Rotating regular black hole solution

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Based on the Newman-Janis algorithm, the Ayón-Beato-García spacetime metric [Phys. Rev. Lett. 80, 5056 (1998)] of the regular spherically symmetric, static, and charged black hole has been converted into rotational form. It is shown that the derived solution for rotating a regular black hole is regular and the critical value of the electric charge for which two horizons merge into one sufficiently decreases in the presence of the nonvanishing rotation parameter a of the black hole.

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I. INTRODUCTION

It is well known that exact solutions of the Einstein equations have one of the "mysterious" properties of the black hole that is called singularity. Singularity has been considered one of the defects of the general relativity because the explanation of singularity cannot be made by the general relativity itself. So-called regular black hole solutions [1–3] can be created in order to eliminate singularity from the spacetime metric.

We know that there are three types of regular black hole solutions: (i) solutions that are continuous throughout spacetime; (ii) solutions with two simple regions, solutions that have boundary surfaces joining the two regions; and (iii) solutions with two separated regions, the solutions that have a surface layer, thin shell, joining the two regions. The Ayón-Beato-García regular black hole solution [1,2] belongs to the first type of the regular black holes.

There are two kinds of singularity: the coordinate singularity (event horizon) and the curvature singularity. We know that at the singularity the curvature of the manifold is becoming infinite. In the case of coordinate singularity, the g_{rr} component of the metric tensor goes to infinity. One can eliminate coordinate singularity by making transformations to the more fortunate coordinate system. Usually, by changing coordinates from the Boyer-Lindquist coordinates to the Eddington-Finkelstein ones, one can remove coordinate singularity from the spacetime metric. Eddington-Finkelstein coordinates are based on the freely falling photons. On the other hand, in the curvature singularity, the Riemann tensor components of the spacetime metric diverge. It is impossible to eliminate curvature singularity from the spacetime metric by coordinate transformations.

In the papers [1–3], the new regular black hole solutions of the Einstein equations have been found by taking into account the coupling to the nonlinear electrodynamic field. Afterwards, this solution has been called the Ayón-Beato-García regular black hole solution. Recently, another regular black hole solution [4] has been considered by introducing a new mass function generalizing the commonly used Bardeen and Hayward mass functions and including the cosmological constant.

The Kerr spacetime metric can be derived from the Schwarzschild one by using the Newman-Janis algorithm [5,6]. The derivation of the Kerr spacetime metric from the Schwarzschild one has been given in several works [5,7] and [8]. Moreover, in the papers [5,8] and [9], the Kerr-Newman solution has been derived from the Reissner-Nordström spacetime metric. The regular Kerr and Kerr-Newman black holes as well as higher dimensional Kerr and Kerr-Newman ones have been derived by Newman-Janis algorithm and their stress-energy tensors and thermodynamics have been studied in the paper [10]. The Newman-Janis algorithm has been used to derive the radiating Kerr-Newman black hole in $f(R)$ gravity [11]. The exact nonstatic charged BTZ-like solutions, in $(\mathrm{N}+1)$-dimensional Einstein gravity, have been found in [12] in the presence of the negative cosmological constant. The Lovelock gravity in the critical spacetime dimension has been studied in Ref. [13].

In order to convert the static, spherically symmetric black hole spacetime metric into a rotational one [if this spacetime metric is given in the Boyer-Lindquist coordinates (t, r, θ, ϕ)] one has to proceed with the following five steps of the Newman-Janis algorithm: (i) a transition from the Boyer-Lindquist coordinates into the advanced Eddington-Filkenstein ones (u, r, θ, ϕ) has to be performed; (ii) a null tetrad ($\mathbf{l}$, $\mathbf{n}$, $\mathbf{m}$, and $\bar{\mathbf{m}}$) (Newman-Penrose tetrad) for a produced metric have to be found; (iii) a complex coordinate transformations has to be applied; (iv) reverse coordinate transformations into the Boyer-Lindquist ones have to be done; and, (v) finally,

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unknown terms of the transformations have to be found based on the reality condition.

Here, we convert the static, spherically symmetric Ayón-Beato-García regular black hole spacetime [1–3] into the rotational one by using the Newman-Janis algorithm [5,6] and by studying some of its basic properties.

II. NEWMAN-JANIS ALGORITHM TO GET A ROTATING REGULAR SOLUTION

In this section, we describe the Newman-Janis algorithm that is used for converting the spherically symmetric static black hole spacetime metric into a rotational one. The Ayón-Beato-García spacetime metric of the regular spherically symmetric black hole is given as [14]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \quad (1)$$

where the lapse function $f(r)$ reads

$$f(r) = 1 - \frac{2Mr^2}{(r^2+Q^2)^{3/2}} + \frac{Q^2 r^2}{(r^2+Q^2)^2}, \quad (2)$$

and M and Q are the total mass and electric charge of the black hole, respectively. The spacetime metric (1) is the solution of the field equations within general relativity, where the nonlinear electrodynamic field satisfying the weak energy condition is considered as a source. As can be seen from the lapse function (2), the spacetime metric (1) has only the coordinate singularity. This is why in order to remove this singularity one has to write the spacetime metric (1) in the advanced Eddington-Finkelstein coordinates. To do this, we make the following transformation for the incoming photon (or ray):

$$v = t - r^*, \quad (3)$$

and for the outgoing photon (or ray),

$$u = t + r^*, \quad (4)$$

where

$$r^* = \int \frac{dr}{f(r)}. \quad (5)$$

Hereafter, we consider only the outgoing photon (4) case. Then the spacetime metric (1) in the advanced Eddington-Finkelstein coordinates takes the form

$$ds^2 = -f(r)du^2 - 2dudr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \quad (6)$$

The Newman-Penrose tetrad consists of four isotropic vectors, $\mathbf{l}$, $\mathbf{n}$, $\mathbf{m}$, and $\bar{\mathbf{m}}$. $\mathbf{l}$ and $\mathbf{n}$ are real vectors, and $\mathbf{m}$ and $\bar{\mathbf{m}}$ are mutually complex conjugate vectors [8].

Newman-Penrose tetrads satisfy the orthogonality condition:

$$l^\mu \cdot m_\mu = l^\mu \cdot \bar{m}_\mu = n^\mu \cdot m_\mu = n^\mu \cdot \bar{m}_\mu = 0, \quad (7)$$

and also the isotropic condition:

$$l^\mu \cdot l_\mu = n^\mu \cdot n_\mu = m^\mu \cdot m_\mu = \bar{m}^\mu \cdot \bar{m}_\mu = 0. \quad (8)$$

Moreover, the basis vectors usually impose the following normalization condition:

$$l^\mu \cdot n_\mu = 1, m^\mu \cdot \bar{m}_\mu = -1, \quad (9)$$

where $\bar{m}^\mu$ is the complex conjugate of m^μ.

The contravariant components of the metric tensor of the spacetime metric (6) are

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & f(r) & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}. \quad (10)$$

We can rewrite (10) with the help of the Newman-Penrose tetrad as

$$g^{\mu\nu} = -l^\mu \cdot n^\nu - l^\nu \cdot n^\mu + m^\mu \cdot \bar{m}^\nu + m^\nu \cdot \bar{m}^\mu, \quad (11)$$

where the components of the null tetrad vectors are

$$l^\mu = [0,1,0,0], \qquad n^\mu = \left[1, -\frac{1}{2}f(r), 0, 0\right],$$
$$m^\mu = \frac{1}{\sqrt{2}r}\left[0,0,1,\frac{i}{\sin\theta}\right],$$
$$\bar{m}^\mu = \frac{1}{\sqrt{2}r}\left[0,0,1,-\frac{i}{\sin\theta}\right]. \quad (12)$$

As the next step, we make the following complex coordinate transformations:

$$\tilde{r} = r + ia\cos\theta, \qquad \tilde{u} = u - ia\cos\theta,$$
$$\tilde{\theta} = \theta, \qquad \tilde{\phi} = \phi. \quad (13)$$

As a result of these transformations, the components of the null tetrad vectors take the form [7]

$$\tilde{l}^\mu = [0,1,0,0], \qquad \tilde{n}^\mu = \left[1, -\frac{1}{2}\tilde{f}(r), 0, 0\right],$$
$$\tilde{m}^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left[ia\sin\theta, -ia\sin\theta, 1, \frac{i}{\sin\theta}\right],$$
$$\tilde{\bar{m}}^\mu = \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left[-ia\sin\theta, ia\sin\theta, 1, -\frac{i}{\sin\theta}\right], \quad (14)$$

where the function

$$\tilde{f}(r) = 1 - \frac{2Mr\sqrt{\Sigma}}{(\Sigma+Q^2)^{3/2}} + \frac{Q^2\Sigma}{(\Sigma+Q^2)^2} \quad (15)$$

is the new form of the lapse function (13) and $\Sigma = r^2 + a^2\cos^2\theta$.

Then the metric tensor $g^{\mu\nu}$ takes new $\tilde{g}^{\mu\nu}$ form,

$$\tilde{g}^{\mu\nu} = -\tilde{l}^\mu \cdot \tilde{n}^\nu - \tilde{l}^\nu \cdot \tilde{n}^\mu + \tilde{m}^\mu \cdot \bar{\tilde{m}}^\nu + \tilde{m}^\nu \cdot \bar{\tilde{m}}^\mu, \tag{16}$$

or

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} \frac{a^2\sin^2\theta}{\Sigma} & -1-\frac{a^2\sin^2\theta}{\Sigma} & 0 & \frac{a}{\Sigma} \\ -1-\frac{a^2\sin^2\theta}{\Sigma} & \tilde{f}(r)+\frac{a^2\sin^2\theta}{\Sigma} & 0 & -\frac{a}{\Sigma} \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ \frac{a}{\Sigma} & -\frac{a}{\Sigma} & 0 & \frac{1}{\Sigma\sin^2\theta} \end{pmatrix}. \tag{17}$$

The covariant components of the metric tensor (17) are

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -\tilde{f}(r) & -1 & 0 & a(\tilde{f}(r)-1)\sin^2\theta \\ -1 & 0 & 0 & a\sin^2\theta \\ 0 & 0 & \Sigma & 0 \\ a(\tilde{f}(r)-1)\sin^2\theta & a\sin^2\theta & 0 & \sin^2\theta[\Sigma - a^2(\tilde{f}(r)-2)\sin^2\theta] \end{pmatrix}, \tag{18}$$

and the spacetime element can be written as

$$d\tilde{s}^2 = g_{uu}du^2 + 2g_{ur}dudr + 2g_{u\phi}dud\phi + 2g_{r\phi}drd\phi + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{19}$$

By using the transformations

$$du = dt + \lambda(r)dr, \qquad d\phi = d\phi + \chi(r)dr, \tag{20}$$

we will turn back into the Boyer-Lindquist coordinates, where functions $\lambda(r)$ and $\chi(r)$ are chosen for eliminating the nondiagonal g_{tr} and $g_{r\phi}$ terms. By putting (20) into (19) and collecting the terms that correspond to the g_{tr} and $g_{r\phi}$ ones, then equalizing the produced expression to zero, one can get two equations for two unknown functions, $\lambda(r)$, $\chi(r)$. By solving these equations simultaneously, one can find expressions for $\lambda(r)$ and $\chi(r)$ in the following way:

$$\lambda(r) = -\frac{\Sigma + a^2\sin^2\theta}{\Sigma\tilde{f}(r) + a^2\sin^2\theta}, \quad \chi(r) = -\frac{a}{\Sigma\tilde{f}(r) + a^2\sin^2\theta}. \tag{21}$$

Finally, the spacetime metric can be expressed in the Boyer-Lindquist coordinates as

$$d\tilde{s}^2 = -\tilde{f}(r)dt^2 + \frac{\Sigma}{\Sigma\tilde{f}(r) + a^2\sin^2\theta}dr^2 - 2a\sin^2\theta(1-\tilde{f}(r))d\phi dt + \Sigma d\theta^2 + \sin^2\theta[\Sigma - a^2(\tilde{f}(r)-2)\sin^2\theta]d\phi^2, \tag{22}$$

where $\tilde{f}(r)$ is given by Eq. (15).

If we consider the black hole as the noncharged one ($Q = 0$), the lapse function (2) takes the same form as one of the Schwarzschild spacetime metrics and the new spacetime metric (22) converts into the Kerr one, namely,

$$ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 - 2\frac{2Mr}{\Sigma}a\sin^2\theta d\phi dt + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \tag{23}$$

where $\Delta = r^2 + a^2 - 2Mr$, $\sum = r^2 + a^2\cos^2\theta$.

In order to investigate properties of the spacetime metric (22), we study here the behavior of the g_{rr} and g_{tt} components of the spacetime metric (22).

III PROPERTIES OF ROTATING REGULAR BLACK HOLE SOLUTION

Now we will analyze the static limit and event horizon defined by the conditions $g_{tt} = 0$ and $1/g_{rr} = 0$, respectively.

The obtained new spacetime metric (22) is also regular everywhere. From Figs. 1 and 2, one can easily see that for

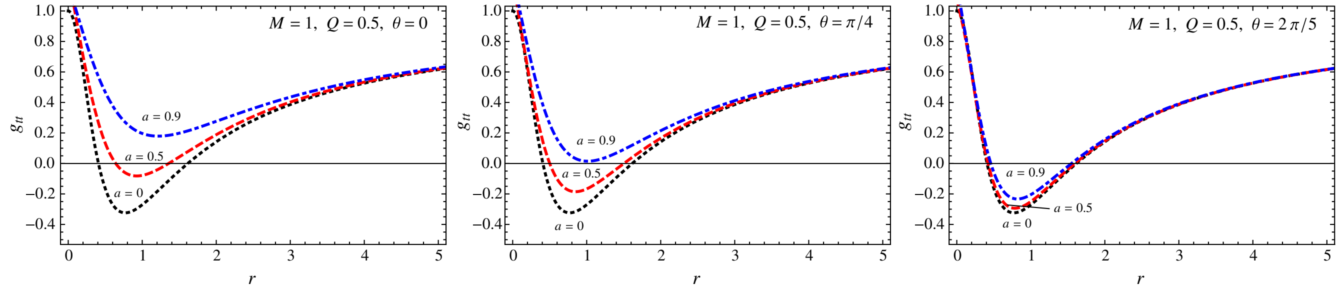

FIG. 1 (color online). Dependence of the g_{tt} component of the metric tensor from the radial coordinate r for the typical values of the rotation parameter a.

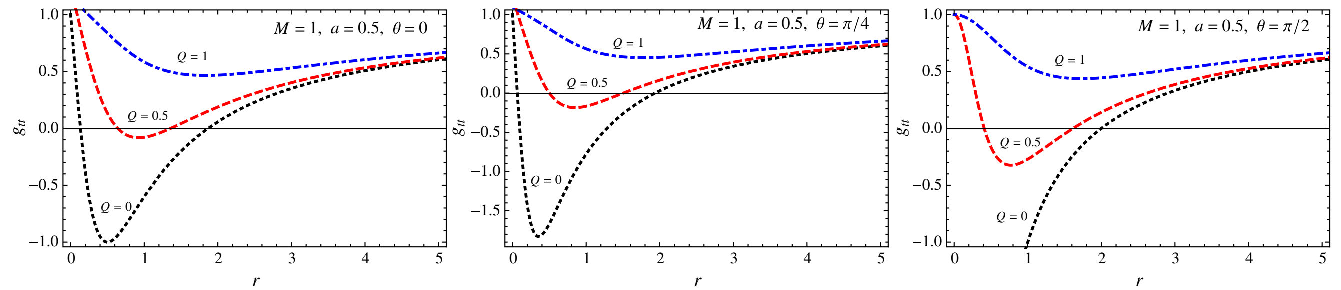

FIG. 2 (color online). Dependence of the g_{tt} component of the metric tensor from the radial coordinate r for the typical values of the electric charge Q.

some set of values of the rotation parameter a and electric charge Q, the solution (22) has coordinate singularity (event horizon).

The radial dependence of the function $\tilde{f}(r)$ presented in Figs. 1 and 2 shows that, with the increase of the value of the rotation parameter a and charge Q, the possibility of the existence of the horizon decreases. In the equatorial plane $(\theta = \pi/2)$ the dependence of the function $\tilde{f}(r)$ on the rotation parameter a vanishes and the existence of the horizon depends only on the value of the charge Q.

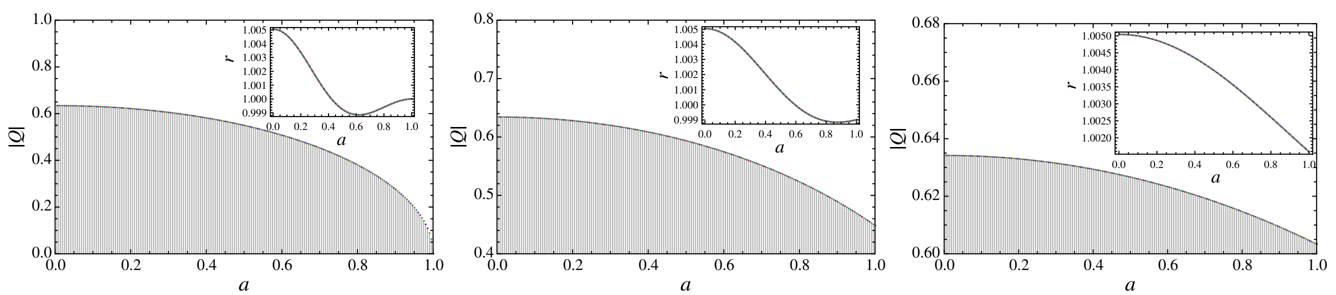

FIG. 3 (color online). Dependence of the critical value of the electric charge Q and radius of the horizon r from the rotation parameter a for the different values of θ: $\theta = 0$, $\theta = \pi/4$, and $\theta = 2\pi/5$ (from left to right, respectively).

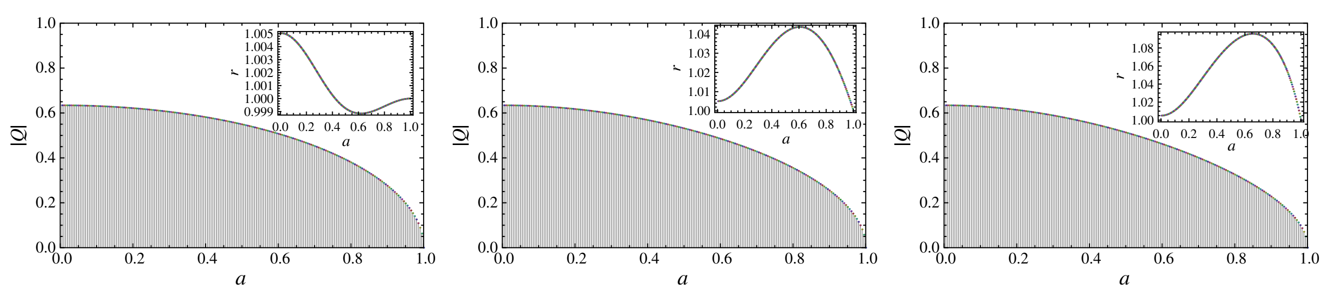

FIG. 4 (color online). Dependence of the critical value of the electric charge Q and radius of the event horizon r from the rotation parameter a for the different values of θ: $\theta = 0$, $\theta = \pi/4$, and $\theta = \pi/2$ (from left to right, respectively).

There is a critical value of the charge Q for which two surfaces described by the solutions of the condition $g_{tt} = 0$ merge into one. In order to find the critical value of Q, the lapse function $g_{tt}(r, a, \theta, Q)$ must satisfy a couple of conditions:

$$g_{tt}(r, a, \theta, Q) = 0, \qquad \partial_r g_{tt}(r, a, \theta, Q) = 0. \quad (24)$$

Since the lapse function g_{tt} is the function of four quantities, r, a, θ, and Q, by solving Eqs. (24) with respect to r and Q one can get the solution as a function of a and θ. In Fig. 3 dependence of the critical value of the electric charge Q and radius of the static limit surface r that is corresponding to the critical state on the rotation parameter a has been shown for several values of θ. The shaded region in the $Q-a$ plot corresponds to the regular black hole with the static limit. The unshaded region in the $Q-a$ plot corresponds to the regular black hole without the static limit. The $r-a$ plot represents the dependence of the

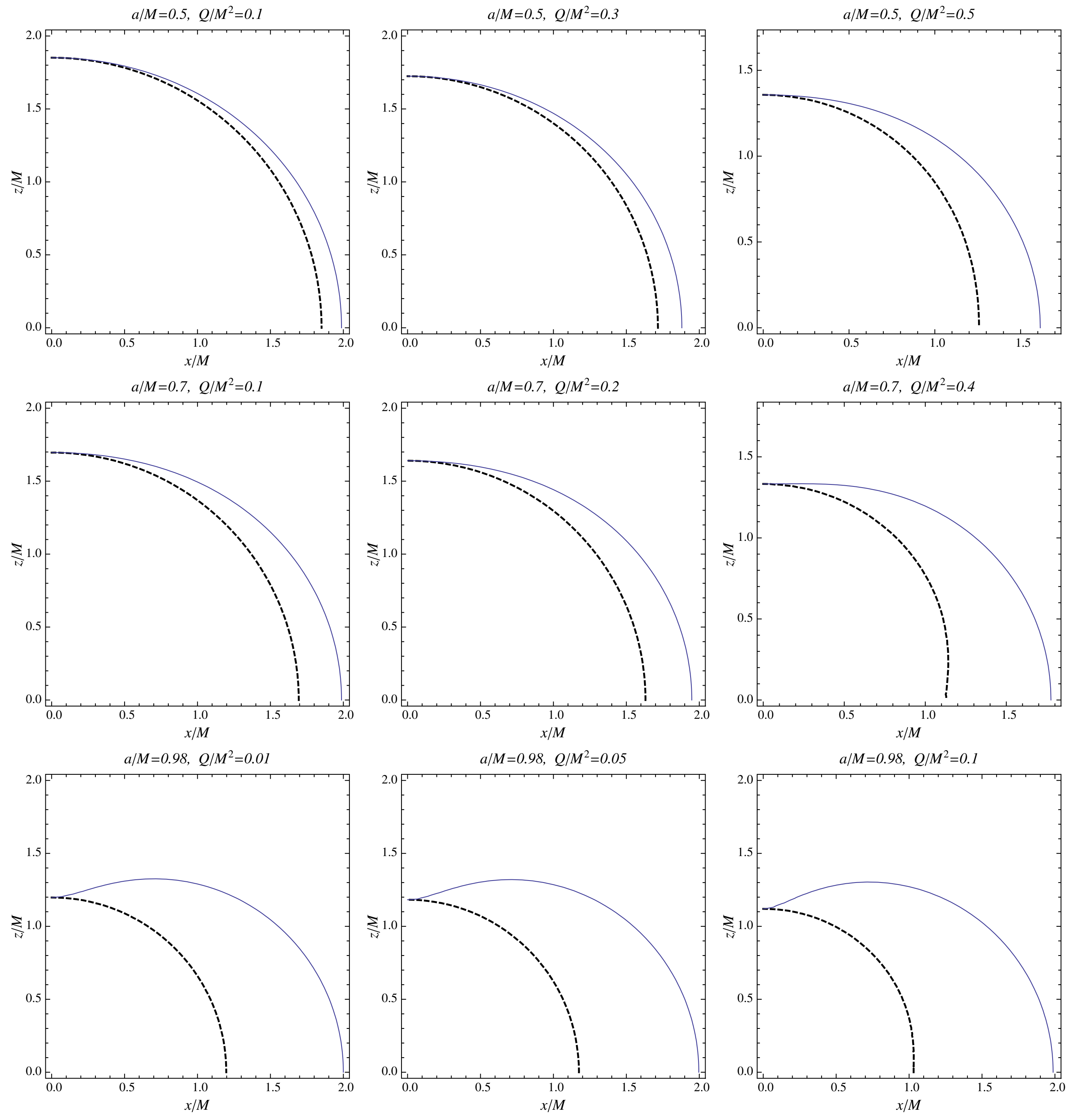

FIG. 5 (color online). Shape and size of the ergosphere for the different values of the rotation parameter a and electric charge Q.

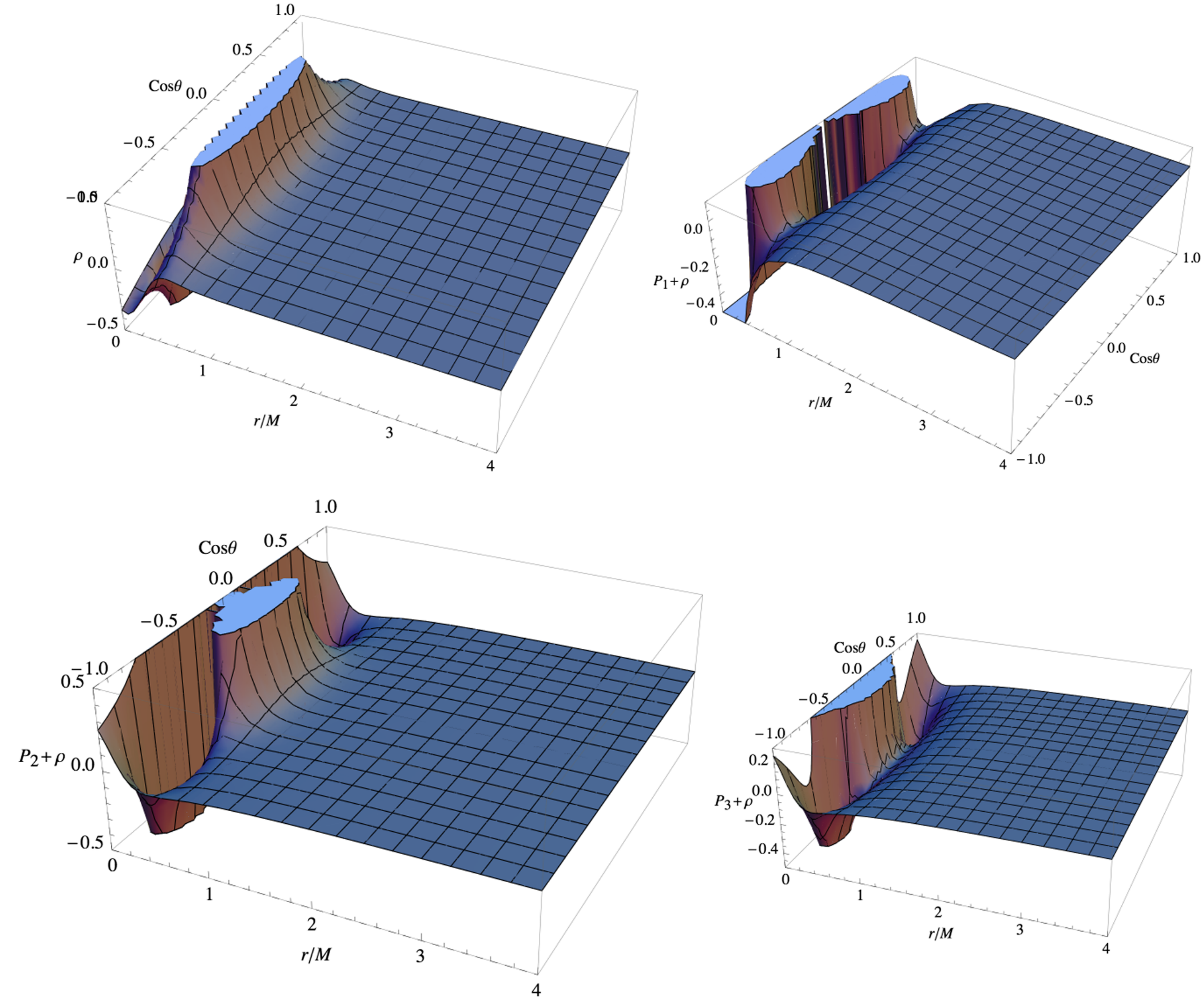

FIG. 6 (color online). Radial and angular dependence of ρ, $P_1+\rho$, $P_2+\rho$, and $P_3+\rho$ for the given value of the rotation parameter $a/M=0.5$ and electric charge $Q/M=0.9$.

radius of the static limit on the rotation parameter that corresponds to the border of shaded and unshaded regions. In order to find the same critical value of the charge, one may use the conditions

$$1/g_{rr}(r,a,\theta,Q)=0, \qquad \partial_r[1/g_{rr}(r,a,\theta,Q)]=0. \quad (25)$$

In Fig. 4 dependence of the critical value of the electric charge Q and radius of the static limit surface r that is corresponding to the critical state on the rotation parameter a has been shown for several values of θ. The shaded region in the $Q-a$ plot corresponds to the regular black hole with two (outer and inner) event horizons. The unshaded region in the $Q-a$ plot corresponds to the regular black hole without an event horizon. The $r-a$ plot represents the dependence of the radius of the static limit on the rotation parameter that corresponds to the border of shaded and unshaded regions. The regular black hole with the rotation parameter $a\sim 1$ can have the horizon in the poles of the black hole ($\theta=0,\ \pi$) even in the case when the value of the charge Q is very small. One may conclude that in the presence of the rotation parameter the small value of the electric charge may cause the elimination of the singularity.

The critical values of the electric charge $Q_{\rm cr}$ are different for the static limit and the event horizon. The critical value of the electric charge for the event horizon is more rapidly decreasing with the increase of the rotation parameter a compared to that of the static limit. This means that the event horizon disappears earlier with the increase of the electric charge for the fixed value of the rotation parameter a.

One can see from Figs. 3 and 4 that when the value of the electric charge $Q\le 0.633$ the static black hole ($a=0$) has a horizon. If $Q\le 0.605$, even an extreme black hole can have the horizon in the equatorial plane ($\theta=\pi/2$).

Figure 5 provides the shape and size of the ergoregion in the $x-z$ plane where $z = r\cos\theta$ and $z = r\sin\theta$. With the increase of the electric charge, one can observe the increase of the relative shape and size of the ergosphere. Note that for the values of an electric charge with $Q \geq Q_{\rm cr}$, the event horizon and static limit both disappear.

As the further step, we study the question of satisfying the weak energy condition and choose the locally nonrotating frame (LNRF) in order to get the stress-energy tensor in diagonal form, namely, $T^{\alpha\beta} = (\rho, P_1, P_2, P_3)$. Then the weak energy condition reads as [6] $\rho \geq 0$, $\rho + P_i \geq 0$, where $i = 1, 2, 3$.

Finally, we express the spacetime geometry in the frame of the LNRF and study the behavior of the angular velocity of these frames. The orthonormal tetrad of the LNRF has the following form:

$$\omega^t = |g_{tt} - g_{\phi\phi}\Omega_{\rm LNRF}^2|^{1/2}dt, \tag{26}$$

$$\omega^r = |g_{rr}|^{1/2}dr, \tag{27}$$

$$\omega^\theta = |g_{\theta\theta}|^{1/2}d\theta, \tag{28}$$

$$\omega^\phi = |g_{\phi\phi}|^{1/2}d\phi - |g_{\phi\phi}|^{1/2}\Omega_{\rm LNRF}dt, \tag{29}$$

where

$$\Omega_{\rm LNRF} = \frac{a(1-\tilde{f}(r,\theta))}{\Sigma - a^2(\tilde{f}(r,\theta)-2)\sin^2\theta} \tag{30}$$

is the angular velocity of the LNRF frame.

In Fig. 6 the three-dimensional plot of the radial and angular dependence of density and pressures is shown for the given value of the rotation parameter $a/M = 0.5$ and electric charge $Q/M = 0.9$. One can easily see that the weak energy condition is violated near the nonsingular origin of the rotating regular black hole.

IV CONCLUSION

In this paper, we used a regular black hole solution with the source with the nature of nonlinear electrodynamics obtained by Ayón-Beato and García [1–3] to generate a rotating regular black hole solution that includes the Ayón-Beato-García and Kerr metrics as special cases. The considered Newman-Janis algorithm uses a static solution to generate rotating solutions without touching the field equation and is very useful in order to get rotating black hole solutions.

The relation between the Einstein vacuum solution and any nonvacuum solution of general relativity opens new directions in studying the properties of the new solution with a nonlinear electrodynamic source. Obviously, when the electric charge is vanishing, the solution reduces to the vacuum one. Here, we have obtained an exact rotating regular black hole (BH) solution in the framework of general relativity. The obtained solution gives an opportunity to study the geometrical and causal structures, as well as to test particle motion around the rotating regular BH, which will be the subject of future projects. On the other hand, it could also be very interesting to compare the rotating Ayon-Beato-Garcia regular spacetimes without horizons to the Kerr naked singularity spacetimes, testing if the interesting and unusual physical phenomena occurring in the Kerr naked singularity spacetimes [15] could arise in the regular rotating spacetimes too.

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