# Vacuum contribution of photons in the theory with Lorentz and *CPT*-violating terms

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The photon contribution to the divergences and conformal anomaly in the theory with Lorentz and *CPT*-violating terms is evaluated. We calculate one-loop counterterms coming from the integration over the electromagnetic field and check that they possess local conformal invariance. Furthermore, conformal anomaly and the anomaly-induced effective action are calculated. It turns out that the new terms do not affect the dynamics of the conformal factor in the anomaly-driven inflation (Starobinsky model) and its extensions. At the same time, one can expect these terms to affect the gravitational wave equation and, in general, cosmic perturbations.

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## I. INTRODUCTION

In the last decades there was a growing interest in the theoretical and experimental aspects of the theories where Lorentz and/or *CPT* symmetries are violated by special terms in the action of quantum fields [1]. Many different tests have been proposed in very different areas of physics, and there are good perspectives to either discover such a violation someday or benefit from a better understanding of physics which will result from the continuously improving upper bounds on these new terms. One of the badly explored aspects of the theories with Lorentz- and *CPT*-violating terms concerns cosmology.[1] The present-day state of art in this area is characterized by rapidly growing precision, especially concerning the cosmic microwave radiation (CMB), coming from the cosmic perturbations in the early universe. Therefore, it would be interesting to evaluate the possibility of such violations, in particular at the inflationary epoch. The early universe can be seen as a subject of very special interest, as far as Lorentz and *CPT* symmetries violation is concerned. According to the formal quantum field theory investigations [3–5], torsion field, which is one of the fields which may produce such a violation, cannot be a propagating degree of freedom, because this would enter into conflict with the unitarity of the theory at the quantum level. At the same time, torsion can exist as a composite field which results from some symmetry breaking in space [5,6]. One can suppose that similar situation holds for other Lorentz and *CPT* symmetries violating parameters, such that they result from certain phase transition. Then the role of these terms may be quite different now and in the inflationary or postinflationary epochs, because some physical processes restoring the space-time symmetries could occur since that time. For example, some of the symmetry violations in the early universe could result in the anisotropy in the CMB, which is apparently observed by Planck [7]. Many of the Lorentz- and *CPT*-violating terms may lead to anisotropy in the cosmological perturbations. Then, after these terms disappear due to some kind of symmetry restoration, their imprint remains in the CMB spectrum. Indeed, theoretical realization of this scheme requires, first of all, a definition of the symmetry-breaking terms.

The natural next question is how to define the form of the possible symmetries violation in the gravitational terms. One of the possibilities is as follows. Assuming that the form of the vacuum corrections should be derived from the quantum effects of matter fields, it becomes obvious that the most relevant are the contributions of photons, since all other particles are massive and should decouple too early to produce a significant effect. Therefore, the vacuum quantum contribution of photons is a natural starting point for the formulation of possible *CPT*- and Lorentz-violating terms in the gravitational sector. One more comment is in order here. Apart from the quantum corrections, one can introduce vacuum terms in *CPT*- and Lorentz-violating theories in many different ways. For example, the general vacuum action of gravity with torsion (small part of *CPT*- and Lorentz-violating terms) includes 168 terms [8]. Such a great ambiguity makes it very difficult to expect any real advances in this area. At the same time one can essentially restrict the number of possible gravitational terms just by introducing only those terms which can emerge as divergences in the theory with Lorentz- and/or *CPT*-violating extensions in the matter-fields sector.

The main purpose of the present work is for contributions coming from the massless photon field. The derivation of one-loop divergences for massless conformal

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invariant fields opens the way to the study of the conformal anomaly [9] and to the anomaly-induced effective action of gravity [10,11]. The last is a useful, compact analytic form of quantum correction, which can be derived also in the presence of other fields, such as torsion [12,13] and scalars [14,15]. In this work the anomaly-induced effective action will be extended to the case of a dimensionless Lorentz- and/or *CPT*-violating parameter in the photon sector. As an important example of cosmological application, one can consider the effect of the new terms to the anomaly-driven inflation (Starobinsky model) [16]. The complete version of this model is based on the anomaly-induced effective action of gravity, and can be extended to the cases when other background fields are present [12,14,15,17,18].

The paper is organized as follows. Section II describes the technique for deriving one-loop divergences in the electromagnetic theory with the new external fields. Let us note that such a calculation is not an easy thing to do, especially in the case of dimensionless fields, as the reader will see in what follows. The method which will be developed here enables one to perform this and similar calculations up to the first order in these fields, but, in principal, one can also go beyond this order. Also in this section we briefly comment on the general structure of renormalization in this theory. For a more extensive discussion of this subject one can consult [19]. Section III is devoted to the technically difficult problem to prove the conformal invariance of the bulky one-loop counterterms in the theory. After this task is accomplished, the derivation of conformal anomaly becomes a simple issue. Furthermore, in Sec. IV we derive the anomaly-induced effective action of gravity and also discuss possible applications to inflation. Finally, in Sec. V we draw our conclusions.

## II. DERIVATION OF ONE-LOOP DIVERGENCES

Let us start with the action describing an extended version of an electromagnetic field with Lorentz and *CPT* symmetry-breaking terms. The corresponding action in flat space was formulated in [20], and the minimal extension to the covariant form is quite simple. The action is

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}k_F^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}k^\alpha_{AF}\epsilon_{\alpha\beta\mu\nu}A^\beta F^{\mu\nu}\right\}, \tag{1}$$

where $F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}$ and parameters $k_F^{\mu\nu\alpha\beta}$, $k^\alpha_{AF}$ describe *CPT* and/or Lorentz violation.

For calculating the one-loop divergences we shall apply the background field method splitting (see, e.g., [18] for an introduction),

$$A_\mu \to A_\mu + B_\mu, \tag{2}$$

where $B_\mu$ is the quantum field. The one-loop effective action is given by the expression

$$\Gamma^{(1)}_{\rm div} = \frac{i}{2}\,\mathrm{Tr}\ln\hat{H}|_{\rm div} - i\,\mathrm{Tr}\,\ln\hat{H}_{gh}|_{\rm div}; \tag{3}$$

here $\hat{H}$ is the operator of the bilinear part of the action in quantum fields and $\hat{H}_{gh}$ is the operator of the gauge (Faddeev-Popov) ghosts term. Let us introduce the gauge-fixing term in the form

$$S_{gf} = \frac{1}{2\alpha}\int d^4x\sqrt{-g}(\nabla_\mu B^\mu)^2, \tag{4}$$

where $\alpha$ is an arbitrary parameter of the gauge fixing. For this choice of the gauge fixing, the corresponding Faddeev-Popov ghosts contribute only to the vacuum (metric-dependent) sector of the theory and these contributions do not depend on the new Lorentz breaking parameters of the theory. We choose $\alpha = -1$ as the simplest option for the practical calculations.

Replacing (2) in the action (1) one can find the bilinear form of the action

$$S^{(2)} = \frac{1}{2}\int d^4x\sqrt{-g}B_\mu H^{\mu\nu}B_\nu, \tag{5}$$

where $H^{\mu\nu} = \hat{H}$ has the form

$$\hat{H} = \hat{H}_0 + \hat{H}_{AF} + \hat{H}_F, \tag{6}$$

$$\hat{H}_0 = g^{\mu\nu}\Box - R^{\mu\nu}, \tag{7}$$

$$\hat{H}_{AF} = -2k^\alpha_{AF}\epsilon_\alpha^{\ \mu\nu\beta}\nabla_\beta - (\nabla_\beta k^\alpha_{AF})\epsilon_\alpha^{\ \mu\nu\beta}, \tag{8}$$

$$\hat{H}_F = -2k_F^{\mu(\alpha\beta)\nu}\nabla_\alpha\nabla_\beta - 2(\nabla_\alpha k_F^{\mu\alpha\beta\nu})\nabla_\beta + k_F^{\mu\alpha\beta\lambda}R^\nu_{\ .\lambda\alpha\beta}. \tag{9}$$

The most important property of these formulas is that operator (6) has a nonminimal structure due to the term $k_F^{\mu(\alpha\beta)\nu}\nabla_\alpha\nabla_\beta$. Then the standard Schwinger-DeWitt technique for deriving the divergences cannot be applied. Next, there is a well-elaborated technique of dealing with nonminimal operators [21], but it works only in the cases when nonminimality can be parametrized by some continuous parameter, such that one can integrate over this parameter from zero (corresponding to the minimal limit) and any given value. However, in the case of (6) one meets a tensor field and not just a parameter. Therefore, since this nonminimal term in (8) has a nonstandard form, the known technique of dealing with nonminimal operators [21] cannot be applied too. We can conclude that the problem of our interest lies beyond the limits of modern possibilities and hence its complete solution is impossible.

In this situation one can try to consider a certain approximation. Let us assume that the parameters $k_F^{\mu\alpha\beta\lambda}$

and $k_{AF}^{\alpha}$ are small, such that the linear order in these parameters will be sufficient for our purposes. Indeed, the expansion can be taken to the next orders. In the case of the dimensional parameter $k_{AF}^{\alpha}$ such an expansion will be finite, but for $k_F^{\mu\alpha\beta\lambda}$ it can be infinite. The general situation concerning renormalization in the presence of parameters such as $k_{AF}^{\alpha}$ has been recently described in [19] and we will not repeat it here completely, only give some necessary comments at the end of this section. On the practical side we will consider only linear order and, as the reader will observe, it will be a technically difficult task.

So, for the sake of calculating the one-loop divergences, let us first split the operator $\hat{H}$ into the minimal part $\hat{H}_m$ and the nonminimal part $\hat{H}_{nm}$ and make the following transformation:

$$\begin{aligned}\mathrm{Tr}\ln\hat{H} &= \mathrm{Tr}\ln(\hat{H}_m+\hat{H}_{nm}) = \mathrm{Tr}\ln\hat{H}_m \\ &\quad + \mathrm{Tr}\ln(\hat{1}+\hat{H}_m^{-1}\hat{H}_{nm}) \\ &= \mathrm{Tr}\ln\hat{H}_m + \mathrm{Tr}\hat{H}_{nm}\hat{H}_0^{-1} + \dots \end{aligned} \tag{10}$$

In the last line we perform the expansion of the logarithm and take into account only terms in the first order in the Lorentz- and $CPT$-violating parameters. One can see that the first term in the last line of Eq. (10) can be directly calculated by the standard Schwinger-DeWitt method [22], while the second term can be calculated by means of the universal functional traces method (generalized Schwinger-DeWitt technique) of Barvinsky and Vilkovisky [21].

The minimal version of the operator (6) has been considered in Ref. [23], with the final result for the divergences was obtained in the form

$$\begin{aligned}&\frac{i}{2}\mathrm{Tr}\ln\hat{H}_m|_{\mathrm{div}} - i\mathrm{Tr}\ln\hat{H}_{gh}|_{\mathrm{div}} \\ &= -\frac{1}{\epsilon}\int d^n x\mu^{n-4}\sqrt{-g}\Big\{R_{\mu\nu}\nabla_\alpha\nabla_\beta k_F^{\beta\mu\alpha\nu} - \frac{1}{6}R\nabla_\alpha\nabla_\beta k_F^{\alpha\beta} \\ &\quad + \frac{1}{3}R_{\mu\nu\alpha\beta}\nabla^\beta\nabla_\tau k_F^{\tau\mu\alpha\nu} - \frac{1}{12}k_F^{\mu\nu\alpha\beta}RR_{\mu\nu\alpha\beta} + \frac{1}{2}k_F^{\mu\alpha\beta\tau}R^{\nu}_{.\alpha\beta\tau}R_{\mu\nu}\Big\} \\ &\quad + \Gamma^{(1)}_{\mathrm{vac}}[g_{\mu\nu}]. \end{aligned} \tag{11}$$

In the last formula we used a standard notation $\epsilon = (4\pi)^2(n-4)$ for the parameter of dimensional regularization and introduced a new notation $k_F^{\mu\lambda\nu}{}_{\lambda} \equiv k_F^{\mu\nu}$. Also, $\Gamma^{(1)}_{\mathrm{vac}}[g_{\mu\nu}]$ is the divergent part of the metric-dependent vacuum effective action of a massless vector field (see, e.g., [18,24]),

$$\Gamma^{(1)}_{\mathrm{vac}}[g_{\mu\nu}] = -\frac{1}{\epsilon}\int d^n x\mu^{n-4}\sqrt{-g}\left\{\frac{1}{10}C^2 - \frac{31}{180}E - \frac{1}{10}\Box R\right\}, \tag{12}$$

with $C^2$ and $E$ representing the square of the Weyl tensor and the Gauss-Bonnet topological term (Euler density), respectively.

In the present work we shall go beyond the results of [23] and perform a calculation for the case of the nonminimal operator. Consider the contribution of the last term in the expression (10) for the divergences. For this calculation we need first the inverse operator of (7). As far as we are interested in the divergences, the critically important observation is that, from the viewpoint of power counting, the presence of the dimensionless parameter $k_F^{\mu\nu\alpha\beta}$ makes no changes. Therefore, even in the presence of this parameter, the counterterms will be given by the terms up to quadratic order in the curvature tensor, and it is safe to ignore higher order terms. Then the inverse operator $\hat{H}_0^{-1}$ can be expressed as

$$\begin{aligned}\hat{H}_0^{-1} &= (H_0^{-1})^{\lambda}_{\nu} = \delta^{\lambda}_{\nu}\frac{1}{\Box} + R^{\lambda}_{\nu}\frac{1}{\Box^2} - 2(\nabla^\rho R^{\lambda}_{\nu})\nabla_\rho\frac{1}{\Box^3} \\ &\quad + R^{\lambda}_{\tau}R^{\tau}_{\nu}\frac{1}{\Box^3} - (\Box R^{\lambda}_{\nu})\frac{1}{\Box^3} \\ &\quad + 4(\nabla^\rho\nabla^\sigma R^{\lambda}_{\nu})\nabla_\rho\nabla_\sigma\frac{1}{\Box^4} + \mathcal{O}(l^{-5}). \end{aligned} \tag{13}$$

In the last formula $1/\Box$ is the inverse of d'Alembert operator and the last term $\mathcal{O}(l^{-5})$ indicates to an infinite series of omitted inessential terms of a higher background dimension $1/l$.

Using Eq. (13) one can obtain the relation

$$\begin{aligned}\mathrm{Tr}\hat{H}_{nm}\hat{H}_0^{-1} &= -2\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}\Big\{g_{\lambda\nu}\nabla_\alpha\nabla_\beta\frac{1}{\Box} + R_{\lambda\nu}\nabla_\alpha\nabla_\beta\frac{1}{\Box^2} + (\nabla_\alpha\nabla_\beta R_{\lambda\nu})\frac{1}{\Box^2} \\ &\quad + 2(\nabla_\alpha R_{\lambda\nu})\nabla_\beta\frac{1}{\Box^2} + R_{\lambda\tau}R^{\tau}_{\nu}\nabla_\alpha\nabla_\beta\frac{1}{\Box^3} - 2(\nabla^\rho R_{\lambda\nu})\nabla_\alpha\nabla_\beta\nabla_\rho\frac{1}{\Box^3} \\ &\quad - 4(\nabla_\alpha\nabla^\rho R_{\lambda\nu})\nabla_\beta\nabla_\rho\frac{1}{\Box^3} - (\Box R_{\lambda\nu})\nabla_\alpha\nabla_\beta\frac{1}{\Box^3} \\ &\quad + 4(\nabla^\rho\nabla^\sigma R_{\lambda\nu})\nabla_\alpha\nabla_\beta\nabla_\rho\nabla_\sigma\frac{1}{\Box^4} + \mathcal{O}(R^3)\Big\}. \end{aligned} \tag{14}$$

Equation (14) is already in the form that allows us to apply the tables of universal functional traces of the generalized Schwinger-DeWitt technique [21]. Using the functional traces formulas of this work, each term of (14) can be directly calculated. As a result we obtain

$$\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}(\nabla_\alpha\nabla_\beta R_{\lambda\nu})\frac{1}{\Box^2}\Big|_{\mathrm{div}} = -\frac{2i}{\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}k_F^{\mu\alpha\beta\nu}\nabla_\alpha\nabla_\beta R_{\mu\nu}, \tag{15}$$

$$\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}R_{\lambda\tau}R^\tau_\nu\nabla_\alpha\nabla_\beta\frac{1}{\Box^3}\Big|_{\mathrm{div}} = \frac{i}{2\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}k_F^{\mu\nu}R_{\mu\lambda}R^\lambda_\nu, \tag{16}$$

$$-\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}(\Box R_{\lambda\nu})\nabla_\alpha\nabla_\beta\frac{1}{\Box^3}\Big|_{\mathrm{div}} = -\frac{i}{2\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}k_F^{\mu\nu}\Box R_{\mu\nu}, \tag{17}$$

$$-4\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}(\nabla_\alpha\nabla^\rho R_{\lambda\nu})\nabla_\beta\nabla_\rho\frac{1}{\Box^3}\Big|_{\mathrm{div}} = \frac{2i}{\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}k_F^{\mu\alpha\beta\nu}\nabla_\alpha\nabla_\beta R_{\mu\nu}, \tag{18}$$

$$4\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}(\nabla^\rho\nabla^\sigma R_{\lambda\nu})\nabla_\alpha\nabla_\beta\nabla_\rho\nabla_\sigma\frac{1}{\Box^4}\Big|_{\mathrm{div}} = -\frac{2i}{\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}\left\{\frac{1}{3}k_F^{\mu\alpha\beta\nu}\nabla_\alpha\nabla_\beta R_{\mu\nu} - \frac{1}{6}k_F^{\mu\nu}\Box R_{\mu\nu}\right\}, \tag{19}$$

$$2\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}(\nabla_\alpha R_{\lambda\nu})\nabla_\beta\frac{1}{\Box^2}\Big|_{\mathrm{div}} = 0, \tag{20}$$

$$-2\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}(\nabla^\rho R_{\lambda\nu})\nabla_\alpha\nabla_\beta\nabla_\rho\frac{1}{\Box^3}\Big|_{\mathrm{div}} = 0, \tag{21}$$

$$\mathrm{Tr}k_F^{\mu(\alpha\beta)\lambda}R_{\lambda\nu}\nabla_\alpha\nabla_\beta\frac{1}{\Box^2}\Big|_{\mathrm{div}} = -\frac{i}{\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}\left\{\frac{1}{3}k_F^{\mu\alpha\beta\nu}R_{\mu\nu}R_{\alpha\beta} + \frac{1}{6}k_F^{\mu\nu}RR_{\mu\nu}\right\}, \tag{22}$$

$$\begin{aligned}\mathrm{Tr}k_F^{\mu(\alpha\beta)\nu}\nabla_\alpha\nabla_\beta\frac{1}{\Box}\Big|_{\mathrm{div}} = -\frac{i}{\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}\Big\{&\frac{1}{45}k_F^{\alpha\beta}R^{\mu\nu}R_{\alpha\mu\beta\nu} + \frac{19}{180}k_F^{\alpha\beta}R_{\alpha\lambda\mu\nu}R_\beta^{\lambda\mu\nu}\\ &-\frac{2}{45}k_F^{\alpha\beta}R_{\alpha\lambda}R^\lambda_\beta + \frac{1}{18}k_F^{\alpha\beta}RR_{\alpha\beta} + \frac{1}{30}k_F^{\alpha\beta}\Box R_{\alpha\beta} + \frac{1}{10}k_F^{\alpha\beta}\nabla_\alpha\nabla_\beta R\\ &+\frac{1}{3}k_F^{\mu(\alpha\beta)\nu}R^\lambda_{.\mu\alpha\tau}R_{\lambda\nu\beta.}^{\ \ \ \ \tau} - k_F\left(\frac{1}{180}R^2_{\mu\nu\alpha\beta} - \frac{1}{180}R^2_{\mu\nu} + \frac{1}{72}R^2 + \frac{1}{30}\Box R\right)\Big\},\end{aligned} \tag{23}$$

where the notation $k_F \equiv g_{\mu\nu}k_F^{\mu\nu}$ has been introduced. By using relations (14)– (23), one can obtain

$$\begin{aligned}\frac{i}{2}\mathrm{Tr}\ln\hat{H}_{nm}\hat{H}_0^{-1}|_{\mathrm{div}} = -\frac{1}{\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}\Big\{&\frac{1}{45}k_F^{\alpha\beta}R^{\mu\nu}R_{\alpha\mu\beta\nu} + \frac{19}{180}k_F^{\alpha\beta}R_{\alpha\lambda\mu\nu}R_\beta^{\lambda\mu\nu}\\ &-\frac{49}{90}k_F^{\alpha\beta}R_{\alpha\lambda}R^\lambda_\beta + \frac{2}{9}k_F^{\alpha\beta}RR_{\alpha\beta} + \frac{1}{5}R_{\alpha\beta}\Box k_F^{\alpha\beta} + \frac{1}{10}R\nabla_\alpha\nabla_\beta k_F^{\alpha\beta} + \frac{1}{3}k_F^{\mu(\alpha\beta)\nu}R^\lambda_{.\mu\alpha\tau}R_{\lambda\nu\beta.}^{\ \ \ \ \tau}\\ &+\frac{2}{3}R_{\mu\nu}\nabla_\alpha\nabla_\beta k_F^{\mu\alpha\beta\nu} + \frac{1}{3}k_F^{\mu\alpha\beta\nu}R_{\mu\nu}R_{\alpha\beta} - k_F\left(\frac{1}{180}R^2_{\mu\nu\alpha\beta} - \frac{1}{180}R^2_{\mu\nu} + \frac{1}{72}R^2 + \frac{1}{30}\Box R\right)\Big\}.\end{aligned} \tag{24}$$

Finally, from Eqs. (10), (11) and (24) we arrive at the result for the one-loop divergences of the effective action,

$$\Gamma^{(1)}_{\mathrm{div}} = -\frac{1}{\epsilon}\int d^nx\mu^{n-4}\sqrt{-g}K(g_{\mu\nu},k_F) + \Gamma^{(1)}_{\mathrm{vac}}[g_{\mu\nu}], \tag{25}$$

where

$$\begin{aligned}K(g_{\mu\nu},k_F) = &\frac{1}{45}k_F^{\alpha\beta}R^{\mu\nu}R_{\alpha\mu\beta\nu} + \frac{19}{180}k_F^{\alpha\beta}R_{\alpha\lambda\mu\nu}R_\beta^{\lambda\mu\nu} - \frac{49}{90}k_F^{\alpha\beta}R_{\alpha\lambda}R^\lambda_\beta + \frac{2}{9}k_F^{\alpha\beta}RR_{\alpha\beta}\\ &+\frac{1}{5}R_{\alpha\beta}\Box k_F^{\alpha\beta} - \frac{1}{15}R\nabla_\alpha\nabla_\beta k_F^{\alpha\beta} + \frac{1}{3}k_F^{\mu(\alpha\beta)\nu}R^\lambda_{.\mu\alpha\tau}R^\tau_{\lambda\nu\beta.} - \frac{1}{3}R_{\mu\nu}\nabla_\alpha\nabla_\beta k_F^{\mu\alpha\beta\nu}\\ &+\frac{1}{3}k_F^{\mu\alpha\beta\nu}R_{\mu\nu}R_{\alpha\beta} + \frac{1}{3}R_{\mu\nu\alpha\beta}\nabla^\beta\nabla_\lambda k_F^{\mu\alpha\lambda\nu} - \frac{1}{12}k_F^{\mu\nu\alpha\beta}RR_{\mu\nu\alpha\beta}\\ &+\frac{1}{2}k_F^{\mu\alpha\beta\lambda}R^\nu_{.\alpha\beta\lambda}R_{\mu\nu} - k_F\left(\frac{1}{180}R^2_{\mu\nu\alpha\beta} - \frac{1}{180}R^2_{\mu\nu} + \frac{1}{72}R^2 + \frac{1}{30}\Box R\right).\end{aligned} \tag{26}$$

The expressions, (25) and (26), represent the final result for the one-loop divergences in the linear order in the parameter (field) $k_F^{\mu\nu\alpha\beta}$. Regardless of its bulky appearance, Eq. (26) satisfies some rigid constraints, as we shall see in the next section, where (25) will be used to calculate the conformal anomaly.

As it was already said before, the result (25), (26) represents only the first term of an infinite series expansion in the external field (space-dependent parameter) $k_F^{\mu\nu\alpha\beta}$. Since the classical term with $k_F^{\mu\nu\alpha\beta}$ is not controlled by some fundamental symmetry, at quantum level the situation here is not the same as with an external metric, which is also dimensionless, as $k_F^{\mu\nu\alpha\beta}$ is. However, in the metric case one can use general covariance and organize an infinite set of metric-dependent counterterms into a small amount of covariant expressions, namely, in the $R^2_{\mu\nu\alpha\beta}$, $R^2_{\mu\nu}$, $R^2$, and $\Box R$ terms (see, e.g., [18,25] and a more formal recent discussion in [26]). In the present case the situation is absolutely different, because $k_F^{\mu\nu\alpha\beta}$ is the parameter of a purely phenomenological origin and there is no fundamental symmetry behind them. Therefore, it is impossible to restore a full set of counterterms from the lower-order expressions such as (25) and (26) and, in case of a real interest, the next order terms should be really calculated in an independent way. At the same time, there are two pieces of exact information about higher order terms. First, it is certain that these terms will have exactly four derivatives, which means they will be quadratic in curvature tensor components or have the structures like $\nabla R\cdot k_F\ldots k_F\nabla k_F$, or $\nabla\nabla R\cdot k_F\ldots k_F$, or $R\cdot k_F\ldots k_F\nabla\nabla k_F$, or $\nabla R\cdot k_F\ldots\nabla k_F\nabla k_F$, or $k_F\ldots k_F(\nabla k_F)^4$, or $k_F\ldots k_F(\nabla k_F)^2(\nabla k_F)^2$, etc. (where we omitted all indices, of course). This feature is due to the power counting-based arguments, which we already mentioned before (see also [19]). The second certain property concerns the local conformal symmetry, which will be checked for (25) and (26) in the next section. A standard general argument shows that this symmetry will hold in all orders in $k_F^{\mu\nu\alpha\beta}$, and can be used for both the verification of quantum calculations and further applications.

The last observation is that, due to the complex calculations, we did not derive the total derivative terms in $\Gamma^{(1)}_{\rm div}$. This means, from the viewpoint of a conformal anomaly, that we will not be able to calculate the local terms of the anomaly-induced effective action [27] and will take care of only about the (most relevant, usually) nonlocal part.

## III. LOCAL CONFORMAL INVARIANCE AND CONFORMAL ANOMALY

The classical action of an electromagnetic field in curved space possesses local conformal invariance. This property is very important, in particular it defined the equation of state $P_r=\rho_r/3$ for the radiation. The breaking of this equation of state occurs only at the quantum level due to the conformal anomaly, and leads to a deformed equation of state for radiation [28,29]. It is very important that the classical action of an electromagnetic field with Lorentz and $CPT$ symmetry-breaking terms (1) also possesses local conformal invariance. In the present case this means that the action of the theory does not change under the following simultaneous transformation of the metric, of the vector $A_\mu$ and of the parameter $k_F^{\mu\nu\alpha\beta}$:

$$g_{\mu\nu}\to g'_{\mu\nu}=g_{\mu\nu}e^{2\sigma},\qquad A_\mu\to A'_\mu=A_\mu,$$
$$k_F^{\mu\nu\alpha\beta}\to k_F'^{\mu\nu\alpha\beta}=k_F^{\mu\nu\alpha\beta}e^{-4\sigma}, \tag{27}$$

where $\sigma=\sigma(x)$. The local conformal invariance of the action (1) implies the vanishing trace of the energy-momentum tensor $T^\mu_\mu=0$ in the on-shell limit. The same is true for the vacuum terms, if we do not put there unnecessary nonconformal terms. However, the situation changes dramatically if we take quantum effects onto account. At the quantum level the classical action of vacuum has to be replaced by the renormalized effective action $\Gamma_R$. Because of the renormalization procedure, the expectation value of the trace $\langle T^\mu_\mu\rangle$ differs from zero, which is called a conformal (trace) anomaly [9].

The renormalized one-loop effective action has the form

$$\Gamma=S+\Gamma^{(1)}+\Delta S, \tag{28}$$

where $\Gamma^{(1)}=\Gamma^{(1)}_{\rm div}+\Gamma^{(1)}_{\rm fin}$ is a direct quantum correction to the classical action and $\Delta S$ is a local counterterm which is called to cancel the divergent part of $\Gamma^{(1)}$. $\Delta S$ is the only source of the noninvariance of the effective action, because classical action and direct quantum contribution are conformal invariant. Then the anomalous trace is

$$\langle T^\mu_\mu\rangle=-\frac{2}{\sqrt{-g}}g_{\mu\nu}\frac{\delta\Gamma_R}{\delta g_{\mu\nu}}\bigg|_{n=4}=-\frac{2}{\sqrt{-g}}g_{\mu\nu}\frac{\delta\Delta S}{\delta g_{\mu\nu}}\bigg|_{n=4}. \tag{29}$$

The calculation of this expression can be done most simply by using the conformal parametrization of the metric,

$$g_{\mu\nu}=g'_{\mu\nu}e^{2\sigma}, \tag{30}$$

where $g'_{\mu\nu}$ is the fiducial metric with fixed determinant (this condition can be seen as purely technical and we can disregard it after the derivation). One can easily prove the relation which provides the simplest way to derive an anomaly for new theories [30],

$$-\frac{2}{\sqrt{-g}}g_{\mu\nu}\frac{\delta A[g_{\mu\nu}]}{\delta g_{\mu\nu}}=-\frac{1}{\sqrt{-g'}}e^{-4\sigma}\frac{\delta A[g'_{\mu\nu}e^{2\sigma}]}{\delta\sigma}\bigg|_{g'_{\mu\nu}\to g_{\mu\nu},\sigma\to 0}. \tag{31}$$

In order to use these general results in our case, we need first to prove that the conformal invariance of the new term,

$$\sqrt{-g'}K(g'_{\mu\nu},k'_F)=\sqrt{-g}K(g_{\mu\nu},k_F), \tag{32}$$

holds in the four dimensional space-time limit. This is not a trivial task, from the technical side, so let us present some details concerning the transformation rules. For the one-parameter Lie group one can safely restrict the consideration by the infinitesimal version of the transformation (27). Then, disregarding the higher orders in $\sigma$ and superficial terms, after some long algebra we arrive at the following transformation rules:

$$(k_F^{\alpha\beta}R^{\mu\nu}R_{\alpha\mu\beta\nu})' = (1-4\sigma)k_F^{\alpha\beta}R^{\mu\nu}R_{\alpha\mu\beta\nu} + 2k_F^{\alpha\beta}R^\lambda_\beta\nabla_\alpha\nabla_\lambda\sigma - k_F^{\alpha\beta}R\nabla_\alpha\nabla_\beta\sigma - k_F R^{\alpha\beta}\nabla_\alpha\nabla_\beta\sigma - k_F^{\alpha\beta}R_{\alpha\beta}\Box\sigma + 2k_F^{\alpha\beta}R_{\mu\alpha\beta\nu}\nabla^\mu\nabla^\nu\sigma + \cdots \tag{33}$$

$$(k_F^{\alpha\beta}R_{\alpha\lambda\mu\nu}R_\beta^{\cdot\lambda\mu\nu})' = (1-4\sigma)k_F^{\alpha\beta}R_{\alpha\lambda\mu\nu}R_\beta^{\cdot\lambda\mu\nu} - 4k_F^{\alpha\beta}R^\lambda_\beta\nabla_\alpha\nabla_\lambda\sigma + 4k_F^{\alpha\beta}R_{\mu\alpha\beta\nu}\nabla^\mu\nabla^\nu\sigma + \cdots \tag{34}$$

$$(k_F^{\alpha\beta}R_{\alpha\lambda}R^\lambda_\beta)' = (1-4\sigma)k_F^{\alpha\beta}R_{\alpha\lambda}R^\lambda_\beta - 4k_F^{\alpha\beta}R^\lambda_\beta\nabla_\alpha\nabla_\lambda\sigma - 2k_F^{\alpha\beta}R_{\alpha\beta}\Box\sigma + \cdots \tag{35}$$

$$(k_F^{\alpha\beta}R_{\alpha\beta}R)' = (1-4\sigma)k_F^{\alpha\beta}R_{\alpha\beta}R - 2k_F^{\alpha\beta}R\nabla_\alpha\nabla_\beta\sigma - 6k_F^{\alpha\beta}R_{\alpha\beta}\Box\sigma - k_F R\Box\sigma + \cdots \tag{36}$$

$$(k_F^{\mu(\alpha\beta)\nu}R^\lambda_{\cdot\mu\alpha\tau}R^\tau_{\lambda\nu\beta\cdot})' = (1-4\sigma)k_F^{\mu(\alpha\beta)\nu}R^\lambda_{\cdot\mu\alpha\tau}R^\tau_{\lambda\nu\beta\cdot} + k_F^{\alpha\beta}R_{\mu\alpha\beta\nu}\nabla^\mu\nabla^\nu\sigma + k_F^{\mu\alpha\beta\nu}R_{\alpha\beta}\nabla_\mu\nabla_\nu\sigma - 6k_F^{\mu\alpha\beta\nu}R_{\lambda\mu\alpha\nu}\nabla^\lambda\nabla_\beta\sigma + \cdots \tag{37}$$

$$(k_F^{\mu\alpha\beta\nu}R_{\mu\nu}R_{\alpha\beta})' = (1-4\sigma)k_F^{\mu\alpha\beta\nu}R_{\mu\nu}R_{\alpha\beta} + 2k_F^{\alpha\beta}R_{\alpha\beta}\Box\sigma - 4k_F^{\mu\alpha\beta\nu}R_{\alpha\beta}\nabla_\mu\nabla_\nu\sigma + \cdots \tag{38}$$

$$(k_F^{\mu\nu\alpha\beta}RR_{\mu\nu\alpha\beta})' = (1-4\sigma)k_F^{\mu\nu\alpha\beta}RR_{\mu\nu\alpha\beta} - 4k_F^{\alpha\beta}R\nabla_\alpha\nabla_\beta\sigma - 6k_F^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\Box\sigma + \cdots \tag{39}$$

$$(k^{\mu\alpha\beta\lambda}R^\nu_{\cdot\alpha\beta\lambda}R_{\mu\nu})' = (1-4\sigma)k_F^{\mu\alpha\beta\lambda}R^\nu_{\cdot\alpha\beta\lambda}R_{\mu\nu} - 2k_F^{\alpha\beta}R^\lambda_\beta\nabla_\alpha\nabla_\lambda\sigma + 2k_F^{\mu\alpha\beta\nu}R_{\alpha\beta}\nabla_\mu\nabla_\nu\sigma + 2k_F^{\mu\alpha\beta\nu}R_{\lambda\mu\alpha\nu}\nabla^\lambda\nabla_\beta\sigma - k_F^{\mu\alpha\beta\nu}R_{\mu\alpha\beta\nu}\Box\sigma + \cdots \tag{40}$$

$$(R\nabla_\alpha\nabla_\beta k_F^{\alpha\beta})' = (1-4\sigma)R\nabla_\alpha\nabla_\beta k_F^{\alpha\beta} - 2k_F^{\alpha\beta}R\nabla_\alpha\nabla_\beta\sigma - 6\nabla_\alpha\nabla_\beta k_F^{\alpha\beta}\Box\sigma - 6k_F^{\alpha\beta}\nabla_\alpha R\nabla_\beta\sigma + k_F\nabla_\lambda R\nabla^\lambda\sigma + \cdots \tag{41}$$

$$(R_{\alpha\beta}\Box k_F^{\alpha\beta})' = (1-4\sigma)R_{\alpha\beta}\Box k_F^{\alpha\beta} - 4k_F^{\alpha\beta}R^\lambda_\beta\nabla_\alpha\nabla_\lambda\sigma - 2k_F^{\alpha\beta}R_{\alpha\beta}\Box\sigma - 4k_F^{\alpha\beta}R_{\mu\alpha\beta\nu}\nabla^\mu\nabla^\nu\sigma - 2\nabla_\alpha\nabla_\beta k_F^{\alpha\beta}\Box\sigma - k_F\Box^2\sigma - 2k_F^{\alpha\beta}\nabla_\alpha R\nabla_\beta\sigma - 2k_F^{\alpha\beta}\nabla_\lambda R_{\alpha\beta}\nabla^\lambda\sigma + 2k_F^{\alpha\beta}\nabla_\alpha R_{\lambda\beta}\nabla^\lambda\sigma - 2k_F^{\alpha\beta}\nabla^\tau R_{\tau\alpha\beta\lambda}\nabla^\lambda\sigma + \cdots \tag{42}$$

$$(R_{\mu\nu}\nabla_\alpha\nabla_\beta k_F^{\mu\alpha\beta\nu})' = (1-4\sigma)R_{\mu\nu}\nabla_\alpha\nabla_\beta k_F^{\mu\alpha\beta\nu} + k_F^{\alpha\beta}R^\lambda_\beta\nabla_\alpha\nabla_\lambda\sigma - k_F^{\mu\alpha\beta\nu}R_{\alpha\beta}\nabla_\mu\nabla_\nu\sigma + 2k_F^{\mu\alpha\beta\nu}R_{\lambda\mu\alpha\nu}\nabla^\lambda\nabla_\beta\sigma + \nabla_\alpha\nabla_\beta k_F^{\alpha\beta}\Box\sigma - k_F^{\alpha\beta}\nabla_\lambda R_{\alpha\beta}\nabla^\lambda\sigma + 2k_F^{\alpha\beta}\nabla_\alpha R_{\lambda\beta}\nabla^\lambda\sigma - 4k_F^{\mu\alpha\beta\nu}\nabla_\alpha R_{\mu\nu}\nabla_\beta\sigma - 2k_F^{\mu\alpha\beta\nu}\nabla_\beta R_{\mu\lambda\alpha\nu}\nabla^\lambda\sigma + \cdots \tag{43}$$

$$(R_{\mu\nu\alpha\beta}\nabla^\beta\nabla_\lambda k_F^{\mu\alpha\lambda\nu})' = (1-4\sigma)R_{\mu\nu\alpha\beta}\nabla^\beta\nabla_\lambda k_F^{\mu\alpha\lambda\nu} + k_F^{\alpha\beta}R^\lambda_\beta\nabla_\alpha\nabla_\lambda\sigma - k_F^{\mu\alpha\beta\nu}R_{\alpha\beta}\sigma_{\mu\nu} + 2k_F^{\mu\alpha\beta\nu}R_{\lambda\mu\alpha\nu}\nabla^\lambda\nabla_\beta\sigma + \nabla_\alpha\nabla_\beta k_F^{\alpha\beta}\Box\sigma + k_F^{\alpha\beta}\nabla_\alpha R_{\lambda\beta}\nabla^\lambda\sigma + k_F^{\alpha\beta}\nabla^\tau R_{\tau\alpha\beta\lambda}\nabla^\lambda\sigma - k_F^{\mu\alpha\beta\nu}\nabla_\alpha R_{\mu\nu}\nabla_\beta\sigma + 3k_F^{\mu\alpha\beta\nu}\nabla^\tau R_{\tau\mu\nu\alpha}\nabla_\beta\sigma - 2k_F^{\mu\alpha\beta\nu}\nabla_\beta R_{\mu\lambda\alpha\nu}\nabla^\lambda\sigma + \cdots \tag{44}$$

$$\left[k_F\left(\frac{1}{180}R^2_{\mu\nu\alpha\beta} - \frac{1}{180}R^2_{\mu\nu} + \frac{1}{72}R^2 + \frac{1}{30}\Box R\right)\right]' = (1-4\sigma)k_F\left(\frac{1}{180}R^2_{\mu\nu\alpha\beta} - \frac{1}{180}R^2_{\mu\nu} + \frac{1}{72}R^2 + \frac{1}{30}\Box R\right) - \frac{1}{45}k_F R^{\alpha\beta}\nabla_\alpha\nabla_\beta\sigma - \frac{2}{9}k_F R\Box\sigma - \frac{1}{5}k_F\Box^2\sigma - \frac{1}{15}k_F\nabla_\lambda R\nabla^\lambda\sigma + \cdots. \tag{45}$$

Substituting these formulas into (26), we find the conformal invariance (32).

By using Eqs. (29), (31), and (32), one can easily find the conformal anomaly,

$$\langle T^\mu_\mu \rangle = -\frac{1}{(4\pi)^2}[wC^2 + bE + c\Box R + K(g_{\mu\nu}, k_F)], \tag{46}$$

where the parameters $w$, $b$, $c$ are, in our case,

$$w = \frac{1}{10}, \qquad b = -\frac{31}{180}, \qquad c = -\frac{1}{10}. \tag{47}$$

## IV. ANOMALY-INDUCED EFFECTIVE ACTION

One can use the conformal anomaly (46) to construct an equation for the finite part of the one-loop correction to the effective action

$$\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta \Gamma_{\text{ind}}}{\delta g_{\mu\nu}} = \frac{1}{(4\pi)^2}[wC^2 + bE + c\Box R + K(g_{\mu\nu}, k_F)]. \tag{48}$$

The solution of this equation is straightforward. The simplest possibility is to parametrize the metric as in (30), separating the conformal factor $\sigma(x)$ and rewrite Eq. (48) using (31). The solution for the effective action is

$$\begin{aligned}\Gamma_{\text{ind}} = S_c[g_{\mu\nu}{}'] + \frac{1}{(4\pi)^2}\int d^4x\sqrt{-g'}\Big\{ w\sigma C^2 \\ + b\sigma\left(E' - \frac{2}{3}\Box' R'\right) + 2b\sigma\Delta_4'\sigma + \sigma K(g'_{\mu\nu}, k'_F) \\ - \frac{3c+2b}{36}[R' - 6(\nabla'\sigma)^2 - 6\Box'\sigma]^2\Big\},\end{aligned} \tag{49}$$

where $\Delta_4$ is a fourth derivative conformal covariant Paneitz operator, acting on dimensionless scalar

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu. \tag{50}$$

$S_c[g'_{\mu\nu}] = S_c[g_{\mu\nu}]$ in Eq. (49) is an arbitrary conformal invariant functional of the metric, which serves as an integration constant of Eq. (48). In the purely metric theory this functional is irrelevant for the dynamics of the conformal factor. Then, for the simplest cosmological applications, the anomaly-induced expression can be seen as an exact effective action. It is important that this term can be also ignored when one is dealing with the black-hole applications [31,32] and gravitational waves [17,33,34]. In both cases the results obtained *without* this term provide a very good fit with the ones obtained by other methods. The reason for this output is that the rest of the action (49) keeps full information about the UV limit of the theory. In other works, it contains all the leading logarithmic corrections, while for $S_c[g_{\mu\nu}]$ only sublogarithmic parts remain.

When other background fields are present, the automatic irrelevance of the term $S_c[g_{\mu\nu}]$ in the zero-order cosmology does not hold, because $S_c[g_{\mu\nu}]$ may depend on these fields, along with the metric. Our present situation belongs to this class of theories [13–15], because this integration constant may depend also on $k_F^{\alpha\beta\mu\nu}$. This means $S_c = S_c[g_{\mu\nu}, k_F^{\alpha\beta\mu\nu}]$. However, taking into account the arguments presented above, we will not really care about this term.

The expression (49) is the quantum correction to the classical action. Let us note that the covariant forms of the anomaly-induced action can be easily calculated on the basis of Eq. (49), in both nonlocal [10,11] and local forms, the last uses auxiliary fields [35,36] (see also [30] for a review).

Let us give just a final result for the local form of the anomaly-induced effective action, with the two auxiliary scalar fields $\varphi$ and $\psi$. Compared to the original formula of [35], this expression has an extra term related to the parameter $k_F^{\alpha\beta\mu\nu}$,

$$\begin{aligned}\Gamma_{\text{ind}} = S_c[g, k_f] - \frac{3c+2b}{36(4\pi)^2}\int d^4x\sqrt{g(x)}R^2(x) + \int d^4x\sqrt{g(x)}\Big\{\frac{1}{2}\varphi\Delta_4\varphi - \frac{1}{2}\psi\Delta_4\psi + \varphi\Big[\frac{\sqrt{-b}}{8\pi}\left(E - \frac{2}{3}\Box R\right) \\ - \frac{1}{8\pi\sqrt{-b}}(aC^2 + K(g_{\mu\nu}, k_F))\Big] + \frac{1}{8\pi\sqrt{-b}}\psi(aC^2 + K(g_{\mu\nu}, k_F))\Big\}.\end{aligned} \tag{51}$$

The last form of the effective action is the most useful one for dealing with Hawking radiation from black holes or exploring the dynamics of gravitational waves on the cosmological background. In both cases one has to solve the equations for the auxiliary fields $\varphi$ and $\psi$ by implementing the appropriate boundary conditions. After that it is possible to study the energy-momentum tensor of the vacuum in case of black holes [31,32] or explore the dynamics of gravitational waves [17]. Indeed, for the homogeneous and isotropic metrics there is no difference between the effective actions (51) and (49); they always give the same dynamics of the conformal factor $\sigma$. Hence, Eq. (49) is completely sufficient for exploring the dynamics of the conformal factor, which we are going to study in the rest of this section.

Consider possible applications of anomaly (46) and the anomaly-induced effective action (49) to inflation. The starting point should be the theory based on the Einstein-Hilbert action with quantum correction (49),

$$S = -\frac{M_P^2}{16\pi}\int d^4x\sqrt{-g}R + \Gamma_{\text{ind}}, \tag{52}$$

where $M_p^2 = 1/G$ is the square of the Planck mass and $\Gamma_{\rm ind}$ is the quantum correction (49). We look for an isotropic and homogeneous solution

$$g_{\mu\nu} = a^2(\eta) g'_{\mu\nu}, \qquad (53)$$

where $\eta$ is the conformal time

$$ds'^2 = g'_{\mu\nu} dx^\mu dx^\nu = d\eta^2 - \frac{dr^2}{1-kr^2} - r^2 d\Omega \qquad (54)$$

and $k$ parametrizes the space-time curvature $k = 0, \pm 1$.

The first observation concerning the effect of the parameter $k_F^{\alpha\beta\mu\nu}$ is its complete irrelevance for the flat-space case $k = 0$. The reason is that the effect of Lorentz- and $CPT$-violating parameters is accumulated in the scalar function $K = K(g'_{\mu\nu}, k_F'^{\alpha\beta\mu\nu})$. From the definition of this function in (26), it directly follows that $K(\eta_{\mu\nu}, k_F'^{\alpha\beta\mu\nu}) = 0$. Therefore, the only chance to observe some effect of the Lorentz-violating parameter $k_F^{\alpha\beta\mu\nu}$ on the dynamics of conformal factor is related to the cases $k = \pm 1$.

The direct calculation of the new term, induced by Lorentz and $CPT$ symmetry-breaking term $K(g'_{\mu\nu}, k'_F)$ requires some long algebra and we shall give only a final result. It is relatively easy to show that all terms which involve $g'_{o\nu}$ for $\nu = 0,1,2,3$ give zero. For the space indices $i, j = 1,2,3$ one can show, by using metric (54) in Eq. (26), the following relation:

$$K(g'_{\mu\nu}, k'_F) = k^2 k_F'^{ij} g'_{ij} - \frac{1}{2} k^2 k_F'^{iklj} g'_{il} g'_{kj} - \frac{1}{2} k^2 k'_F. \qquad (55)$$

At this point we have to remember that the tensor $k_F^{\alpha\beta\mu\nu}$ has the same algebraic symmetries as the Riemann tensor. According to the definitions, $k'_F = k_F'^{\mu\nu\alpha\beta} g'_{\mu\alpha} g'_{\nu\beta}$ and $k_F'^{\nu\beta} = k_F'^{\mu\nu\alpha\beta} g'_{\mu\alpha}$, it is not difficult to check that, finally, $K(g'_{\mu\nu}, k'_F) = 0$. This means that the new term with $K(g_{\mu\nu}, k_F)$ gives no contribution to the dynamics of the conformal factor in the theory (52).

The negative result concerning the effect of the new terms on the behavior of the conformal factor of the metric does not mean that there cannot be other relevant effects. In particular, one can expect the modifications of equations for cosmic perturbations [37] and especially for the gravitational waves. An important result concerning the dynamics of traceless and transverse perturbations of the metric in the theory (52) without the term $K(g_{\mu\nu}, k_F)$ is that there are no growing modes in this theory [17,33,34]. This fact has important phenomenological consequences, including the relatively small role of tensor perturbations compared to the scalar one (see, e.g., [38]). It would be interesting to check whether the situation remains the same or gets changed in the theory by Lorentz and $CPT$ breaking terms $K(g_{\mu\nu}, k_F)$.

## V. CONCLUSIONS

Quantum effects and, in particular, renormalization, represent an essential part of the development of the theories with Lorentz and $CPT$ breaking. In the first papers [39,40] the calculations have been performed by means of Feynman diagrams. Later on, the functional methods, such as Schwinger-DeWitt and heat-kernel techniques, have been used in [23]. In this paper the renormalization has been carried out in curved space-time and some general features of the renormalization were established. However, the calculations were not complete, because only the dimensional symmetry-violating parameters were considered. In the present paper we go beyond the framework of Ref. [23] and derive, for the first time, the contribution of the dimensionless parameter $k_F^{\mu\nu\alpha\beta}$ in the photon sector to the renormalization of the vacuum.

The performed calculations are new in the sense that we had to work out the new type of nonminimal operator (6), which is different from the standard ones which were considered before [21]. In these standard cases the nonminimality was caused by the choice of gauge-fixing parameters. The corresponding operator can be always studied by integrating over such parameters starting from the special minimal operator case. In the case of the nonminimal operator (6) the nonminimality is caused by the presence of an external dimensionless *function* and this makes a direct application of the methods of [21] impossible. The problem has been solved by a trick of inverting the minimal operator and by working in the first order in the symmetry-violating function $k_F^{\mu\nu\alpha\beta}$. As a result of this procedure one can start using the functional traces of [21] and finally arrive at the first-order counterterms. The obtained expression, Eq. (25), represents only a part of an infinite expansion, according to a general analysis given in [19]. The result also passed a technically complicated test related to the local conformal invariance.

The derivation of anomaly and anomaly-induced effective actions did not meet serious obstacles, and finally the expression (51) was obtained. It turns out that the dimensionless parameter $k_F^{\mu\nu\alpha\beta}$ makes no contribution to the dynamics of the conformal factor of the metric. At the same time, depending on the choice of this parameter, one can expect relevant contributions and maybe even the growth of the tensor modes of metric perturbations during the inflationary epoch. The study of this potentially interesting problem will require significant efforts, but finally it can lead to some constraints on the parameter $k_F^{\mu\nu\alpha\beta}$.

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