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Coupled quintessence with double exponential potentials

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We study flat Friedmann–Robertson–Walker models with a perfect fluid matter source and a scalar field nonminimally coupled to matter having a double exponential potential. It is shown that the scalar field almost always diverges to infinity. Under conditions on the parameter space, we show that the model is able to give an acceptable cosmological history of our Universe, that is, a transient matter era followed by an accelerating future attractor. It is found that only a very weak coupling can lead to viable cosmology. We study in the Einstein frame the cosmological viability of the asymptotic form of a class of $f(R)$ theories predicting acceleration. The role of the coupling constant is briefly discussed.

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I. INTRODUCTION

The standard inflationary idea requires that there be a period of slow-roll evolution of a scalar field (the inflaton) during which its potential energy drives the Universe in a quasiexponential expansion. Besides a cosmological constant, a nearly massless scalar field (quintessence) provides the simplest mechanism to obtain accelerated expansion of the Universe within general relativity. Therefore, scalar fields play a prominent role in the construction of cosmological scenarios aiming to describe the evolution of the early and the present Universe.

Earlier investigations in scalar-field cosmology assumed a minimal coupling of the scalar field (see, for example, Refs. [1] and [2] for models containing both a perfect fluid of ordinary matter and a scalar field with an exponential potential, the so-called "scaling" cosmologies, [3] and references therein for scalar-tensor theories with exponential potential and [4] for a phase-space analysis of the qualitative evolution of cosmological models with a scalar field with positive or negative exponential potentials). Inclusion of nonminimal coupling increases the mathematical difficulty of the analysis, yet it is important to consider nonminimal coupling in scalar field cosmology [5]. As is stressed in Ref. [6], the introduction of nonminimal coupling is not a matter of taste; a large number of physical theories predict the presence of a scalar field coupled to matter, and we mention a few important examples.

In the string effective action, the dilaton field is generally coupled to matter in the Einstein frame [7]. In scalar-tensor theories of gravity [5,8], the action in the Einstein frame takes the form

$$S=\int d^4x\sqrt{-g}\{R-[(\partial\phi)^2+2V(\phi)]+2\chi^{-2}L_{\mathrm{m}}(\tilde{g}_{\mu\nu},\Psi)\}, \tag{1}$$

with

$$\tilde{g}_{\mu\nu}=\chi^{-1}g_{\mu\nu},$$

where $\chi=\chi(\phi)$ is the coupling function and matter fields are collectively denoted by Ψ. In particular, for higher-order gravity (HOG) theories derived from Lagrangians of the form

$$f(\tilde{R})+2L_{\mathrm{m}}(\tilde{g}_{\mu\nu},\Psi), \tag{2}$$

it is well known that under the conformal transformation, $g_{\mu\nu}=f'(\tilde{R})\tilde{g}_{\mu\nu}$, the field equations reduce to the Einstein field equations with a scalar field ϕ as an additional matter source. The conformal equivalence can be formally obtained by conformally transforming the Lagrangian (2), and the resulting action becomes [9],

$$S=\int d^4x\sqrt{-g}\{R-[(\partial\phi)^2+2V(\phi)]+2e^{-2\sqrt{2/3}\phi}L_{\mathrm{m}}(e^{-\sqrt{2/3}\phi}g_{\mu\nu},\Psi)\}.$$

Therefore, the Lagrangian of HOG theories is a particular case of the general scalar-tensor Lagrangian with $\chi(\phi)=e^{\sqrt{2/3}\phi}$, in Eq. (1). Nonminimal coupling occurs also in models of chameleon gravity [10], [11],

$$S=\int d^4x\sqrt{-g}\{R-[(\partial\phi)^2+2V(\phi)]+2L_{\mathrm{m}}(\tilde{g}_{\mu\nu},\Psi)\},$$

with

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$$\tilde{g}_{\mu\nu} = e^{2\beta\phi} g_{\mu\nu},$$

where β is a coupling constant. The same form of coupling has been proposed in models of the so-called coupled quintessence [12] (see also Ref. [13] for more general couplings and Ref. [14] for a generalization involving a scalar field coupled both to matter and a vector field).

Variation of the action (1) with respect to the metric g yields the field equations,

$$G_{\mu\nu} = T_{\mu\nu}(g, \phi) + T^{\mathrm{m}}_{\mu\nu}(g, \Psi), \tag{3}$$

where $T^{\mathrm{m}}_{\mu\nu}$ is the matter energy-momentum tensor. The Bianchi identities imply that the total energy-momentum tensor is conserved, and therefore there is an energy exchange between the scalar field and ordinary matter. In all the above examples, the conservation of their sum is provided by the equations (compare to Ref. [12])

$$\nabla^{\mu} T^{\mathrm{m}}_{\mu\nu}(g, \Psi) = Q T^{\mathrm{m}} \nabla_{\nu}\phi, \qquad \nabla^{\mu} T_{\mu\nu}(g, \phi) = -Q T^{\mathrm{m}} \nabla_{\nu}\phi,$$

where $Q \coloneqq d \ln \chi / d\phi$, depends in general on ϕ and T^{m} is the trace of the matter energy-momentum tensor, i.e., $T^{\mathrm{m}} = g^{\mu\nu} T^{\mathrm{m}}_{\mu\nu}(g, \Psi)$. Variation of S with respect to ϕ yields the equation of motion of the scalar field,

$$\Box\phi - \frac{dV}{d\phi} = -Q T^{\mathrm{m}}. \tag{4}$$

In this paper, we study the late-time evolution of initially expanding flat Friedmann–Robertson–Walker (FRW) models, with a scalar field coupled to matter and having a potential of the form

$$V(\phi) = V_1 e^{-\alpha\phi} + V_2 e^{-\beta\phi}, \tag{5}$$

where α, β are positive constants and V_1, V_2 are constants of arbitrary sign. Without loss of generality, we assume $0 < \alpha < \beta$. For $0 < \alpha = \beta$, the case reduces to a single exponential potential. We also assume that the coupling coefficient is a constant, of order $Q \lesssim 1$. The double exponential potential is usually the asymptotic form of other potentials. For example, in Kaluza–Klein theories with d extra dimensions reformulated in the Einstein frame, α and β are $\sqrt{2d/(d+2)}$ and $\sqrt{2(d+2)/d}$, respectively [5]. The physical reason for the choice (5) is that in quintessence models the dark energy is the energy of a slowly varying scalar field ϕ with equation of state $p_{\phi} = w\rho_{\phi}$, $w \simeq -1$. In most of the models of dark energy, it is assumed that the cosmological constant is zero and the potential energy, $V(\phi)$, of the scalar field driving the present stage of acceleration slowly decreases and eventually vanishes as the field approaches the value $\phi = \infty$, [15]. In this case, after a transient accelerating stage, the speed of expansion of the Universe decreases, and the Universe reaches the Minkowski regime. Double exponential potentials of the form (5) were investigated in Refs. [16,17]. Solutions were obtained in Ref. [18] with the ansatz $\dot{\phi} = \lambda H$; see also Ref. [19] for more general couplings. A scalar field with a double exponential potential without coupling to matter was investigated in Ref. [20]. For exact solutions of a scalar field noncoupled to dust with single and double exponential potentials, see Ref. [21]. Quintessence cosmologies of double exponential potentials in the absence of matter were studied in Ref. [22] with the techniques of phase-space analysis. A coupled quintessence field with a double exponential potential and Galileon-like correction were considered in Ref. [23].

The plan of the paper is as follows. In the next section, we write the field equations for flat FRW models as a constrained four-dimensional dynamical system. Assuming an initially expanding Universe, we show that for potentials (5) the scalar field almost always diverges to plus or minus infinity as $t \to \infty$, depending on the signs of V_1, V_2. Using expansion-normalized variables, the system is written as a polynomial three-dimensional system. In Sec. III, we study the equilibrium points and analyze the structure of the solutions. It is shown that under conditions on the parameter space the model is able to give an acceptable cosmological history of our Universe: a transient matter era followed by an accelerating future attractor. In particular, if we assume that ordinary matter satisfies plausible energy conditions, i.e., $\gamma \gtrsim 1$, the scale factor during the matter era evolves approximately as $a \sim t^{2/3}$, provided that the coupling constant, Q, takes very small values. In Sec. IV, we examine the asymptotic form of a popular class of $f(R)$ theories predicting acceleration; in the Einstein frame, this theory is equivalent to a scalar field with a double exponential potential, and we discuss its cosmological viability. Section V is a brief discussion on the acceptable range of the coupling constant.

II. COUPLED SCALAR FIELD MODEL

For homogeneous and isotropic flat spacetimes, the field equations (3) and (4) (ordinary matter is described by a perfect fluid with equation of state $p = (\gamma - 1)\rho$, where $0 < \gamma < 2$) reduce to the Friedmann equation,

$$3H^2 = \rho + \frac{1}{2}\dot{\phi}^2 + V(\phi); \tag{6}$$

the Raychaudhuri equation,

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 - \frac{\gamma}{2}\rho; \tag{7}$$

the equation of motion of the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{4 - 3\gamma}{2} Q\rho; \tag{8}$$

and the conservation equation,

$$\dot{\rho} + 3\gamma\rho H = -\frac{4-3\gamma}{2} Q\rho\dot{\phi}. \quad (9)$$

We adopt the metric and curvature conventions of Ref. [24]. $a(t)$ is the scale factor, an overdot denotes differentiation with respect to time t, $H = \dot{a}/a$, and units have been chosen so that $c = 1 = 8\pi G$. Here, $V(\phi)$ is the potential energy of the scalar field, and $V'(\phi) = dV/d\phi$. Interaction terms between the two matter components of the form $-\alpha\rho\dot{\phi}$ as in Eq. (9) with a simple exponential potential were first considered in Ref. [25] (see also Ref. [26]). Although there is an energy exchange between the fluid and the scalar field, it is easy to see that the set, $\rho > 0$, is invariant under the flow of Eqs. (7)–(9); therefore, ρ is nonzero if initially $\rho(t_0)$ is nonzero, and this trivial physical demand is not satisfied if one assumes arbitrary interaction terms—cf. Ref. [27].

As is explained in the last paragraph of the Appendix, the physically interesting cases are $V_1, V_2 > 0$ or $V_1 > 0$, $V_2 < 0$. The dynamical system (7)–(9) has for $V_1 > 0$, $V_2 < 0$ only one finite equilibrium point, $(\phi = \phi_m, \dot{\phi} = 0, \rho = 0, H = \sqrt{V_{\max}/3})$; see Fig. 1. It represents de Sitter solutions, and it is easy to see that it is unstable. It is known that for potentials having a maximum the field near the top of the potential corresponds to the tachyonic (unstable) mode with negative mass squared [15,28,29]. The other asymptotic states of the system correspond to the points at infinity, $\phi \to \pm\infty$.

For potentials (5) with $V_1, V_2 > 0$, the global result that, for expanding flat models, $\phi \to \infty$ as $t \to \infty$ can be shown. In fact, the following slightly stronger result holds, which generalizes Proposition 4 in Ref. [30].

Proposition 1: Let V be a potential function with the following properties: 1) V is non-negative. 2) V' is continuous and $V'(\phi) < 0$. 3) If $A \subseteq \mathbb{R}$ is such that V is bounded on A, then V' is bounded on A. Then, $\lim_{t\to+\infty} \dot{\phi} = 0 = \lim_{t\to+\infty}\rho$, and $\lim_{t\to+\infty}\phi = +\infty$.

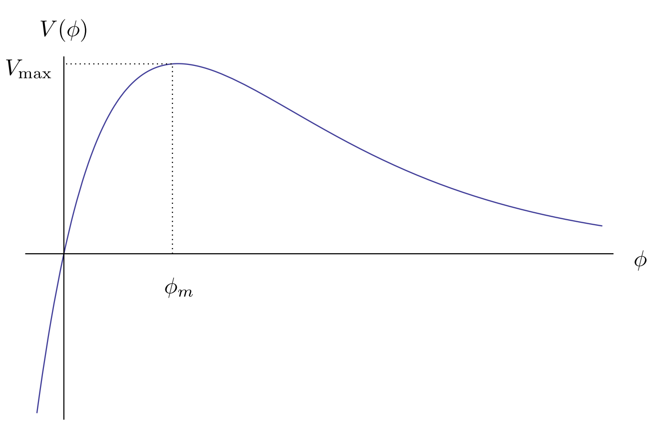

FIG. 1 (color online). Potentials (5) with $V_1 > 0$, $V_2 < 0$ have a local maximum at some ϕ_m and diverge to minus infinity as $\phi \to -\infty$. In this figure, $V_2 = -V_1 < 0$.

Proof: Since $V(\phi) \geq 0$, it follows from Eq. (6) that H is never zero, and thus it cannot change sign. Hence, H is always non-negative if $H(t_0) > 0$. Furthermore, H is decreasing in view of Eq. (7), and thus $H(t) \leq H(t_0)$, for all $t \geq t_0$. We then deduce from Eq. (6) that each of the terms ρ, $\frac{1}{2}\dot{\phi}^2$, and V is bounded by $3H(t_0)^2$. Since H is decreasing, $\exists \lim_{t\to+\infty} H = \eta \geq 0$; therefore, Eq. (7) implies that

$$\frac{1}{2}\int_{t_0}^{+\infty} (\dot{\phi}^2 + \gamma\rho)dt = H(t_0) - \eta < +\infty. \quad (10)$$

In general, if f is a non-negative function, the convergence of $\int_{t_0}^{\infty} f(t)dt$ does not imply that $\lim_{t\to\infty} f(t) = 0$, unless the derivative of f is bounded. In our case and setting $\lambda = (4-3\gamma)Q$,

$$\frac{d}{dt}(\dot{\phi}^2 + \gamma\rho) = -6H\dot{\phi}^2 - 2\dot{\phi}V'(\phi) - 3\gamma^2\rho H + \lambda\left(1-\frac{\gamma}{2}\right)\rho\dot{\phi}$$
$$\leq -2\dot{\phi}V'(\phi) + \lambda\left(1-\frac{\gamma}{2}\right)\rho\dot{\phi}.$$

As we already remarked, $\dot{\phi}$ and ρ are bounded; also, by our assumption on V, $V'(\phi)$ is bounded. We conclude that the derivative of the function $\dot{\phi}^2 + \gamma\rho$ is bounded from above, and therefore Eq. (10) implies that $\lim_{t\to\infty} \dot{\phi}(t)^2 = 0$ and $\lim_{t\to\infty}\rho(t) = 0$.

The proof that $\lim_{t\to+\infty}\phi = +\infty$ follows after suitable adaptation of the arguments used in Proposition 4 in Ref. [30]. ■

If in addition $\lim_{\phi\to+\infty} V(\phi) = 0$, as is the case of the double exponential potential (5), then we conclude that $H \to 0$ as $t \to \infty$.

The case $V_1 > 0$, $V_2 < 0$ is more delicate, and the asymptotic state depends on the initial conditions: (i) If initially $\phi_0 > \phi_m$, and $3H(t_0)^2 < V_{\max}$, then from Eq. (6), $V(\phi)$ remains less than $V_{\max}$ since H is decreasing. We conclude that $V(\phi(t)) < V_{\max}$ for all $t \geq t_0$, and thus ϕ cannot pass to the left of ϕ_m. In the interval $(\phi_m, +\infty)$, the potential satisfies the assumptions of the above Proposition, and therefore $\phi \to \infty$ as $t \to \infty$. (ii) If initially $\phi_0 < \phi_m$, and $\dot{\phi}_0$ is larger than the critical value $\dot{\phi}_{\text{crit}} > 0$, which allows for ϕ to pass on the right of ϕ_m, then the conclusions of case (i) hold. (iii) Finally, suppose that initially $\phi_0 < \phi_m$, and $\dot{\phi}_0$ is less than the critical value $\dot{\phi}_{\text{crit}} > 0$, i.e., $-\infty < \dot{\phi}_0 < \dot{\phi}_{\text{crit}}$. From Eq. (7), H is monotonically decreasing and not bounded below from zero; hence, eventually H may change sign. We cannot use the same argument as in Proposition 1 concerning the asymptotic behavior of $\dot{\phi}(t)^2$ and $\rho(t)$, since V and V' are not bounded. Suppose, first, that $\lim_{t\to+\infty} H = \eta$, where η is finite. But an asymptotic state of the form $\mathbf{p} = (H = \eta, \rho = \rho_*, \dot{\phi} = \dot{\phi}_*, \phi = \phi_*)$ is impossible; i.e., the point $\mathbf{p}$ cannot be an equilibrium point of the dynamical system (7)–(9) for $\phi_* < \phi_m$. Although we cannot exclude periodic orbits, or strange attractors as ω−limit sets for our system, numerical experiments suggest that H diverges to $-\infty$. If this is the case, it can be shown that H diverges to $-\infty$, in a finite time. Suppose, on the contrary that, $\lim_{t\to+\infty} H = -\infty$. Since $\gamma < 2$,

$$3H^2 = \frac{\dot{\phi}^2}{2} + \rho + V(\phi) < \frac{\dot{\phi}^2 + \gamma\rho}{\gamma} + V(\phi) = -\frac{2\dot{H}}{\gamma} + V(\phi);$$

hence,

$$3 < -\frac{2\dot{H}}{\gamma H^2} + \frac{V(\phi)}{H^2}. \tag{11}$$

Taking limits as $t \to +\infty$, and since $V(\phi)$ is bounded from above, $\lim_{t\to+\infty} V(\phi)/H^2 \leq 0$. Inequality (11) implies that $\lim_{t\to+\infty}(-\dot{H}/H^2) \geq 3\gamma/2$, which is impossible, since $-\dot{H}/H^2 = d/dt(1/H)$ and $1/H \to 0$. In view of Eq. (6), $\dot{\phi}^2 + \gamma\rho$ also diverges to infinity. Again, an asymptotic state of the form $H = -\infty$, $\dot{\phi}^2 + \gamma\rho = \infty$ and $\phi =$ finite is impossible, and therefore ϕ diverges to $-\infty$ in a finite time. The above arguments, supported by numerical investigation, establish the following result, although we were unable to prove it rigorously.

Proposition 2: Let V be a C^1 potential function with the following properties: 1) V is negative and monotonically increasing for $\phi < 0$, with $\lim_{\phi\to-\infty} V(\phi) = -\infty$. 2) V has a global maximum at some $\phi_m > 0$. Suppose that the initial conditions hold, $H(t_0) > 0$, $\phi(t_0) < \phi_m$, and $-\infty < \dot{\phi}(t_0) < \dot{\phi}_{\text{crit}}$, where $\dot{\phi}_{\text{crit}} > 0$ is the critical value that allows for ϕ to pass to the right of ϕ_m. Then, H and ϕ diverge to $-\infty$ in a finite time.

This result generalizes previous investigations indicating that negative potentials may drive a flat initially expanding Universe to recollapse; see Refs. [4,31,32]. Negative potentials appear also in ekpyrotic models (see, for example, Ref. [33] and references therein and Ref. [34] with multiple fields).

The function (5) belongs to the class of multiexponential potentials of the form

$$V(\phi) = \sum_{i=1}^{N} V_i e^{-k_i\phi},$$

which arise as a special case of generalized models with multiple fields studied in the context of assisted inflation (see, for example, Ref. [35]; for an elegant mathematical generalization, see Ref. [36]). There exists a well-established mathematical procedure for the investigation of scalar-field cosmologies with exponential potentials in the context of dynamical systems theory [1,24]. It consists of the introduction of the so-called expansion-normalized variables by defining

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \qquad y = \sqrt{\frac{V_1 e^{-\alpha\phi}}{3H^2}}, \qquad z = \sqrt{\frac{V_2 e^{-\beta\phi}}{3H^2}},$$
$$\Omega = \frac{\rho}{3H^2} \tag{12}$$

and a new time variable $\tau = \ln a$. The Friedmann equation (6) imposes the constraint

$$\Omega = 1 - (x^2 + y^2 + z^2) \tag{13}$$

to the state vector (x, y, z, Ω). This equation can be used to eliminate Ω from the evolution equations, and we end up with a three-dimensional dynamical system,

$$\begin{aligned} x' &= \sqrt{6}Q - \frac{3}{2}\sqrt{\frac{3}{2}}\gamma Q + \left(\frac{3\gamma}{2} - 3\right)x + \left(\frac{3}{2}\sqrt{\frac{3}{2}}\gamma - \sqrt{6}\right)Qx^2 \\ &\quad + \left(3 - \frac{3\gamma}{2}\right)x^3 + \left(\sqrt{\frac{3}{2}}\alpha - \sqrt{6}Q + \frac{3}{2}\sqrt{\frac{3}{2}}\gamma Q\right)y^2 \\ &\quad + \left(\sqrt{\frac{3}{2}}\beta - \sqrt{6}Q + \frac{3}{2}\sqrt{\frac{3}{2}}\gamma Q\right)z^2 - \frac{3}{2}\gamma xy^2 - \frac{3}{2}\gamma xz^2, \\ y' &= y\left(\frac{3\gamma}{2} - \sqrt{\frac{3}{2}}\alpha x + \left(3 - \frac{3\gamma}{2}\right)x^2 - \frac{3\gamma}{2}y^2 - \frac{3\gamma}{2}z^2\right), \\ z' &= z\left(\frac{3\gamma}{2} - \sqrt{\frac{3}{2}}\beta x + \left(3 - \frac{3\gamma}{2}\right)x^2 - \frac{3\gamma}{2}y^2 - \frac{3\gamma}{2}z^2\right), \end{aligned} \tag{14}$$

where

$$x^2 + y^2 + z^2 \leq 1, \tag{15}$$

and a prime denotes derivative with respect to τ. Note that y and z can take both real and pure imaginary values, depending on the signs of V_i. With this choice, we avoid having four different dynamical systems (see, however, Ref. [4] in which real normalized variables are used). For V_1, $V_2 > 0$, the phase space (15) is the closed unit ball in $\mathbb{R}^3$. For $V_1 > 0$ and $V_2 < 0$, the phase space is the one sheet hyperboloid $x^2 + y^2 - (\mathrm{Im}z)^2 = 1$ and its interior. The resulting dynamical system depends on four parameters $(\gamma, \alpha, \beta, Q)$. Using Eq. (7), the effective equation of state,

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2},$$

is written in terms of the new variables as

$$w_{\text{eff}} = -1 + 2x^2 + \gamma\Omega.$$

III. COSMOLOGICALLY ACCEPTABLE SOLUTIONS

By inspection, system (14) is symmetric under reflection, with respect to the planes x-z and x-y. The planes $y = 0$ and $z = 0$ are invariant sets for the system (14). The full list and analysis of the critical points of our system are presented in the Appendix. In this section, we discuss only these equilibria, which allow for a viable cosmological history of the Universe. In Table I are shown the equilibria for $V_1 > 0$ and

$$\alpha < \sqrt{2}, \qquad \gamma \le 1,$$
$$(4-3\gamma)Q \in (\max\{0, 2(\alpha^2 - 3\gamma)/\alpha\}, \sqrt{6}(2-\gamma)).$$

The two critical points $\mathcal{A}_\pm$ correspond to kinetic dominated solutions, which are unstable and are only expected to be relevant at early times. Point $\mathcal{B}$ represents a type of scaling solution; i.e., the kinetic energy density of the scalar field remains proportional to that of the perfect fluid. Points $\mathcal{C}_\pm$ are accelerated only for $V_1 > 0$. They correspond to scalar-field-dominated solutions, which exist for sufficiently flat potentials, $\alpha < \sqrt{6}$. These are the same conclusions as in Ref. [37] for an exponential potential and $Q = \sqrt{2/3}$ and also in Refs. [1], [4], and [20] and in the case of a scalar-field noncoupled to matter, although the ranges of the parameters (α, γ) are different. Points $\mathcal{D}_\pm$ exist only in models with $V_1 > 0$, $V_2 < 0$. They correspond to the unstable state $(\phi=\phi_m, \dot{\phi}=0, \rho=0, H=\sqrt{V_{\max}/3})$ and represent de Sitter solutions.

A successful cosmological model should comprise an accelerating solution as a future attractor. It is evident that points $\mathcal{C}_\pm$ could satisfy the condition for acceleration, $w_{\rm eff} < -1/3$, provided that $\alpha < \sqrt{2}$ (compare with the conclusions in Ref. [1]). From now on, we assume this range for the parameter α. Moreover, the equilibria $\mathcal{C}_\pm$ are stable for all physically interesting values of γ. For a cosmological theory to be acceptable, it has to possess a matter-dominated epoch followed by a late-time accelerated attractor. The saddle character of point $\mathcal{B}$ implies that it represents a transient phase, and therefore it is a good candidate for a matter point, provided that Ω is close to 1. This happens only for very small values of the coupling parameter Q and for γ close to 1. Another way to see this is the following. During the matter era, the scale factor has to expand approximately as $a \sim t^{2/3}$. The scale factor near $\mathcal{B}$ evolves as $a \sim t^{\frac{2}{3(w_{\rm eff}+1)}}$; therefore, $w_{\rm eff}$, has to be close to zero. As seen in Table I, $a(t)$ at $\mathcal{B}$ evolves as $t^{2/3}$ when Q takes the values

$$Q = \frac{\sqrt{6(2-\gamma)(1-\gamma)}}{(4-3\gamma)}, \qquad \gamma \le 1. \tag{16}$$

Therefore, the realistic value $\gamma = 1$, corresponding to dust, is incompatible to a scalar field coupled to matter; i.e., the coupling parameter Q must be zero (see also Ref. [38]). On the other hand, Eqs. (8) and (9) imply that for $\gamma = 4/3$ the value of Q is undetermined. Below, we summarize our results for the particular values $\gamma = 1$, $4/3$, $2/3$:

(A) Dust ($\gamma = 1$): The critical points of our system are those of Table I for $\alpha < \sqrt{2}$, $\beta > \alpha$, and $Q = 0$. Note that the future attractors $\mathcal{C}_\pm$ have nonphantom acceleration for every value of α in the interval $(0, \sqrt{2})$. A cosmologically acceptable trajectory should pass near $\mathcal{B}$ and finally land on one of the points $\mathcal{C}_\pm$, depending on the initial conditions. Note that $\mathcal{A}_\pm$, $\mathcal{B}$, and $\mathcal{C}_\pm$ lie on the invariant plane $z = 0$, and $\mathcal{C}_\pm$ exist only in potentials with $V_1 > 0$. We consider the projection of the system (14) on that plane. The phase portrait is shown in Fig. 2 and is the same in both cases in which the phase space is a sphere ($V_2 > 0$), or a one-sheet hyperboloid ($V_2 < 0$).

(B) Radiation ($\gamma = 4/3$): The case of $\gamma = 4/3$ corresponds to radiation, and therefore there is no matter point with a scale factor $a \sim t^{2/3}$. Instead, point $\mathcal{B}$, which coincides with the origin (0, 0, 0), now represents the well-known radiation-dominated solution, $a \sim t^{1/2}$, as a transient phase. $\mathcal{C}_\pm$ are future attractors for $\alpha < \sqrt{2}$.

(C) The value $\gamma = 2/3$ corresponds to ordinary matter marginally satisfying the strong energy condition. Equation (16) implies $Q = \sqrt{2/3}$. An acceptable

TABLE I. Equilibrium Points.

Label	(x, y, z)	Ω	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	Unstable	$t^{1/3}$
$\mathcal{B}$	$(\frac{(4-3\gamma)Q}{\sqrt{6}(2-\gamma)}, 0, 0)$	$1 - \frac{(4-3\gamma)^2 Q^2}{6(2-\gamma)^2}$	Saddle	$t^{4(2-\gamma)/(6\gamma(2-\gamma)+(4-3\gamma)^2 Q^2)}$
$\mathcal{C}_\pm$	$(\frac{\alpha}{\sqrt{6}}, \pm\sqrt{1-\frac{\alpha^2}{6}}, 0)$	0	Stable	t^{2/α^2}
$\mathcal{D}_\pm$	$(0, \pm\sqrt{\frac{\beta}{\beta-\alpha}}, \pm\sqrt{\frac{\alpha}{\alpha-\beta}})$	0	Saddle	e^t

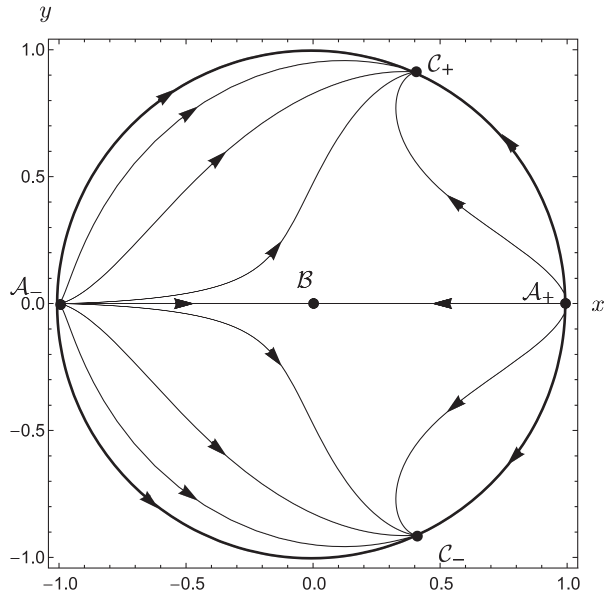

FIG. 2. Phase portrait of the projected three-dimensional system on the invariant set $z = 0$.

trajectory exists for $\alpha < \sqrt{2}$. For these values of α and Q, points $\mathcal{A}_\pm$ are always unstable. Point $\mathcal{B} \equiv (1/2, 0, 0)$ corresponds to the transient matter era, with $\Omega = 3/4$. The accelerated points $\mathcal{C}_\pm$ are future attractors.

Throughout this paper, we do not consider the case $\alpha\beta < 0$ for the potentials (5). The reason is that for $\alpha\beta < 0$ and V_1, $V_2 > 0$ the function $V(\phi)$ in Eq. (5) has a strictly positive minimum, say $V_{\min}$, and the de Sitter solution with $H = \sqrt{V_{\min}/3}$ is the future attractor for the system, Ref. [37]. This follows directly either from the original equations (7)–(9) or from the system (14) written in the new variables. Moreover, it is easy to see that a matter era represented by a saddle equilibrium $\mathcal{B}$ precedes the final accelerated epoch.

IV. ASYMPTOTIC FORM OF SOME $f(R)$ THEORIES PREDICTING ACCELERATION

A large class of dynamical dark energy models is based on the large-distance modification of gravity (see Ref. [39] for recent reviews). For example, in the context of $f(R)$ gravity theories, the models $f(R) = R - \mu^{2(n+1)}/R^n$, where $\mu > 0$, $n > 1$, were proposed to explain the late-time cosmic acceleration [40,41]. The obvious idea is the introduction of modifications to the Einstein–Hilbert Lagrangian, which become important at low curvatures. For these models, the potential functions in the Einstein frame have the form

$$V_n(\phi) = \frac{\mu^2(n+1)n^{1/(n+1)}(e^{\sqrt{2/3}\phi} - 1)^{n/(n+1)}}{2ne^{2\sqrt{2/3}\phi}}. \quad (17)$$

These functions are defined only for $\phi \geq 0$, and their behavior is similar to that indicated in Fig. 1; i.e., they have a local maximum at some ϕ_m depending on n, and for large ϕ, they approach zero exponentially. As $n \to \infty$, the potentials (17) approach the function

$$V(\phi) = \frac{\mu^2}{2}(e^{-\sqrt{2/3}\phi} - e^{-2\sqrt{2/3}\phi}), \quad (18)$$

corresponding to the asymptotic form of these theories, [41]. Thus, Eq. (18) is a particular case of Eq. (5) with $\beta = 2\alpha = 2\sqrt{2/3}$, $V_1 = -V_2 = \mu^2/2 > 0$; cf. Fig. 1. Note that for large ϕ V in Eq. (18) behaves similarly to V_n in Eq. (17). In contrast to the family (17), V in Eq. (18) has the nice property that it is defined for all $\phi \in \mathbb{R}$. As mentioned in the introduction, the coupling coefficient takes the value $Q = \sqrt{2/3}$, regardless of the form of $f(R)$ [42].

The constraint (15) implies that the phase space is the set $x^2 + y^2 - (\mathrm{Im}z)^2 \leq 1$. There are up to seven critical points for that system, depending on the value of γ.

Label	(x, y, z)	Ω	Existence	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	Always	Unstable	$t^{1/3}$
$\mathcal{B}$	$(\frac{4-3\gamma}{3(2-\gamma)}, 0, 0)$	$\frac{4(5-3\gamma)}{9(2-\gamma)^2}$	$\gamma \leq 5/3$	Saddle	$t^{3(2-\gamma)/(8-3\gamma)}$
$\mathcal{C}_\pm$	$(\frac{1}{3}, \pm\frac{2\sqrt{2}}{3}, 0)$	0	Always	Stable	t^3
$\mathcal{D}_\pm$	$(0, \pm\sqrt{2}, \pm i)$	0	Always	Saddle	e^t

Points $\mathcal{C}_\pm$ are future attractors and have nonphantom acceleration with $w_{\text{eff}} = -7/9$. However, in the case of dust, $\gamma = 1$, the scale factor at matter point $\mathcal{B}$ evolves as $a \sim t^{3/5}$, rather than the usual $a \sim t^{2/3}$. The scale factor evolves "correctly" only for $\gamma = 2/3$. The absence of the standard matter epoch is associated with the fact that matter is strongly coupled to gravity. This result is in agreement with the general conclusions in Refs. [42], [43], and [44] that these $f(R)$ dark-energy models are not cosmologically viable.

V. CONCLUSION

In this paper, we have focused on a general treatment of a scalar field with a double exponential potential nonminimally coupled to a perfect fluid. A full analysis of the equilibrium points of the resulted dynamical system is quite complicated, yet it reveals that the model predicts a late accelerated phase of the Universe for a wide range of the parameters, α, β, γ, and Q. Moreover, there exists transient solutions representing a matter era, preceding the accelerating attractor. However, in most cases, the scale factor near these transient phases evolves as $a(t) \sim t^{q(Q)}$, where the exponent q is in general different from the usual $2/3$. The "wrong" matter epoch is associated with the fact that for values of Q of order unity matter is strongly coupled to gravity. A coupling constant of order unity means that matter feels an additional scalar force as strong as gravity itself; cf. Ref. [42]. Assuming that ordinary matter satisfies plausible energy conditions, i.e., $\gamma \gtrsim 1$, the coupling constant, Q, has to be very small; more precisely, $q(Q) \to 2/3$, only for $Q \to 0$. Therefore, only a very weak coupling of the scalar field to ordinary matter can lead to acceptable cosmological histories of the Universe. This surprising result indicates that cosmological evolution imposes strict constraints on the choice of the correct Lagrangian of a gravity theory. In this study, we restricted ourselves to constant couplings; had we let Q be a function of ϕ, the dimension of the dynamical system would have increased by 1. In that case, it would be very interesting to see if the dynamics leads to a very tiny value of Q at late times. Such a result could lead to a generalization of the attractor mechanism of scalar-tensor theories toward general relativity, found by Damour and Nordtvedt in the case of a massless scalar field [45].

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APPENDIX

We present here the full analysis of the stability of the system (14). The critical points are listed in Table II.

We assume that $0 < \alpha < \beta$. The case $0 < \beta < \alpha$ is a mere renaming of some of the equilibrium points. According to the definition (12), the modulus of z lies between 0 and the absolute value of y. Therefore, points $\mathcal{D}'_\pm$, $\mathcal{F}_\pm$, and $\mathcal{G}_\pm$ are not acceptable. The eigenvalues of the remaining equilibria are presented in the Table III.

As mentioned in the main text, a cosmologically acceptable trajectory passes near a matter point and lands at an accelerated point. A critical point is a good candidate for a matter point if it (i) satisfies the matter condition, $\Omega > 0$; (ii) satisfies the "right" scale factor condition, $a \sim t^{2/3}$ (or, equivalently, $w_{\rm eff}$ close to zero); and (iii) is a saddle point, i.e., represents a transient phase. On the other hand, an acceptable late attractor has to be (iv) accelerated, $w_{\rm eff} < -1/3$, and (v) stable. Points $\mathcal{B}$ and $\mathcal{E}_\pm$ could be used as matter points, and $\mathcal{B}$, $\mathcal{C}_\pm$, and $\mathcal{E}_\pm$ could be used as accelerated attractors. We are going to determine under which conditions on the parameters α, β, γ, and Q there exist at the same time at least one matter point with $w_{\rm eff}$ close to 1, followed by at least one accelerated future attractor.

(i) $\mathcal{C}_\pm$: Following the terminology of Ref. [4], these are kinetic-potential scaling solutions and exist in potentials with $V_1 > 0$ for $\alpha < \sqrt{6}$ and in potentials with $V_1 < 0$ for $\alpha > \sqrt{6}$. They are stable and accelerated whenever

$$(4-3\gamma)Q > \frac{2(\alpha^2-3\gamma)}{\alpha} \quad \text{and} \quad \alpha < \sqrt{2}. \tag{A1}$$

Hence, they are good candidates as accelerated late attractors only in potentials with $V_1 > 0$.

(ii) $\mathcal{E}_\pm$: Points $\mathcal{E}_\pm$ are fluid-kinetic-potential scaling solutions (see also Ref. [4] for the uncoupled case). They enter the phase space when

$$(4-3\gamma)Q \leq \frac{2(\alpha^2-3\gamma)}{\alpha}. \tag{A2}$$

Points $\mathcal{E}_\pm$ may be used for the matter epoch if they satisfy conditions (i), (ii), and (iii), i.e., if

$$Q = 2\alpha\frac{1-\gamma}{4-3\gamma}, \qquad \gamma \leq 1, \qquad \alpha > \sqrt{\frac{3}{2}}\sqrt{\frac{2-\gamma}{1-\gamma}}. \tag{A3}$$

TABLE II. Critical points.

Label	(x, y, z)	Ω	$w_{\rm eff}$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	1
$\mathcal{B}$	$(\frac{(4-3\gamma)Q}{\sqrt{6}(2-\gamma)}, 0, 0)$	$1-\frac{(4-3\gamma)^2Q^2}{6(2-\gamma)^2}$	$-1+\gamma+\frac{(4-3\gamma)^2Q^2}{6(2-\gamma)}$
$\mathcal{C}_\pm$	$(\frac{\alpha}{\sqrt{6}}, \pm\sqrt{1-\frac{\alpha^2}{6}}, 0)$	0	$-1+\frac{\alpha^2}{3}$
$\mathcal{D}_\pm$	$(0, \pm\sqrt{\frac{\beta}{\beta-\alpha}}, \pm\sqrt{\frac{\alpha}{\alpha-\beta}})$	0	-1
$D'_\pm$	$(0, \pm\sqrt{\frac{\beta}{\beta-\alpha}}, \mp\sqrt{\frac{\alpha}{\alpha-\beta}})$	0	-1
$\mathcal{E}_\pm$	$(u_\alpha, \pm v_\alpha, 0)$	ω_α	$-1+\sqrt{\frac{2}{3}}\alpha u_\alpha$
$\mathcal{F}_\pm$	$(\frac{\beta}{\sqrt{6}}, 0, \pm\sqrt{1-\frac{\beta^2}{6}})$	0	$-1+\frac{\beta^2}{3}$
$\mathcal{G}_\pm$	$(u_\beta, 0, \pm v_\beta)$	ω_β	$-1+\sqrt{\frac{2}{3}}\beta u_\beta$

where $u_\alpha = \frac{\sqrt{6}\gamma}{2\alpha-(4-3\gamma)Q}$, $v_\alpha = \sqrt{\frac{(4-3\gamma)^2Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma)}{(2\alpha-(4-3\gamma)Q)^2}}$,$\omega_\alpha = \frac{2(2\alpha^2-6\gamma-\alpha(4-3\gamma)Q)}{(2\alpha-(4-3\gamma)Q)^2}$, and similarly for u_β, v_β, ω_β.

TABLE III. Eigenvalues.

Label	Eigenvalues
$\mathcal{A}_+$	$3-\sqrt{\frac{3}{2}}\alpha,\ 3-\sqrt{\frac{3}{2}}\beta,\ 6-3\gamma-\sqrt{\frac{3}{2}}(4-3\gamma)Q,$
$\mathcal{A}_-$	$3+\sqrt{\frac{3}{2}}\alpha,\ 3+\sqrt{\frac{3}{2}}\beta,\ 6-3\gamma+\sqrt{\frac{3}{2}}(4-3\gamma)Q,$
$\mathcal{B}$	$\frac{(4-3\gamma)^2Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma)}{4(2-\gamma)},\ \frac{(4-3\gamma)^2Q^2-2\beta(4-3\gamma)Q+6\gamma(2-\gamma)}{4(2-\gamma)},\ \frac{(4-3\gamma)^2Q^2-6(2-\gamma)^2}{4(2-\gamma)}$
$\mathcal{C}_\pm$	$\frac{\alpha^2-6}{2},\ \frac{\alpha(\alpha-\beta)}{2},\ \frac{2\alpha^2-6\gamma-\alpha(4-3\gamma)Q}{2}$
$\mathcal{D}_\pm$	$\frac{1}{2}(-3\pm\sqrt{9+12\alpha\beta}),\ -3\gamma$
$\mathcal{E}_\pm$	$\frac{3(\alpha-\beta)\gamma}{2\alpha-(4-3\gamma)Q},\ \frac{\sigma\pm\sqrt{\sigma^2-4\delta}}{2(2\alpha-(4-3\gamma)Q)^2}$

where $\sigma=3(2\alpha-(4-3\gamma)Q)((4-3\gamma)Q-\alpha(2-\gamma))$, and $\delta=\frac{3}{2}(2\alpha-(4-3\gamma)Q)^2(2\alpha^2-6\gamma-\alpha(4-3\gamma)Q)\ ((4-3\gamma)^2Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma))$.

In that case, points $\mathcal{E}_\pm$ exist only for potentials with $V_1<0$. Hence, when $\mathcal{E}_\pm$ are used as matter points, points $\mathcal{C}_\pm$ cannot be used as the accelerated attractors. The only candidate left for the accelerated epoch is $\mathcal{B}$, but as we will see, $\mathcal{B}$ cannot be accelerated for Q given in Eq. (A3). For points $\mathcal{E}_\pm$ to be used for the accelerated epoch, they have to satisfy conditions (iv) and (v). This happens for

$$(4-3\gamma)Q<(2-3\gamma)\alpha \ \text{ and}$$
$$(4-3\gamma)^2Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma)>0. \quad \text{(A4)}$$

Whenever $\mathcal{E}_\pm$ are accelerated attractors, the only remaining candidate for the matter epoch is point $\mathcal{B}$, but as we shall see right below, $\mathcal{B}$ does not satisfy the matter point conditions for the range of the parameters given in Eq. (A4).

(iii) $\mathcal{B}$: This is a fluid-kinetic scaled solution. Point $\mathcal{B}$ enters the phase space when

$$Q\le\sqrt{6}\frac{2-\gamma}{|4-3\gamma|} \quad \text{(A5)}$$

for $\gamma\neq 4/3$ and lies always in the phase space for $\gamma=4/3$, irrespective of the nature of the potential. For $\gamma<4/3$, condition (A5) is always satisfied for sufficiently small values of Q, e.g., $Q\lesssim 1$. Matter point conditions (i), (ii), and (iii) are satisfied whenever

$$Q=\frac{\sqrt{6(2-\gamma)(1-\gamma)}}{4-3\gamma},\quad \gamma\le 1,\quad \alpha<\sqrt{\frac{3}{2}}\sqrt{\frac{2-\gamma}{1-\gamma}}. \quad \text{(A6)}$$

On the other hand, point $\mathcal{B}$ may be an accelerated attractor if (iv) and (v) hold, provided that Eq. (A5) is satisfied. The condition for acceleration (iv) gives

$$Q<\frac{\sqrt{2(2-\gamma)(2-3\gamma)}}{4-3\gamma}, \quad \text{(A7)}$$

with $\gamma<2/3$. Assuming Eq. (A7), the stability condition, (v), gives

$$(4-3\gamma)^2Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma)<0.$$

Nevertheless, Q given in Eq. (A3) does not satisfy Eq. (A7). Hence, matter points $\mathcal{E}_\pm$ cannot be combined with accelerated attractor $\mathcal{B}$.

We conclude that there is only one case in which we have at the same time at least one matter point and at least one accelerated attractor. This happens whenever $\mathcal{B}$ represents the matter solution and $\mathcal{C}_\pm$ stand for attractors. In that case, the potential has $V_1>0$, and the parameters take the values

$$\alpha<\sqrt{2},\qquad \gamma\le 1,\qquad Q=\frac{\sqrt{6(2-\gamma)(1-\gamma)}}{4-3\gamma}, \quad \text{(A8)}$$

leading to Table I in the main text.

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