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Large-N running of the spectral index of inflation

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We extend previous classifications of inflationary models by means of their behavior at large N, where N is the number of e-foldings. In addition to the perturbative $1/N$ case, whose slow-roll parameters fall off as powers of $1/N$, we introduce the constant, nonperturbative and logarithmic classes. This covers the large majority of inflationary models. Furthermore, we calculate the running of the spectral tilt for all these classes. Remarkably, we find that the tilt's runnings essentially cluster around the per-mil level. We comment on the implications for future experiments.

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I. INTRODUCTION

After measuring the cosmic microwave background (CMB) with unprecedented precision, Planck has presented us with a pristine image of the early Universe. This has resulted in strong constraints on the inflationary phase. In particular, the spectral index has been measured to be [1]

$$n_s = 0.9603 \pm 0.0073, \tag{1}$$

although subsequent analyses have argued for a slightly higher value [2]. Similarly, the running of the spectral index has been constrained with some precision:

$$\alpha_s = -0.0134 \pm 0.0090. \tag{2}$$

On the other hand, the BICEP2 Collaboration [3] has recently announced the detection of the primordial gravitational wave background produced during inflation, with an amplitude responsible for a tensor-to-scalar ratio

$$r = 0.20^{+0.07}_{-0.05}, \tag{3}$$

with $r = 0$ ruled out at more than 5σ. This bound is in slight tension with the reported Planck upper bound for a six-parameter ΛCDM model, $r < 0.11$ at a 95% C.L., although foreground subtraction could decrease to some extent the reported value. If confirmed, such detection would lead to important consequences. In particular, many inflationary scenarios are ruled out by these bounds. Indeed, Planck and BICEP2 together seem to prefer large-field models of inflation [4,5], with predictions different from the plateau potentials of Starobinsky-like models [6], preferred before BICEP2 results.

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In this paper, we aim to understand the values quoted above in terms of the number of e-foldings N between the horizon exit of the quantum modes that have resulted in the CMB anisotropies and the end of inflation. As will be discussed in more detail below, this approach allows for a classification of different inflationary models in terms of universality classes with distinct large-N behavior. It builds on preceding works [7,8], where the deviation from a Harrison-Zeldovich scale-invariant spectrum with $n_s = 1$ was attributed to $1/N$ effects. We will extend this with different leading terms in the $1/N$ expansion. Moreover, we will also investigate the running of the spectral index as a function of N. The running is the second-order term in the expansion of the scale dependence of the power spectrum, and hence, it is natural to consider it in this framework as well. Finally, the values of this running are considered for a wide range of models and shown to cluster in a very limited range around a central1 value of $\log_{10}|\alpha_s| = -3.2$.

The outline of the paper is as follows: We introduce the N formalism in Sec. II, where all slow-roll parameters are expressed in terms of N. Section III contains the different universality classes of large-N behavior. The running of the spectral index is discussed in Sec. IV. Finally, we conclude in Sec. V. In the Appendix, we consider also models of k inflation as a separate class of its own.

II. THE N FORMALISM

In this section, we will express the time evolution of all cosmological observables through their N dependence. In particular, we will demonstrate that it suffices to specify the equation-of-state parameter $\epsilon(N)$ to fully determine the inflationary phase. We will refer to this approach as the N formalism. It can be seen as the background complement to the δN formalism for cosmological perturbations [6,9–11]. Moreover, in the next section we will demonstrate its use in providing a perturbative expansion for inflationary models.

The evolution of a general scalar field is described by the Hamiltonian and momentum constraints

$$3H^2(\phi) = 2H'(\phi)^2 + V(\phi), \qquad \dot{\phi} = -2H'(\phi) \tag{4}$$

plus the evolution equations

$$\dot{H} = -2H'(\phi)^2, \qquad \ddot{\phi} + 3H(\phi)\dot{\phi} + V'(\phi) = 0, \tag{5}$$

where dots denote derivatives with respect to cosmic time and primes with respect to the field ϕ. Throughout this paper, we set $\kappa^2 = 8\pi G \equiv 1$. The only quantity needed to specify the whole evolution is therefore the Hubble function $H(\phi)$. The Hamilton-Jacobi formalism allows one to compute the evolution in terms of a new time variable, the scalar field ϕ. In general, these equations are too difficult to solve without specifying a potential. However, we can characterize the whole evolution via the equation-of-state parameter

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}(1+w) \tag{6}$$

and its derivatives, the so-called Hubble slow-roll parameters:

$$\begin{aligned} \epsilon &= 2\left(\frac{H'(\phi)}{H(\phi)}\right)^2 = \frac{\dot{\phi}^2}{2H^2}, \\ \delta &= 2\left(\frac{H''(\phi)}{H(\phi)}\right) = -\frac{\ddot{\phi}}{H\dot{\phi}}, \\ \xi &= 4\left(\frac{H'(\phi)H'''(\phi)}{H^2(\phi)}\right) = \frac{\dddot{\phi}}{H^2\dot{\phi}} - \delta^2. \end{aligned} \tag{7}$$

In the slow-roll approximation, one is assuming $H^2(\phi) \propto V(\phi)$. Therefore, the Hubble slow-roll parameters are related to the slow-roll parameters $\epsilon_V, \eta_V, \ldots$ defined in terms of derivatives of the potential by replacing $H^2(\phi)$ with $V(\phi)$ in the above.

The two sets of slow-roll parameters are equivalent in the slow-roll approximation. However, the Hubble slow-roll parameters offer a number of advantages for our purposes. First of all, they are more accurate, since they do not ignore the scalar kinetic term in Eq. (4). Moreover, they can be derived from the equation-of-state parameter:

$$\begin{aligned} H(N) &= H_0 \exp \int \epsilon(N) dN, \\ \delta(N) &= \epsilon(N) + \frac{1}{2}\frac{\epsilon'}{\epsilon}(N), \\ \xi(N) &= \frac{3}{2}\epsilon'(N) + \epsilon^2(N) + \frac{1}{2}\left(\frac{\epsilon'}{\epsilon}\right)'(N), \end{aligned} \tag{8}$$

where primes denote differentiation with respect to N, the number of e-folds:

$$N = \ln\frac{a_{\rm end}}{a_{\rm CMB}} = \int_{t_{\rm CMB}}^{t_{\rm end}} H dt = \int_{\phi_{\rm CMB}}^{\phi_{\rm end}} \frac{d\phi}{\sqrt{2\epsilon}}. \tag{9}$$

We can thus describe the whole evolution in a new time unit related to the scale factor, and not directly to cosmic time. The field ϕ therefore no longer plays the role of the clock; instead, time is measured by the logarithmic growth of the scale factor; see e.g. also Ref. [12]. We can also write the conformal time τ as an integral:

$$\begin{aligned} -\tau\mathcal{H} &= 1 + e^{-\int dN(1-\epsilon)} \int dN \epsilon(N) e^{\int dN(1-\epsilon)} \\ &\simeq \frac{1}{1-\epsilon(N)}, \end{aligned} \tag{10}$$

where $\mathcal{H} = aH$, and the last expression is valid only in the slow-roll approximation, where $\epsilon \ll 1$.

Note, therefore, that the equation-of-state parameter $\epsilon(N)$ is all we need to specify the dynamics. In particular, this parameter determines the spectra of metric perturbations, both scalar and tensor:

$$P_s(k) = \frac{H^2}{8\pi^2\epsilon}, \qquad P_t(k) = \frac{2H^2}{\pi^2}. \tag{11}$$

The scale dependence of the power spectra can be expanded as

$$\begin{aligned} \ln P_s(k) &= \ln P_s(k_0) + (n_s - 1)\ln\frac{k}{k_0} + \frac{1}{2}\alpha_s \ln^2\frac{k}{k_0}, \\ \ln P_t(k) &= \ln P_s(k_0) + n_t \ln\frac{k}{k_0} + \frac{1}{2}\alpha_t \ln^2\frac{k}{k_0}. \end{aligned} \tag{12}$$

This translates into the following definitions for the scale-dependence coefficients in terms of $\epsilon(N)$:

$$\begin{aligned} n_s - 1 &= -2\epsilon(N) + \frac{\epsilon'}{\epsilon}(N), \qquad n_t = -2\epsilon(N), \\ \alpha_s &= 2\epsilon' - \left(\frac{\epsilon'}{\epsilon}\right)', \qquad \alpha_t = 2\epsilon', \end{aligned} \tag{13}$$

which can be computed straightforwardly in the N formalism. Note that the *only* difference between the scalar and tensor spectral indices is the term ϵ'/ϵ. In models where ϵ is constant in N, they are identical, and this can be used to constrain the amplitude of the tensor contribution, while in general ϵ'/ϵ is not constant. Finally,

$$r = 16\epsilon(N) \tag{14}$$

denotes the ratio between the scalar and tensor power spectra.

III. UNIVERSALITY CLASSES

In this section, we will discuss the N dependence of the equation-of-state parameter ϵ of a number of inflationary models. In particular, we will classify such models according to the leading behavior at large values of N. Although nontrivial to determine observationally, typical values for N range around 50 and 60. Furthermore, this number does not constitute an upper bound on N but only corresponds to the portion of the inflationary trajectory that we have observational access to at the moment via the CMB. The total number of e-foldings could thus be much larger. Barring the apparent hints of power loss at large angular scales, we have no reason to assume that $N = 50$–60 is by any means special, and hence it seems reasonable to assume that inflation has taken place over more than the observed number of e-foldings.

Moreover, the recently measured value of the tensor-to-scalar ratio by the BICEP2 Collaboration, $r \sim 0.2$, suggests that inflation took place at high-energy scales, close to 10^{16} GeV, and thus the number of e-folds of inflation has a robust lower limit of $N \gtrsim 50$, which generically indicates large N values for cosmological quantities that crossed the Hubble scale during inflation and reenter today. It therefore seems natural to consider a perturbative expansion of inflationary observables in terms of $1/N$. In the case of a polynomial expansion, as was analyzed in Refs. [7,8], one can argue that only the leading contributions are relevant.[1] Subleading corrections will be very difficult to measure observationally. Moreover, strictly within the slow-roll approximation, these are not unambiguously defined for the following reasons[2]: First of all, the approximation itself is precisely obtained by neglecting this type of subleading terms and equating the Hubble parameter and the scalar potential. Moreover, N can only be defined up to order-1 errors even within the slow-roll approximation. Therefore, any subleading coefficients of a polynomial expansion of the slow-roll result are physically meaningless; these can only be obtained by performing a full Hamilton-Jacobi analysis.

The large-N limit therefore seems to be a powerful discriminant between different models. We will introduce a number of new classes of models, where each class has a specific $1/N$ dependence. Physically different models in the same class will thus give rise to the same large-N behavior, and they will generically agree on their cosmological predictions.[3] The reason for this universality between different models is that the large-N limit only probes a limited region of the inflationary potential; for instance, for models where inflation takes place on a plateau, the details of the valley are washed out in this limit. Similarly, for chaotic-like inflationary models with polynomial potentials, the large-N limit only depends on the highest power in the inflaton field.

Our discussion extends the aforementioned classification, where only a leading $1/N^p$ behavior for ϵ was considered, which we will refer to as the *perturbative* class. In addition to this possibility, we also introduce the *constant*, *nonperturbative* and *logarithmic* classes. For each of these, we will give the (leading-order approximation to) the spectral tilt, the tensor-to-scalar ratio, and the running. Moreover, for each case, examples of inflationary models and (when possible) their exact expressions are provided. We also assess the accuracy of the leading $1/N$ approximation.

A. Constant class

The constant class is characterized by a *constant*, N-independent equation-of-state parameter

$$\epsilon(N) = \epsilon_0. \tag{15}$$

The spectral index, tensor ratio and scalar running are given by

$$n_s = 1 - 2\epsilon_0, \qquad r = 16\epsilon_0, \qquad \alpha_s = 0. \tag{16}$$

An example of this class is power-law inflation, with an exponential scalar potential:

$$V = V_0 e^{\alpha\phi}. \tag{17}$$

Although this model does not have a scenario for inflation to end by itself, one can study its predictions during the inflationary phase. These give rise to an exactly constant equation of state with $\epsilon_0 = \frac{1}{2}\alpha^2$. This class is ruled out, since in fixing the constant to agree with the scalar spectral index $n_s = 0.96$, one finds that $r = 0.32$, which exceeds the upper limit set by Planck. This is illustrated by the upper line in Fig. 1.

B. Perturbative class

This class is characterized by a *perturbative* equation-of-state parameter

$$\epsilon(N) = \frac{\epsilon_p}{N^p}, \tag{18}$$

with ϵ_p constant. We will assume that $p \geq 1$, as we do not know any viable model with $p < 1$. This leads to the following observables for $p = 1$:

[1]In particular, this class was analyzed in terms of the Hubble slow-roll parameters by Ref. [7], while Ref. [8] employs the slow-roll parameters in terms of the scalar potential. In this paper we will follow the former approach.

[2]We thank Andrei Linde for an interesting discussion on this issue.

[3]See Ref. [13] for a similar classification from a holographic viewpoint.

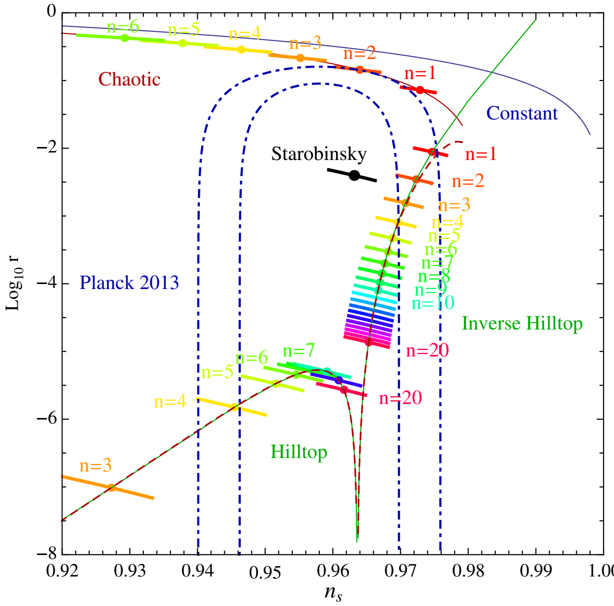

FIG. 1 (color online). The plane $(n_s, \log_{10} r)$ with the different perturbative models: chaotic, inverse hilltop and hilltop (both with $\mu = 1$), as well as the constant model. The colored bars correspond to the range of $N \in [50, 60]$. The solid lines are exact, while the dashed lines are the leading-N contributions. Note that these essentially agree inside the Planck 2013 region. Also shown for reference is the model of Starobinsky.

$$n_s = 1 - \frac{2\epsilon_1 + 1}{N}, \qquad r = \frac{16\epsilon_1}{N}, \qquad \alpha_s = -\frac{2\epsilon_1 + 1}{N^2}, \tag{19}$$

while $p > 1$ gives rise to a qualitatively different behavior:

$$n_s = 1 - \frac{p}{N}, \qquad r = \frac{16\epsilon_p}{N^p}, \qquad \alpha_s = -\frac{p}{N^2}. \tag{20}$$

An attractive feature of these universality classes is that the $1/N$ term provides a natural explanation for the percent deviation from scale invariance ($n_s \simeq 0.96$). The coefficients in this expansion, which are ϵ_1 or p, respectively, are therefore of order 1, giving a natural perturbative expansion. Examples of corresponding inflationary models are as follows:

(1) $p = 1$: Chaotic inflation [14] with

$$V = M^4 \left(\frac{\phi}{\mu}\right)^n$$

has an equation-of-state parameter given by

$$\epsilon = \frac{\frac{1}{4}n}{N + \frac{1}{4}n}, \tag{21}$$

which is exact in the slow-roll approximation (with inflation ending at $\epsilon = 1$). This set of models is almost ruled out, since in fixing the constant to agree with the scalar spectral index $n_s = 0.96$, one finds that the models are just outside the 2σ contour allowed by Planck; see the second-highest line in Fig. 3.

(2) $1 < p < 2$: Another type of model, which we will refer to as "inverse hilltop," is characterized by a potential

$$V = V_0\left[1 - \left(\frac{\mu}{\phi}\right)^n\right], \tag{22}$$

where n is a positive power. It leads to an equation of state [4]

$$\epsilon(N) = \frac{\frac{1}{2}n^2/\mu^2}{(Nn(n+2)/\mu^2)^{\frac{2(n+1)}{n+2}}} \tag{23}$$

at lowest order in $1/N$. Different positive values of n interpolate from $p = 1$ to $p = 2$. Note that this model, for $n \geq 4$, gets a spectral index close to $n_s = 0.96$, well within the 2σ contour of Planck; see the lower line in Fig. 2.

(3) $p = 2$: The Whitt potential corresponding to the Starobinsky model [6,15],

$$V = V_0(1 - e^{-\sqrt{2/3}\phi})^2, \tag{24}$$

is characterized by an equation-of-state parameter

$$\epsilon(N) = \frac{3}{4N^2} \tag{25}$$

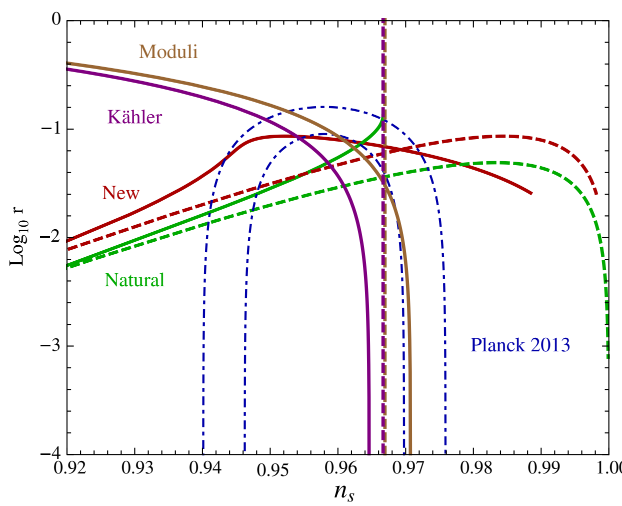

FIG. 2 (color online). The plane $(n_s, \log_{10} r)$ with the different nonperturbative models (new and natural), as well as the logarithmic models (moduli and Kähler). The solid lines correspond to the exact expressions, while the dashed lines are the leading $1/N$ contributions. Note that these differ significantly even inside the Planck 2013 region.

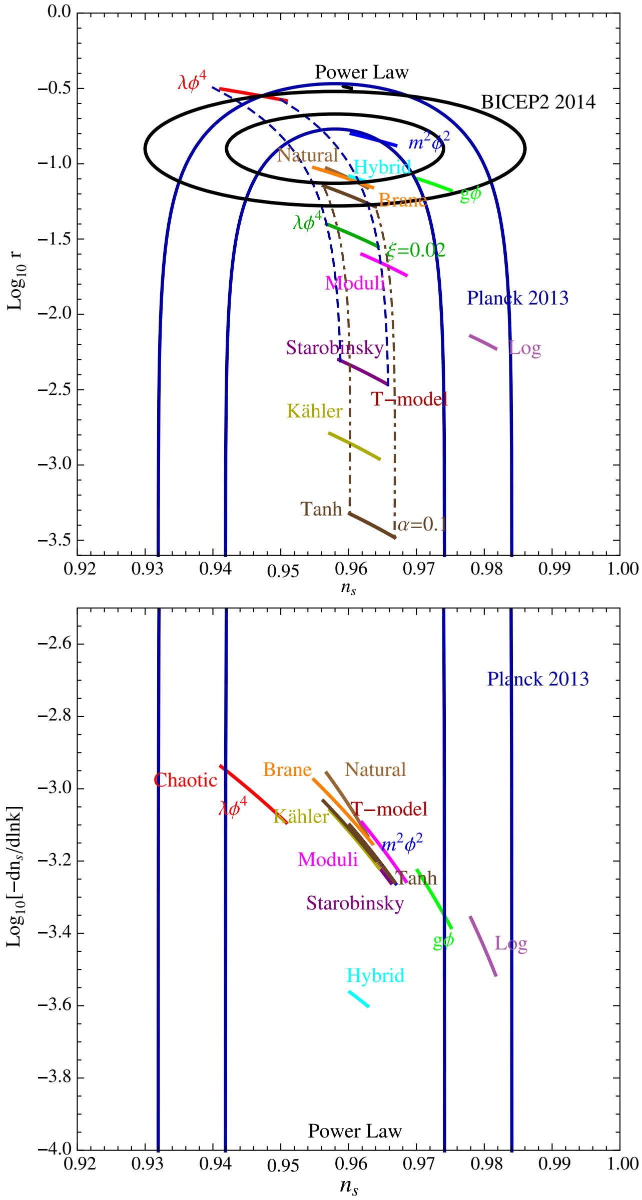

FIG. 3 (color online). The planes $(n_s, \log_{10} r)$ and $[n_s, \log_{10}(-\alpha_s)]$ with all the models discussed in the text. The ranges of values correspond to the interval $N \in [50, 60]$. The solid blue (black) lines indicate the Planck (BICEP2) constraints at 1σ and 2σ. The dashed line corresponds to a nonminimally coupled chaotic $\lambda\phi^4$ model, $\xi = 0 \to 10$. The dot-dashed line corresponds to T models in a range of values of $\alpha = 0.1 \to 10$.

at lowest order in $1/N$. This model is not ruled out but precisely agrees with the scalar spectral index $n_s = 0.96$, and one finds that the model is well inside the 2σ contour allowed by Planck.

An interesting generalization in the same universality class is provided by T models of inflation with a potential [16]

$$V = V_0 \tanh^{2n}(\phi/\sqrt{6\alpha}),$$

with α arbitrary, which corresponds to an equation-of-state parameter

$$\epsilon(N) = \frac{3\alpha n}{4nN^2 + 2N\sqrt{3\alpha(3\alpha + 4n^2)} + 3\alpha n}. \tag{26}$$

This is an exact expression within the slow-roll approximation, with inflation ending at $\epsilon = 1$. Note that this model gets a spectral index $n_s = 0.96$ for $n = 2$, almost independent of α. As we vary the parameter α from 0.1 to 10, we move up in the plane (n_s, r), as can be seen by the dot-dashed lines in Fig. 3.

(4) $p > 2$: In contrast to the previous models, hilltop inflation takes place near the origin [17]. Its scalar potential is

$$V = V_0\left[1 - \left(\frac{\phi}{\mu}\right)^n\right], \tag{27}$$

with $n > 2$. It leads to an equation of state

$$\epsilon(N) = \frac{\frac{1}{2}n^2/\mu^2}{(Nn(n-2)/\mu^2)^{\frac{2n-2}{n-2}}}. \tag{28}$$

Different values of n therefore fill out the range $p > 2$. For $n \geq 4$, this model is well within the Planck 2σ contour.

We have illustrated the accuracy of the $1/N$ expansion in Fig. 2, where we plot the various perturbative models (chaotic, hilltop and inverse hilltop inflation) as a function of their parameter (the leading power of the potential) in the plane $(n_s, \log_{10} r)$. It is clear that, for models within the Planck 2013 2σ region, the leading $1/N$ contribution correctly describes the model.

C. Nonperturbative class

This class is characterized by an equation-of-state parameter that asymptotes to

$$\epsilon(N) = \epsilon_0 e^{-2cN}, \tag{29}$$

which is nonperturbative around $1/N \to 0$. The resulting spectral index, tensor ratio and scalar running at lowest order are given by

$$n_s = 1 - 2c, \qquad \alpha_s = -4c\epsilon_0 e^{-2cN}, \qquad r = 16\epsilon_0 e^{-2cN}. \tag{30}$$

In contrast to the previous class, the nonperturbative class has a constant shift of the spectral index. The corresponding parameter thus has to be percent level to account for the observed deviation from the spectral index. This spells somewhat of a problem inperturbation theory, as the effective parameter in which we are expanding ϵ is not $1/N$ but $1/(cN)$. While the former is naturally small, the latter is not necessarily. We therefore expect subleading terms to have some relevance in this region of parameter space. Indeed, this will be confirmed by an analysis of the following two examples.

For the new inflation model [18] with

$$V = V_0\left(1 - \frac{\phi^2}{\mu^2}\right)^2, \qquad (31)$$

where $c = 4/\mu^2$ and $\epsilon_0 = 2c(1-\sqrt{c})^2$, the above expression for ϵ is exact and not a large-N approximation. This model is almost ruled out, since in fixing the constant c to agree with the scalar spectral index $n_s = 0.96$, one finds that the models are just inside the 2σ contour allowed by Planck; see Fig. 2.

Another example of this class is natural inflation [19], with

$$V = V_0\left(1 + \cos\frac{\phi}{\mu}\right). \qquad (32)$$

The equation-of-state parameter can be computed exactly as

$$\epsilon = \frac{c}{e^{2cN} - 1}, \qquad (33)$$

with $c = 1/(2\mu^2)$. This model is on the verge of being ruled out, since in fixing the constant to agree with the scalar spectral index $n_s = 0.96$, one finds that the model is just inside the 2σ contour allowed by Planck; see Fig. 2. It is different from that of new inflation, since $\alpha_s = -cr/4$ is not satisfied in this case $\forall\ N$, since $(\epsilon'/\epsilon)' \neq 0$.

We show in Fig. 2 the validity of the $1/N$ expansion for the various nonperturbative models (new and natural inflation) as a function of their parameter (the vacuum expectation value of the field) in the plane $(n_s, \log_{10} r)$. It is clear that, for models within the Planck 2013 2σ region, the leading $1/N$ contribution gives a relatively poor description of the model dependence. In these models, the full nonperturbative expression is needed, and we cannot rely on the leading $1/N$ contribution.

D. Logarithmic class

This class is characterized by an equation-of-state parameter with logarithmic terms:

$$\epsilon(N) = \epsilon_p \frac{\ln^q N}{N^p}. \qquad (34)$$

We will mainly be interested in the case with $p = 2$. Then the leading expressions for the cosmological observables are

$$n_s = 1 - \frac{2}{N}, \qquad \alpha_s = -\frac{2}{N^2}, \qquad r = 16\epsilon_p \frac{\ln^q N}{N^2}. \qquad (35)$$

Remarkably, the logarithmic dependence drops out of the scalar power spectrum in the large-N limit. The only remnant can be found in the ratio of the tensor-to-scalar power.

The Kähler moduli inflation [20] class, in particular, is characterized by a potential

$$V = V_0(1 - \alpha\phi^{4/3} e^{-\beta\phi^{4/3}}), \qquad (36)$$

which corresponds to an equation-of-state parameter

$$\epsilon(N) = \frac{c}{2N^2\sqrt{\ln N}}, \qquad (37)$$

with $c = 9/16\beta^{3/2}$. A toy version of this model has the scalar potential [4]

$$V = V_0(1 - \alpha\phi e^{-\phi}). \qquad (38)$$

Its inflationary trajectory can be characterized by the equation-of-state parameter

$$\epsilon(N) = \frac{\ln^2 N}{2N^2}. \qquad (39)$$

Note that both models predict a spectral index $n_s = 0.96$ and a negligible tensor-to-scalar ratio, much smaller even than that of the Starobinsky model, well within the 2σ contour of Planck; see Fig. 2.

IV. RUNNING OF THE SPECTRAL TILT

In this section, we will take a somewhat more phenomenological approach to study the possible values of the running α_s of the spectral index, independent of the N formalism and universality classes. In the preceding sections, we studied the parameter dependence of the (n_s, r) values for different models. We will now fix these parameters to their best values to agree with the Planck and BICEP2 data on (n_s, r), and subsequently calculate the resulting running α_s within that model.

As we have given all relevant expressions in the previous section, we will only quote the results here. First, in Fig. 3(a), the spectral index and the tensor-to-scalar ratio are plotted for all models that we have discussed. In case these models have free parameters, we have fixed these in order to comply with the Planck data. Moreover, we have varied N from 50 to 60 to give an indication of the N dependence of these models. Second, in Fig. 3(b), we show the spectral index and its running for the same parameter values.

While the ratio of tensor to scalar amplitudes in all models discussed above, which fall into various universality classes, can vary significantly over several orders of magnitude [see Fig. 3(a)], it is surprising that the actual running of the scalar spectral index n_s does not, as shown in Fig. 3(b). All models seem to cluster around

$$\alpha_s \simeq -0.001.$$

PRISM [21,22] is possibly capable of detecting the running of such models, but unfortunately Planck has not enough resolution.

One could try to understand this clustering from the fact that we are only considering models that fall within the Planck bounds for ranges of e-folds from 50 to 60, i.e. $\Delta N \sim 10$, while the variation of the spectral index in all these models is never larger than their deviation from Harrison-Zel'dovich, i.e. $\Delta n_s \sim 0.01$. Therefore, it is not surprising that all of these models seem to fall in the same ballpark, irrespective of which universality class they belong to. This is a purely phenomenological observation. For instance, if we had considered supergravity hybrid inflation, which has a large variation of the spectral index over CMB scales, from $n_s > 1$ on horizon scales down to $n_s < 0.8$ on small scales, then we would have predicted a $dn_s/d\ln k \sim 0.1$, orders of magnitude larger. However, those models seem to be ruled out by Planck already, due to their scalar spectral index alone, and therefore it is the remaining models which have such a small running.

In the not so far future, with larger a range of scale coverage—e.g., with the measurement of the spectral index of linear fluctuations in neutral hydrogen clouds emitting in the 21 cm line at and after reionization—by probes like the Square Kilometer Array [23], we may be able to bring down the precision on α_s to the level of a few parts in 10,000, which could allow us to differentiate among these models. Otherwise, we will not have a significant lever arm to tell them apart.

V. DISCUSSION

In this paper, we have introduced the N formalism, which can be thought of as the background complement to the δN formalism, and where all inflationary observables are expressed in terms of the number of e-foldings. This is particularly useful given the generic requirement that N should exceed 50 and suggests a large-N expansion.

We have demonstrated that the majority of models allow for such an expansion in $1/N$; the leading terms for n_s are either constant, perturbative, nonperturbative or logarithmic. A quick overview of the properties of these classes can be found in Table I. Moreover, we have shown that subleading terms are generically irrelevant (in addition to ambiguously defined): the leading approximation at large N agrees excellently with the exact expression. The only exceptions to this behavior are the nonperturbative examples of the natural and quadratic hilltop models. In these models, the effective expansion is in terms of cN, which does not allow for a leading approximation when c is sufficiently small.

TABLE I. A sketch of the various classes and their corresponding asymptotic potentials and equations of state at large N. More details can be found in Sec. III.

Class	$V(\phi)$	$\epsilon(N)$
Constant	e^{ϕ}	constant
Perturbative $p = 1$	ϕ^n with $n > 0$	$1/N$
Perturbative $1 < p < 2$	$1 - \phi^n$ with $n < 0$	$1/N^p$
Perturbative $p = 2$	$1 - e^{-\phi}$	$1/N^2$
Perturbative $p > 2$	$1 - \phi^n$ with $n > 0$	$1/N^p$
Nonperturbative	$1 - \phi^2$	e^{-N}
Logarithmic	$1 - \phi e^{-\phi}$	$\ln(N)/N$

We should also mention that there are inflationary models that do not allow for a large-N expansion. The chief example is supergravity hybrid inflation [24], with a potential

$$V(\phi) = \mu^4 \left(1 + \frac{\kappa^2}{8\pi^2} \ln\frac{\phi}{\phi_c} + \frac{\phi^4}{8}\right). \tag{40}$$

The resulting equation-of-state parameter reads

$$\epsilon = \frac{\kappa^3}{16\pi^3 \sin\frac{\kappa N}{\pi}\left(1 + \cos\frac{\kappa N}{\pi}\right)}. \tag{41}$$

Due to the trigonometric functions, this parameter does not have a definite large-N behavior and hence defies the proposed classification.

Building on the N dependence of these universality classes, we have investigated the values of the running for the different inflationary models. Using the Planck constraints for the spectral index as input, it turns out that essentially all models give a similar prediction for the running of n_s as a function of scale, centered around $\log_{10}|\alpha_s| = -3.2$ and with a range of only half a decade. This remarkable feature seems to have gone unnoticed so far. As we have pointed out, one can gain a first understanding of the limited range of α_s from the preference of Planck for inflationary models with correct spectral indices for both $N = 50$ as well as $N = 60$.

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Note added.—Recently, the BICEP2 Collaboration presented their groundbreaking results, and we had to adapt to the situation, changing slightly our perspective, without modifying our conclusions. In particular, a high-scale model of inflation makes the large-N expansion more plausible, since a large reheating temperature imposes a robust lower limit on the number of e-folds. On the other hand, it shifts the attention from Starobinsky-like models to chaotic-type models of inflation. But most importantly, it imposes very stringent constraints on a large array of models that fall under the various universality classes, as can be appreciated in the new Fig. 3.

APPENDIX: *k* INFLATION CLASS

This class is characterized by both an equation-of-state parameter ϵ and a speed of sound $c_s^2 = \epsilon\rho/(3X\rho_{,X})$ different from 1 with $X = 1/2(\partial\phi)^2$. The amplitude of the scalar power spectrum changes, since it arises from a curvature fluctuation, $v = z\zeta$, satisfying a new mode equation

$$v_k'' + \left(c_s^2 k^2 - \frac{z''}{z}\right) v_k = 0, \tag{A1}$$

which gives

$$P_s(k) = \frac{H^2}{8\pi^2 c_s \epsilon}, \tag{A2}$$

while the tensor spectrum does not change. Therefore, the spectral indices, tensor ratio and scalar running are given by

$$n_s - 1 = -2\epsilon + \frac{(c_s\epsilon)'}{c_s\epsilon}, \tag{A3}$$

$$n_t = -2\epsilon, \tag{A4}$$

$$\frac{d\ln n_s}{d\ln k} = 2\epsilon' - \left(\frac{(c_s\epsilon)'}{c_s\epsilon}\right)', \tag{A5}$$

$$r = 16 c_s \epsilon = -8 c_s n_g. \tag{A6}$$

What is new in this model is a possibly large non-Gaussianity, $\zeta = \zeta_L - 3/5 f_{NL}\zeta_L^2$, which contributes mostly to the equilateral three-point correlation function of fluctuations in the CMB. The present Planck data constrain f_{NL}^{eq} to satisfy

$$f_{NL}^{\mathrm{eq}} = \frac{35}{108}\left(\frac{1}{c_s^2} - 1\right)\langle 33 \Rightarrow c_s\rangle 0.1. \tag{A7}$$

We can combine these results with any of the previous models and find that, at most, we can reduce the tensor contribution by 1 order of magnitude, but we should not expect a very big change in the value of the spectral indices.

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