Two dimensional hydrodynamics with gauge and gravitational anomalies

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We present a new approach to discuss two-dimensional chiral and nonchiral hydrodynamics with gauge and gravitational anomalies. Exact constitutive relations for the stress tensor and charge current are obtained. For the chiral theory, the constitutive relations may be put in the ideal (*chiral*) fluid form, whereas the constitutive relations corresponding to nonchiral case do not take the ideal fluid form. The constitutive relations in the presence of both gravity and gauge sectors are new. These expressions, in the absence of the gauge sector, reproduce the results obtained in the gradient expansion approach.

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I. INTRODUCTION

Hydrodynamics is basically a finite temperature quantum field theory in the limit of large time and length scales [1]. Certain conservation laws govern it that are manifested by the global symmetries of the underlying theory. For instance, the conservation of the energy-momentum tensor invokes the spacetime symmetry, while the conservation of charge current is yielded by the charge symmetry. The energy-momentum tensor and the U(1) current have to satisfy the conservation equations. The constitutive equations give an explicit form for the one-point functions of the energy-momentum tensor and the U(1) current in terms of the hydrodynamic variables, e.g., velocity, temperature, chemical potential, etc. These constitutive equations are traditionally required to make the local production of entropy positive, in accordance with a local version of the second law of thermodynamics.

Recently, investigations on fluid dynamics in the presence of anomalies have received considerable attention [2–13]. This is important because one expects that the standard picture might nontrivially be modified in the presence of the quantum anomalies. In this paper we analyze the structure of constitutive relations in the presence of both gauge and gravitational anomalies. The latter includes a diffeomorphism anomaly (violation of general coordinate invariance) as well as a trace anomaly (violation of conformal invariance). The present analysis is confined to $(1+1)$ dimensions. Incidentally two dimensions have certain peculiarities which make it feasible to abstract various results that may not be possible in higher dimensions. It is well known that the two-dimensional metric, in general, can be expressed in a conformally flat form. Therefore, the effective action for a field under this background is exactly obtainable, and one does not need the derivative expansion method. This ensures the transparency of the physics of the problem at each stage of the computation.

In this paper, we discuss both the nonchiral and chiral theories. At the classical level, there are no anomalies. For a quantum theory, a one-loop computation leads to anomalies. In the case of a nonchiral theory (i.e., where ordinary fermions are coupled to the gauge and/or gravity sector), the anomaly is a manifestation of the clash between general coordinate invariance and conformal invariance. For a gravitational theory, usually the former is retained at the expense of the latter. Hence, one has to appropriately account for the conformal (trace) anomaly. A chiral theory, on the other hand, has both diffeomorphism and conformal anomalies, since it is not possible to preserve any of the symmetries at the quantum level.

We develop a new method to discuss anomalous hydrodynamics in two dimensions that includes both gauge and gravity sectors. We start from the anomalous energy-momentum tensor and current obtained from the corresponding effective action, and then the components are evaluated explicitly for a general static metric in null coordinates. Solutions of the anomalous expressions involve integration constants. These constants crop up in the various expressions for the constitutive relations, which are found by introducing the appropriate fluid variables, for the stress tensor and the gauge current. The constitutive relations found here are exact and do not require any gradient expansion as happens in higher dimensions. These relations are new and have not been posited earlier in the literature [5–12]. Moreover, for the chiral theory, the constitutive relations may be put in a form that resembles the structure for an *ideal chiral* fluid. The constitutive relation for an ideal chiral fluid, it may be recalled, has a form similar to the usual ideal fluid [1] but with the velocity vector replaced by the chiral velocity vector [13]. It is

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reassuring to note that our results agree with those found by the derivative expansion approach when the gauge sector is absent [5]. Incidentally, in the presence of this sector, results are not available in the gradient expansion method.

The organization of our paper is as follows. A general setup for our main purpose is presented in the next section. In Sec. III, we provide a brief discussion, some of which may not be that well known, on two-dimensional gauge and gravitational anomalies. Then we find the constitutive relations for the nonchiral case in Sec. IV. The chiral case is analyzed in the next section. Section VI is devoted to a comparison with the gradient expansion approach. The constitutive relations obtained by us contained an arbitrary constant. For a particular value of this constant, our expressions (in the absence of a gauge field) reproduce those found by the derivative expansion method. Inclusion of gauge fields has not been considered in full in the derivative expansion method; hence, a comparison with our result is not feasible. Finally, we conclude in Sec. VII. We also added an Appendix.

II. METRIC AND GENERAL SETUP

Here we shall consider a $(1+1)$ general static spacetime. Also the explicit components of the comoving velocity vector will be calculated under this background, which is necessary for our main purpose. All the required expressions will sometimes be expressed in null coordinates to make them more transparent.

The static metric in $(1+1)$ dimensions can be taken in the following form:

$$ds^2 = -e^{2\sigma(r)}dt^2 + g_{11}dr^2. \tag{1}$$

It has a timelike Killing vector, and the Killing horizon is given by the solution of the equation $e^{2\sigma(r)}|_{r_0} = 0$. The $U(1)$ gauge fields A_a are of the linear form

$$A_a = (A_t(r), 0). \tag{2}$$

The null coordinates (u, v) are defined as

$$u = t - r_*; \qquad v = t + r_*, \tag{3}$$

where the tortoise coordinate r_* is given by $dr_* = -\sqrt{g_{11}}e^{-\sigma}dr$. In these coordinates, the metric takes the following off-diagonal form:

$$ds^2 = -\frac{e^{2\sigma}}{2}(dudv + dvdu). \tag{4}$$

To express the anomalous energy-momentum tensor in terms of fluid variables, we adopt the comoving frame. In this frame, the velocity vector of a fluid u^a is normalized as

$$u^a u_a = -1. \tag{5}$$

Note that the norm of the velocity has to be negative since we are considering timelike trajectories. The absolute normalization is fixed to unity by choosing the comoving frame. Subjected to the above condition and the metric (1), the usual ansatz for the velocity follows,

$$u^a = e^{-\sigma}(1,0); \qquad u_a = -e^{\sigma}(1,0), \tag{6}$$

with $a = t, r$. Correspondingly, in null coordinates (u, v), this transforms to the following form:

$$u^a = e^{-\sigma}(1,1); \qquad u_a = -\frac{e^{\sigma}}{2}(1,1). \tag{7}$$

With the above expressions for the velocity vector, the following combinations of fluid variables can be explicitly expressed in terms of the metric coefficients,

$$\begin{aligned} u^a\nabla^b\nabla_a u_b &= \frac{1}{2g_{11}^2}(2g_{11}\sigma'' + 2g_{11}\sigma'^2 - \sigma' g'_{11}); \\ u^a\nabla^b\nabla_b u_a &= \frac{\sigma'^2}{g_{11}}, \end{aligned} \tag{8}$$

which are necessary to find the constitutive relations.

To discuss the $(1+1)$-dimensional chiral theory, in which both the diffeomorphism and trace anomalies appear, we define chiral velocity as

$$u^{(c)}{}_a = u_a - \tilde{u}_a, \tag{9}$$

where $\tilde{u}_a = \bar{\epsilon}_{ab}u^b$ is the dual to u_a and $\bar{\epsilon}_{ab}$ is an antisymmetric tensor with $\bar{\epsilon}_{ab} = \sqrt{-g}\epsilon_{ab}$ and $\bar{\epsilon}^{ab} = \epsilon^{ab}/\sqrt{-g}$. In null coordinates the components are given by

$$\epsilon_{uv} = 1; \quad \epsilon^{uv} = -1; \quad u^{(c)}{}_a = -e^{\sigma}(1,0). \tag{10}$$

Note that the definition (9) of the chiral velocity $u^{(c)}{}_a$ ensures that it satisfies the familiar chiral property in two dimensions:

$$u^{(c)}{}_a = -\bar{\epsilon}_{ab}u^{(c)b}. \tag{11}$$

III. REVIEW OF GAUGE AND GRAVITATIONAL ANOMALIES

An anomaly is a breakdown of some classical symmetry upon quantization. It may have different manifestations, but generally speaking these are connected. A violation of gauge symmetry is revealed by a nonconservation of the gauge current (gauge anomaly) or, alternatively, by the presence of anomalous terms in the algebra of currents. These anomalous terms are related to the gauge anomaly. Likewise, a violation of diffeomorphism symmetry leads to the nonconservation of the stress tensor.

Of particular significance are theories in which chiral symmetries are gauged. The equations of motion show that the chiral currents are covariantly conserved. However, quantum effects destroy this feature as may be checked by doing a one-loop computation. Algebraically, this may be expressed as

$$\langle D_\mu J^\mu \rangle^a = \partial_\mu \langle J^{\mu a} \rangle - f^{abc} A^b_\mu \langle J^{\mu c} \rangle = G^a, \tag{12}$$

where the average is interpreted to be taken over the fermionic degrees of freedom appearing in the chiral current J^a_μ and the other symbols have their usual meaning. G^a is called the anomaly.

There are different ways of defining the averaged current corresponding to different regularization choices. Among the various possibilities, two are outstanding. The first and perhaps the more common way is to interpret the averaged current as the functional derivative of an effective action W,

$$\langle J^a_\mu(x) \rangle = \frac{\delta W}{\delta A^{a\mu}(x)}. \tag{13}$$

This current satisfies the integrability condition,

$$\frac{\delta J^{\mu a}(x)}{\delta A^b_\nu(x')} = \frac{\delta J^{\nu b}(x')}{\delta A^a_\mu(x)}. \tag{14}$$

A direct consequence of this relation is that the anomaly of this current satisfies the Wess–Zumino consistency condition. To obtain this condition, let us introduce the operator

$$L^a(x) = \partial_\mu \frac{\delta}{\delta A^a_\mu(x)} - f^{abc} A^b_\mu(x) \frac{\delta}{\delta A^c_\mu(x)}, \tag{15}$$

which is basically the generator of the infinitesimal gauge transformation. From Eqs. (12), (13), and (15), it follows that

$$L^a(x) W = G^a(x) \tag{16}$$

so that the existence of the anomaly is a statement about the lack of gauge invariance of the one-loop effective action. The generators satisfy the closed algebra,

$$[L^a(x), L^b(y)] = f^{abc} \delta(x-y) L^c(x). \tag{17}$$

Acting on W and using Eq. (16) immediately yields

$$L^a(x) G^b(y) - L^b(y) G^a(x) = f^{abc} \delta(x-y) G^c(x). \tag{18}$$

This is the Wess–Zumino condition. It is now clear that the anomaly of the current (13) defined through the effective action must satisfy this consistency condition and is hence called the consistent anomaly.

The other way of defining the averaged current is to regularize it by a gauge covariant method. In that case the current transforms covariantly under the gauge transformation so that

$$L^a(x) J^b_\mu(y) = f^{abc} \delta(x-y) J^c_\mu. \tag{19}$$

It is then simple to show, by taking the covariant divergence of both sides of this equation, that the corresponding anomaly G^a also transforms covariantly,

$$L^a(x) G^b_{\rm cov}(y) = f^{abc} \delta(x-y) G^c_{\rm cov}. \tag{20}$$

This anomaly is called the covariant anomaly. It is easy to see, by an appropriate change of indices, that this anomaly satisfies the condition

$$L^a(x) G^b_{\rm cov}(y) - L^b(y) G^a_{\rm cov}(x) = 2 f^{abc} \delta(x-y) G^c_{\rm cov}(x). \tag{21}$$

Comparison with Eq. (18) immediately shows that the covariant anomaly is incompatible with the Wess–Zumino condition; it is off by a factor of 2. This analysis illustrates the difference between covariant and consistent expressions. While the covariant anomaly transforms covariantly under a gauge transformation but does not satisfy the Wess–Zumino condition, the behavior of the consistent anomaly is just the reverse. It satisfies the Wess–Zumino condition but does not transform covariantly. Since currents and/or stress tensors are only defined modulo local counterterms manifesting the regularization ambiguities, covariant and consistent expressions are also related by such counterterms. These local polynomials were obtained by using either differential geometric methods [14] or by dynamical means [15,16].

The above discussion is simply illuminated by means of the two-dimensional example, which is the case considered here. Using a covariant regularization, the covariant (gauge) anomaly is found as[1]

$$G_{\rm cov} = \frac{1}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu}, \tag{22}$$

where we have considered the gauge group to be Abelian.

The Euclidean effective action is defined as [15,16]

$$W = \int_0^1 dg \int d^2x A_\mu(x) J^{(g)}_\mu(x), \tag{23}$$

[1] A word about the notation. Spacetime indices are denoted in this section by Greek letters μ, ν, etc. In other sections, it is denoted by Latin a, b, etc. Here a, b stand for the non-Abelian group indices.

where the superscript "g" indicates that this is the coupling constant to be used in the construction of the current $J_\mu(x)$. The above equation is a formal definition of W. To concretize it, one has to regularize the current. Let us regularize it covariantly. Then, under an Abelian gauge transformation, $A_\mu \to A_\mu - \partial_\mu \alpha$,

$$\int d^2x \left\{ \partial_\mu \frac{\delta W}{\delta A_\mu} \right\} \alpha = \int_0^1 dg \int d^2x \alpha \partial_\mu J_\mu^{(g)}(x). \tag{24}$$

Since α is arbitrary, equating the integrands yields

$$\partial_\mu \frac{\delta W}{\delta A_\mu} = \int_0^1 dg \partial_\mu J_\mu^{(g)}(x). \tag{25}$$

Inserting the value of the covariant anomaly (22) yields the consistent anomaly,

$$\partial_\mu \frac{\delta W}{\delta A_\mu} = \frac{1}{8\pi} \epsilon_{\mu\nu} F_{\mu\nu}, \tag{26}$$

where the half factor comes from the integration over g since it occurs linearly in $F_{\mu\nu}(= g(\partial_\mu A_\nu - \partial_\nu A_\mu))$. Indeed, in any arbitrary even $d = 2n$ dimensions the (Abelian) consistent and covariant anomalies are related as

$$G_{\rm cons} = \frac{1}{n+1} G_{\rm cov} \tag{27}$$

since F involves g homogeneously to the nth power. It is now straightforward to check that the local polynomial connecting the consistent and covariant currents is given by

$$J_\mu^{\rm cons} = \frac{\delta W}{\delta A_\mu} = J_\mu^{\rm cov} - \frac{1}{4\pi} \epsilon_{\mu\nu} A_\nu \tag{28}$$

such that compatibility with Eqs. (22) and (26) is established.

We now make an important point. Initially Eq. (23) was defined by taking J_μ to be covariant. If we take J_μ to be consistent, then the result remains unaffected since the difference vanishes,

$$\int_0^1 dg \int d^2x A_\mu (\epsilon_{\mu\nu} A_\nu) = 0. \tag{29}$$

This shows that the effective action W remains the same whether the current is regularized covariantly or consistently. It is a general result valid in any dimensions [15,16].

Let us next consider gravitational anomalies. Such anomalies occur in $4n+2$ dimensions $(n = 0, 1, 2\ldots)$ in contrast to gauge anomalies that occur in $2n$ dimensions $(n = 1, 2\ldots)$ [17,18]. This shows that two dimensions are slightly special. It is the simplest spacetime dimension in which both gauge and gravitational anomalies may be present. Since our subsequent analysis will be done for two dimensions, we will restrict our discussion on gravitational anomalies also for this case. Although the results are known, this presentation is given primarily for two reasons. First, the results are obtained in a simple and elementary way using special properties of two dimensions. Second, this approach, discussed previously [19,20] in fragmented parts, is not particularly well known.

We shall derive the result for the two-dimensional gravitational (diffeomorphism) anomaly that is known to exist in a chiral theory. The anomaly will be obtained directly in the covariant form, which will be exploited in later sections, to derive the constitutive relations. As mentioned previously different manifestations of anomalies are related. Here we show the obtention of the diffeomorphism anomaly in a chiral theory from the conformal (trace) anomaly in a nonchiral (vectorlike) theory. In the latter theory, it is well known that it is not possible, at the quantum level, to simultaneously preserve both diffeomorohism and conformal symmetries that are present classically. Since diffeomorphism invariance is regarded as more fundamental in a gravitational theory, this is retained at the quantum level at the expense of conformal invariance. The breakdown of conformal invariance leads to the trace anomaly,

$$T_\mu^\mu = \frac{R}{24\pi}, \tag{30}$$

where R is the Ricci scalar.

Now the energy-momentum tensor in two dimensions can decomposed into a traceful and a traceless parts as

$$T_{\mu\nu} = \frac{R}{48\pi} g_{\mu\nu} + \theta_{\mu\nu}, \tag{31}$$

where $\theta_{\mu\nu}$ is symmetric $(\theta_{\mu\nu} = \theta_{\nu\mu})$ to preserve the symmetric property of $T_{\mu\nu}$ and traceless $\theta^\mu{}_\mu = 0$. Taking the trace of Eq. (31) then yields Eq. (30). Furthermore, since general coordinate invariance is preserved $(\nabla^\mu T_{\mu\nu} = 0)$, it implies the following constraint on $\theta_{\mu\nu}$:

$$\nabla^\mu \theta_{\mu\nu} = -\frac{1}{48\pi} \nabla_\nu R. \tag{32}$$

The stress tensor (31) may be interpreted as the sum of the contributions from the right and left moving modes. Moreover, the left-right symmetry implies that the contribution from one mode is equal to that from the other mode, except that the u and v variables have to be interchanged. Since $T_{\mu\nu}$ is symmetric, we have $T_{\mu\nu} = T_{\mu\nu}^{(R)} + T_{\mu\nu}^{(L)}$ with

$$T_{\mu\nu}^{R(L)} = \frac{R}{96\pi} g_{\mu\nu} + \theta_{\mu\nu}^{R(L)}, \tag{33}$$

where $\theta_{\mu\nu}=\theta^{(R)}_{\mu\nu}+\theta^{(L)}_{\mu\nu}$ (in analogy with $T_{\mu\nu}$). The chirality (or holomorphy) condition on $T^{R(L)}_{\mu\nu}$ implies the following equality [19,20]:

$$T^{R(L)}_{\mu\nu} = -(+)\frac{1}{2}(\bar{\epsilon}_{\mu\sigma}T^{\sigma\,R(L)}_{\ \nu} + \bar{\epsilon}_{\nu\sigma}T^{\sigma\,R(L)}_{\ \mu}) + \frac{1}{2}g_{\mu\nu}T^{\alpha\,R(L)}_{\ \alpha}. \tag{34}$$

Hence, we have the following equations for the right and left modes:

$$T^R_{vv} = T^L_{uu} = 0, \qquad T^R_{uu} = T^L_{vv} \neq 0. \tag{35}$$

Expectedly, the $L \leftrightarrow R$ symmetry under the interchange $u \leftrightarrow v$ is preserved. The above conditions along with the tracelessness of $\theta_{\mu\nu}$ yield further relations that follow from Eq. (33):

$$\theta^R_{uv} = \theta^R_{vv} = 0, \qquad \theta^R_{uu} \neq 0, \qquad \theta^L_{uv} = \theta^L_{uu} = 0, \qquad \theta^L_{vv} \neq 0. \tag{36}$$

With this input it is possible to deduce the various anomalies. The trace anomaly for the chiral theory is quickly obtained from (33)

$$T^{\mu\,(R)}_{\mu} = T^{\mu\,(L)}_{\mu} = \frac{R}{48\pi}, \tag{37}$$

which is half the result for the usual theory. It needs a little bit more algebra to derive the diffeomorphism anomaly for the chiral components. Taking the right mode and applying the covariant derivative on Eq. (33) yields

$$\nabla^\mu T^R_{\mu\nu} = \frac{1}{96\pi}\nabla_\nu R + \nabla^\mu \theta^R_{\mu\nu}. \tag{38}$$

Next, using Eqs. (32) and (36) for the R mode, we find

$$\nabla^\mu \theta^R_{\mu u} = \frac{1}{48\pi}\nabla_u R, \qquad \nabla^\mu \theta^R_{\mu v} = 0. \tag{39}$$

Inserting these expressions in Eq. (38), we obtain for $\nu = u$ and $\nu = v$

$$\nabla^\mu T^R_{\mu u} = \frac{1}{96\pi}\nabla_u R - \frac{1}{48\pi}\nabla_u R = -\frac{1}{96\pi}\nabla_u R \tag{40}$$

and

$$\nabla^\mu T^R_{\mu v} = \frac{1}{96\pi}\nabla_v R. \tag{41}$$

Combining them yields

$$\nabla^\mu T^R_{\mu\nu} = \frac{1}{96\pi}\bar{\epsilon}_{\nu\sigma}\nabla^\sigma R, \tag{42}$$

which is the cherished covariant gravitational anomaly. One can repeat the calculation for the left mode or directly obtain the final result by using the $L \leftrightarrow R$ symmetry under $u \leftrightarrow v$. The result is the same as Eq. (42) except for a change in sign. Since covariant expressions have been used throughout, the final result is also covariant. It is possible to obtain the consistent anomaly by adding local polynomials. However, this issue need not concern us since we will be dealing with the covariant anomaly only.

To summarize, whereas the nonchiral theory has a trace anomaly (30) but no diffeomorphism anomaly, the chiral theory admits both types of anomalies (37), (42). Physically, this is related to the unidirectional property of a chiral theory [20,21]. This distinction has an important role in the structure of the constitutive relations.

IV. ANOMALOUS CONSTITUTIVE RELATIONS IN HYDRODYNAMICS

As elaborated in the previous section, for the nonchiral theory, after quantization, either the trace or the diffeomorphism anomaly exists. Usually, one likes to retain the diffeomorphism symmetry at the cost of the conformal symmetry. In that case, one has the trace anomaly. In $(1+1)$ dimension the corresponding effective action is given by the Polyakov form [22],

$$S^{(g)}_P = \frac{1}{96\pi}\int d^2x d^2y \sqrt{-g}R(x)\frac{1}{\Box}(x,y)\sqrt{-g}R(y), \tag{43}$$

$$S^{U(1)}_P = \frac{e^2}{2\pi}\int d^2x d^2y \sqrt{-g}\bar{\epsilon}^{ab}\partial_a A_b(x) \times \frac{1}{\Box}(x,y)\sqrt{-g}\bar{\epsilon}^{cd}\partial_c A_d(y), \tag{44}$$

where $S^{(g)}_P$ is the effective action for the gravity sector, whereas $S^{U(1)}_P$ is that for the gauge sector. The total action is $S_P = S^{(g)}_P + S^{U(1)}_P$. $\frac{1}{\Box}$ is the inverse of d'Alembertian $\Box = \nabla^a\nabla_a = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b)$. This action is nonlocal, but it can be written in a local form by introducing auxiliary fields Φ and B, defined as

$$\Phi(x) = \int d^2y \frac{1}{\Box}(x,y)\sqrt{-g}R(y), \tag{45}$$

$$B(x) = \int d^2y \frac{1}{\Box}(x,y)\sqrt{-g}\bar{\epsilon}^{ab}\partial_a A_b(y). \tag{46}$$

Then

$$S^{(g)}_P = \frac{1}{96\pi}\int d^2x\sqrt{-g}(-\Phi\Box\Phi + 2\Phi R), \tag{47}$$

$$S^{U(1)}_P = \frac{e^2}{2\pi}\int d^2x\sqrt{-g}(-B\Box B + 2\bar{\epsilon}^{ab}\partial_a A_b B). \tag{48}$$

The above structure of the effective action is the general form for theories in which usual (and not chiral) fermions or scalars are coupled to the gauge and gravitational fields. The chiral case is considered in the next section. We prefer to discuss the models, case by case, in order to highlight the interplay between the conformal and diffeomorphism anomalies, apart from generating exact results.

The two-dimensional anomalous energy-momentum tensor and the $U(1)$ current are given by

$$T^{(g)}_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S^{(g)}_P}{\delta g^{ab}} = \frac{1}{48\pi}\left[\nabla_a\Phi\nabla_b\Phi - 2\nabla_a\nabla_b\Phi + g_{ab}\left(2R - \frac{1}{2}\nabla_c\Phi\nabla^c\Phi\right)\right]; \quad (49)$$

$$T^{U(1)}_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S^{U(1)}_P}{\delta g^{ab}} = \frac{e^2}{\pi}\left[\nabla_a B\nabla_b B - \frac{1}{2}g_{ab}\nabla^c B\nabla_c B\right] \quad (50)$$

and

$$J^a = \frac{1}{\sqrt{-g}}\frac{\delta S^{U(1)}_p}{\delta A_a} = \frac{e^2}{\pi}\bar{\epsilon}^{ab}\partial_b B, \quad (51)$$

respectively. The auxiliary fields Φ and B satisfy the following equations of motion:

$$\Box\Phi = R; \qquad \Box B = \bar{\epsilon}^{ab}\partial_a A_b. \quad (52)$$

The total energy-momentum tensor is $T_{ab} = T^{(g)}_{ab} + T^{U(1)}_{ab}$. Here R is the two-dimensional Ricci scalar, which, for the metric (1), is given by

$$R = \frac{1}{g_{11}^2}(g'_{11}\sigma' - 2g_{11}\sigma'^2 - 2g_{11}\sigma''). \quad (53)$$

It turns out that Eq. (49) leads to $\nabla_a T^{ab(g)} = 0$ and nonvanishing of the trace, whereas Eq. (50) yields $\nabla_b T^{ab\,U(1)} = J_b F^{ab}$ and vanishing of the trace of the stress tensor. Therefore, the explicit form of the trace anomaly of the theory is solely given by the gravity part. Hence, we have

$$T^{a(g)}_a = \frac{R}{24\pi}; \qquad T^{a\,U(1)}_a = 0. \quad (54)$$

Next, we shall evaluate the explicit expressions for the components of the energy-momentum tensor and $U(1)$ current in null coordinates for the background (4).

The (uv) component of stress tensor is easy to find because in null coordinates it is proportional to the trace. Therefore, from Eq. (54) we obtain

$$T^{(g)}_{uv} = -\frac{e^{2\sigma}R}{96\pi}; \qquad T^{U(1)}_{uv} = 0. \quad (55)$$

To find the other components, we need to start from Eqs. (49) and (50), where the auxiliary fields are determined by the solutions of Eq. (52). The solutions of Eq. (52), for the background (1), turn out to be

$$\Phi = \Phi_0(r) - 4pt + q; \qquad \partial_r\Phi_0 = -2\sigma' + z\sqrt{g_{11}}e^{-\sigma}, \quad (56)$$

$$B = B_0(r) + Pt + Q; \qquad \partial_r B_0(r) = -e^{-\sigma}\sqrt{g_{11}}(A_t(r) + C), \quad (57)$$

where p, q, z, P, Q, and C are constants. A detailed analysis to find the solutions is given in the Appendix. Then Eqs. (49) and (50) yield

$$T^{(g)}_{uu} = \frac{e^{2\sigma}}{96\pi g_{11}^2}(2\sigma'' g_{11} - \sigma' g'_{11}) + C_{uu}, \quad (58)$$

$$T^{(g)}_{vv} = \frac{e^{2\sigma}}{96\pi g_{11}^2}(2\sigma'' g_{11} - \sigma' g'_{11}) + C_{vv}, \quad (59)$$

and

$$T^{U(1)}_{uu} = \frac{e^2}{4\pi}(A_t - P + C)^2; \qquad T^{U(1)}_{vv} = \frac{e^2}{4\pi}(A_t + P + C)^2, \quad (60)$$

where C_{uu} and C_{vv} are constants, made out of p and z. Similarly, the components of the current are

$$J_u = \frac{e^2}{2\pi}(A_t - P + C); \qquad J_v = \frac{e^2}{2\pi}(A_t + P + C). \quad (61)$$

We now construct the constitutive relation for the energy-momentum tensor of anomalous hydrodynamics. This will be done as follows. In the comoving frame, the fluid velocity components are given in Eq. (7). Using Eqs. (7), (8), and (53), apart from also using the Tolman relation for the temperature $T = T_0 e^{-\sigma}$, where T_0 is the equilibrium temperature, and the chemical potential, $\mu = A_t e^{-\sigma}$, the constitutive relations may be determined. The various components can be written in the covariant forms

$$T^{(g)}_{ab} = \left[\frac{1}{12\pi}(u^c\nabla^d - u^d\nabla^c)\nabla_c u_d + 4\bar{C}T^2\right]u_a u_b - \left[\frac{1}{24\pi}u^c\nabla^d\nabla_d u_c - 2\bar{C}T^2\right]g_{ab}, \quad (62)$$

$$T_{ab}^{U(1)} = \left[\frac{e^2}{2\pi}(\mu^2 + \bar{C_1}^2 T^2)\right](2u_a u_b + g_{ab}) + \left[\frac{e^2}{\pi}\mu\bar{C}_1 T\right](u_a\tilde{u}_b + \tilde{u}_a u_b), \quad (63)$$

$$J_a = -\frac{e^2}{\pi}\left(\mu + C\frac{T}{T_0}\right)u_a - \left(\frac{e^2 P}{\pi}\right)\frac{T}{T_0}\tilde{u_a} = -\frac{e^2}{\pi}(\mu + \bar{C}_1 T)u_a, \quad (64)$$

where $\bar{C} = C_{uu}T_0{}^{-2}$, $\bar{C}_1 = (C - P)T_0{}^{-1}$ for the uu component and $\bar{C} = C_{vv}T_0{}^{-2}$, $\bar{C}_1 = (P + C)T_0{}^{-1}$ for the vv component.

The above constitutive relations (62–64) are new findings. For the chargeless case, the relations (62–64) reproduce the results obtained by the derivative expansion approach [11], for the choice $\bar{C} = \frac{\pi}{12}$. Inclusion of charges in that approach requires the consideration of higher-order terms, which has not been done. We will further elaborate on this in the next section.

V. ANOMALOUS CONSTITUTIVE RELATIONS IN CHIRAL HYDRODYNAMICS

In this section we use the energy-momentum tensor and the gauge current to derive anomalous constitutive relations in chiral hydrodynamics. For the chiral gauge theory, as already mentioned, after quantization we have both the trace and diffeomorphism anomaly, where the trace anomaly comes only from the gravity part.

The two-dimensional chiral effective action with $U(1)$ gauge field is given by [23,24]

$$\Gamma_{(H)} = -\frac{1}{3}z(\omega) + z(A), \quad (65)$$

where

$$z(v) = \frac{1}{4\pi}\int d^2x d^2y \epsilon^{ab}\partial_a v_b(x)\frac{1}{\Box}(x,y) \times \partial_c[(\epsilon^{cd} + \sqrt{-g}g^{cd})v_d(y)]. \quad (66)$$

The spin connection and the gauge field are denoted, respectively, by ω_a and A_a. From a variation of the effective action, the consistent forms for the energy-momentum tensor and the gauge current are obtained. However, we are interested in the covariant forms, so appropriate local polynomials have to be added. This is possible because energy-momentum tensors and currents are only defined modulo local polynomials. Then we have

$$\delta\Gamma_H = \int d^2x\sqrt{-g}\left(\frac{1}{2}\delta g_{ab}T^{ab} + \delta A_a J^a\right) + l, \quad (67)$$

where the local polynomial is given by [23,24]

$$l = \frac{1}{4\pi}\int d^2x\epsilon^{ab}\left(A_a\delta A_b - \frac{1}{3}\omega_a\delta\omega_b - \frac{1}{24}Re_a^z\delta e_b^z\right). \quad (68)$$

Here e_a^z is the zweibein vector, which fixes the metric g_{ab}.

Including the effect of the local polynomial, the two-dimensional covariant chiral energy-momentum tensor can easily be determined by variation of the effective action (67) [24],

$$T_{ab}^{(g)} = \frac{1}{4\pi}\left(\frac{1}{48}D_a\Phi D_b\Phi - \frac{1}{24}D_a D_b\Phi + \frac{1}{24}g_{ab}R\right), \quad (69)$$

$$T_{ab}^{U(1)} = \frac{e^2}{4\pi}(D_a B D_b B), \quad (70)$$

whereas the covariant chiral gauge current is given by

$$J^a = -\frac{e^2}{2\pi}D^a B. \quad (71)$$

The total energy-momentum tensor is $T_{ab} = T_{ab}^{(g)} + T_{ab}^{U(1)}$. The auxiliary fields, Φ and B, are again determined by the solutions of Eq. (52), which are Eqs. (56) and (57). The chiral nature of T_{ab} and J_a are manifested by the presence of the chiral covariant derivative D_a that is defined in terms of usual covariant derivative ∇_a,

$$D_a = \nabla_a - \bar{\epsilon}_{ab}\nabla^b = -\bar{\epsilon}_{ab}D^b. \quad (72)$$

Based on the above identity, it is possible to show the properties

$$T_{ab} = -\frac{1}{2}(\bar{\epsilon}_{ac}T_b^c + \bar{\epsilon}_{bc}T_a^c) + \frac{1}{2}g_{ab}T_c^c, \qquad J_a = -\bar{\epsilon}_{ab}J^b, \quad (73)$$

which manifest the chiral nature of T_{ab} and J_a.

It is easy to check that in null coordinates $D_u = 2\nabla_u$ and $D_v = 0$, and hence this corresponds to the outgoing modes. The above stress tensor leads to both trace and diffeomorphism anomalies. The trace anomaly again comes from the gravity part alone:

$$T_a^{a(g)} = \frac{R}{48\pi}; \qquad T_a^{aU(1)} = 0. \quad (74)$$

These results are simply obtained by exploiting the chirality condition (72). The stress tensor satisfies the covariant conservation law,

$$\nabla_b T^{ab} = \frac{\bar{\epsilon}^{ab}}{96\pi}\nabla_b R + J_b F^{ab}, \quad (75)$$

where $F^{ab} = \nabla^a A^b - \nabla^b A^a$ is the gauge field strength. The first term on the right is the covariant diffeomorphism anomaly, while the second is the usual Lorentz force term.

Now we will determine the expressions for the components of T_{ab} in null coordinate (u, v). Here, as earlier, the (uv) components are determined from the trace expression (74), whereas the other components are found out from Eqs. (69) and (70) with the use of Eqs. (56) and (57):

$$T^{(g)}_{uu} = \frac{e^{2\sigma}}{96\pi g_{11}^2}(2\sigma'' g_{11} - \sigma' g'_{11}) + C_{uu};$$
$$T^{(g)}_{uv} = -\frac{e^{2\sigma}R}{192\pi}; \qquad T^{(g)}_{vv} = 0, \tag{76}$$

$$T^{U(1)}_{uu} = \frac{e^2}{4\pi}(P - A_t - C)^2; \qquad T^{U(1)}_{vv} = 0; \qquad T^{U(1)}_{uv} = 0. \tag{77}$$

Similarly, the components of the current are

$$J_u = \frac{e^2}{2\pi}(A_t - P + C), \qquad J_v = 0. \tag{78}$$

Relations given in Eqs. (76) and (77) yield the expression for the covariant energy-momentum tensor with the U(1) current in chiral hydrodynamics.

Finally we deduce the constitutive relations of a chiral fluid in the comoving frame. In this frame the chiral fluid velocity is given in Eq. (9) and fluid variables determined in Eq. (8). Therefore, the components can be expressed in the forms

$$T^{(g)}_{ab} = \left[\frac{1}{48\pi}(u^c\nabla^d - u^d\nabla^c)\nabla_c u_d + e^{-2\sigma}C_{uu}\right] \times (2u_a u_b - u_a\tilde{u}_b - \tilde{u}_a u_b) - \left[\frac{1}{48\pi}(u^c\nabla^d\nabla_d u_c) - e^{-2\sigma}C_{uu}\right]g_{ab}, \tag{79}$$

$$T^{U(1)}_{ab} = \frac{e^2}{4\pi}\left(\mu^2 + C_1\mu e^{-\sigma} + \frac{{C_1}^2}{4}e^{-2\sigma}\right) \times (2u_a u_b - u_a\tilde{u}_b - \tilde{u}_a u_b + g_{ab}), \tag{80}$$

$$J_a = -\frac{e^2}{2\pi}\left(\mu + \frac{C_1}{2}e^{-\sigma}\right)u^{(c)}_a, \tag{81}$$

where, at an intermediate step, we have used the identity

$$\tilde{u}_a\tilde{u}_b - u_a u_b = g_{ab}. \tag{82}$$

It is now shown that the above constitutive relations can be put in the corresponding forms for an ideal chiral fluid.

Let us first recall that the constitutive relation for an ideal chiral fluid is different from the usual expression. To account for the chiral property, it is necessary to replace the velocity vectors by chiral velocity vectors [13]. Once this is done, the relevant constitutive relation becomes

$$T_{ab} = (\epsilon^c + \mathcal{P}^c)u^{(c)}_a u^{(c)}_b + \mathcal{P}^c g_{ab}, \tag{83}$$

where $u^{(c)}_a$ is defined in Eq. (9). The quantities ϵ^c and $\mathcal{P}^c$ may be regarded as mimicking the energy density and pressure, respectively, that appear in the (standard) contribution to the ideal stress tensor,

$$T^0{}_{ab} = (\epsilon + \mathcal{P})u_a u_b + \mathcal{P}g_{ab}, \tag{84}$$

satisfying $\nabla^a T^0{}_{ab} = 0$. The total stress tensor,[2] is a sum of the contributions from the diffeomorphism-invariant part ($T^0{}_{ab}$) and the anomalous part (T_{ab}),

$$T^{(\text{total})}_{ab} = T^0{}_{ab} + T_{ab}. \tag{85}$$

It is straightforward to verify the holomorphy condition (34) for Eq. (83). Using Eq. (9) the ideal chiral constitutive relation (83) is written as

$$T_{ab} = (\epsilon^c + \mathcal{P}^c)(2u_a u_b - u_a\tilde{u}_b - \tilde{u}_a u_b) + (\epsilon^c + 2\mathcal{P}^c)g_{ab}. \tag{86}$$

As done in the last section, we introduce the Tolman relation $T = T_0 e^{-\sigma}$ and exploit Eqs. (79) and (80) to write the energy-momentum tensor as

$$T_{ab} = T^{(g)}_{ab} + T^{U(1)}_{ab} = \left[\frac{1}{48\pi}(u^c\nabla^d - u^d\nabla^c)\nabla_c u_d + \bar{C}T^2 + \frac{e^2}{4\pi}\left(\mu^2 + \bar{C}_1\mu T + \frac{{\bar{C}_1}^2 T^2}{4}\right)\right](2u_a u_b - u_a\tilde{u}_b - \tilde{u}_a u_b) + \left[\frac{e^2}{4\pi}\left(\mu^2 + \bar{C}_1\mu T + \frac{{\bar{C}_1}^2 T^2}{4}\right) + \bar{C}T^2 - \frac{1}{48\pi}(u^c\nabla^d\nabla_d u_c)\right]g_{ab}, \tag{87}$$

[2]The constitutive relations in the previous section have to be interpreted similarly. Expressions (62–64) yield only the anomalous part. However, contrary to the chiral case, these cannot be expressed in the form (83).

where $\bar{C}=C_{uu}T_0{}^{-2}$ and $\bar{C}_1=2(C-P)T_0{}^{-1}$. Equation (87) reproduces Eq. (86) with the following identifications:

$$\epsilon^c = \frac{1}{48\pi}(2u^c\nabla^d - u^d\nabla^c)\nabla_c u_d + \bar{C}T^2 + \frac{e^2}{4\pi}\left(\mu^2 + \bar{C}_1\mu T + \frac{\bar{C}_1{}^2T^2}{4}\right), \quad (88)$$

$$\mathcal{P}^c = -\frac{1}{48\pi}(u^c\nabla^d\nabla_c u_d). \quad (89)$$

Note that in the chiral case the contribution to $\mathcal{P}^c$ from the gauge field is zero. Also, the constitutive relation (81) for the current is manifestly in the form of an ideal chiral fluid relation. Constitutive relations done in this section are general and exact. In the absence of a gauge field, the relation (87) reproduces the result of Ref. [13] as well as that found by the gradient expansion method [11].

We note that although ϵ^c and $\mathcal{P}^c$ contain derivative/dissipative terms we refer to the relation (83) as ideal because it has a structural resemblance with the usual ideal form. The quantities ϵ^c and $\mathcal{P}^c$ are not ideal in the sense that they include dissipation.

VI. COMPARISON WITH DERIVATIVE EXPANSION APPROACH

In this section we make a brief comparison with the derivative expansion approach that is a favored approach in the context of anomalous hydrodynamics. This would also help in putting our analysis in a proper perspective.

The one-point function of the covariant stress tensor is given by

$$T^{ab} = \epsilon u^a u^b + \mathcal{P}\tilde{u}^a\tilde{u}^b + \theta(\tilde{u}^a u^b + u^a\tilde{u}^b), \quad (90)$$

which is the general form for a symmetric second rank tensor constructed from the velocity vector u^a and its dual $\tilde{u}^a = \bar{\epsilon}^{ab}u_b$. The explicit values for ϵ, $\mathcal{P}$, and θ are provided by the gradient expansion scheme as [11]

$$\begin{aligned} \epsilon &= p_0T^2 + C_w(u^b\nabla^a\nabla_a u_b) + 2C_w(u^a\nabla^b - u^b\nabla^a)\nabla_a u_b \\ \mathcal{P} &= p_0T^2 - C_w(u^b\nabla^a\nabla_a u_b) \\ \theta &= CT^2 - 2C_g(u^a\nabla^b - u^b\nabla^a)\nabla_a u_b, \end{aligned} \quad (91)$$

where, for simplicity, we consider the chargeless case. The problems of including charge within this scheme will be highlighted later. Here C_w and C_g are the normalization factors of the conformal (trace) and gravitational (diffeomorphism) anomalies. Likewise, p_0 and C are certain response parameters that are undetermined for the moment. To determine them it is essential to use earlier results from various $(1+1)$-dimensional conformal field theories. One obtains

$$p_0 = 4\pi^2 C_w, \qquad C = -8\pi^2 C_g. \quad (92)$$

It is now feasible to deduce the constitutive relation for the stress tensor. Let us first consider the chiral case. Here $C_w = \frac{1}{48\pi} = 2C_g$ as seen from Eqs. (74) and (75). The response parameters are found to be

$$p_0 = \frac{\pi}{12} = -C. \quad (93)$$

Inserting these values in Eq. (91) and using the resulting expressions in Eq. (90) reproduces Eq. (87) for $e = 0$ (chargeless case) and for the specific choice of $\bar{C} = \frac{\pi}{12}$.

For the nonchiral case, there is no diffeomorphism anomaly (so that $C_g = 0$), and there is only a conformal anomaly with $C_w = \frac{1}{24\pi}$ (54). The response parameters (92) are thus given by

$$p_0 = \frac{\pi}{6}, \qquad C = 0. \quad (94)$$

Putting these values in Eq. (91) and inserting the resulting forms for ϵ, $\mathcal{P}$, and θ in Eq. (90) reproduces the constitutive relation (62) for the particular choice $\bar{C} = \frac{\pi}{12}$.

Some comments are now in order. It is seen that the final constitutive relation can be obtained provided the additional information (92) is known. This is not required in our analysis. Also, the constitutive relation found in the gradient expansion approach corresponds to a specific value ($\bar{C} = \frac{\pi}{12}$) of our results. We may compare this with our approach in which the actual value of $\bar{C}$ is left open. Finally, inclusion of charge is quite nontrivial in the gradient expansion approach. The relations (91) are no longer exact. There are nonleading (higher-derivative) corrections. Likewise, the first relation in Eq. (92) also gets modified. These corrections and/or modifications have not been discussed in the literature. Thus, the form of the constitutive relation in the presence of both gauge and gravitational anomalies is not clearly spelled out. Hence, a one-to-one comparison with our general form is not possible.

VII. CONCLUSIONS

Gauge and gravitational anomalies in two dimensions have played a significant role in different contexts. While such anomalies can and do occur in various theories, perhaps their most dramatic appearance happens for chiral theories, i.e., where chiral fermions are coupled to the gauge and/or gravitational field. In such theories, due to the lack of a chiral invariant regularization, the one-loop effective action always yields anomalies. Depending upon the regularization, anomalies may be covariant or consistent. The use of the covariant or consistent structure depends upon the needs of the problem; however, it is useful to note that the anomaly vanishing condition is identical in both cases. Since the covariant form preserves the correct transformation property of the anomalous current or stress tensor, it is usually favored over the consistent form. Indeed, besides their recent influence in hydrodynamics [2–13], covariant anomalies have been used to study such diverse phenomena such as the Hawking effect in black holes [24,25] or the thermal Hall effect in topological insulators [26].

In this paper we have developed a new approach to study the consequences of both gauge and gravitational anomalies in fluid dynamics. Explicit computations were done in two dimensions, leading to the obtention of exact results. To highlight the competing role of different anomalies, usual (i.e., nonchiral) and chiral hydrodynamics were treated separately.

Exact closed-form expressions for the constitutive relations for the current and stress tensor were obtained. For chiral hydrodynamics these relations could be put in the form of an ideal (chiral) fluid. The constitutive relation for an ideal (chiral) fluid, it may be recalled, is derived from the corresponding relation for an ideal fluid by replacing the velocity vector by the chiral velocity vector. Only then does the current or stress tensor satisfy the special properties of two-dimensional chirality [13].

Despite the fact that several papers [2–13] discuss anomalous hydrodynamics including some that are solely devoted to two dimensions [9–13], the results presented here are both new and from a different approach. The novelty of our approach consists of the inclusion of charge for which no constitutive relation was earlier provided. Since the two-dimensional metric is conformally flat, the effective action is exactly known. This leads to exact constitutive relations for either the current or the stress tensor. Chirality imposes additional restrictions that eventually justify the structures for the constitutive relations. In other words, not only do we provide new results, but we also explain their appearance. However, it would be interesting if our result is compared with the derivative expansion approach by expanding our exact results. But this is beyond the scope of the paper. We leave this exercise for the future.

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APPENDIX: SOLUTIONS FOR THE AUXILIARY FIELDS

The solutions for Φ and B can be obtained from Eq. (52) in the following way. Let us first concentrate on the solution for Φ. Under the background metric (1), we obtain

$$\begin{aligned}\Box\Phi &= \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b)\Phi \\ &= -e^{-2\sigma}\partial_t^2\Phi + \frac{1}{e^\sigma\sqrt{g_{11}}}\partial_r\left(\frac{e^\sigma}{\sqrt{g_{11}}}\partial_r\right)\Phi, \quad \text{(A1)}\end{aligned}$$

and R is given by Eq. (53). Now since the metric (1) is static, it has a timelike Killing vector, and so the ansatz for Φ can be taken as Eq. (56). Then Eq. (A1) reduces to

$$\Box\Phi = \frac{1}{e^\sigma\sqrt{g_{11}}}\partial_r\left(\frac{e^\sigma}{\sqrt{g_{11}}}\partial_r\right)\Phi_0(r). \quad \text{(A2)}$$

The Ricci scalar (53), multiplied by $e^\sigma\sqrt{g_{11}}$, can be expressed as a total derivative with respective to the radial coordinate r:

$$e^\sigma\sqrt{g_{11}}R = -\partial_r\left(\frac{2e^\sigma\sigma'}{\sqrt{g_{11}}}\right). \quad \text{(A3)}$$

Hence, $\Box\Phi = R$ yields the solution for $\Phi_0(r)$ as given in Eq. (56).

Similarly, the solution for B can be obtained. Substitution of the ansatz in Eq. (57) leads to

$$\Box B = \frac{1}{e^\sigma\sqrt{g_{11}}}\partial_r\left(\frac{e^\sigma}{\sqrt{g_{11}}}\partial_r\right)B_0, \quad \text{(A4)}$$

whereas $\bar{\epsilon}^{ab}\partial_a A_b = -\frac{1}{e^\sigma\sqrt{g_{11}}}\partial_r A_t$. Hence, we obtain the solution for $B_0(r)$ as given there.

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