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Perils of analytic continuation

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A nice paper by Morrison [arXiv:1302.1860] demonstrates the recent convergence of opinion that has taken place concerning the graviton propagator on de Sitter background. We here discuss the few points which remain under dispute. First, the inevitable decay of tachyonic scalars really does result in their 2-point functions breaking de Sitter invariance. This is obscured by analytic continuation techniques which produce formal solutions to the propagator equation that are not propagators. Second, Morrison's de Sitter invariant solution for the spin two sector of the graviton propagator involves derivatives of the scalar propagator at $M^2 = 0$, where it is not meromorphic unless de Sitter breaking is permitted. Third, de Sitter breaking does not require zero modes. Fourth, the ambiguity Morrison claims in the equation for the spin two structure function is fixed by requiring it to derive from a mode sum. Fifth, Morrison's spin two sector is not "physically equivalent" to ours because their coincidence limits differ. Finally, it is only the noninvariant propagator that gets the time independence and scale invariance of the tensor power spectrum correctly.

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I. INTRODUCTION

The increasingly compelling evidence for primordial inflation [1] has transformed quantum field theory on de Sitter space from an esoteric exercise in mathematical physics to the essential framework for deriving the initial conditions of observational cosmology [2]. The best understood of these initial conditions take the form of primordial scalar and tensor perturbations [3]. It is important to understand that these perturbations were not present before inflation; they formed through a time-dependent process in which virtual particles were ripped out of the vacuum by the accelerated expansion of spacetime, and they have preserved a memory of conditions at the time they were formed.

The basic perturbations are tree order effects and their power spectra can be expressed in terms of plane wave mode functions, evaluated after they have experienced first horizon crossing [4]. Of course the laws of physics are governed by an interacting quantum field theory so primordial perturbations must affect one another, at some level, and they must also affect other particles. Loop effects of this sort could be expressed in terms of mode sums but it is simplest to recognize these mode sums as propagators. Hence the interest in scalar and tensor propagators on a nearly de Sitter background.

The intrinsically time-dependent process through which inflationary perturbations are generated would seem to preclude de Sitter invariant propagators for either the massless, minimally coupled (MMC) scalar or for the graviton. This was recognized quite early for the MMC scalar by exhibiting the time dependence of its coincidence limit [5], and a formal proof was given soon afterwards [6]. However, opinions about the graviton propagator have been divided between cosmologists—who argue that it must break de Sitter invariance because free gravitons in transverse-traceless-spatial gauge obey the same equation as MMC scalars [7]—and mathematical physicists who argue that this is a gauge artifact [8,9].

Explicit constructions of the graviton propagator have produced equivocal results. On the one hand, adding a de Sitter invariant gauge fixing term to the action yields propagator equations for which explicit de Sitter invariant solutions have been given [10], except for an infinite set of discrete choices of the two gauge fixing parameters for which infrared divergences preclude a de Sitter invariant solution [11]. On the other hand, why should *any* choice of arbitrary gauge fixing parameters lead to de Sitter breaking? And it was shown early on that the solution with a noncovariant gauge fixing parameter [12] manifests de Sitter breaking even when the compensating gauge transformation is added [13].[1] This has led to a curious state of affairs in which every complete, dimensionally regulated graviton loop correction [15–20] has been made using a propagator that mediates

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[1]The same de Sitter breaking occurs using the different field variables favored by Kitamoto and Kitazawa [14].

plausible de Sitter breaking effects—for example, that scattering with inflationary gravitons induces secular growth of fermion wave functions [16]—which mathematical physicists suspect to be gauge artifacts.

Four recent insights have partially resolved this unsatisfactory situation:

(i) There is an obstacle to adding invariant gauge fixing terms on any manifold, such as de Sitter, with a linearization instability [21];[2]

(ii) Power law infrared divergences are incorrectly subtracted off by the analytic continuation techniques routinely employed by mathematical physicists, leading to formal solutions of the propagator equation which are not true propagators [23];

(iii) When de Sitter invariant gauges are imposed as strong operator conditions—as opposed to the average conditions effected by adding gauge fixing functions—the resulting propagators show de Sitter breaking [24–26]; and

(iv) The old, noncovariant gauge propagator [18], and all of the new covariant gauge ones [27], give a result that mathematical physicists accept as correct for the linearized Weyl-Weyl correlator [28].[3]

The second point also explains the isolated infrared divergences long encountered in constructions of the graviton propagator [11]. Analytic continuation techniques only register *logarithmic* divergences [30,31], and the problematic parameter values are just those for which a power law infrared divergence happens to become logarithmic.

A recent paper by Morrison [32] reveals how close the two sides have grown. In particular, he has exploited the formalism of covariant projection operators acting on scalar structure functions which was developed to represent the graviton self-energy [33] and later applied to the propagator in exact de Donder gauge [24]. Using this formalism he has explained carefully what must be done to extract a de Sitter invariant solution, and he has exhibited the rather small differences in the structure functions which distinguish the de Sitter breaking solution from the invariant one. He has also demonstrated that the two propagators agree when smeared with transverse-traceless test functions in the sense of Fewster and Hunt [34].

It would be reasonable to infer from Morrison's work that mathematical physicists have no further objections to the noncovariant gauge propagator [12] which has been used for every complete, dimensionally regulated graviton loop so far computed [15–20]. Nevertheless, there are a few issues that remain controversial. These concern the validity of the analytic continuation techniques used in Morrison's construction and the distinction between formal solutions of the propagator equation and true propagators.

This paper contains five sections of which the first is this Introduction. In Sec. II we consider the contention by mathematical physicists [32,35] that minimally coupled scalars with tachyonic masses $M^2 < 0$ nonetheless possess de Sitter invariant propagators on D-dimensional de Sitter space with Hubble constant H, except for the discrete values $M^2 = -N(N+D-1)H^2$, where $N = 0, 1, 2, \ldots$ In Sec. III we revisit classic work on the MMC scalar [6,36] to debunk the more recent argument that its de Sitter breaking derives from the isolated zero mode in global coordinates, which gravitons lack. In fact the MMC scalar's infrared finite de Sitter breaking derives from the late time approach to scale invariance and time independence of its power spectrum in the ultraviolet, not the infrared. Both of these features are shared by the graviton. The infrared divergences of open coordinates derive from an infinite number of modes being in this saturated state at finite times. Section IV discusses what is wrong with the construction which leads to a de Sitter invariant spin two structure function, why the coincidence limit [25] shows that the de Sitter invariant solution [32] is not physically equivalent to the de Sitter breaking one [24], and why the time independence and scale invariance of the tensor power spectrum imply that the de Sitter breaking solution is correct. Minor points are that the coincidence limit of the graviton propagator appears in even simple one loop diagrams [15–20]—so it cannot be dismissed—and that no sequence of the transverse-traceless test functions of Fewster and Hunt [34] approaches a delta function—so they are not analogous to the scalar smearing functions long employed by mathematical physicists. Our conclusions are summarized in Sec. V.

II. SCALAR TACHYONS

Some mathematical physicists believe strongly that tachyonic scalars possess de Sitter invariant propagators for any mass-squared which avoids the discrete values $M^2 = -N(N+D-1)H^2$, where N is a nonnegative integer [32,35]. They believe this because, except for those special masses, the scalar propagator equation,

$$\sqrt{-g}[\square - M^2]i\Delta(x;x') \equiv [\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - M^2\sqrt{-g}]i\Delta(x;x') = i\delta^D(x-x'), \tag{1}$$

has a de Sitter invariant solution,

$$i\Delta^{\rm dS}(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{D}{2})} \times {}_2F_1\left(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y}{4}\right), \tag{2}$$

[2]Ignoring this problem in scalar quantum electrodynamics leads to on-shell singularities in the scalar self-mass-squared [22].

[3]The mathematical physics computation [28] had a number of significant errors that were discovered by comparison with the cosmological result [18] and then corrected [29].

where the index ν and the de Sitter length function $y(x;x')$ depend the upon M^2 and the invariant length $\ell(x;x')$ as,

$$\nu \equiv \sqrt{\left(\frac{D-1}{2}\right)^2 - \frac{M^2}{H^2}}, \qquad y(x;x') \equiv 4\sin^2\left(\frac{1}{2}H\ell(x;x')\right). \tag{3}$$

This belief is perplexing to cosmologists who feel that tachyonic particles must decay, even on de Sitter space, and that this decay is an inherently time-dependent process which depends upon when the state is released and hence must break de Sitter invariance because the result does not depend only on the observation time. Much of the observed phenomenology of the standard model would not make sense otherwise. In this section we first explain, in very simple terms, why tachyonic scalars decay, and then how the use of analytic continuation techniques can lead to employing formal solutions to the scalar propagator equation which are not true propagators. The key point is that the quantum mechanical requirement that states have finite, positive norm imposes restrictions on analytic continuation which are being violated to discard the de Sitter breaking secular terms. The section closes with a discussion of the unacceptable phenomenology implied by this practice.

A. Why tachyonic scalars decay

The first thing to understand is that tachyonic scalars decay equally in open coordinates,

$$ds^2 = -dt^2 + e^{2Ht} d\vec{x}\cdot d\vec{x}, \tag{4}$$

and in closed coordinates (which we specialize to $D = 4$),

$$ds^2 = -d\tau^2 + H^{-2}\cosh^2(H\tau)[d\chi^2 + \sin^2(\chi)d\theta^2 + \sin^2(\chi)\sin^2(\theta)d\phi^2]. \tag{5}$$

Another important point is that the decay occurs for each mode separately, so one need never worry about more than a single degree of freedom evolving in whatever is the appropriate time. This single degree of freedom is known as a mode. The natural modes for open coordinates are spatial plane waves $e^{\vec{k}\cdot\vec{x}}$ and the associated mode functions are $u(t,k)$; for closed coordinates the natural modes are the 4-dimensional spherical harmonics $Y_{k\ell m}(\chi,\theta,\phi)$ and the associated mode functions are $u_k(\tau)$. Finally, it is important to understand that there is only one scalar field operator, $\varphi(x)$, and it can be expanded in either coordinate system,

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3}\{u(t,k)e^{i\vec{k}\cdot\vec{x}}a(\vec{k}) + u^*(t,k)e^{-i\vec{k}\cdot\vec{x}}a^\dagger(\vec{k})\}, \tag{6}$$

$$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=-\ell}^{\ell}\{u_k(\tau)Y_{k\ell m}(\chi,\theta,\phi)a_{k\ell m} + u_k^*(\tau)Y_{k\ell m}^*(\chi,\theta,\phi)a_{k\ell m}^\dagger\}. \tag{7}$$

The massive scalar Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} - \frac{1}{2}M^2\varphi^2\sqrt{-g}. \tag{8}$$

The Euler-Lagrange equation derived from (8) implies the equations obeyed by $u(t,k)$ and $u_k(\tau)$,

$$[\partial_t^2 + 3H\partial_t + k^2e^{-2Ht} + M^2]u(t,k) = 0, \tag{9}$$

$$[\partial_\tau^2 + 3H\tanh(H\tau)\partial_\tau + H^2k(k+2)\mathrm{sech}^2(H\tau) + M^2]u_k(\tau) = 0. \tag{10}$$

Similarly, the canonical commutation relations (which ensure positive norm states) derived from (8) fix the normalizations of the respective Wronskians,

$$u(t,k)\dot{u}^*(t,k) - \dot{u}(t,k)u^*(t,k) = ie^{-3Ht}, \tag{11}$$

$$u_k(\tau)u_k'^*(\tau) - u_k'(\tau)u_k^*(\tau) = i\,\mathrm{sech}^3(H\tau). \tag{12}$$

The close relation between the open coordinate mode equations (9) and (11) and their closed coordinate analogs (10) and (12) is evident. This is why claims for de Sitter invariance are no better justified in closed coordinates than in open coordinates. We shall have more to say about this in Sec. II D and Sec. III.

Equations (9) and (10) make it quite apparent both why tachyonic scalars decay, and why the decay is much stronger on de Sitter than it is in flat space. Each equation contains three terms which we can identify as an "acceleration," a "friction force" and a "restoring force." For open coordinates these three terms are

$$\text{Acceleration} \longleftrightarrow \ddot{u}(t,k), \tag{13}$$

$$\text{Friction Force} \longleftrightarrow -3H\dot{u}(t,k), \tag{14}$$

$$\text{Restoring Force} \longleftrightarrow -(M^2 + k^2e^{-2Ht})u(t,k). \tag{15}$$

For $M^2 > 0$ the restoring force makes the mode accelerate in the direction opposite to its current value. So if $u(t,k)$ is positive, its acceleration is negative; while if $u(t,k)$ is negative, its acceleration is positive. The behavior in that case is either underdamped or overdamped oscillations, depending upon the relation between the friction force and the restoring force.

Tachyonic scalars have $M^2 < 0$, which tends to make the mode accelerate in the same direction it already is: if $u(t,k)$ is positive, a tachyonic mass term tends to make it accelerate in the positive direction; and if $u(t,k)$ is negative, a tachyonic mass term makes it accelerate in the negative direction. This effect is resisted by the spatial gradient term k^2e^{-2Ht}, but that redshifts to zero at late times, so that all modes eventually experience an antirestoring force. That is why the instability is worse than in flat space. The effect of a tachyonic mass is to make the mode function $u(t,k) \rightarrow K(k) \times e^{\lambda t}$ grow exponentially at late times with time constant,

$$\lambda = -\frac{3}{2}H + \sqrt{\left(\frac{3}{2}H\right)^2 - M^2}. \tag{16}$$

The asymptotic late time behavior of the closed coordinate mode function $u_k(\tau)$ is not a bit different for $\tau \rightarrow +\infty$, but it also shows exponential growth for all modes as $\tau \rightarrow -\infty$.

There should be nothing surprising about this analysis; it is elementary Newtonian mechanics. We are describing the modes in terms of point particle motions and it is a fact that balls in uniform gravitational fields tend to roll down parabolic hills with ever-increasing speed. The Hubble friction term decreases the exponential time constant somewhat but it cannot prevent the decay.

There *are* classical configurations which fail to experience exponential growth for $M^2 < 0$. For example, if one chooses the initial conditions so that $u(0,k) = 0$ and $\dot{u}(0,k) = 0$ then the solution is $u(t,k) = 0$ for all time. One can also arrange the initial conditions so that the mode approaches zero at late times. If we were doing classical mechanics—*or just trying to solve the propagator equation*—there would be no objection to these solutions. However, the quantum mechanical requirement of positive norm states (11) and (12) precludes the mode function having just the exponentially falling solution.

A true propagator is the expectation value in some normalized state we might call $|\Omega\rangle$ of the time-order product of two field operators. Substituting the open coordinate and closed coordinate free field expansions (6) and (7) gives,[4]

$$\langle\Omega|T[\varphi(x)\varphi(x')]|\Omega\rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\{\theta(t-t')u(t,k) \times u^*(t',k) + \theta(t'-t)u^*(t,k) \times u(t',k)\}, \tag{17}$$

$$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{m=-\ell}^{\ell} Y_{k\ell m}Y^*_{k\ell m}\{\theta(\tau-\tau') \times u_k(\tau)u_k^*(\tau') + \theta(\tau'-\tau)u_k^*(\tau) \times u_k(\tau')\}. \tag{18}$$

Based on the analysis we have given of the mode equations (9) and (10), these mode sums cannot be de Sitter invariant because the mode functions all show the same exponential growth at late times,

$$u(t,k)u^*(t',k) \rightarrow |K(k)|^2 \times e^{\lambda(t+t')}, \tag{19}$$

$$u_k(\tau)u_k^*(\tau') \rightarrow |K_k|^2 \times e^{\lambda(\tau+\tau')}. \tag{20}$$

The positive-definite constant of proportionality depends upon the time at which the tachyonic mass term $M^2 < 0$ dominates the gradient term [k^2e^{-2Ht} for open coordinates, $H^2k(k+2)\text{sech}^2(H\tau)$ for closed coordinates]. For k so small that the tachyonic mass term always dominates, the constant of proportionality depends upon the time at which the state was released.[5]

At this stage one might wonder how anyone can extract a de Sitter invariant result (2) and (3) from mode sums (17) and (18) which must obviously break de Sitter invariance for $M^2 < 0$. This is especially curious if, at the same time, they admit there is de Sitter breaking for the special values of $M^2 = -N(N+D-1)H^2$, where N is a nonnegative integer [32,35]. This implies believing that, while a tachyonic scalar at one of those values decays, making the mass a little *more* tachyonic would prevent the decay.

We will see that the truth is less paradoxical. They have analytically continued the mode sums (17) and (18) from positive mass-squared—for which there is no de Sitter breaking decay—to negative mass-squared. These analytic continuations preserve the equation of motion (9) and (10) but not the canonical normalization (11) and (12) which ensures that states have positive norm. The resulting mode sum is a formal solution to the propagator equation (1) which is not a true propagator in the sense of being the expectation value of the time-ordered product of two fields in the presence of some positive norm state [38].

The problematic nature of analytic continuations which avoid de Sitter breaking is the same for both open and closed coordinates. In both cases de Sitter breaking derives from more and more large k modes reaching the saturated condition (19) and (20) as time progresses, and this can only be avoided by including negative norm states in the mode sum. However, there are interesting differences between the two coordinate systems as regards infrared (small k) modes. In open coordinates an infinite number of infrared modes are in the saturated condition (19) even at early times, whereas only a finite number of closed coordinate modes have reached the analogous form (20) at any finite time. This results in the open coordinate mode sum possessing an infrared divergence which is absent from the closed coordinate mode sum. We will explore this infrared divergence further in the next two subsections to

[4]Note that these expansions assume the Heisenberg field equation is obeyed in the strong operator sense which is standard for quantum field theory. There have been attempts to avoid de Sitter breaking for tachyons by weakening the sense in which the field equations hold [37].

[5]The massless limit is also interesting. In that case all modes approach a constant $(H/\sqrt{2k^3})$ at late times, which also precludes a de Sitter invariant result.

see that the special thing about $M^2 = -N(N+D-1)H^2$ is one of the power law infrared divergences happens to become logarithmic, and hence visible to analytic regularization techniques. What happens in closed coordinates will be discussed in Sec. III.

B. Analytic continuations miss power law divergences

In quantum field theory we are familiar with two sorts of divergences: ultraviolet and infrared [39]. Both require regularization in order to render potentially divergent expressions well defined so that they can be analyzed. And it can happen that a regularization technique fails to register the presence of a certain class of divergences. For example, dimensionally regulating [40] the quartically divergent vacuum energy of free photons gives zero,

$$2 \times \int \frac{d^3k}{(2\pi)^3} \frac{k}{2} \rightarrow (D-2) \times \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{k}{2} = 0. \tag{21}$$

String theorists are familiar with how the application of zeta function regularization [41] to the central charge of the Virasoro algebra converts the sum of positive integers into a finite, negative number,

$$1 + 2 + 3 + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} \rightarrow \lim_{s \rightarrow -1} \zeta(s) = -\frac{1}{12}. \tag{22}$$

The suppression of obviously positive divergences in expressions (21) and (22) is known as an "automatic subtraction." The origin of the name can be understood if one regulates the ill-defined sum on the left-hand side of (22) using an exponential cutoff,

$$f(\epsilon) \equiv \sum_{n=1}^{\infty} n \times e^{-\epsilon n} = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2} = \frac{1}{4\sinh^2(\frac{\epsilon}{2})}. \tag{23}$$

This sort of method is known as a "physical regularization" because it shows the quadratic divergence in the unregulated limit of $\epsilon \rightarrow 0$,

$$f(\epsilon) = \frac{1}{\epsilon^2} - \frac{1}{12} + O(\epsilon^2). \tag{24}$$

The zeta function result (22) is the $\epsilon \rightarrow 0$ limit of $f(\epsilon)$ with the power law divergence $1/\epsilon^2$ subtracted off. Hence the name, "automatic subtraction." Dimensional regularization can be similarly derived by automatically subtracting power law divergences from a nonlocal exponential cutoff which produces incomplete Gamma functions [42].

Dimensional regularization and zeta function regularization are known as "analytic regularizations" because they work by considering the divergent expression to be an analytic function of some parameter—the spacetime dimension D or the power s to which the eigenvalues of some operator are raised. The function makes sense for certain parameter values and is then defined for all others by analytic continuation. A hallmark of analytic continuation techniques is that they fail to register power law divergences such as those in expressions (21) and (22). On the other hand, they do show the presence of logarithmic divergences. One can see this by comparing the dimensionally regulated result for the vacuum energy of a massive scalar,

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sqrt{k^2+m^2} \rightarrow \mu^{4-D} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{2} \sqrt{k^2+m^2}, \tag{25}$$

$$= -\frac{\Gamma(-\frac{D}{2})}{2(4\pi)^{\frac{D}{2}}} \left(\frac{\mu}{m}\right)^{4-D} m^4, \tag{26}$$

$$= -\frac{m^4}{32\pi^2} \left[\frac{1}{4-D} - \frac{\gamma}{2} + \frac{3}{4} + \frac{1}{2} \ln\left(\frac{4\pi\mu^2}{m^2}\right) + O(4-D)\right]. \tag{27}$$

with the same quantity evaluated using a physical regularization such as a momentum cutoff,

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sqrt{k^2+m^2} \rightarrow \frac{1}{4\pi^2} \int_0^{\Lambda} dk k^2 \sqrt{k^2+m^2}, \tag{28}$$

$$= \frac{1}{32\pi^2} \left[(2\Lambda^3 + m^2\Lambda)\sqrt{\Lambda^2+m^2} - m^4 \ln\left(\frac{\Lambda}{m} + \sqrt{\frac{\Lambda^2}{m^2}+1}\right)\right], \tag{29}$$

$$= \frac{1}{32\pi^2} \left[2\Lambda^4 + 2m^2\Lambda^2 - m^4 \ln\left(\frac{2\Lambda}{m}\right) + \frac{1}{4} m^4 + O\left(\frac{m^2}{\Lambda^2}\right)\right]. \tag{30}$$

The dimensionally regulated result (27) agrees with the logarithmic divergence of the momentum cutoff (30) under the correspondence,

$$\ln\left(\frac{2\Lambda}{m}\right) \leftrightarrow \frac{1}{4-D}. \tag{31}$$

However, the quartic and quadratic divergences in the physical regularization (30) have been automatically subtracted from the analytic regularization (27).

C. Infrared divergences must not be subtracted

Many features of infrared divergences are the same as for ultraviolet divergences. In particular, both require regularization for careful analysis, and analytic regularization techniques automatically subtract off power law divergences from both. However, what this means differs greatly.

As the name indicates, ultraviolet divergences originate from short distance dynamics, far beyond the reach of any experiment. The cure for an ultraviolet divergence is renormalization. In the sense of Bogoliubov-Parasiuk-Hepp-Zimmerman [43] (BPHZ) which is relevant to quantum gravity, this means systematically adding new local interactions to absorb primitive divergences, order-by-order as they occur in the loop expansion. This can always be done, so the only effect of an unphysical regularization technique which happens to automatically subtract off a certain class of divergences is to spare one the effort of constructing the relevant counterterms and using them to absorb the subtracted divergences. In that case no physical error results from using an analytic regularization technique; indeed, it is the faster and simpler way to calculate.

Infrared divergences are very different. They come from dynamical laws that have been thoroughly tested and are not subject to change. The appearance of an infrared divergence is a quantum field theory's way of indicating that something is physically wrong with the question that is being asked of it. The proper course of action in this case is to consider carefully what unphysical assumption led to the divergence, and then pose a more physically meaningful question. *It is a mistake to ignore the problem and continue to ask the same, unphysical question.* However, the naive use of an analytic regularization technique makes it difficult to recognize this mistake, unless the infrared divergence happens to be logarithmic.

Flat space quantum electrodynamics (QED) provides the classic example. Infrared divergences in that theory derive from the exchange of soft photons between the external legs of exclusive amplitudes. The physical problem with exclusive amplitudes is that any real detector has a finite energy resolution, so there is no experimental way to distinguish final states which differ by the inclusion of a very low energy photon [39]. When the question being asked is made physically meaningful by including the emission of arbitrary soft photons whose total energy is less than some fixed detector resolution, the result becomes infrared finite [44]. It also depends upon the detector resolution in a way that agrees with experiment.

Quantum gravity with zero cosmological constant manifests very similar infrared divergences, whose resolution is also achieved by accounting for soft graviton emission [45]. On the other hand, infrared divergences can indicate the breakdown of unphysical assumptions about the symmetry of a state. Veneziano considered the perturbative massless limit of a real scalar field theory with cubic self-interactions in flat space [46]. He showed that including the emission of soft scalars will not cure that theory's infrared divergences. One must instead allow the vacuum to decay, which of course breaks time translation invariance [47].

Ford and Parker discovered an example of great relevance to our discussion in 1977 [48]. They considered a massless, minimally coupled scalar on a spatially flat, FRW background,

$$ds^2 = -dt^2 + a^2(t)d\vec{x}\cdot d\vec{x} \Rightarrow H(t) \equiv \frac{\dot{a}}{a}, \qquad \epsilon(t) \equiv -\frac{\dot{H}}{H^2}. \tag{32}$$

The Fourier plane wave modes of this system are harmonic oscillators with a time dependent mass ($m(t) \sim a^3(t)$) and frequency ($\omega(t) = k/a(t)$). At any instant the ground state energy of each mode is $k/2a(t)$, although which state is the instantaneous ground state changes with time. Ford and Parker specialized to power law scale factors for which the slow roll parameter ϵ is an arbitrary constant. A natural vacuum state—and the one analogous to Bunch-Davies vacuum for de Sitter [49]—is the state which was minimum energy in the distant past. Ford and Parker worked in $D = 4$ spacetime dimensions but it is useful to give the mode function for general D,

$$u(t,k) = a^{-(\frac{D-1}{2})}\sqrt{\frac{\pi}{4(1-\epsilon)H}}H_\nu^{(1)}\left(\frac{-k}{(1-\epsilon)Ha}\right) \quad \text{with} \quad \nu = \frac{D-1-\epsilon}{2(1-\epsilon)}. \tag{33}$$

The Fourier mode sum for the corresponding propagator would be[6]

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\{\theta(t-t')u(t,k)u^*(t',k) + \theta(t'-t)u^*(t,k)u(t',k)\}. \tag{34}$$

[6]We might note that setting $\epsilon = 0$ and $\nu = \sqrt{(\frac{D-1}{2})^2 - \frac{M^2}{H^2}}$ in expression (33) gives the mode function for a massive scalar in open de Sitter coordinates. The closed coordinate mode functions [50] were derived by analytically continuing from Euclidean de Sitter [51].

Ford and Parker showed that expression (34) suffers from infrared divergences throughout the range from $\epsilon = 0$ (de Sitter) to $\epsilon = \frac{3}{2}$ (matter domination). This is obvious from the small k form of the mode functions (33),

$$u(t,k)u^*(t',k) \to \frac{4^{|\nu|}(1-\epsilon)^{|2\nu|}\Gamma^2(|\nu|)}{4\pi(1-\epsilon)\sqrt{Ha^{D-1}H'a'^{D-1}}} \times \left[\frac{HaH'a'}{k^2}\right]^{|\nu|}\{1+O(k^2)\}. \tag{35}$$

For most values of ϵ and D the infrared divergences Ford and Parker found are of the power law type that would be invisible to an analytic regularization. One can see this by evaluating (34) for the infrared-finite case of $\epsilon > 2(D-1)/D$, and noting that it produces an analytic function [52,53],

$$[(1-\epsilon)^2HH']^{\frac{D}{2}-1}\frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{(4\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2})} \times {}_2F_1\left(\frac{D-1}{2}+\nu,\frac{D-1}{2}-\nu;\frac{D}{2};1-\frac{y}{4}\right), \tag{36}$$

where the constant ϵ length function (with infinitesimal δ to fix the branch) is

$$y(x;x') \equiv HaH'a'\left[(1-\epsilon)^2\|\vec{x}-\vec{x}'\|^2 - \left(-\frac{1}{Ha}+\frac{1}{H'a'}-i\delta\right)^2\right]. \tag{37}$$

Expressions (36) and (37) are well defined for all values of ϵ and D except for the two discrete series,

$$\epsilon = \frac{2N}{D-2+2N} \quad \text{or} \quad \epsilon = 1+\frac{D-2}{D+2N}. \tag{38}$$

Comparison with expression (35) reveals that these are just the values for which either the leading small k contribution from the mode functions—or one of its k^{2N} corrections—combines with the $k^{D-2}dk$ from the measure to give the dk/k needed to produce a logarithmic infrared divergence [31].

This example is perfect for our purposes because expressions (36) and (37) are how a mathematical physicist would define the MMC scalar propagator (*and the spin two structure function of the graviton propagator*) for a constant ϵ cosmology. For $\epsilon > 0$ this example also avoids the charged issue of de Sitter invariance: there are no symmetries but homogeneity and isotropy, nor can there be any appeal to the full de Sitter manifold. And the analogy with expressions (17) and (2) and (3) could hardly be clearer: even though the original mode sum (34) harbors infrared divergences throughout the range $0 \le \epsilon \le 2(D-1)/D$, these have been automatically subtracted in (36)–(37), except for the discrete cases (38). At those values our mathematically-minded colleague would say that there is an infrared divergence which precludes the assumption (33) of minimum energy in the distant past. But he would insist that (36) must be the correct propagator because it solves the propagator equation away from these values. Before discussing what is wrong with that view we should follow our own injunction by describing why the infrared divergence of Ford and Parker happens and how to fix it.

The unphysical thing about (34) is that it assumes each mode of the initial state was prepared so that it is minimum energy in the distant past (33). This is possible for initially subhorizon modes but causality precludes a local observer from having much effect on modes which are *initially* superhorizon. Of course the same obstacle exists for a local observer to guarantee to prepare the initially superhorizon modes in any other state. However, it is especially problematic to reach the state which was minimum energy in the distant past because the occupation number (relative to the instantaneous ground state) tends to grow like $N(t,k) \sim [H(t_k)a(t)/2k]^2$ [54], so it is quite highly excited for small k, and becoming more so rapidly. The chance of accidentally hitting this state for a dense set of superhorizon modes is zero. Hence the naive mode sum (34) represents an unphysical question which should be reformulated sensibly rather than being blindly defined by analytic continuation.

Two plausible fixes have been proposed to the infrared problem of Ford and Parker:

(i) One can retain infinite space in (34) but assume initial values for the superhorizon modes which are less singular than (33) [55]. Of course the time dependence of the mode functions is determined by the scalar field equation but their initial values and those of their first time derivatives can be freely specified. As long as there is no infrared divergence initially then none can develop [56].

(ii) One can also work on a compact spatial manifold, such as the torus T^{D-1}, for which there are no initially super-horizon modes [57]. Doing this changes (34) from an integral to a sum, but it is generally valid to approximate this sum as an integral with a nonzero lower limit.

In practice there is not much difference between the two fixes in that they both cut off initially superhorizon modes. Note that both fixes endow the propagator with a secular dependence associated with the time elapsed from the initial value surface [31].

D. Distinguishing Green's functions from propagators

The problem with using expression (36) and (37) for the propagator of a MMC scalar on a constant ϵ geometry is that it does not represent the expectation value of the time-ordered product of $\varphi(x)\varphi(x')$ in the presence of any normalizable state. The same comments apply to using

(2) and (3) for the propagator of a tachyonic scalar on de Sitter. Neither statement should come as any surprise. In both cases direct examination of the mode sums—(34) and (17)—show infrared divergences, and in both cases analytic regularizations were used to automatically subtract the power law divergences. *These infrared divergences are not required to solve the propagator equation but they are necessary to make the result a propagator.* And note that the real problem is making *subtractions*, which corresponds to adding negative (and sometimes imaginary) norm states. The subtractions are finite in closed coordinates, but they are equally invalid.

It should be obvious that there are many solutions to the propagator equation which are not true propagators. For example, consider $i/2$ times the sum of the advanced and retarded propagators. Distinguishing a solution to the propagator equation from a true propagator can sometimes be difficult, especially if one is more interested in certain symmetries than in physics.

An illuminating example is the one-dimensional point particle $q(t)$ of mass m in an inverted potential $V(q) = -\frac{1}{2}m\Omega^2 q^2$. This system is especially important because it possesses no infrared divergence, just like the closed coordinate mode sums. One can clearly see how the analytic continuation $\omega \to -i\Omega$ from the conventional harmonic oscillator violates the quantum mechanical requirement that states have positive norm.

The full solution of the Heisenberg operator equation of motion is

$$q(t) = q_0 \cosh(\Omega t) + \frac{\dot{q}_0}{\Omega} \sinh(\Omega t). \tag{39}$$

Of course this system must show quantum mechanical spread. That is evident from the way its propagator breaks time translation invariance,

$$\langle\psi|T[q(t)q(t')]|\psi\rangle = -\frac{i\hbar}{2m\Omega}\sinh[\Omega|t-t'|] + \langle\psi|q_0^2|\psi\rangle\cosh(\Omega t)\cosh(\Omega t') + \langle\psi|\frac{q_0\dot{q}_0 + \dot{q}_0 q_0}{2\Omega}|\psi\rangle\sinh[\Omega(t+t')] + \langle\psi|\frac{\dot{q}_0^2}{\Omega^2}|\psi\rangle\sinh(\Omega t)\sinh(\Omega t'). \tag{40}$$

Let us denote the three expectation values by the letters A, B and C,

$$A \equiv \langle\psi|q_0^2|\psi\rangle, \qquad B \equiv \left\langle\psi\left|\frac{q_0\dot{q}_0 + \dot{q}_0 q_0}{2\Omega}\right|\psi\right\rangle, \qquad C \equiv \left\langle\psi\left|\frac{\dot{q}_0^2}{\Omega^2}\right|\psi\right\rangle. \tag{41}$$

For large t and t' we can express the real part of (40) as,

$$A\cosh(\Omega t)\cosh(\Omega t') + B\sinh[\Omega(t+t')] + C\sinh(\Omega t)\sinh(\Omega t') \to \frac{1}{4}e^{\Omega(t+t')}\{A + 2B + C\}. \tag{42}$$

It is simple to see that the constant $A + 2B + C$ must be positive. First, note $A > 0$ and $C > 0$ because they are the expectation values of positive operators. Only the constant B might be negative. Hence we can write,

$$A + 2B + C \geq A - 2\sqrt{B^2} + C. \tag{43}$$

Now note that A, B and C are constrained by the uncertainty principle and by the Schwarz inequality (with the requirement of normalizability),

$$A \times C \geq \frac{\hbar^2}{4m^2\Omega^2}, \qquad A \times C > B^2. \tag{44}$$

Using the second relation of (44) in (43) allows us to reach the desired conclusion after some simple algebra,

$$A + 2B + C > A - 2\sqrt{AC} + C = (\sqrt{A} - \sqrt{C})^2 \geq 0. \tag{45}$$

Hence the propagator (40) must show secular growth which violates time translation invariance for any valid quantum mechanical state $|\psi\rangle$.

The preceding paragraph is how a quantum physicist would go about solving the propagator equation,

$$-m\left[\frac{d^2}{dt^2} - \Omega^2\right] i\Delta(t;t') = i\hbar\delta(t-t'). \tag{46}$$

However, a mathematical physicist might consider (46) to be simply a second order differential equation he can solve at will. He might be very attracted to the solution that follows from making the analytic continuation $\omega \to -i\Omega$ in the propagator of the simple harmonic oscillator,

$$\frac{\hbar}{2m\omega}e^{-i\omega|t-t'|} \to \frac{i\hbar}{2m\Omega}e^{-\Omega|t-t'|}. \tag{47}$$

The right-hand side of expression (47) solves the propagator equation (46); it is also time translation invariant, and it falls off exponentially as the difference between t and t' grows. However, comparison with expression (40) reveals some very peculiar quantum mechanics,

$$\frac{i\hbar}{2m\Omega}e^{-\Omega|t-t'|} \Rightarrow A = -C = \frac{i\hbar}{2m\Omega}, \qquad B = 0. \tag{48}$$

There is no normalizable state $|\psi\rangle$ for which positive operators such as q_0^2 and $\dot{q}_0^2$ can have imaginary expectation values. Expression (47) is a formal solution to the propagator equation which is not a true propagator, even though it derives from analytic continuation ($\omega \to -i\Omega$) of a true propagator. The same comments pertain to expressions (36) and (37), for $0 \leq \epsilon \leq 2(D-1)/D$, and to expressions (2) and (3), for $M^2 \leq 0$.

E. Math versus physics

We have seen that the application of analytic continuation to an infrared singular mode sum—such as (17) or (34)—whose order of divergence depends upon some free parameter—such as the mass-squared [23], the dimension of spacetime [23,30], or the cosmological deceleration parameter [31]—will only reveal divergences for the discrete, special values of the parameter that happen to produce logarithmic divergences. These special values always abut, at least on one side, a continuum for which analytic continuation is equally invalid on account of power law divergences that simply fail to show up in the analytic continuation. In this case the use of analytic continuation produces a formal solution to the propagator equation which is not a propagator in the sense of being the expectation value of the time-ordered product of two fields in the presence of a normalizable state [38].

This has been pointed out before [23] but the conclusion is not uniformly accepted [35]. Indeed, Morrison claims to have verified the results of analytic continuation in the signature by demonstrating that they agree with analytic continuation in the mass-squared [32]. Of course what he actually showed is that both analytic continuations make the same error of automatically subtracting power law infrared divergences. Some mathematical physicists also attempt to justify their analytic continuations in closed coordinates, where there are no infrared divergences. We will treat that in detail in the next section.

To close this section it is interesting, and in a sense more powerful, to briefly discuss the phenomenological consequences that would follow if the mathematical viewpoint was to be accepted. The basic problem is that tachyonic scalars roll down their potentials, even in de Sitter space. While this is admitted for $M^2 = -N(N+D-1)H^2$, it is denied for all other tachyonic masses. The conclusion would be that we have a physical theory with the bizarre feature that a scalar with one of these special masses does decay, but (it is claimed that) the decay can be stabilized by making the mass a little *more* tachyonic.

Much worse follows when we turn to the standard model Higgs scalar whose tachyonic mass term is responsible for spontaneous symmetry breaking. Consider the Gedanken experiment of formulating the standard model in the symmetric vacuum on de Sitter background. Does the Higgs field roll down its tachyonic potential to break $SU(2) \times U(1)$ and give mass to the $W^{\pm}$ and Z^0 bosons, along with the quarks and charged leptons? Note that we can make the de Sitter Hubble constant enormously smaller than the magnitude of the tachyonic Higgs mass. For example, in the current universe it would be 44 orders of magnitude smaller. In this context the claim implies that the Higgs would not roll down its potential unless the tachyonic mass happens to agree with one of the discrete values $M^2 = -N(N+D-1)H^2$. So spontaneous symmetry breaking would be controlled by the gravitational parameter H whose scale is 44 orders of magnitude below the electroweak scale. Furthermore, minuscule fractional changes in the Higgs mass would lead to completely different physics.

III. ZERO MODES ARE NOT THE PROBLEM

Mathematical physicists distrust open coordinates because they do not cover the full de Sitter manifold. They suspect that the naive use of open coordinates has misled cosmologists into making subtle errors, and that a clearer picture emerges in closed coordinates. In particular, the closed coordinate mode functions are discrete so there can be nothing like the accumulation of very small Fourier k modes which leads to the infrared divergence of the open coordinate mode sums for the MMC scalar and graviton propagators. Moreover, the belief is that the infrared divergence of the MMC scalar propagator is reflected, in closed coordinates, by the fact that there is a discrete zero mode. Because the graviton has no such zero mode, they argue that there can be no problem with the graviton propagator.

This view of the genesis of de Sitter breaking in the MMC scalar has long been recognized as false. Let us quote from the classic 1985 discussion by Bruce Allen [36]:

> *It is often believed that "what goes wrong" when $m^2 = 0$ has something to do with the fact that the wave equation has a constant solution, which is often called a "zero mode." This is simply not true.*

In fact the problem with the massless, minimally coupled scalar is not its single zero mode but rather the way *all* modes behave at late times. This emerges clearly from Eq. (4.5) of the paper by Allen and Folacci [6], in which the zero mode is excluded from the scalar mode sum in $D = 4$ closed coordinates. In our notation this relation reads,

$$G^{(1)}_{\rm NZM}(x;x') \equiv 2\,{\rm Re}\left[\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\sum_{m=-\ell}^{\ell} u_k(\tau){\rm Y}_{k\ell m}(\chi,\theta,\phi)\times u_k^*(\tau'){\rm Y}^*_{k\ell m}(\chi',\theta',\phi')\right], \tag{49}$$

$$= \frac{H^2}{4\pi^2}\left[\frac{4}{y} - \ln\left[\frac{y(x;x')}{\cosh(H\tau)\cosh(H\tau')}\right] - {\rm sech}^2(H\tau) - {\rm sech}^2(H\tau')\right]. \tag{50}$$

The de Sitter breaking logarithm in the coincidence limit does not arise from the zero mode but rather from the late time limiting form which *all* modes approach,

$$u_k(\tau) \to \frac{H}{2\pi k^{\frac{3}{2}}}. \tag{51}$$

This form sets in for $H\tau \gtrsim \ln(k)$, before which there is destructive interference from oscillations, so one is effectively summing $1/k$ up to $k \sim e^{H\tau}$,

$$\sum_{k=1}^{e^{H\tau}}\sum_{\ell=0}^{k}\sum_{m=-\ell}^{\ell}|u_k(\tau){\rm Y}_{k\ell m}(\chi,\theta,\phi)|^2 \approx \int_1^{e^{H\tau}} dk k^2 \times \frac{H^2}{4\pi^2 k^3} = \frac{H^2}{4\pi^2}H\tau. \tag{52}$$

Because $u_k(\tau)$ and $u_k^*(\tau')$ both approach the same form (51), each mode contributes *positively.* One can only avoid the de Sitter breaking growth by including negative norm states.

It will be seen that the de Sitter breaking, secular growth of expressions (50) and (52) derives from the fact that more and more modes approach the saturated condition (51) as time progresses. Hence de Sitter breaking originates in the large (but still finite) k end of the mode sum, where there is not even any distinction between open and closed coordinates. In particular, the presence or absence of a zero mode is irrelevant.

Cosmologists ascribe two properties to modes which obey (51):

(i) *Freezing in*; and
(ii) *Scale invariance*.

These two properties are why the primordial scalar power spectrum can be observed during the current epoch, so no amount of clever mathematics can make them disappear. Graviton mode functions possess the same key features of freezing in and scale invariance. Hence the closed coordinate mode sum for the coincidence limit of the graviton propagator must possess the same de Sitter breaking infrared logarithm. This is not some sort of gauge artifact, it is precisely why the tensor power spectrum from primordial inflation can be observed during the current epoch. In Sec. IV we will see that the presence of this de Sitter breaking infrared logarithm is not being recognized by mathematical physicists partly because they substitute formal solutions for the original, de Sitter breaking mode sums and partly because they employ analytic continuation techniques to evaluate those mode sums they do consider.

Before concluding we should comment on the closed coordinate mode sum (18) for tachyons. This is free of infrared divergences, which has prompted some mathematical physicists to claim that it fails to show de Sitter breaking [35]. That is incorrect. The de Sitter breaking of the closed coordinate mode sum derives from more and more modes approaching the saturated condition (20) as time progresses. Because each mode contributes positively, there is no way to avoid this without violating the canonical normalization condition (12) that all states have positive norm. Analytically continuing from M^2 positive to negative represents such a violation, just as we saw with the equally illegitimate analytic continuation $\omega \to -i\Omega$ for the inverted harmonic potential (48). In fact, the use of negative norm states to produce a de Sitter invariant solution to the tachyon propagator equation is admitted by Faisal and Higuchi [35]:

> *We note in passing that the modes $\Phi^{(\ell\ell_2 m)}$ with positive norm form a unitary representation of the de Sitter group if L_0 is an integer whereas for a positive non-integer value of L_0 no unitary representation exists because of the negative norm modes.*

The authors do not seem to have realized that this admission precludes their solution from being a true propagator.

Faisal and Higuchi are also wrong in employing the term "infrared divergence" to describe what happens for $M^2 = -N(N+D-1)H^2$, which is the case of their quantity L_0 being a non-negative integer. The problem actually arises from their analytically continued mode functions becoming degenerate. These mode functions consist of powers of the scale factor times associated Legendre polynomials $P^\mu_\nu(z)$ evaluated at $z = i\sinh(H\tau)$ [50,51]. Because the associated Legendre polynomial is evaluated at an imaginary argument, the mode function and its complex conjugate are linearly independent for most values of M^2, leading to a nonzero (although negative) Wronskian. When $M^2 = -N(N+D-1)H^2$ the mode function becomes proportional to its complex conjugate so that the Wronskian between them vanishes the same way it does for $J_\nu(z)$ and $J_{-\nu}(z)$ when ν becomes an integer. That could have been avoided by employing the second linearly independent solution, $Q^\mu_\nu(z)$, along with $P^\mu_\nu(z)$, whose peculiar time dependence would make de Sitter breaking even more obvious.

IV. SPIN 2 SECTOR OF THE GRAVITON PROPAGATOR

One of the nice features of Morrison's paper is that he has identified precisely where the two approaches diverge in constructing the graviton propagator when a de Sitter invariant gauge condition is imposed as a strong operator equation,[7]

$$g^{\rho\sigma}\left[h_{\mu\rho;\sigma}-\frac{\beta}{2}h_{\rho\sigma;\mu}\right]=0. \tag{53}$$

In that case the propagator consists of a spin zero part which derives from the constrained part of the gravitational field and a transverse-traceless (spin two) sector which derives from the $\frac{1}{2}D(D-3)$ dynamical gravitons and the remaining $(D-1)$ constrained fields. The spin zero structure function involves the scalar propagator for $M^2=-2(D-1)H^2/(2-\beta)$, which is infrared singular and de Sitter breaking for all $\beta<2$ [26]. The comments of Sec. II have already addressed the curious contention that there is no de Sitter breaking except for the discrete values of $\beta=2-2(D-1)/N(N+D-1)$ [32,35]. In this section we discuss what Morrison's work says about the difference between the two approaches regarding the spin two sector.

A. The price of de Sitter invariance

This subsection begins by summarizing notation. Then we review the derivation employed [24] for the de Sitter breaking solution to the spin two sector of the propagator. The subsection closes by identifying the two points at which Morrison's de Sitter invariant construction deviates from ours.

In any coordinate system we define the graviton field $h_{\mu\nu}$ by subtracting the de Sitter background $g_{\mu\nu}$ from the full metric,

$$g^{\rm full}_{\mu\nu}\equiv g_{\mu\nu}+\kappa h_{\mu\nu},\qquad \kappa^2\equiv 16\pi G. \tag{54}$$

By convention its indices are raised and lowered with the de Sitter background metric. Covariant derivatives with respect to the de Sitter background are represented by D_α and $\square\equiv g^{\alpha\beta}D_\alpha D_\beta$.

The spin two part of the graviton propagator takes the form [24],

$$i[{}_{\mu\nu}\Delta^2_{\rho\sigma}](x;x')=\frac{1}{4H^4}\mathbf{P}^{\alpha\beta}_{\mu\nu}(x)\times\mathbf{P}_{\rho\sigma}{}^{\kappa\lambda}(x')\times[\mathcal{R}_{\alpha\kappa}(x;x')\mathcal{R}_{\beta\lambda}(x;x')\mathcal{S}_2(x;x')], \tag{55}$$

where $\mathcal{S}_2(x;x')$ is the spin two structure function, $\mathcal{R}_{\alpha\kappa}(x;x')$ is a mixed second derivative of the de Sitter length function $y(x;x')$ (3), normalized to give $g_{\alpha\kappa}$ in the coincidence limit,

$$\mathcal{R}_{\alpha\kappa}(x;x')\equiv-\frac{1}{2H^2}\frac{\partial^2y(x;x')}{\partial x^\alpha\partial x'^\kappa}\Rightarrow\mathcal{R}_{\alpha\kappa}(x;x)=g_{\alpha\kappa}(x), \tag{56}$$

and $\mathbf{P}_{\mu\nu}{}^{\alpha\beta}(x)$ is the transverse-traceless projector,

$$\begin{aligned}\mathbf{P}_{\mu\nu}{}^{\alpha\beta}\equiv\frac{1}{2}\left(\frac{D-3}{D-2}\right)\Big\{&-\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}[\square-DH^2][\square-2H^2]\\&+2D_{(\mu}[\square+H^2]\delta^{(\alpha}_{\nu)}D^{\beta)}-\left(\frac{D-2}{D-1}\right)D_{(\mu}D_{\nu)}D^{(\alpha}D^{\beta)}\\&+g_{\mu\nu}g^{\alpha\beta}\left[\frac{\square^2}{D-1}-H^2\square+2H^4\right]\\&-\frac{D_{(\mu}D_{\nu)}}{D-1}[\square+2(D-1)H^2]g^{\alpha\beta}\\&-\frac{g_{\mu\nu}}{D-1}[\square+2(D-1)H^2]D^{(\alpha}D^{\beta)}\Big\}.\end{aligned} \tag{57}$$

Our form (55) is preferable to the representation employed in the mathematical physics literature [10] because its tensor structure makes no assumption of de Sitter invariance and because the essential spacetime dependence is represented using only a *single* scalar structure function $\mathcal{S}_2(x;x')$, rather than having a distinct scalar coefficient function for each of the five de Sitter invariant tensor factors.[8] It is also worth pointing out that this representation could be generalized to any background if we note that the transverse-traceless projector is $\mathbf{P}_{\mu\nu}{}^{\alpha\gamma}\equiv\mathcal{P}_{\mu\nu}{}^{\alpha\beta\gamma\delta}D_\beta D_\delta$ [24], where the second order differential operator $\mathcal{P}_{\mu\nu}{}^{\alpha\beta\gamma\delta}$ can be read off from the linearized Weyl tensor [33],

$$C^{\alpha\beta\gamma\delta}=\mathcal{P}_{\mu\nu}{}^{\alpha\beta\gamma\delta}\times h^{\mu\nu}+O(h^2). \tag{58}$$

The operator $\mathbf{P}_{\mu\nu}{}^{\alpha\beta}$ has four important properties. The first two are transversality and tracelessness on each of its index groups [24],

$$\begin{aligned}g^{\mu\nu}\times\mathbf{P}_{\mu\nu}{}^{\alpha\beta}=0=\mathbf{P}_{\mu\nu}{}^{\alpha\beta}\times g_{\alpha\beta},\\D^{\mu}\times\mathbf{P}_{\mu\nu}{}^{\alpha\beta}=0=\mathbf{P}_{\mu\nu}{}^{\alpha\beta}\times D_{\alpha}.\end{aligned} \tag{59}$$

The third property is commuting with the d'Alembertian [24],

[7] For $\beta=2$ condition (53) cannot be imposed because it implies the vanishing of the linearized Ricci scalar, which is gauge invariant at linearized order.

[8] If one imposes only the cosmological symmetries of homogeneity and isotropy, the number of tensor factors rises to 14 [25].

$$\Box \times \mathbf{P}_{\mu\nu}{}^{\alpha\beta} = \mathbf{P}_{\mu\nu}{}^{\alpha\beta} \times \Box. \tag{60}$$

And the final property concerns its square [24],

$$\mathbf{P}_{\mu\nu}{}^{\gamma\delta} \times \mathbf{P}_{\gamma\delta}{}^{\alpha\beta} = -\frac{1}{2}\left(\frac{D-3}{D-2}\right)[\Box - 2H^2][\Box - DH^2]\mathbf{P}_{\mu\nu}{}^{\alpha\beta}. \tag{61}$$

The product of the transverse projector and two factors of $\mathcal{R}$ also obeys an important commutation relation [24],

$$\Box\mathbf{P}_{\mu\nu}{}^{\alpha\beta}(x)\mathcal{R}_{\alpha\kappa}(x;x')\mathcal{R}_{\beta\lambda}(x;x') = \mathbf{P}_{\mu\nu}{}^{\alpha\beta}(x)\mathcal{R}_{\alpha\kappa}(x;x')\mathcal{R}_{\beta\lambda}(x;x')[\Box + 2H^2]. \tag{62}$$

The spin two part of the propagator equation reads [24],

$$\frac{1}{2}\left[\Box - 2H^2\right] i\left[{}_{\mu\nu}\Delta^2_{\rho\sigma}\right](x;x') = i\left[{}_{\mu\nu}P^2_{\rho\sigma}\right](x;x'). \tag{63}$$

The quantity on the right-hand side of (63) is i times the transverse-traceless projection operator. It takes the same form (55) as the spin two part of the graviton propagator but with a different structure function $\mathcal{P}_2(x;x')$,

$$i\left[{}_{\mu\nu}P^2_{\rho\sigma}\right](x;x') = \frac{1}{4H^4}\mathbf{P}_{\mu\nu}{}^{\alpha\beta}(x) \times \mathbf{P}_{\rho\sigma}{}^{\kappa\lambda}(x')[\mathcal{R}_{\alpha\kappa}(x;x') \times \mathcal{R}_{\beta\lambda}(x;x')\mathcal{P}_2(x;x')]. \tag{64}$$

It can also be expressed as,

$$i[{}_{\mu\nu}P^2_{\rho\sigma}](x;x') = g_{\mu(\rho}g_{\sigma)\nu} \times \frac{i\delta^D(x-x')}{\sqrt{-g}} + (\text{Traces and Gradients}), \tag{65}$$

where the traces and gradients enforce transversality and traceless.

Acting transverse-traceless projectors on the x^μ and x'^μ dependence of expression (65) and exploiting relations (59–62) gives an equation for the structure function $\mathcal{P}_2(x;x')$,

$$\mathbf{P}_{\mu\nu}{}^{\alpha\beta}(x) \times \mathbf{P}_{\rho\sigma}{}^{\kappa\lambda}(x')\{\mathcal{R}_{\alpha\kappa}\mathcal{R}_{\beta\lambda}\Box[\Box - (D-2)H^2]\Box'[\Box' - (D-2)H^2]\mathcal{P}_2\} = \mathbf{P}_{\mu\nu}{}^{\alpha\beta}(x) \times \mathbf{P}_{\rho\sigma}{}^{\kappa\lambda}(x')\left\{\mathcal{R}_{\alpha\kappa}\mathcal{R}_{\beta\lambda} \times 16\left(\frac{D-2}{D-3}\right)^2 H^4 \frac{i\delta^D(x-x')}{\sqrt{-g}}\right\}. \tag{66}$$

If it is valid to drop the projectors from both sides of (66) we would have a scalar equation for $\mathcal{P}_2(x;x')$ [24],

$$\Box[\Box - (D-2)H^2]\Box'[\Box' - (D-2)H^2]\mathcal{P}_2(x;x') = 16\left(\frac{D-2}{D-3}\right)^2 H^4 \frac{i\delta^D(x-x')}{\sqrt{-g}}. \tag{67}$$

Equation (67) implies that $[\Box - (D-2)H^2]\Box'[\Box' - (D-2)H^2]\mathcal{P}_2(x;x')$ is proportional to the de Sitter breaking propagator of the MMC scalar, so $\mathcal{P}_2(x;x')$ would necessarily break de Sitter invariance as well.

It is straightforward to derive explicit solutions to equations such as (67). The trick is to consider the scalar propagator $\Delta_i(x;x')$ for an arbitrary mass-squared M_i^2,

$$[\Box - M_i^2]i\Delta_i(x;x') = \frac{i\delta^D(x-x')}{\sqrt{-g}}. \tag{68}$$

Then if acting $[\Box - M_1^2]$ on $i\Delta_{12}(x;x')$ produces the propagator $i\Delta_2(x;x')$, the solution is straightforward [23],

$$[\Box - M_1^2]i\Delta_{12}(x;x') = i\Delta_2(x;x') \Rightarrow i\Delta_{12} = \frac{i\Delta_1 - i\Delta_2}{M_1^2 - M_2^2}. \tag{69}$$

The same trick works when the source is an integrated propagator,

$$[\Box - M_1^2]i\Delta_{123}(x;x') = i\Delta_{23}(x;x') \Rightarrow i\Delta_{123} = \frac{i\Delta_{12} - i\Delta_{13}}{M_2^2 - M_3^2}. \tag{70}$$

When two of the masses coincide one gets a derivative with respect to mass-squared. *Note, however, that these relations require one to consider the scalar propagator $i\Delta_i(x;x')$ as an analytic function of its mass-squared M_i^2.* As we have explained in Sec. II, that assumption of analyticity in M_i^2 is only valid if one allows the propagators to break de Sitter invariance when M_i^2 goes from positive to negative, so de Sitter breaking must be evident even for M_i^2 slightly positive.

One can show that the de Sitter breaking of $\mathcal{P}_2(x;x')$ which is implied by Eq. (67) does not drop out of the spin two projection operator (64) [25]. At this point it is obvious from Eq. (63) that the spin two sector of the graviton propagator must break de Sitter invariance as well. Indeed, the same manipulations that led to (67) give,

$$\frac{1}{2}\Box\mathcal{S}_2(x;x') = \mathcal{P}_2(x;x'). \tag{71}$$

and acting $\square[\square - (D-2)H^2]\square'[\square' - (D-2)H^2]$ on both sides gives,

$$\square^2[\square - (D-2)H^2]\square'[\square' - (D-2)H^2]\mathcal{S}_2(x;x') = 32\left(\frac{D-2}{D-3}\right)^2 H^4 \frac{i\delta^D(x-x')}{\sqrt{-g}}. \quad (72)$$

The de Sitter breaking implied for $i[_{\mu\nu}\Delta^2_{\rho\sigma}](x;x')$ by (67–72) has been explicitly worked out [25] and shown to agree with both the noncovariant gauge propagator [12] and with the result in transverse-traceless-spatial gauge [9].

This completes our review of how the de Sitter breaking solution was constructed [24]. Morrison has demonstrated that the de Sitter invariant solutions [10,35] follow by deviating from this procedure at two points [32]:

(1) One must add a special constant to the right-hand side of Eq. (72)—or equivalently, to the right-hand side of Eq. (67); and
(2) One must solve integrated propagator equations of the form (69) and (70) by assuming that the de Sitter invariant scalar propagator is a meromorphic function of its mass-squared, with simple poles at $M^2 = -N(N+D-1)H^2$.

We have already explained in Sec. II that the second deviation produces formal solutions to the desired equations which are not true propagators. That single observation would suffice to invalidate the mathematical physics solutions, but it happens that the first deviation is also problematic.

B. Why no constant can be added

The motivation for the first deviation is the fact, noted in earlier work [25], that the transverse-traceless projectors annihilate constant shifts in the structure functions,

$$\mathbf{P}_{\mu\nu}{}^{\alpha\beta}(x) \times \mathbf{P}_{\rho\sigma}{}^{\kappa\lambda}(x')[\mathcal{R}_{\alpha\kappa}(x;x')\mathcal{R}_{\beta\lambda}(x;x')] = 0. \quad (73)$$

Hence, it is claimed, one cannot pass from Eq. (66) to (67) [32]. If we were only interested in solving differential equation (63) this conclusion would be correct. However, what we really seek is a propagator, and Sec. II has already demonstrated that propagator equations have many solutions that are not true propagators [38]. When constructing a propagator one can indeed pass from Eq. (66) to (67).

The simplest way to see what is wrong with adding a constant to Eq. (67) is by taking the flat space limit. In that limit the two structure functions become translation invariant,

$$\lim_{H\to 0}\frac{\mathcal{S}_2(x;x')}{4H^4} \equiv S_{\rm flt}(x-x'), \qquad \lim_{H\to 0}\frac{\mathcal{P}_2(x;x')}{4H^4} \equiv P_{\rm flt}(x-x'). \quad (74)$$

The spin two part of the graviton propagator (55) also takes the simple form,

$$i[_{\mu\nu}\Delta^{\rm flt}_{\rho\sigma}](x;x') \equiv \lim_{H\to 0} i[_{\mu\nu}\Delta^2_{\rho\sigma}](x;x'), \quad (75)$$

$$= \frac{1}{4}\left(\frac{D-3}{D-2}\right)^2\left[\Pi_{\mu(\rho}\Pi_{\sigma)\nu} - \frac{\Pi_{\mu\nu}\Pi_{\rho\sigma}}{D-1}\right]\partial^4 S_{\rm flt}(x-x'), \quad (76)$$

where indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$, parenthesized indices are symmetrized, $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat space d'Alembertian and $\Pi_{\mu\nu}$ is the transverse projector,

$$\Pi_{\mu\nu} \equiv \eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu. \quad (77)$$

The 8th order differential operator acting upon $S_{\rm flt}(x-x')$ in expression (76) appears so frequently in this discussion that we will denote it by the symbol $\mathbf{T}_{\mu\nu\rho\sigma}$,

$$\mathbf{T}_{\mu\nu\rho\sigma} \equiv \frac{1}{4}\left(\frac{D-3}{D-2}\right)^2\left[\Pi_{\mu(\rho}\Pi_{\sigma)\nu} - \frac{\Pi_{\mu\nu}\Pi_{\rho\sigma}}{D-1}\right]\partial^4. \quad (78)$$

Of course the factor of $(D-3)^2$ derives from the two Weyl tensors (58) involved in the construction of $\mathbf{T}_{\mu\nu\rho\sigma}$.

The flat space limits of the graviton propagator equation (63) and the defining relation (66) for the transverse-traceless projection operator are

$$\mathbf{T}_{\mu\nu\rho\sigma} \times \frac{\partial^2}{2} S_{\rm flt} = \mathbf{T}_{\mu\nu\rho\sigma} \times P_{\rm flt}, \quad (79)$$

$$\mathbf{T}_{\mu\nu\rho\sigma} \times \partial^8 P_{\rm flt} = \mathbf{T}_{\mu\nu\rho\sigma} \times 4\left(\frac{D-2}{D-3}\right)^2 i\delta^D(x-x'). \quad (80)$$

The point the mathematical physicists dispute is the validity of removing the factors of $\mathbf{T}_{\mu\nu\rho\sigma}$ from Eqs. (79) and (80) to conclude,

$$\partial^{10} S_{\rm flt}(x-x') = 8\left(\frac{D-2}{D-3}\right)^2 i\delta^D(x-x'). \quad (81)$$

If Eq. (81) is accepted, the spin two structure function obeys,

$$\partial^4 S_{\rm flt}(x-x') = 8\left(\frac{D-2}{D-3}\right)^2 \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \frac{(\Delta x^{6-D} - \mu^{D-4}\Delta x^2)}{8(D-6)(D-4)}, \quad (82)$$

where $\Delta x^2 \equiv \eta_{\mu\nu}(x-x')^\mu(x-x')^\nu$. Substituting this form for the structure function into (76) gives the recognized spin two part of the graviton propagator in flat space [58],

$$i[_{\mu\nu}\Delta^{\rm flt}_{\rho\sigma}](x;x') = \frac{1}{4}\Big(\frac{D-2}{D-1}\Big)\Big\{3\eta_{(\mu\nu}\eta_{\rho\sigma)} - (D+2)\frac{[\eta_{\mu\nu}\Delta_\rho\Delta x_\sigma + \Delta x_\mu\Delta x_\nu\eta_{\rho\sigma}]}{\Delta x^2} + 4D\frac{\Delta x_{(\mu}\eta_{\nu)(\rho}\Delta x_{\sigma)}}{\Delta x^2} + D(D-2)\frac{\Delta x_\mu\Delta x_\nu\Delta x_\rho\Delta x_\sigma}{\Delta x^4}\Big\}\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}\Delta x^{D-2}}. \tag{83}$$

Let us see what happens if we exploit what the mathematical physicists assert to be the freedom to add a constant to Eq. (81),

$$\partial^{10}S_{\rm flt}(x-x') = 8\Big(\frac{D-2}{D-3}\Big)^2 i\delta^D(x-x') + M^D. \tag{84}$$

At this point some mathematical physicists object that the only dimensionful constant on de Sitter is H, so any constant we add to the propagator equation must be proportional to H^D, which vanishes in the flat space limit. This is sophistry. Morrison's argument is based on the vanishing of expression (73) so it applies to an *arbitrary* constant. If the ambiguity is real then we must be able to add any constant to the propagator equation, including one which fails to vanish in the flat space limit.

The result of changing the flat space propagator equation to (84) is to change the structure function by a term we might call $\Delta S(x-x')$ which obeys,

$$\partial^4\Delta S(x-x') = \frac{M^D\Delta x^6}{48D(D+2)(D+4)}. \tag{85}$$

The resulting change in the spin two part of the propagator is

$$\mathbf{T}_{\mu\nu\rho\sigma}\Delta S = \frac{(D+1)(D-3)^2M^D}{8(D+4)(D+2)D(D-1)(D-2)^2}\{(D^2+2D-4)\eta_{\mu(\rho}\eta_{\sigma)\nu}\Delta x^2 - (D+2)\eta_{\mu\nu}\eta_{\rho\sigma}\Delta x^2 - 4D\Delta x_{(\mu}\eta_{\nu)(\rho}\Delta x_{\sigma)} + 4[\Delta x_\mu\Delta x_\nu\eta_{\rho\sigma} + \eta_{\mu\nu}\Delta x_\rho\Delta x_\sigma]\}. \tag{86}$$

It is difficult to understand what sort of state could give rise to the long range correlations evident in expression (86).

The addition of ill-behaved terms such as (86) is typical when one solves the propagator equation (79) without demanding that the solution be a propagator. The structure function must be a mode sum in order to give a true propagator, which precludes the addition of constants, or any other function annihilated by the projectors. This is obvious in the spatial Fourier basis appropriate to flat space, and to de Sitter in open coordinates. The case for an extra constant seems better for de Sitter in closed coordinates, because the constant can be represented as $Y_{000}(\chi,\theta,\phi)\times Y^*_{000}(\chi',\theta',\phi')$. However, it will be noted that the temporal dependence does not quite work out. There are two linearly independent zero modes, only one of which is constant, and a proper mode sum must involve both of them.

For those mathematical physicists who still insist on $M\sim H$ we should note that it is not necessary to take the flat space limit to see that adding the constant is problematic. The fact that graviton modes in transverse-traceless-spatial gauge agree with those of the MMC scalar [7], and the analysis of Ford and Parker for the latter [48], imply that transverse-traceless-spatial gravitons suffer from infrared problems for all FRW geometries whose first slow roll parameter $\epsilon\equiv -\dot{H}/H^2$ is constant and in the range $0\le\epsilon\le 2(D-1)/D$. As we have already pointed out, the spin two sector of the graviton propagator for all these cases can be represented as (55), with only a slight generalization of the transverse-traceless projector $\mathbf{P}_{\mu\nu}{}^{\rho\sigma}$. (Indeed, the "generalization" consists of undoing the specialization of the original operator to de Sitter [33].) In particular, the propagator equation for all these cases would allow the addition of terms annihilated by $\mathbf{P}_{\mu\nu}{}^{\rho\sigma}$, and the consequent additions to the graviton propagator would be as unphysical as the one (86) we found for flat space. Only for the de Sitter case of $\epsilon=0$ does the siren call of additional symmetry beguile the mathematically inclined to dispute the passage from (66) to (67).

Consideration of the photon propagator on de Sitter background makes the argument even stronger. The spin one sector of the photon can be given a representation comparable to (55), for which it was indeed the paradigm [24],

$$i[_\mu\Delta^1_\rho](x;x') = -\frac{1}{2H^2}\mathbf{P}_\mu{}^\nu(x) \times \mathbf{P}_\rho{}^\sigma(x')[\mathcal{R}_{\nu\sigma}(x;x')\mathcal{S}_T(x;x')]. \tag{87}$$

The transverse projector $\mathbf{P}_\mu{}^\nu$ in (87) is constructed from the field strength tensor the same way as $\mathbf{P}_{\mu\nu}{}^{\alpha\beta}$ was constructed from the Weyl tensor [33],

$$F^{\alpha\beta} = \mathcal{P}_\mu{}^{\alpha\beta} \times A^\mu \Rightarrow \mathbf{P}_\mu{}^\nu \equiv \mathcal{P}_\mu{}^{\nu\alpha} D_\alpha. \tag{88}$$

The transverse projectors annihilate constants in the spin one sector the same way that the transverse-traceless projectors do (73) in the spin two sector,

$$\mathbf{P}_\mu{}^\nu(x) \times \mathbf{P}_\rho{}^\sigma(x')[\mathcal{R}_{\nu\sigma}(x;x')] = 0. \tag{89}$$

So there is equal justification for adding a constant to the equation for the spin one structure function [24],

$$[\square - (D-2)H^2]^2[\square' - (D-2)H^2]\mathcal{S}_T(x;x') = -2H^2 \frac{i\delta^D(x-x')}{\sqrt{-g}}. \tag{90}$$

The only problem is: *Eq. (90) already gives a de Sitter invariant propagator* [59] *which mathematical physicists accept* [60]. Adding any nonzero constant to (90) would produce a different, and incorrect result. The freedom Morrison claims to have discovered is simply not present.

C. Inequivalence of the two propagators

The coincidence limit provides a very simple way of seeing that no de Sitter invariant solution to the propagator equation (1) can be physically equivalent to our de Sitter breaking propagator. The coincidence limit of our result is [25],

$$\lim_{x'\to x} i[{}_{\mu\nu}\Delta^2_{\rho\sigma}](x;x') = (\text{Const})\left[2g_{\mu(\rho}g_{\sigma)\nu} - \frac{2}{D}g_{\mu\nu}g_{\rho\sigma}\right] + \left(\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}2Ht + \text{Const}\right) \times \left[2g^\perp_{\mu(\rho}g^\perp_{\sigma)\nu} - \frac{2}{D-1}g^\perp_{\mu\nu}g^\perp_{\rho\sigma}\right], \tag{91}$$

where $g^\perp_{\mu\nu}$ is the purely spatial part of the metric. By contrast, the coincidence limit of a de Sitter invariant solution to the propagator equation could only have the de Sitter invariant first line of (91); it could never contain the explicitly time dependent factor of Ht, or the de Sitter breaking tensor structure of the second line. These de Sitter breaking features agree with the traceless part of the noncovariant graviton propagator [12], and with the result in transverse-traceless-spatial gauge [9]. The physical origin of the secular growth evident in (91) is the same as for the coincidence limit of the MMC scalar [5]: as time progresses, more and more modes experience first horizon crossing and become constant. This is not a gauge artifact but rather the mechanism by which quantum fluctuations from primordial inflation become fossilized so that they can be observed at late times.

A mathematical physicist might be tempted to dismiss the coincidence limit of a propagator as too singular to provide a good comparison but it makes perfect sense in dimensional regularization. Figures 1 and 2 also show that the coincident graviton propagator contributes to every single one of the graviton loops for which fully dimensionally regulated results have so far been obtained on de Sitter background [15–20]. And the de Sitter breaking evident in the coincidence limit (91) is of course present as well for $x'^\mu \neq x^\mu$. Taking the coincidence limit is just the most obvious way of demonstrating that de Sitter breaking is a real effect.

Morrison argues that our de Sitter breaking propagator is nonetheless "physically equivalent" to his de Sitter invariant solution to the propagator equation. The argument consists of showing that smearing with the transverse-traceless test functions of Fewster and Hunt [34] makes the de Sitter breaking difference drop out [32],

$$\int d^Dx f_1^{\mu\nu}(x) \int d^Dx' f_2^{\rho\sigma}(x') \times i[{}_{\mu\nu}\Delta^{2\text{br}}_{\rho\sigma}](x;x') = 0. \tag{92}$$

Morrison interprets (92) to mean that the de Sitter breaking contributions to our propagator are pure gauge. That cannot be so because the identical time dependence occurs in the completely fixed transverse-traceless-spatial gauge [9]. The actual explanation is that transverse-traceless smearing test functions do not completely scrutinize free graviton fields because no sequence of them can be made to approach a delta function. In this feature they show a critical difference from the scalar test functions [62] on which they were surely based.

To see the problem we may as well consider $D=4$ flat space. Mathematical physicists working in constructive quantum field theory consider a scalar field $\varphi(x)$ to be an operator-valued distribution which is too singular to be studied in its original form, as a function of spacetime [62]. They instead "smear" $\varphi(x)$ against smooth test functions $f(x)$,

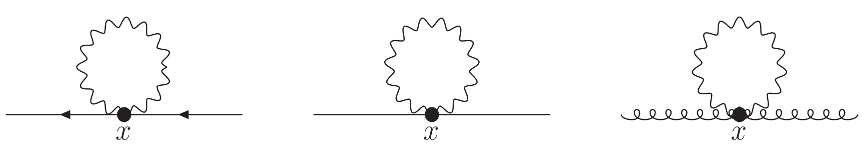

FIG. 1. Coincident graviton propagator contributions to various matter field 1PI 2-point functions. The leftmost diagram is from the one loop fermion self-energy [16,19]; the center figure is from the one loop scalar self-mass-squared [17]; and the rightmost diagram is from the one loop vacuum polarization [20].

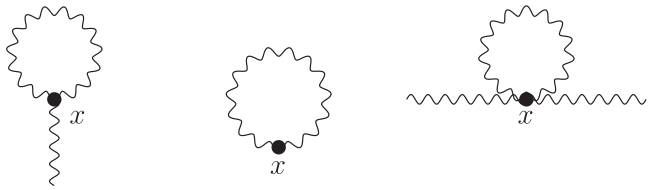

FIG. 2. Coincident graviton propagator contributions to various one loop 1PI functions and expectation values in pure gravity. The leftmost diagram is from the one loop graviton 1-point function [15]; the center figure is from the one loop expectation value of the square of the Weyl tensor [18,27]; and the rightmost diagram is from an old computation of the graviton self-energy with a momentum cutoff [61] which is being redone with dimensional regularization.

$$\varphi(x) \to \varphi[f] \equiv \int d^4x f(x)\varphi(x). \tag{93}$$

The point is to be able to prove nonperturbative theorems. Of course there is no formalism of quantum gravity which makes sense beyond the realm of regulated perturbation theory, and there is absolutely no need for smearing when using regulated perturbation theory as we are. However, nothing is lost by using the smearing formalism for scalars because one can form delta sequences of test functions which approach a delta function,

$$f_n(x;x') \Rightarrow \delta^4(x-x'). \tag{94}$$

Were we to represent the test function in Fourier space the analogous statement would be

$$f_n(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{f}_n(t,\vec{k}) \Rightarrow \tilde{f}_n(t,\vec{k}) \to \delta(t-t')e^{-i\vec{k}\cdot\vec{x}'}. \tag{95}$$

Let us now consider how smearing works for linearized gravitons. A general transverse-traceless test function might be defined as,

$$f^{\mu\nu}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \sum_{\lambda=\pm} \epsilon^{\mu\nu}(\vec{k},2\lambda)\tilde{f}(t,\vec{k},\lambda). \tag{96}$$

The graviton polarization tensors $\epsilon^{\mu\nu}(\vec{k},2\lambda)$ can be expressed by taking products of photon polarization vectors,

$$\epsilon^{\mu\nu}(\vec{k},2\lambda) = \epsilon^{\mu}(\vec{k},\lambda) \times \epsilon^{\nu}(\vec{k},\lambda). \tag{97}$$

We define the latter to be purely spatial and transverse. Transversality can be explicitly enforced by expressing the Fourier wave vector in spherical coordinates $\vec{k} = k\hat{r}$ with the spherical unit vectors defined as usual,

$$\hat{r} \equiv \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}, \tag{98}$$

$$\hat{\theta} \equiv \cos(\theta)\cos(\phi)\hat{x} + \cos(\theta)\sin(\phi)\hat{y} - \sin(\theta)\hat{z}, \tag{99}$$

$$\hat{\phi} \equiv -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}. \tag{100}$$

Because $\lambda = \pm 1$ we can define a general transverse polarization vector as,

$$\epsilon^i(\vec{k},\lambda) \equiv \frac{1}{\sqrt{2}}(\hat{\theta}^i + i\lambda\hat{\phi}^i) \Rightarrow \eta_{\mu\nu}\epsilon^{\mu}(\vec{k},\lambda)\epsilon^{\nu}(\vec{k},\lambda') = \delta_{\lambda\lambda'}. \tag{101}$$

The rest of the derivation is straightforward. If any sequence of $\tilde{f}(t,\vec{k},\lambda)$ did lead to a delta function comparison with (95) suggests that it would be

$$\tilde{f}_n(t,\vec{k},2\lambda) \to \delta(t-t')e^{-i\vec{k}\cdot\vec{x}'}(\delta_{\lambda+} + \delta_{\lambda-}). \tag{102}$$

To see that the Fourier transform of (102) cannot produce a 4-dimensional delta function, simply choose the $\hat{z}$ axis of $\vec{k}$ parallel to $\Delta\vec{x} \equiv \vec{x} - \vec{x}'$, which leaves only the polarization tensors depending upon the azimuthal angle ϕ. For either polarization one finds,

$$\int_0^{2\pi} d\phi \epsilon^i \epsilon^j = \frac{\pi}{2}\sin^2(\theta)\left(3\frac{\Delta x^i \Delta x^j}{\Delta x^2} - \delta^{ij}\right). \tag{103}$$

Performing the θ and r integrations gives,

$$f_n^{ij}(t,\vec{x}) \to \frac{\delta(t-t')}{8\pi\Delta x^3}\left(3\frac{\Delta x^i \Delta x^j}{\Delta x^2} - \delta^{ij}\right). \tag{104}$$

Expression (104) is transverse, traceless and purely spatial, but it does not let us recover the original graviton field. In fact, expression (104) vanishes at $\vec{x}' = \vec{x}$ if one employs it inside an integral for which the angular average gives $3\Delta x^i \Delta x^j/\Delta x^2 \to \delta^{ij}$. In particular, one can never approach the coincidence limit by smearing two transverse-traceless test functions as in Morrison's identity (92). So it is not correct to say that our de Sitter breaking propagator is physically equivalent to his de Sitter invariant solution of the propagator equation; rather he has not permitted himself to scrutinize the difference between them with sufficient resolution. And we might note that it was already obvious from the agreement of the linearized Weyl-Weyl correlators [18] that a high resolution probe is needed to detect the difference.

D. Tensor power spectrum gives the coincidence limit

A probe with the required sensitivity is at hand in the form of the primordial tensor power spectrum. The tensor power spectrum derives from the late time limit of the spatial Fourier transform, in open coordinates, of the

2-point correlator of the graviton field in transverse-traceless-spatial gauge [4]. In our notation (54) everything but the late time limit would be

$$\Delta_h^2(k,t) \equiv \frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k}\cdot\vec{x}} \left\langle \Omega \middle| \frac{\sqrt{2}\kappa}{a^2(t)} h_{ij}(t,\vec{x}) \times \frac{\sqrt{2}\kappa}{a^2(t)} h_{ij}(t,\vec{0}) \middle| \Omega \right\rangle, \tag{105}$$

$$= \frac{k^3}{2\pi^2} \times 32\pi G \times 2 \times |u(t,k)|^2. \tag{106}$$

The actual tensor power spectrum is defined by evolving the tensor mode function $u(t,k)$ past the 1st horizon crossing time (which is $t_k = H^{-1}\ln(k/H)$ for de Sitter) at which they "freeze in" to $u(t,k) \sim H/\sqrt{2k^3}$. The result for de Sitter is

$$\lim_{t \gg t_k} \Delta_h^2(k) \to \frac{16}{\pi} GH^2. \tag{107}$$

The absence of any dependence on the wave number k is known as "scale invariance." It is these two features of freezing in and scale invariance which enable us to observe the primordial power spectra.

Although the "tensor power spectrum" is defined as the late time limit of expressions (105) and (106), it is better to retain the time dependent formulas for our current discussion. The point of this subsection is that there is a simple relation between the power spectrum and the trace of the coincident spin two propagator we have been debating,

$$16\pi G g^{\mu\rho}(x) g^{\nu\sigma}(x) \times i[{}_{\mu\nu}\Delta^2_{\rho\sigma}](x;x) = \frac{5}{2} \times \int \frac{d^3k}{(2\pi)^3} 32\pi G \times 2 \times |u(t,k)|^2, \tag{108}$$

$$= \frac{5}{2} \times \int \frac{d^3k}{(2\pi)^3} \frac{2\pi^2}{k^3} \Delta_h^2(k,t), \tag{109}$$

$$= \frac{5}{2} \times \int \frac{dk}{k} \Delta_h^2(k,t). \tag{110}$$

The factor of $\frac{5}{2}$ derives from the contribution of three constrained fields to the gauge-fixed but unconstrained propagator $i[{}_{\mu\nu}\Delta^2_{\rho\sigma}](x;x')$, whereas the tensor power spectrum has only the two dynamical gravitons.

Because the tensor power spectrum is a gauge invariant observable, relation (110) provides an enormously powerful insight into the de Sitter breaking time dependence of the coincidence limit (91) of the spin two sector of the graviton propagator. First, we note that the naive mode sum is infrared divergent. The physical origin of this infrared divergence is the same as the analogous scalar infrared divergence which was discussed at the end of Sec. II C. With either of the two standard fixes [55,57] the naive mode sum is effectively cut off at some fixed lower limit corresponding to the comoving wave number of the longest wave length which is initially in Bunch-Davies vacuum. The time dependence of the result (110) arises because the time dependent power spectrum $\Delta_h^2(k,t)$ assumes its asymptotic form (107) at the time of first horizon crossing, which is $t_k \sim H^{-1}\ln(k/H)$ for de Sitter. So the integral becomes,

$$\frac{5}{2} \times \int_H^{He^{Ht}} \frac{dk}{k} \times \frac{16}{\pi} GH^2 = \frac{40}{\pi} GH^2 \times Ht. \tag{111}$$

Substituting relation (91) to the left-hand side of (108) gives complete agreement with (111). Note again the complete impossibility of accommodating a de Sitter invariant solution to the propagator equation.

A closely related point has been made before in the context of the totally gauge fixed and constrained propagator in transverse-traceless-spatial gauge. An on-shell field redefinition which carries this de Sitter breaking propagator to a de Sitter invariant one has been given in [8]. Of course it is not possible to change the propagator, while preserving the gauge-fixed and constrained field equations, without altering the canonical commutation relations [9], so their construction is really an excursion into noncanonical quantization. One consequence of the altered quantization scheme is that the usual definition of the tensor power spectrum produces a result which breaks scale invariance [9]. Mathematical physicists retort that one must employ a new, "gauge invariant" definition of the tensor power spectrum which recovers the usual, scale invariant result [8]. They have so far neglected to specify this definition but one might observe first, that any quantity becomes invariant when defined in a unique gauge such as transverse-traceless-spatial gauge [63], and second, that any revised definition of the power spectrum which amounts to using the old, de Sitter breaking propagator in the old way is indistinguishable from simply conceding that free gravitons break de Sitter invariance.

We close by anticipating an objection which might be raised against appealing to the observability of the tensor power spectrum in the context of de Sitter results. The argument goes that perfect de Sitter inflation never ends, therefore modes which have experienced first horizon crossing will never reenter the horizon, which is necessary for them to produce a detectable spatial variation. Hence the power spectrum of de Sitter is unobservable and it cannot be invoked to prove de Sitter breaking. We ask those who attempt to escape the inevitability of de Sitter breaking through recourse to this argument to consider a multiscalar inflation model in which the usual decline of $H(t)$ ceases for a period of time which is controlled by a "clock" provided by one of the other scalars. During the $\dot{H}(t) = 0$ phase the geometry is locally de Sitter and modes which experience first horizon crossing should have the scale invariant amplitude. However, inflation eventually ends so these modes can experience second horizon crossing and become observable to a late time observer. Would such a late time observer measure their power spectrum to be scale invariant? If the answer is conceded to be "yes" then it must be admitted that the coincidence limit of the graviton propagator shows de Sitter breaking.

V. DISCUSSION

The recent paper by Morrison [32] demonstrates the remarkable convergence of opinion on the graviton propagator which has taken place over the past few years. In particular, the allowed gauges are universally agreed, as is almost all of the spacetime dependence and tensor structure in any allowed gauge. The remaining points of disagreement have been narrowed to just seven issues, which we summarize from our perspective:

(1) It is no more valid to define tachyonic mode sums by analytic continuation in the scalar mass-squared than to analytically continue in the dimension, in the signature or in the deceleration parameter. Demonstrating that these analytic continuations all give the same result only shows that they all make the same error of incorporating negative norm states.

(2) The massive scalar propagator breaks de Sitter invariance for all $M^2 \leq 0$ and, by continuity, de Sitter breaking shows up even for $M^2 > 0$ in the solution which is truly an analytic function of M^2. (See also [64,65].) Denying this leads to the nonsensical conclusion that a tachyonic scalar with $M^2 = -N(N + D - 1)H^2$ decays, but making M^2 slightly *more* tachyonic stabilizes it.

(3) The de Sitter breaking of the massless, minimally coupled scalar propagator is not due to its isolated zero mode but rather to the fact that all its mode functions approach scale invariant constants. Graviton mode functions approach the same scale invariant constants. These are physical effects, not gauge artifacts, and they show up in closed coordinates as well as on the cosmological patch. This behavior is why the scalar and tensor power spectra from primordial inflation can be observed during the current epoch.

(4) There is no ambiguity in the equation for the spin two structure function if one requires that the propagator and the projection operator be positive norm mode sums. Violating this precept in flat space would compromise unitarity.

(5) Because no sequence of the transverse-traceless smearing functions proposed by Fewster and Hunt [34] recovers the pointwise graviton field, equality of two smeared propagators does not imply their physical equivalence.

(6) The coincidence limit of the graviton propagator—which shows up in every fully dimensionally regulated graviton loop that has so far been computed [15–20]—reveals that our de Sitter breaking propagator is not physically equivalent to any de Sitter invariant solution to the propagator equation.

(7) The time independence and scale invariance of the tensor power spectrum require that the graviton propagator breaks de Sitter invariance.

Rather than regarding the continuing debate over these points as a distasteful controversy to be deplored and avoided, we view it as the embodiment of the scientific method. We hope this paper will continue the process, and we foresee complete concurrence in the near future.

Morrison has carefully laid out the procedures necessary to extract a de Sitter invariant solution from the graviton propagator equation. These are:

(i) One must add a constant to the equation which defines the structure function of the spin two projection operator; and

(ii) One must consider the scalar propagator to be both de Sitter invariant and a meromorphic function of the scalar mass-squared, with simple poles at $M^2 = -N(N + D - 1)H^2$.

Morrison has also derived the precise difference between our de Sitter breaking structure functions and the de Sitter invariant structure functions that result from following his procedures. These differences are rather small for the spin two sector—which had to be the case in view of the fact that the Weyl-Weyl correlators agree [18,27]—but they have the significant effect of making the coincident propagator time dependent. And we emphasize that the coincident propagator enters every single one of the dimensionally regulated graviton loops which have so far been computed [15–20]. It is obvious the two solutions to the propagator equation mediate different physics, and it is important to resolve which one is the true propagator.

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