{ij}}{r^3}\int_0^R dyy^4A(y) - \frac{8\pi G}{r}\delta^{ij}\int_0^R dyy^2C(y). \end{aligned} \tag{65}$$

All integrals in Eq. (65) are constants characterizing the matter distribution under consideration in addition to the mass M. Such additional constants do not appear in usual forms of the external Schwarzschild metric, and either can be eliminated by some coordinate transformation or vanish in virtue of the local equations of motion.

The dependence of $q_{ij}^{+} = q_{ij}^{+}(t,\mathbf{x})$ on $\mathbf{x}$ in Eq. (65) is fully explicit. There are terms of the following type: i) δ^{ij}/r, ii) δ^{ij}/r^2, iii) $\hat{n}^{ij}/r^2$, and iv) $\hat{n}^{ij}/r^3$. The additional constants appear in terms of types i and iv. We demonstrate first that the terms of type i cancel with the term in Eq. (47) proportional to $\ddot{N}$ and coming from $\Lambda_{,t}$ in Eq. (28). Collecting all terms of this type in Eq. (65), one gets

$$\begin{aligned} q_{ij}^{+}|_{1/r} &= \frac{\delta^{ij}G}{3r}\left(-24\pi\int_0^R dyy^2C(y) + GM^2\int_0^R dy\frac{f^2(y)}{y^2} + \frac{GM^2}{R}\right) \\ &= -8\pi\frac{\delta^{ij}G}{3r}\int_0^R dy\left(3y^2C(y) + y^3\sigma(y)\frac{dw(y)}{dy}\right) \\ &= -8\pi\frac{\delta^{ij}G}{3r}\frac{d}{dt}\int_0^R dyy^3B(t,y) \\ &= -\frac{\delta^{ij}G}{3r}\ddot{N}, \end{aligned} \tag{66}$$

where we used the Newtonian local equations of motion:

$$\frac{\partial}{\partial t}\sigma + \frac{\partial}{\partial x^i}\sigma^i = \mathcal{O}(c^{-2}), \tag{67}$$

$$\frac{\partial}{\partial t}\sigma^i + \frac{\partial}{\partial x^j}T^{ij} = \sigma\frac{\partial}{\partial x^i}w + \mathcal{O}(c^{-2}). \tag{68}$$

The second equation, in the case of spherical symmetry (14)–(17), can be simplified to

$$\begin{aligned} \frac{2}{3}\frac{\partial}{\partial r}A(t,r) + \frac{2A(t,r)}{r} + \frac{\partial}{\partial t}B(t,r) + \frac{\partial}{\partial r}C(t,r) \\ = \sigma\frac{\partial}{\partial r}w(t,r) + \mathcal{O}(c^{-2}). \end{aligned} \tag{69}$$

The quantity N appearing in the final result in Eq. (66) is just the moment of inertia of the body defined by Eq. (40). Comparing Eq. (66) with the last term in Eq. (47), we conclude that the $1/r$ terms of order $\mathcal{O}(c^{-4})$ in g_{ij} cancel exactly.

Finally, let us note that the terms of type iv in Eq. (65) can be eliminated by a gauge transformation:

$$t' = t, \qquad x'^i = x^i + \frac{1}{c^4}\partial_i h. \tag{70}$$

This transformation changes the metric tensor according to

$$g'_{00} = g_{00} + \mathcal{O}(c^{-5}), \tag{71}$$

$$g'_{0i} = g_{0i} + \mathcal{O}(c^{-5}), \tag{72}$$

$$g'_{ij} = g_{ij} - \frac{2}{c^4}\partial_{ij}h + \mathcal{O}(c^{-5}). \tag{73}$$

One can see that the coordinate gauge remains harmonic if the function h satisfies the condition $\partial_{kk}h = \mathcal{O}(c^{-1})$. Taking

$$h = \frac{1}{30}\frac{G}{r}\left(GM^2\int_0^R dyf^2(y) - GM^2R + 4\pi\int_0^R dyy^4A(y)\right), \tag{74}$$

one can eliminate the terms of type iv in Eq. (65) and in the metric. The transformation (70) with h given by Eq. (74) augments the definition of the skeletonized harmonic gauge for a spherically symmetric matter distribution in the post-linear approximation. Note that both Λ appearing in Eq. (35) and h in Eq. (70) depend on the internal structure of the body, while the resulting external metric does not. Indeed, omitting the primes again, we can see that the metric tensor at the external point $(t,\mathbf{x})$ with $|\mathbf{x}| \geq R$ takes the form

$$g_{00} = -1 + \frac{2}{c^2}\frac{GM}{r} - \frac{2}{c^4}\frac{G^2M^2}{r^2} + \mathcal{O}(c^{-5}), \tag{75}$$

$$g_{0i} = \mathcal{O}(c^{-5}), \tag{76}$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}\frac{GM}{r} + \frac{1}{c^4}\frac{G^2M^2}{r^2}\right) + \frac{1}{c^4}\frac{G^2M^2}{r^2}n^in^j + \mathcal{O}(c^{-5}). \tag{77}$$

This metric fully agrees with the well-known external Schwarzschild metric in harmonic coordinates in the corresponding approximation.

III. CASE OF A STATIC INCOMPRESSIBLE FLUID

The case of a static body composed of an incompressible fluid is often discussed in the literature when dealing with the internal Schwarzschild solution [6]. It is well known that for a static incompressible fluid the four functions describing the matter distribution in Eqs. (14)–(17) are time independent and read

$$\sigma(r) = \kappa\left(1 + \frac{1}{c^2}(2w + 3p)\right) + \mathcal{O}(c^{-4}) = \kappa\left(1 + \frac{1}{2c^2}\frac{GM}{R}(9 - 5\eta^2)\right) + \mathcal{O}(c^{-4}), \quad (78)$$

$$A(r) = \mathcal{O}(c^{-2}), \quad (79)$$

$$B(r) = \mathcal{O}(c^{-2}), \quad (80)$$

$$C(r) = p + \mathcal{O}(c^{-2}) = \frac{1}{2}\frac{GM}{R}(1 - \eta^2)\kappa + \mathcal{O}(c^{-2}), \quad (81)$$

where $\eta \equiv r/R$, $\kappa = \text{const}$ is the invariant density (rest mass plus internal energy density) and $p = p(r)$ is the isotropic pressure that can be computed from the condition of hydrostatic equilibrium $dp/dr = \kappa dw/dr + \mathcal{O}(c^{-2})$ with the boundary condition $p(R) = 0$. The well-known Newtonian formula for the internal potential, $w = \frac{1}{2}\frac{GM}{R}(3 - \eta^2) + \mathcal{O}(c^{-2})$, was used here and in Eq. (78).

The equations that define the gravitational potentials simplify for the static incompressible fluid case to

$$w(t, \mathbf{x}) = \begin{cases} \frac{1}{2}\frac{GM}{R}(3 - \eta^2) + \frac{3}{8c^2}\frac{G^2M^2}{R^2}(1 - \eta^2)^2 + \mathcal{O}(c^{-4}), & r \le R, \\ \frac{GM}{r} + \mathcal{O}(c^{-4}), & r \ge R, \end{cases} \quad (82)$$

$$w^i(t, \mathbf{x}) = \mathcal{O}(c^{-2}), \quad (83)$$

$$q^{ij}(t, \mathbf{x}) = G^2M^2 \times \begin{cases} \frac{1}{R^2}\eta^2\left(\frac{3}{20} - \frac{1}{14}\eta^2\right)\hat{n}^{ij} - \frac{1}{2R^2}\left(1 - \eta^2 + \frac{1}{3}\eta^4\right)\delta^{ij} + \mathcal{O}(c^{-1}), & r \le R, \\ \frac{1}{4r^2}\left(\hat{n}^{ij} - \frac{2}{3}\delta^{ij}\right) - \frac{2}{35}R\partial_{ij}\left(\frac{1}{r}\right) + \mathcal{O}(c^{-1}), & r \ge R, \end{cases} \quad (84)$$

where the mass M is defined by

$$M = 4\pi\int_0^R dy y^2\sigma(t, y) + \mathcal{O}(c^{-4}) = \frac{4}{3}\pi R^3\kappa\left(1 + \frac{3GM}{c^2R}\right) + \mathcal{O}(c^{-4}). \quad (85)$$

Here, we used the fact that for a static incompressible fluid $f(r) = \eta^3 + \mathcal{O}(c^{-1})$.

It is important to see that for a static incompressible fluid $\mathcal{P} = 0$, and therefore $\Lambda = 0$ [see Eqs. (41) and (30)]. It means that no additional time transformation (35) is needed to bring the external metric in the usual form of the Schwarzschild solution in harmonic coordinates. In the gauge transformation of spatial coordinates (70)–(74), one should take $h = -\frac{G^2M^2R}{35r}$. This eliminates the last term in Eq. (84) for q^{ij} for an external point. In this way, the metric outside of the body again coincides with Eqs. (75)–(77) and agrees with the well-known Schwarzschild solution.

For a point inside the body with $r \le R$, Eqs. (82)–(84) together with the definitions (2)–(4) allow us to write

$$g_{00} = -1 + \frac{1}{c^2}\frac{GM}{R}(3 - \eta^2) - \frac{1}{4c^4}\frac{G^2M^2}{R^2}(15 - 6\eta^2 - \eta^4) + \mathcal{O}(c^{-5}), \quad (86)$$

$$g_{0i} = \mathcal{O}(c^{-5}), \quad (87)$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{1}{c^2}\frac{GM}{R}(3 - \eta^2) + \frac{1}{12c^4}\frac{G^2M^2}{R^2}(39 - 30\eta^2 + 7\eta^4)\right) + \frac{1}{35c^4}\frac{G^2M^2}{R^2}\eta^2(21 - 10\eta^2)\hat{n}^{ij} + \mathcal{O}(c^{-5}). \quad (88)$$

As expected, we see that the internal metric depends on the radius of the body R as it is the case also in the Newtonian limit.

IV. DERIVATION OF THE METRIC FROM THE EXACT SOLUTION IN THE CASE OF A STATIC INCOMPRESSIBLE FLUID

It is well known that both internal and external Schwarzschild solutions for the case of the static incompressible fluid can be written as exact solutions. In this section, we will compare our results (82)–(88) for the static incompressible fluid case with those that can be found in the literature (e.g., Ref. [6] where standard Schwarzschild coordinates are used).

A. Metric tensor in standard coordinates

For this Schwarzschild problem, the metric tensor in standard coordinates $(t, \rho, \vartheta, \lambda)$ is of the form

$$ds^2 = -\mathcal{B}(\rho)c^2dt^2 + \mathcal{A}(\rho)d\rho^2 + \rho^2(d\vartheta^2 + \sin^2\vartheta d\lambda^2). \tag{89}$$

Let the radius of the body be at $\rho = a$. Then, the internal metric for $\rho \leq a$ is given by (e.g., see Secs. 8.2 and 11.9 of Ref. [6])

$$\mathcal{A}^-(\rho) = \left(1 - \frac{2m\rho^2}{a^3}\right)^{-1}, \tag{90}$$

$$\mathcal{B}^-(\rho) = \frac{1}{4}\left[3\left(1 - \frac{2m}{a}\right)^{1/2} - \left(1 - \frac{2m\rho^2}{a^3}\right)^{1/2}\right]^2, \tag{91}$$

and the external metric for $\rho \geq a$ reads

$$\mathcal{A}^+(\rho) = \left(1 - \frac{2m}{\rho}\right)^{-1}, \tag{92}$$

$$\mathcal{B}^+(\rho) = 1 - \frac{2m}{\rho}. \tag{93}$$

Here, $m = GM/c^2$, and

$$M = 4\pi\int_0^a \kappa\rho^2 d\rho = \frac{4\pi}{3}\kappa a^3. \tag{94}$$

Below, we will show that this expression for M is in accordance with Eq. (85) above.

B. Transformation to harmonic coordinates

Our goal now is to transform this solution into harmonic coordinates. It is well known that the transformation between standard and harmonic coordinates only affects the radial coordinate $r = r(\rho)$. The transformation of the radial coordinate brings the metric (89) into the form

$$g_{00} = -\mathcal{B}, \tag{95}$$

$$g_{0i} = 0, \tag{96}$$

$$g_{ij} = \mathcal{D}\delta_{ij} + \mathcal{N}n^i n^j, \tag{97}$$

where

$$\mathcal{D} = \frac{\rho^2}{r^2}, \qquad \mathcal{N} = \left[\left(\frac{dr}{d\rho}\right)^{-2}\mathcal{A} - \frac{\rho^2}{r^2}\right]. \tag{98}$$

The transformation from the standard radial coordinate ρ to some harmonic coordinate $r = r(\rho)$ is determined by the second-order differential equation (e.g., Sec. 8.1 of Ref. [6]):

$$\frac{d}{d\rho}\left(\rho^2\mathcal{B}^{1/2}\mathcal{A}^{-1/2}\frac{dr}{d\rho}\right) = 2\mathcal{A}^{1/2}\mathcal{B}^{1/2}r. \tag{99}$$

Clearly, this gives two distinct differential equations for $\rho \leq a$ and $\rho \geq a$ according to Eqs. (90)–(93). We will now determine the function $r(\rho)$ satisfying these two equations such that both $r(\rho)$ and its derivative $dr/d\rho$ are continuous at the stellar surface at $\rho = a$ or $r(a) = R$. According to Eq. (99), this is needed to keep the metric in harmonic coordinates continuous at $r(a) = R$. In the following, we will consistently neglect all terms proportional to m^3 (or equivalently c^{-6}).

For both internal and external solutions, we start with the ansatz

$$r = \rho(1 + mb(\rho) + m^2c(\rho)) + \mathcal{O}(m^3), \tag{100}$$

where $b(\rho)$ and $c(\rho)$ are unknown functions to be determined from the differential equation (99) and boundary conditions.

For the external case, we substitute the external metric (92)–(93) and the ansatz (100) into Eq. (99), expand in powers of m, neglect terms $\mathcal{O}(m^3)$, and get the general solutions of the resulting second-order differential equations for $b(\rho)$ and $c(\rho)$,

$$b^+(\rho) = C_2^+ - \frac{1}{\rho} - \frac{C_1^+}{3\rho^3}, \tag{101}$$

$$c^+(\rho) = C_4^+ - \frac{C_2^+}{\rho} - \frac{C_3^+}{3\rho^3} - \frac{2C_1^+}{3\rho^4}, \tag{102}$$

where C_i^+ are four arbitrary constants. A similar procedure for the internal metric (90)–(91) gives

$$b^-(\rho) = \frac{\rho^2}{2a^3} + C_2^- - \frac{C_1^-}{3\rho^3}, \tag{103}$$

$$c^-(\rho) = \frac{15}{28}\frac{\rho^4}{a^6} - \frac{3}{20}\frac{\rho^2}{a^4} + \frac{C_2^-}{2}\frac{\rho^2}{a^3} + C_4^- + \frac{C_1^-}{3a^3\rho} - \frac{C_3^-}{3\rho^3}, \tag{104}$$

where C_i^- are four arbitrary constants (generally speaking, different from C_i^+). Note that both C_i^- and C_i^+ are not dimensionless. Any values of C_i^- and C_i^+ can be taken to satisfy the differential equation (99). However, all these constants can be fixed from four boundary conditions:

(1) $r(\rho)$ is equal to ρ at spatial infinity: $\lim_{\rho\to\infty} r(\rho) = \rho$ or, equivalently $\lim_{\rho\to\infty} b^+(\rho) = 0$ and $\lim_{\rho\to\infty} c^+(\rho) = 0$;
(2) $r(\rho)$ is regular for $\rho = 0$: $r(0) = 0$ or, equivalently, $\lim_{\rho\to 0} \rho b^-(\rho) = 0$ and $\lim_{\rho\to 0} \rho c^-(\rho) = 0$;
(3) $r(\rho)$ is continuous at $\rho = a$: $b^-(a) = b^+(a)$ and $c^-(a) = c^+(a)$;
(4) the derivative of $r(\rho)$ is continuous at $\rho = a$: $\frac{db^-(\rho)}{d\rho}\big|_{\rho=a} = \frac{db^+(\rho)}{d\rho}\big|_{\rho=a}$ and $\frac{dc^-(\rho)}{d\rho}\big|_{\rho=a} = \frac{dc^+(\rho)}{d\rho}\big|_{\rho=a}$.

With these boundary conditions, we have

$$r^- = \rho\left(1 - \frac{3}{2}\frac{m}{a} + \frac{1}{2}\frac{m\rho^2}{a^3} + \frac{1}{4}\frac{m^2}{a^2} - \frac{9}{10}\frac{m^2\rho^2}{a^4} + \frac{15}{28}\frac{m^2\rho^4}{a^6}\right) + \mathcal{O}(m^3), \tag{105}$$

$$r^+ = \rho\left(1 - \frac{m}{\rho} - \frac{4}{35}\frac{a}{\rho}\frac{m^2}{\rho^2}\right) + \mathcal{O}(m^3). \tag{106}$$

This also determines the relation between a (the stellar radius in ρ) and R (the stellar radius in r):

$$R = a - m - \frac{4}{35}\frac{m^2}{a} + \mathcal{O}(m^3),$$
$$a = R + m + \frac{4}{35}\frac{m^2}{R} + \mathcal{O}(m^3). \tag{107}$$

From this, we see that both definitions for the mass M, (85) and (94) are in accordance with each other.

Note that the first derivative of $\mathcal{A}$ is not continuous at $\rho = a$: $\frac{d\mathcal{A}^+}{d\rho}\big|_{\rho=a} \neq \frac{d\mathcal{A}^-}{d\rho}\big|_{\rho=a}$. Interestingly, this is compensated by a discontinuity of the second derivative of $r(\rho)$ at $\rho = a$ so that the resulting harmonic metric and its first derivatives are continuous.

Here, we have only worked in an approximation neglecting terms $\mathcal{O}(m^3)$. Let us note that the differential equation (99) outside the star has the solution (e.g., Ref. [14])

$$r = C_1(\rho - m) + C_2 F(\rho), \qquad C_i = \text{const}, \tag{108}$$

$$F(\rho) \equiv [(\rho - m)\ln(1 - 2m/\rho) + 2m] = -m\sum_{i=2}^{\infty}\frac{2^i(i-1)}{i(i+1)}(m/\rho)^i. \tag{109}$$

Inside the star, Eq. (99) can be transformed into a Heun equation for which the solutions are also known (see also Ref. [15]).

Using the coordinate transformations (105)–(106) and the metric tensor (95)–(98), we can now derive the explicit expressions for the metric tensor in harmonics coordinates.

C. Internal metric

With these results, the internal metric is given by

$$g_{00}^- = -\mathcal{B}^- = -1 + \frac{m}{R}(3 - \eta^2) - \frac{m^2}{4R^2}(15 - 6\eta^2 - \eta^4) + \mathcal{O}(m^3), \tag{110}$$

from which we derive Eq. (82) for $r \le R$ in virtue of $w/c^2 = -\frac{1}{2}\ln(-g_{00}) + \mathcal{O}(m^3)$. Here, again, $\eta \equiv r/R$. Furthermore,

$$\mathcal{D}^-(r) = 1 + \frac{m}{R}(3 - \eta^2) + \frac{m^2}{2R^2}\left(\frac{13}{2} - \frac{27}{5}\eta^2 + \frac{19}{14}\eta^4\right) + \mathcal{O}(m^3) \tag{111}$$

$$\mathcal{N}^-(r) = \frac{1}{35}\frac{m^2}{R^2}\eta^2(21 - 10\eta^2) + \mathcal{O}(m^3). \tag{112}$$

These equations and Eq. (97) allows one to recover our previous result (88).

D. External metric

The metric component $g_{00}^+ = -\mathcal{B}^+$ coincides with Eq. (45) and

$$\mathcal{D}^+(r) = 1 + \frac{2m}{r} + \frac{m^2}{r^2}\left(1 + \frac{8}{35}\frac{R}{r}\right) + \mathcal{O}(m^3), \tag{113}$$

$$\mathcal{N}^+(r) = \frac{m^2}{r^2}\left(1 - \frac{24}{35}\frac{R}{r}\right) + \mathcal{O}(m^3). \tag{114}$$

These equations and Eq. (97) agree with our previous result (77). It is easy to see that the metric in harmonics coordinates, and its first derivative is continuous at $r = R$.

E. Computation of q_{ij}

It is interesting to check if we can also recover our expression (84) for q_{ij} for a static incompressible fluid. From the definition of q_{ij}, Eq. (4), one obtains

$$q_{ij} = \frac{c^4}{4}\left[\left(\mathcal{D} - 1 - \frac{2w}{c^2} - \frac{2w^2}{c^4}\right)\delta_{ij} + \mathcal{N}\frac{x^i x^j}{r^2}\right] + \mathcal{O}(m^3). \tag{115}$$

which immediately gives Eq. (84).

V. CONCLUSIONS

We have treated the gravitational field of some spherically symmetric matter distribution in harmonic coordinates to post-linear order. We started with the general case in which the matter distribution might be time dependent and left the form of the energy-momentum tensor open. The metric tensor was derived explicitly for both the interior and the exterior region. In the exterior region, the metric tensor can be expanded in terms of the Blanchet–Damour mass, and it is demonstrated explicitly how the usual external Schwarzschild field can be derived from the field equations. Terms depending on the internal structure appear in several places in intermediate calculations, and it was shown how they can be removed with additional gauge transformations or how such terms cancel exactly in virtue of the local equations of motion.

The results of this paper should be considered as an intermediate step in the derivation of the post-linear metric (2)–(4) for a body possessing a full multipole structure, i.e., having arbitrary mass and spin moments, M_L and S_L. This would be a generalization of the well-known post-Newtonian multipole expansions of Blanchet and Damour [7] (discussed also in Sec. II D 3) and the post-Minkowskian ones derived by Damour and Iyer [8]. Such a metric is, e.g., required for relativistic modeling of future space astrometric projects aiming at nanoarcsecond accuracies.

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