PHYSICAL REVIEW D **89,** 103008 (2014)

Cosmology of the proxy theory to massive gravity

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(Received 21 March 2014; published 13 May 2014)

In this paper, we scrutinize very closely the cosmology in the proxy theory to massive gravity obtained in de Rham and Heisenberg [Phys. Rev. D 84, 043503 (2011)]. This proxy theory was constructed by covariantizing the decoupling limit Lagrangian of massive gravity, and it represents a subclass of Horndeski scalar-tensor theory. Thus, this covariantization unifies two important classes of modified gravity theories, namely, massive gravity and Horndeski theories. We go beyond the regime which was studied in de Rham and Heisenberg [Phys. Rev. D 84, 043503 (2011)] and show that the theory does not admit any homogeneous and isotropic self-accelerated solutions. We illustrate that the only attractor solution is the flat Minkowski solution; hence, this theory is less appealing as a dark energy model. We also show that the absence of de Sitter solutions is tightly related to the presence of shift symmetry breaking interactions.

DOI: 10.1103/PhysRevD.89.103008 PACS numbers: 98.80.-k, 04.50.Kd

I. INTRODUCTION

Whether the law of gravitation at cosmological distances can be described by general relativity or not will provide us with rich information of dark energy, which is responsible for the present accelerated expansion of the Universe. One such candidate for alternative theories of gravity is massive gravity, originally proposed by Fierz and Pauli [1]. They introduced a mass term in the linearized theory of general relativity in the context of Lorentz invariant theory. Unfortunately, once Fierz-Pauli massive gravity is extended to a nonlinear theory, the 6th degree of freedom, called the Boulware-Deser ghost, appears [2]. This problem was recently solved by de Rham and Gabadadze by adding higher order potential terms, removing the 6th degree of freedom [3]. It turns out that this infinite potential can be resummed by introducing the tensor, which has a square-root structure [4], and this theory is now referred to as de Rham-Gabadadze-Tolley (dRGT) massive gravity, which has been shown to be technically natural [5,6]. Since the inception of the dRGT theory, there has been a flurry of investigations related to the self-accelerating solutions in the full theory. In dRGT theory, the Universe cannot be of a flat or closed Friedmann-Robertson-Walker (FRW) form [7]; nonetheless, an open FRW universe is still allowed [8]. In this solution, the mass term behaves exactly as the cosmological constant, which allows a self-accelerating universe. However, the perturbations suffer from the instabilities, and the kinetic terms in the scalar and vector sectors vanish, which signals a strong coupling at a certain scale [9–11]. On the other hand, in Ref. [12] it has been shown that there are exact de Sitter solutions in the decoupling limit theory, which is only valid within a certain region in the Universe. This solution, however, suffers from ghost instabilities of the vector modes [12–14]. In any case, it is very interesting that the mass of the graviton can drive an accelerated expansion of the Universe [15].

As an alternative to massive gravity, one can covariantize the decoupling limit theory [16], and this "proxy theory" is not a massive gravity theory any longer but rather a nonminimally coupled subclass of Horndeski scalar-tensor theory [17]. Horndeski theory is the scalar-tensor theory whose equations of motion remain second-order differential equations, while the Lagrangian contains second derivatives with respect to space-time. It has been shown that Horndeski theory is equivalent to generalized Galileon theory [18], which is the general extension of the Galileon theory [19], and these theories contain four arbitrary functions in the Lagrangian.[1] In the proxy theory, these arbitrary functions can be automatically determined by covariantization, and it shares the same decoupling limit with dRGT massive gravity. In Ref. [16], the authors found a self-accelerating solution in a given approximated regime driven by the scalar field, which originally represented the helicity-0 mode in massive gravity. In contrast to the pure Galileon models, generalized Galileons do not impose

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[1]See Refs. [20,21] for the generalized vector Galileons.

Galileon symmetry. The naive covariantization of the Galileon interactions on nonflat backgrounds breaks the Galileon symmetry explicitly; however, one can successfully generalize the Galileon interactions to maximally symmetric backgrounds while keeping the corresponding symmetries [22]. Inspired by these Horndeski scalar-tensor interactions, one can, in a similar way, construct the most general vector-tensor interactions with nonminimal couplings with only second-order equations of motion [20,21]. The cosmology of these theories has been explored in [23].

In the present paper, we study the cosmological evolution in the proxy theory in more detail beyond the approximations used in [16] and show the absence of de Sitter attractor solutions, which renders the theory not suitable as a dark energy model. In Sec. II, we briefly review dRGT massive gravity and the derivation of proxy theory. In Sec. III, we first investigate the de Sitter solution; then we study the dynamical system of cosmological solutions by using phase analysis. In Sec. IV, we summarize our results.

Throughout the paper, we use units in which the speed of light and the Planck constant are unity, $c = \hbar = 1$, and $M_{\rm Pl}$ is the reduced Planck mass related to Newton's constant by $M_{\rm Pl} = 1/\sqrt{8\pi G}$. We follow the metric signature convention $(-,+,+,+)$. Some contractions of rank-2 tensors are denoted by $\mathcal{K}^\mu{}_\mu = [\mathcal{K}]$, $\mathcal{K}^\mu{}_\nu \mathcal{K}^\nu{}_\mu = [\mathcal{K}^2]$, $\mathcal{K}^\mu{}_\alpha \mathcal{K}^\alpha{}_\beta \mathcal{K}^\beta{}_\mu = [\mathcal{K}^3]$, and so on.

II. PROXY THEORY TO MASSIVE GRAVITY

A. dRGT massive gravity and the decoupling limit

In massive gravity, one has to introduce the fluctuation tensor $h_{\mu\nu}$, which measures the mass of the graviton, and it is usually defined by the difference between the physical metric and the Minkowski metric, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Once we introduce a mass term in a gravitational theory, the theory does not preserve the diffeomorphism invariance; however, the diffeomorphism invariance can be restored by introducing the Stückelberg field ϕ^a [24], through the relation $H_{\mu\nu} = g_{\mu\nu} - \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$, where $H_{\mu\nu}$ is the covariant version of the fluctuation tensor $h_{\mu\nu}$.[2] Then the action for massive gravity is, in general, given by

$$S_{\rm MG} = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left(R - \frac{m^2}{4}\mathcal{U}(g,H)\right) + S_m(g_{\mu\nu},\psi), \tag{2.1}$$

where m is the mass of the graviton, $U(g,H)$ are the potential terms, and S_m is the action for the matter fields ψ living on the geometry. The candidate of the potential is the Fierz-Pauli mass term, which is the ghost-free term at quadratic order in $H_{\mu\nu}$ [1]. However, this term produces an extra ghostly degree of freedom at nonlinear level, found by Boulware and Deser [2]. In order to eliminate this Boulware-Deser ghost, one has to add the infinite nonlinear corrections in addition to the quadratic potential [3]. These infinite nonlinear potentials can be remarkably simplified by using the new tensor $\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu} = \delta^\mu{}_\nu - \sqrt{\eta_{ab}g^{\mu\alpha}\partial_\alpha\phi^a\partial_\nu\phi^b}$, and then the resummed potential for ghost-free massive gravity becomes [4]

$$\mathcal{U}(g,H) = -4(\mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4), \tag{2.2}$$

where $\alpha_{3,4}$ are model parameters and

$$\begin{aligned}
\mathcal{U}_2 &= -\frac{1}{2}\varepsilon_{\mu\alpha\rho\sigma}\varepsilon^{\nu\beta\rho\sigma}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta = [\mathcal{K}]^2 - [\mathcal{K}^2],\\
\mathcal{U}_3 &= -\varepsilon_{\mu\alpha\gamma\rho}\varepsilon^{\nu\beta\delta\rho}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta\mathcal{K}^\gamma{}_\delta = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3],\\
\mathcal{U}_4 &= -\varepsilon_{\mu\alpha\gamma\rho}\varepsilon^{\nu\beta\delta\sigma}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta\mathcal{K}^\gamma{}_\delta\mathcal{K}^\rho{}_\sigma = [\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2]\\
&\quad + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4].
\end{aligned} \tag{2.3}$$

The 6th degree of freedom is absent in massive gravity with this potential, and this theory has 5 degrees of freedom, which are the proper degrees of freedom in massive gravity [25,26]. Note that we are interested in the decoupling limit about flat space-time in this work. In other words, we are interested in the case where Minkowski is a vacuum solution to the equations of motion. Therefore, we neglect the contributions of the cosmological constant and the tadpole in this work, i.e., $\alpha_0 = \alpha_1 = 0$. Furthermore, we fix $\alpha_2 = 1$ in order to have the right renormalization for the kinetic terms in the decoupling limit. The five polarization modes in the ghost-free massive gravity can be decomposed into the scalar, vector, and tensor modes by taking the decoupling limit, which is very convenient to capture the dynamics of each mode within the scale m^{-1}. In order to decompose these modes, we usually expand the Stückelberg field around the unitary gauge[3] as

$$\phi^a = \delta^a_\mu x^\mu - \eta^{a\mu}\partial_\mu\pi/M_{\rm Pl}m^2, \tag{2.4}$$

and the physical metric around the Minkowski background as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{\rm Pl}$, where π describes the scalar mode of a massive graviton. Then the decoupling limit can be taken by the following limits,

$$M_{\rm Pl} \to \infty, \qquad m \to 0, \qquad \Lambda_3 = (M_{\rm Pl}m^2)^{1/3} = \text{fixed}. \tag{2.5}$$

The Lagrangian in the decoupling limit takes the following simple form:

[2]The choice of the Stückelberg field is arbitrary, and fixing the unitary gauge, $\phi^a = \delta^a_\mu x^\mu$, reduces $H_{\mu\nu}$ to the original fluctuation tensor $h_{\mu\nu}$.

[3]The vector modes are disregarded for simplicity. For details of the complete derivation, see [14,27].

$$\mathcal{L} = -\frac{1}{2}h^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + h^{\mu\nu}\sum_{n=1}^{3}\frac{a_n}{\Lambda_3^{3(n-1)}}X^{(n)}_{\mu\nu}[\Pi] + \frac{1}{2M_{\rm Pl}}h^{\mu\nu}T_{\mu\nu}, \tag{2.6}$$

where the first term represents the usual kinetic term for the graviton defined in the standard way, with the Lichnerowicz operator given by

$$\mathcal{E}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = -\frac{1}{2}(\Box h_{\mu\nu} - 2\partial_\alpha\partial_{(\mu}h^\alpha_{\nu)} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta})), \tag{2.7}$$

whereas $a_1 = -1/2$, $a_{2,3}$ are two arbitrary constants related to the model parameters $\alpha_{3,4}$, and the tensors $X^{(1,2,3)}_{\mu\nu}$ denote the interactions with the helicity-0 mode [3]:

$$X^{(1)}_{\mu\nu} = -\frac{1}{2}\varepsilon_\mu{}^{\alpha\rho\sigma}\varepsilon_\nu{}^\beta{}_{\rho\sigma}\Pi_{\alpha\beta}, \tag{2.8}$$

$$X^{(2)}_{\mu\nu} = -\frac{1}{2}\varepsilon_\mu{}^{\alpha\gamma\rho}\varepsilon_\nu{}^{\beta\delta}{}_\rho\Pi_{\alpha\beta}\Pi_{\gamma\delta}, \tag{2.9}$$

$$X^{(3)}_{\mu\nu} = \varepsilon_\mu{}^{\alpha\gamma\rho}\varepsilon_\nu{}^{\beta\delta\sigma}\Pi_{\alpha\beta}\Pi_{\gamma\delta}\Pi_{\rho\sigma}. \tag{2.10}$$

Here we defined $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi$, and Λ_3 represents the strong coupling scale of this theory. One can easily check that this Lagrangian possesses the diffeomorphism invariance, $x^\mu \to x^\mu + \xi^\mu$, and the Galileon symmetry, $\partial_\mu\pi \to \partial_\mu\pi + c_\mu$. The structure of $X^{(1,2,3)}_{\mu\nu}$ is the same as the Galileon theory, which ensures that the equation of motion remains a second-order differential equation (i.e., this theory is free of the Boulware-Deser ghost) and which also guarantees the existence of a nonrenormalization theorem [5].

B. Proxy theory from the decoupling limit

We now want to covariantize the decoupling limit theory. The decoupling limit theory is only valid within the Compton wavelength of the massive graviton.[4] Once we covariantize the decoupling limit theory, the proxy theory is no longer massive gravity; however, they share the same decoupling limit. It would be very interesting to study the cosmology of the proxy theory in order to see the differences from the original massive gravity theory. After covariantizing the decoupling limit interactions, the resulting interactions become [16]

$$h^{\mu\nu}X^{(1)}_{\mu\nu} \longleftrightarrow \frac{1}{2}\sqrt{-g}\pi\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta}{}_{\rho\sigma}R_{\mu\alpha\nu\beta} = -\sqrt{-g}\pi R, \tag{2.11}$$

$$h^{\mu\nu}X^{(2)}_{\mu\nu} \longleftrightarrow -\frac{1}{2}\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_\sigma R_{\mu\alpha\nu\beta}\partial_\rho\pi\partial_\gamma\pi = -\sqrt{-g}\partial_\mu\pi\partial_\nu\pi G^{\mu\nu}, \tag{2.12}$$

$$h^{\mu\nu}X^{(3)}_{\mu\nu} \longleftrightarrow \sqrt{-g}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}R_{\mu\alpha\nu\beta}\partial_\rho\pi\partial_\gamma\pi\Pi_{\sigma\delta} = -\sqrt{-g}\partial_\mu\pi\partial_\nu\pi\Pi_{\alpha\beta}L^{\mu\alpha\nu\beta}. \tag{2.13}$$

Here we use the fact that

$$[\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}R_{\mu\alpha\nu\beta}]_h = -\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\partial_\mu\partial_\alpha h_{\nu\beta}, \tag{2.14}$$

and the tensors $G_{\mu\nu}$ and $L^{\mu\alpha\nu\beta}$ are the Einstein and the dual Riemann tensors respectively,

$$G^{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{2.15}$$

$$L^{\mu\alpha\nu\beta} = 2R^{\mu\alpha\nu\beta} + 2(R^{\mu\beta}g^{\nu\alpha} + R^{\nu\alpha}g^{\mu\beta} - R^{\mu\nu}g^{\alpha\beta} - R^{\alpha\beta}g^{\mu\nu}) + R(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\nu\alpha}). \tag{2.16}$$

Thus, the covariantization of the decoupling limit Lagrangian (2.6) gives birth to the following proxy theory:

$$\mathcal{L} = \sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}R + \mathcal{L}^\pi(\pi, g_{\mu\nu}) + \mathcal{L}^{\rm matter}(\psi, g_{\mu\nu})\right), \tag{2.17}$$

where the Lagrangian for π is

$$\mathcal{L}^\pi = M_{\rm Pl}\left(-\pi R - \frac{a_2}{\Lambda^3}\partial_\mu\pi\partial_\nu\pi G^{\mu\nu} - \frac{a_3}{\Lambda^6}\partial_\mu\pi\partial_\nu\pi\Pi_{\alpha\beta}L^{\mu\alpha\nu\beta}\right). \tag{2.18}$$

These correspondences relate the decoupling limit of massive gravity to the subclass of Horndeski scalar-tensor interactions. This proxy theory represents a theory of general relativity on top of which a new scalar degree of freedom is added, which is nonminimally coupled to gravity.[5] The Galileon symmetry is broken by covariantizing the decoupling limit Lagrangian as in the most general second-order scalar-tensor theory. Furthermore, the constant shift symmetry, $\pi \to \pi + c$, is not even preserved by covariantization. Note that the πR term satisfies the constant shift symmetry at linear level; however, the nonlinear corrections in the πR term break the shift symmetry.

C. Proxy theory as a subclass of Horndeski scalar-tensor theories

As mentioned above, the proxy theory is a subclass of Horndeski scalar-tensor theories which describes the most

[4]In order to explain the current accelerated expansion of the Universe driven by the mass of the graviton, it has to be of the order of the present Hubble horizon H_0.

[5]See also [28] where similar interactions were considered, even though they are unrelated to massive gravity.

general scalar-tensor interactions with second-order equations of motion. The general functions of the Horndeski interactions can be related with the proxy theory. The Horndeski action is given by the following action:

$$S = \int d^4x\sqrt{-g}\left(\sum_{i=2}^{5}\mathcal{L}_i + \mathcal{L}_m\right), \tag{2.19}$$

with

$$\begin{aligned}
\mathcal{L}_2 &= K(\pi, X)\\
\mathcal{L}_3 &= -G_3(\pi, X)[\Pi]\\
\mathcal{L}_4 &= G_4(\pi, X)R + G_{4,X}([\Pi]^2 - [\Pi^2])\\
\mathcal{L}_5 &= G_5(\pi, X)G_{\mu\nu}\Pi^{\mu\nu} - \frac{1}{6}G_{5,X}([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]),
\end{aligned} \tag{2.20}$$

where the arbitrary functions K, G_3, G_4 and G_5 depend on the scalar field π and its derivatives $X = -\frac{1}{2}(\partial\pi)^2$, and furthermore, $G_{i,X} = \partial G_i/\partial X$ and $G_{i,\pi} = \partial G_i/\partial\pi$. The proxy theory corresponds to the case for which the above functions take the following concrete forms [29,30]:

$$\begin{aligned}
K(\pi, X) &= 0\\
G_3(\pi, X) &= 0\\
G_4(\pi, X) &= \frac{M_{\rm Pl}^2}{2} - M_{\rm Pl}\pi - \frac{M_{\rm Pl}}{\Lambda^3}a_2 X\\
G_5(\pi, X) &= 3\frac{M_{\rm Pl}}{\Lambda^6}a_3 X.
\end{aligned} \tag{2.21}$$

The Horndeski scalar-tensor theories represent an interesting class of modified gravity models. However, with the general functions K, G_3, G_4 and G_5 it is hard to study the entire class at once. In the literature, there have been some attempts at parametrizing the theory in a way that would allow one to investigate the theory as a whole in order to be favored or ruled out by observations [31,32]. The interesting point in the proxy theory is that it has its original motivation in massive gravity, and it has to have its explicit form constructed out of the decoupling limit. Thus, this construction relates two important classes of modified gravity theories, namely, massive gravity and Horndeski theories.

III. DYNAMICAL SYSTEM ANALYSIS

A. Field equations

From now on, we discuss the properties of cosmological solutions. We work with the spatially flat Friedmann-Robertson-Walker metric, $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, and assume that a universe is filled with dust, for later convenience. Then the gravity equations are given by

$$3M_{\rm Pl}^2 H^2 = \rho_\pi + \rho_m, \tag{3.1}$$

$$-M_{\rm Pl}^2(2\dot{H} + 3H^2) = p_\pi, \tag{3.2}$$

where $H(=\dot{a}/a)$ is the Hubble parameter, ρ_m is the energy density of matter, and the energy density and the pressure of the Galileon field are defined by

$$\rho_\pi = M_{\rm Pl}\left(6H^2\pi + 6H\dot{\pi} - \frac{9a_2}{\Lambda^3}H^2\dot{\pi}^2 - \frac{30a_3}{\Lambda^6}H^3\dot{\pi}^3\right), \tag{3.3}$$

$$\begin{aligned}
p_\pi = M_{\rm Pl}\Bigg(&-2(2\dot{H} + 3H^2)\pi - 4H\dot{\pi} - 2\ddot{\pi}\\
&+ \frac{a_2}{\Lambda^3}(3H^2\dot{\pi}^2 + 2\dot{H}\dot{\pi}^2 + 4H\dot{\pi}\ddot{\pi})\\
&+ \frac{6a_3}{\Lambda^6}(2H^3\dot{\pi}^3 + 2H\dot{H}\dot{\pi}^3 + 3H^2\dot{\pi}^2\ddot{\pi})\Bigg),
\end{aligned} \tag{3.4}$$

and the equation of motion for π in the FRW space-time is given by

$$\begin{aligned}
&\frac{6a_2}{\Lambda^3}\left(3H^3\dot{\pi} + 2H\dot{H}\dot{\pi} + H^2\ddot{\pi}\right)\\
&\quad + \frac{18a_3}{\Lambda^6}\left(3H^2\dot{H}\dot{\pi}^2 + 3H^4\dot{\pi}^2 + 2H^3\dot{\pi}\ddot{\pi}\right) = \bar{R},
\end{aligned} \tag{3.5}$$

where $\bar{R}$ is the Ricci scalar evaluated in the FRW metric, $\bar{R} = 6(\dot{H} + 2H^2)$. This field equation for π can be recast in a compact form,

$$\ddot{\phi} + 3H\dot{\phi} - \bar{R} = 0, \tag{3.6}$$

where the new field ϕ is defined by

$$\dot{\phi} = H^2\left(\frac{6a_2}{\Lambda^3}\dot{\pi} + \frac{18a_3}{\Lambda^6}\dot{\pi}^2 H\right). \tag{3.7}$$

B. de Sitter regime

The de Sitter solutions which were found in [16] are only valid in the approximation $H\pi \ll \dot{\pi}$. In [16], it was shown that de Sitter is a legitimate solution when such an approximation holds. In the following, we will study the validity of this approximation in more detail. In a pure de Sitter background with a constant expansion rate H_{dS}, the exact homogeneous field equation reads

$$\begin{aligned}
&\frac{6H_{dS}^2}{\Lambda^3}\left(a_2 + 6a_3\frac{H_{dS}}{\Lambda^3}\dot{\pi}\right)\ddot{\pi}\\
&\quad + 18\frac{H_{dS}^3}{\Lambda^3}\left(a_2 + 3a_3\frac{H_{dS}}{\Lambda^3}\dot{\pi}\right)\dot{\pi} = 12H_{dS}^2.
\end{aligned} \tag{3.8}$$

In [16], this equation, together with the Friedmann equation, was solved by using the approximation $\pi H \ll \dot{\pi}$ and the Ansatz of constant $\dot{\pi}$. However, this equation can actually be exactly solved without making such an approximation, and the corresponding solution exhibits the two following branches for $\dot{\pi}$:

$$\dot{\pi} = \frac{-a_2\Lambda^3 \pm e^{-\frac{3}{2}H_{dS}t}\sqrt{4a_3e^{C_1} + (a_2^2 + 8a_3)e^{3H_{dS}t}\Lambda^6}}{6a_3H_{dS}}, \tag{3.9}$$

with C_1 an integration constant. At late times, one can easily see that $\dot{\pi}$ evolves towards the constant value

$$\dot{\pi}(t \gg H_{dS}^{-1}) \simeq -\frac{\Lambda^3}{6a_3H_{dS}}\left[a_2 \pm \sqrt{a_2^2 + 8a_3}\right]. \tag{3.10}$$

This coincides with the finding in [16] when assuming the Ansatz $\dot{\pi} = q\Lambda^3/H_{dS}$, showing that such a solution is indeed the attractor solution in a de Sitter background. It is important to notice that this solution has been obtained by assuming that the de Sitter background is not driven by the π field but by some other independent effective cosmological constant. Now we study if such an effective cosmological constant can be generated by the π field itself so that de Sitter is an actual solution of the system. From the above solution for $\dot{\pi}$, it is straightforward to obtain the solution for π by means of a simple integration

$$\pi(t \gg H_{dS}^{-1}) \simeq -\frac{\Lambda^3}{6a_3H_{dS}}\left[a_2 \pm \sqrt{a_2^2 + 8a_3}\right]t + C_2, \tag{3.11}$$

where C_2 is another integration constant. If we plug this solution into the energy density of π (which gives the right-hand side of the Friedmann equation), we obtain

$$\rho_\pi \simeq \frac{M_p\Lambda^3}{18}\left[108C_2\frac{H_{dS}^2}{\Lambda^3} + \left(\frac{a_2^3}{a_3^2} + 6\frac{a_2}{a_3}\right) \pm \left(\frac{a_2^2 + 2a_3}{a_3^2}\right)\sqrt{a_2^2 + 8a_3}\right] - \frac{M_p\Lambda^3}{a_3}(a_2 \pm \sqrt{a_2^2 + 8a_3})H_{dS}t. \tag{3.12}$$

At early times, when $H_{dS}t \ll 1$, we can neglect the second term in this expression; the energy density of the π field is approximately constant, as it corresponds to a de Sitter solution. However, we must keep in mind that this solution is actually valid at late times; in that case, the second term growing linearly with time drives the energy density evolution, and thus, de Sitter cannot be the solution. This also agrees with the fact that the condition $\pi H \ll \dot{\pi}$ will eventually be violated at late times because the scalar field grows in time, whereas H and $\dot{\pi}$ are assumed to be constant. One might think that a solution would be to tune the parameters so that $a_2 \pm \sqrt{a_2^2 + 8a_3} = 0$. However, the only solution to this equation is $a_3 = 0$, which represents a singular value. In fact, if we take the limit $a_3 \to 0$ in the above solution, we obtain $\rho_\pi \to 6C_2H_{dS}M_p + 4M_p\Lambda^3H_{dS}t/a_2$, so the growing term remains. From this simple analysis, it seems that de Sitter cannot exist as an attractor solution of the phase map; it can only represent transient regimes. This can, in turn, be useful for inflationary models where the accelerated expansion needs to end, but it is less appealing as a dark energy model.

C. Phase analysis without a matter component

In the following, we will make this simple analysis more rigorous and look at it in more detail. In order to obtain a general overview of the class of cosmological solutions that one can expect to find in the proxy theory, we shall perform a dynamical system analysis. This will give us the critical points of the cosmological equations as well as their stability. The first step to perform the dynamical system analysis will be to obtain the equations to be analyzed. Since we are interested in cosmological solutions, we assume that the metric takes the FLRW form with flat spatial sections. The most convenient time variable for the analysis will be the number of e-folds, $N \equiv \ln a$. The equation of motion for the π field in terms of this time variable is given by

$$\left(a_2 + 6a_3H^2\frac{\pi'}{\Lambda^3}\right)\pi'' + 3\left[a_2\left(1 + \frac{H'}{H}\right) + \frac{a_3H^2}{\Lambda^3}\left(3 + 5\frac{H'}{H}\right)\pi'\right]\pi' = 2\frac{\Lambda^3}{H^2}\left(1 + \frac{H'}{2H}\right), \tag{3.13}$$

where the prime denotes a derivative with respect to N. In addition to this equation, we also need the corresponding Einstein equations, which in our case are given by

$$H^2 = \frac{1}{6M_p^2}\rho_\pi, \tag{3.14}$$

$$2HH' + 3H^2 = -\frac{1}{2M_P^2}p_\pi, \tag{3.15}$$

where we have used $dN = Hdt$ and ρ_π and p_π are the energy density and pressure of the π field expressed in terms of N. We now have three equations for the two variables π and H. Of course, not all of these equations are independent. In order to reduce these equations to the form of an autonomous system, we will first use the Friedmann constraint to obtain an expression for π in terms of π' and

H. The resulting expression will constitute a constraint for π and will allow us to get rid of its dependence in the remaining equations so that we end up with dependence only on H, H', π' and π''. This will prove very useful since it reduces the number of variables in our autonomous system. In fact, we can use $y \equiv \pi'$ as one of our dynamical variables, and then we have a system of two first-order differential equations for y and H. After some simple algebra, one can reduce the equations to the following autonomous system:

$$\frac{dy}{dN} = -\frac{1 + 3b_2H^2y + (25b_3 - 9b_2^2)H^4y^2 - 87b_2b_3H^6y^3 - 180b_3^2H^8y^4}{1 - 6b_2H^2y + 6(b_2^2 - 5b_3)H^4y^2 + 52b_2b_3H^6y^3 + 105b_3^2H^8y^4}y$$
$$\frac{dH}{dN} = -\frac{2 - 8b_2H^2y + (9b_2^2 - 33b_3)H^4y^2 + 72b_2b_3H^6y^3 + 135b_3^2H^8y^4}{1 - 6b_2H^2y + 6(b_2^2 - 5b_3)H^4y^2 + 52b_2b_3H^6y^3 + 105b_3^2H^8y^4}H, \tag{3.16}$$

where we have introduced the rescaled parameters $b_2 \equiv a_2M_p^3/\Lambda^3$ and $b_3 \equiv a_3M_p^6/\Lambda^6$. One can immediately see that $H = y = 0$ is a stable critical point which is independent of the parameters and corresponds to the vacuum Minkowski solution. For the remaining critical points, we need to solve the equations

$$1 + 3b_2H^2y + (25b_3 - 9b_2^2)H^4y^2 - 87b_2b_3H^6y^3 - 180b_3^2H^8y^4 = 0,$$
$$2 - 8b_2H^2y + (9b_2^2 - 33b_3)H^4y^2 + 72b_2b_3H^6y^3 + 135b_3^2H^8y^4 = 0.$$

To solve these equations, it will be convenient to introduce the new rescaling $\hat{y} \equiv H^2yb_2$ and the new constant $c_3 \equiv b_3/b_2^2 = a_3/a_2^2$. Then, the previous equations can be written in the simpler form

$$1 + 3\hat{y} + (25c_3 - 9)\hat{y}^2 - 87c_3\hat{y}^3 - 180c_3^2\hat{y}^4 = 0, \tag{3.17}$$

$$2 - 8\hat{y} + (9 - 33c_3)\hat{y}^2 + 72c_3\hat{y}^3 + 135c_3^2\hat{y}^4 = 0. \tag{3.18}$$

As we can see, we have an overdetermined system of equations, so solutions cannot be found for arbitrary c_3. In fact, the above equations can be solved for $\hat{y}$ and c_3 in order to obtain the models with additional critical points. Remarkably, there is only one real solution for these equations, and it is given by $c_3 \simeq 0.094$ and $\hat{y} \simeq -3.99$. Notice that this, in fact, does not represent one single critical point for the autonomous system but a curve of critical points in the plane (y, H). The obtained result implies that pure de Sitter does not correspond to a critical point of the proxy theory and can only exist as a transient regime, as we had anticipated from our previous simple analysis.

Another interesting feature of the autonomous system is the existence of separatrices in the phase map determined by the curve along which the denominators in (3.16) vanish, i.e.,

$$1 - 6b_2H^2y + 6(b_2^2 - 5b_3)H^4y^2 + 52b_2b_3H^6y^3 + 105b_3^2H^8y^4 = 0. \tag{3.19}$$

This curve can be simplified if we use our previously defined rescaled variable $\hat{y}$ and parameter c_3, in terms of which the separatrix is determined by

$$1 - 6\hat{y} + 6(1 - 5c_3)\hat{y}^2 + 52c_3\hat{y}^3 + 105c_3^2\hat{y}^4 = 0, \tag{3.20}$$

which is a quartic polynomial equation. Since the independent term and the highest power coefficient are both positive, this equation does not always have real solutions, so the separatrix does not exist for arbitrary parameters. Indeed, the previous equation determines a curve in the plane $(\hat{y}, c_3)$, which can be regarded as the function

$$c_3 = \frac{15 - 26\hat{y} \pm \sqrt{2}\sqrt{60 - 75\hat{y} + 23\hat{y}^2}}{105\hat{y}^2}. \tag{3.21}$$

This function has been plotted in Fig. 1. As we can see in that figure, the value of c_3 determines the number of real solutions and, therefore, the number of separatrices in the phase map of the autonomous system. We find that for $c_3 > 0$, the system always exhibits four separatrices. When $c_3 = 0$, the cubic and quartic terms of the separatrix equation vanish, so we only have two real solutions. In the cases with $0 > c_3 > -0.093$, the system has four separatrices again. When $-0.093 > c_3 > -0.215$, there are only two separatrices. Finally, for $c_3 < -0.215$, the equation has no real solutions and, therefore, it does not generate any separatrix. Special cases are $c_3 = -0.093$ with

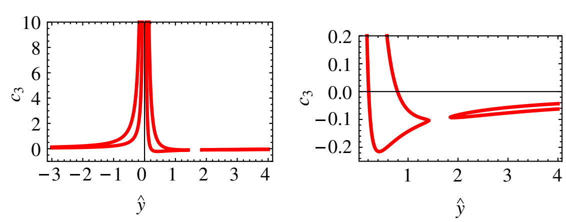

FIG. 1 (color online). In this plot we show the curve determined by Eq. (1) in the plane $(\hat{y} = H^2\pi'b_2, c_3 = a_3/a_2^2)$. The right panel shows more clearly the structure of the corresponding area. As explained in the main text, the value of the parameter c_3 determines the number of separatrices in the phase map.

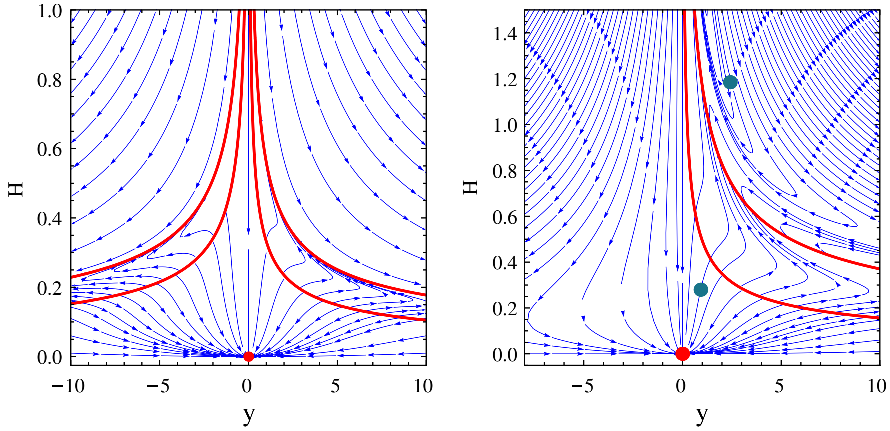

FIG. 2 (color online). In this figure, we show two examples of phase map portraits of the dynamical autonomous system for $b_2 = 1$ and $b_3 = 1.5$ (with $c_3 = 1.5$) in the left panel and $b_2 = 1$ and $b_3 = -0.1$ (with $c_3 = -0.1$) in the right panel. These values have been chosen to show examples with $c_3 > 0$ (always with four separatrices) and $c_3 < 0$ with two separatrices (see main text and Fig. 1). The red lines represent the corresponding separatrices, and the red point denotes the Minkowski vacuum solution. We can see that this solution is indeed an attractor. Concerning the attracting behavior of the separatrices, we can see that the upper ones behave as attractors, whereas the lower ones act as repellers. In the right panel, we additionally indicate with green points the analytical solutions found in [16] under the approximation $\pi H \ll \dot{\pi}$.

three separatrices and $c_3 = -0.215$ with only one separatrix. All this can be clearly seen in Fig. 1.

If the solutions of Eq. (3.21) are denoted by $\hat{y} = y_i^*$, then the separatrices are given by the curves $y = b_2 y_i^*/H^2$ or, equivalently, $H = \pm\sqrt{b_2 y_i^*/y}$ in the phase map. Notice that, depending on the sign of $b_2 y_i^*$, the corresponding separatrix will only exist in the semiplane $y > 0$ or $y < 0$ for $b_2 y_i^* > 0$ or $b_2 y_i^* < 0$, respectively. This can be seen in the examples shown in Fig. 2 where we have plotted the phase maps corresponding to two characteristic cases, namely, one with $c_3 = 1.5$ (which has four separatrices and positive c_3) and one with $c_3 = -0.1$ (which has only two separatrices and negative c_3). One interesting feature that we can observe in both cases is the attracting nature of the upper separatrices, whereas the lower ones behave as repellers. Remarkably, the attracting separatrices do not behave as asymptotic attractors, but the trajectories actually hit the separatrix and the universe encounters a singularity.

The phase map shown in the right panel corresponds to parameters satisfying all the existence and stability requirements obtained in [16] from the approximate analytical solutions. The green points in the phase map denote the solutions that had been identified in [16] with stable self-accelerating solutions. However, we can see now that the eventual attractor solution is not actually de Sitter but rather the Minkowksi vacuum solution. The stability condition for such a solution actually corresponds to the convergence of the nearby trajectories.

It is worthwhile pointing out once more that, although (quasi-) de Sitter solutions do not exist as critical points in the phase maps, it is possible to have transient regimes with quasi-de Sitter expansion. One possibility where such transient regimes can be found corresponds to the trajectories above the upper separatrix in the right panel of Fig. 2. These trajectories initially evolve towards large values of y, but, at some point, there is a turnover where they go towards smaller values of y. While this turnover is taking place, the value of H can remain nearly constant for some time and, thus, we can have a period of quasi-de Sitter expansion. The number of e-folds corresponding to this transient regime depends on the parameters and the initial conditions, but it is generally quite small (see Fig. 3 where we plot the evolution of one particular solution).

In order to study the properties of the dynamical system near the separatrix, we will rewrite the autonomous system in terms of the variable $\hat{y}$, since, as suggested from our previous analysis, the equations will look simpler. In particular, the separatrices will become straight vertical lines in this variable, and the behavior of the trajectories near them can be straightforwardly studied. In such variables, the autonomous system reads

$$\frac{d\hat{y}}{dN} = -\frac{5 - 13\hat{y} + (9 - 41c_3)\hat{y} + 57c_3\hat{y}^3 + 90c_3^2\hat{y}^4}{1 - 6\hat{y} + 6(1 - 5c_3)\hat{y}^2 + 52c_3\hat{y}^3 + 105c_3^2\hat{y}^4}\hat{y}$$
$$\frac{dH}{dN} = -\frac{2 - 8\hat{y} + (9 - 33c_3)\hat{y}^2 + 72c_3\hat{y}^3 + 135c_3^2\hat{y}^4}{1 - 6\hat{y} + 6(1 - 5c_3)\hat{y}^2 + 52c_3\hat{y}^3 + 105c_3^2\hat{y}^4}H. \tag{3.22}$$

As we anticipated, the equations look simpler in these variables. In particular, the equation for $\hat{y}$ completely decouples from the equation for the Hubble expansion rate. Near the separatrix located at y_s, we can expand $\hat{y} = \hat{y}_s + \delta\hat{y}$ and obtain the leading terms of the above equations, given by

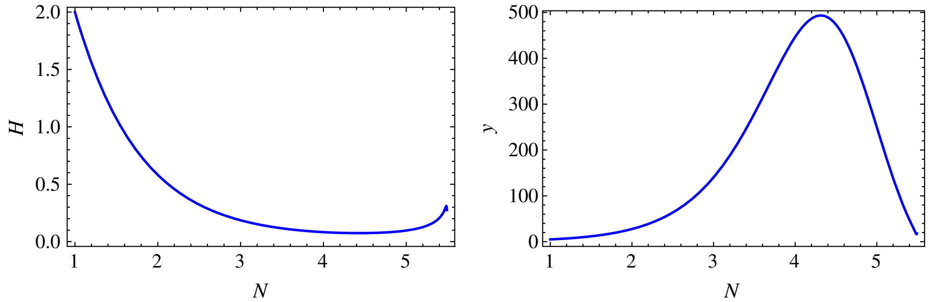

FIG. 3 (color online). In this figure, we show the numerical solution for H (left panel) and y (right panel) with the initial conditions $H_{\rm ini} = 2$ and $y_{\rm ini} = 5$. We can see the transient period of quasi-de Sitter expansion in the evolution of H corresponding to the turnover and how it lasts for barely 1–2 e-folds. In addition, we can see the singularity corresponding to the moment when the trajectory reaches the separatrix at a finite number of e-folds.

$$\frac{d\delta\hat{y}}{dN} = \frac{k_y}{\delta\hat{y}}, \qquad \frac{dH}{dN} = \frac{k_H}{\delta\hat{y}}H, \tag{3.23}$$

with

$$k_y \equiv -\frac{5 - 13\hat{y}_s + (9 - 41c_3)\hat{y}_s + 57c_3\hat{y}_s^3 + 90c_3^2\hat{y}_s^4}{1 - 6 + 12(1 - 5c_3)\hat{y}_s + 156c_3\hat{y}_s^2 + 420c_3^2\hat{y}_s}\hat{y}_s, \tag{3.24}$$

$$k_H \equiv -\frac{2 - 8\hat{y}_s + (9 - 33c_3)\hat{y}_s^2 + 72c_3\hat{y}_s^3 + 135c_3^2\hat{y}_s^4}{1 - 6 + 12(1 - 5c_3)\hat{y}_s + 156c_3\hat{y}_s^2 + 420c_3^2\hat{y}_s}. \tag{3.25}$$

Now, it is straightforward to read the conditions for the separatrix to attract the trajectories. Notice that the attracting or repelling nature of the separatrix will be the same on both sides. Thus, whenever k_y is negative, the separatrix will represent an attractor of the phase map, whereas it will be a repeller for positive k_y.

The equation for $\delta\hat{y}$ near the separatrix can be easily integrated to give

$$\delta\hat{y}(N) \simeq \pm\sqrt{2k_yN + C_y}, \tag{3.26}$$

with C_y an integration constant, and the two branches correspond to both sides of the separatrix. If the separatrix is an attractor, we have that k_y is negative and, therefore, the solution only exists until $N_s = -\frac{C_y}{2k_y}$, confirming our previous statement that the trajectories do not asymptotically approach the separatrix, but they hit it and end there. On the other hand, with the solution for $\delta\hat{y}$, we can also obtain the solution for H, which is given by

$$H(N) = C_H e^{\pm\frac{k_H}{k_y}\sqrt{2k_yN + C_y}}, \tag{3.27}$$

with C_H being another integration constant. We see that the Hubble expansion rate does not diverge at the separatrix, but it goes to the constant value C_H so that the energy density of the field remains finite. However, the derivative of the Hubble expansion rate near the separatrix evolves as

$$\dot{H} \simeq H^2 \frac{k_H}{\sqrt{2k_yN + C_y}}, \tag{3.28}$$

so it goes to infinity as it approaches the separatrix. This signals a divergence in the pressure of the scalar field when the trajectory hits the separatrix, so we find a future sudden singularity. This kind of singularity was first studied in [33] and corresponds to the type II according to the classification performed in [34].

D. Phase analysis with matter component

So far, in our study we have focused on the case when only the π field contributes to the energy density of the universe, and we have neglected any other possible component that might be present. We have shown that the only critical point is the pure vacuum Minkowski solution with $H = y = 0$. Moreover, we have shown that the separatrices can also act as attractors of the phase map and, when this happens, the evolution ends in a singularity where the derivative of the Hubble expansion rate diverges. In order to have a more realistic scenario, at least a dustlike matter component (pressureless matter) should be included. This will add a new dimension to the phase space, and thus a new phenomenology is expected to arise. In particular, it could change some stability requirements, and additional critical points might appear. Therefore, let us discuss in the following the case with matter fields.

If we include a pressureless matter component and use the variables H, $\hat{y}$ and[6] $\Omega_m \equiv \rho_m b_2/(6H^2)$ to describe the extended cosmological evolution, the corresponding autonomous system reads

[6]Notice the factor b_2 in this definition of the matter density parameter that does not appear in the usual definition.

$$\frac{d\hat{y}}{dN} = -\frac{(5 - 13\hat{y} + (9 - 41c_3)\hat{y}^2 + 57c_3\hat{y}^3 + 90c_3^2\hat{y}^4)\hat{y} + (1 - 3\hat{y} - 9c_3\hat{y}^2)H^2\Omega_m}{1 - 6\hat{y} + 6(1 - 5c_3)\hat{y}^2 + 52c_3\hat{y}^3 + 105c_3^2\hat{y}^4 - 2(1 + 6c_3\hat{y})H^2\Omega_m}$$
$$\frac{dH}{dN} = -\frac{2 - 8\hat{y} + (9 - 33c_3)\hat{y}^2 + 72c_3\hat{y}^3 + 135c_3^2\hat{y}^4 - 3(1 + 6c_3\hat{y})H^2\Omega_m}{1 - 6\hat{y} + 6(1 - 5c_3)\hat{y}^2 + 52c_3\hat{y}^3 + 105c_3^2\hat{y}^4 - 2(1 + 6c_3\hat{y})H^2\Omega_m}H$$
$$\frac{d\Omega_m}{dN} = \frac{1 + 2\hat{y} + 24c_3\hat{y}^2 - 12c_3\hat{y}^3 - 45c_3^2\hat{y}^4}{1 - 6\hat{y} + 6(1 - 5c_3)\hat{y}^2 + 52c_3\hat{y}^3 + 105c_3^2\hat{y}^4 - 2(1 + 6c_3\hat{y})H^2\Omega_m}\Omega_m. \tag{3.29}$$

Since we are seeking critical points with $\Omega_m \neq 0$, we can solve for this system by using the vanishing of $d\hat{y}/dN$ to obtain the expression

$$\Omega_m H^2 = \frac{5 - 13\hat{y} + (9 - 41c_3)\hat{y}^2 + 57c_3\hat{y}^3 + 90c_3^2\hat{y}^4}{-1 + 3\hat{y} + 9c_3\hat{y}^2}\hat{y}, \tag{3.30}$$

for the potential new critical points. Then, we can plug this relation into the remaining two equations given by the vanishing of dH/dN and $d\Omega_m d/N$ to obtain the critical points. However, when doing so we end up with the solutions

$$c_3 = \frac{4\hat{y}^2 - 2\hat{y}^3 \pm \sqrt{21\hat{y}^4 - 6\hat{y}^5 + 4\hat{y}^6}}{15\hat{y}^4}, \tag{3.31}$$

which is incompatible for any value of c_3 since the solutions one finds for c_3 and $\hat{y}$ are singular, meaning that they sit on top of the separatrix with the singularity. Therefore, the inclusion of matter does not introduce new critical points in the phase map.

E. Covariantization of the new kinetic interactions

Above, we have seen that the only critical point existing in the phase map of the proxy theory (even if we include a dust component) is the vacuum Minkowski solution. The proxy theory was constructed from the decoupling limit of the potential interactions of massive gravity. The mass and potential interactions of the graviton break the diffeomorphism invariance. Therefore, one might wonder whether or not there exist derivative interactions for the graviton which break diffeomorphism invariance but still give rise to only five propagating physical degrees of freedom. In the literature this exact question about the existence of new kinetic interactions was investigated [35–37]. Possible terms of the form

$$\mathcal{K}_{\mu\nu}G^{\mu\nu}$$
$$\mathcal{K}_{\mu\nu}\mathcal{K}_{\alpha\beta}L^{\mu\nu\alpha\beta} \tag{3.32}$$

have been considered and unfortunately shown to contain a ghost degree of freedom. Nevertheless, we can consider the decoupling limit of these interactions and covariantize them in a similar way as for the potential interactions. To first order in h these interactions do not give any nontrivial interactions and are identically zero up to total derivatives. The second-order interaction in h of the interaction $K_{\mu\nu}G^{\mu\nu}$ gives rise to a ghost degree of freedom after covariantization, and therefore, we will not consider this contribution. On the other hand, from the interaction $K_{\mu\nu}K_{\alpha\beta}L^{\mu\nu\alpha\beta}$, the only second-order contribution in h which gives rise to a ghost-free interaction is $\mathcal{L}_{\rm DI} = \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\partial_\mu\partial_\alpha h_{\nu\beta}h_{\rho\gamma}\partial_\sigma\partial_\delta\pi$ [38,39]. Covariantization of this decoupling limit Lagrangian $\mathcal{L}_{\rm DI}$ of the derivative interactions in dRGT massive gravity gives rise to the nonminimally coupled Gauss-Bonnet term[7]:

$$\mathcal{L}_{\pi\rm GB} = M_{\rm Pl}^2\frac{a_4}{\Lambda^3}\pi(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2). \tag{3.33}$$

As is known, Gauss-Bonnet terms can give rise to accelerated expansion, so we will now modify the original proxy theory to include this new coupling of the scalar field to the Gauss-Bonnet term. Since we construct this additional Gauss-Bonnet term by covariantizing the decoupling limit of the derivative interactions of the dRGT theory, the resulting theory can still be considered as a proxy theory to massive gravity. The additional contributions in the energy density, pressure, and scalar field equation coming from $\mathcal{L}_{\pi\rm GB}$ are given by

$$\rho_{\pi\rm GB} = M_{\rm Pl}^2\frac{24a_4}{\Lambda^6}H^3\dot{\pi}, \tag{3.34}$$

$$p_{\pi\rm GB} = -M_{\rm Pl}^2\frac{8a_4}{\Lambda^3}(2H^3\dot{\pi} + 2H\dot{H}\dot{\pi} + H^2\ddot{\pi}), \tag{3.35}$$

$$\dot{\phi}_{\pi\rm GB} = -M_{\rm Pl}\frac{8a_4}{\Lambda^3}H. \tag{3.36}$$

The cosmological equations in this case can be expressed as the following autonomous system:

[7]Note that this interaction itself produces the second-order differential equation of motion. However, in the context of massive gravity, the nonlinear derivative interactions unfortunately contain a Boulware-Deser ghost [35,36].

$$\frac{d\hat{y}}{dN} = -\frac{(5-13\hat{y}+(9-41c_3)\hat{y}^2+57c_3\hat{y}^3+90c_3^2\hat{y}^4)+4\epsilon\hat{H}^2(3-3\hat{y}-10c_3\hat{y}^2)}{1-6\hat{y}+6(1-5c_3)\hat{y}^2+52c_3\hat{y}^3+105c_3^2\hat{y}^4+16\hat{H}^4+8\epsilon\hat{H}^2(1-2\hat{y}-9c_3\hat{y}^2)}\hat{y}$$
$$\frac{d\hat{H}}{dN} = -\frac{2-8\hat{y}+(9-33c_3)\hat{y}^2+72c_3\hat{y}^3+135c_3^2\hat{y}^4+16\hat{H}^4+12\epsilon\hat{H}^2(1-2\hat{y}+8c_3\hat{y}^2)}{1-6\hat{y}+6(1-5c_3)\hat{y}^2+52c_3\hat{y}^3+105c_3^2\hat{y}^4+16\hat{H}^4+8\epsilon\hat{H}^2(1-2\hat{y}-9c_3\hat{y}^2)}\hat{H}, \tag{3.37}$$

where $\epsilon \equiv \mathrm{sign}(b_4)$ and $\hat{H} \equiv H\sqrt{|b_4|}$, with $b_4 \equiv a_4 M_p^3/\Lambda^3$ (and the number of e-folds is defined with such a rescaled Hubble expansion rate). In order to look for critical points with $H \neq 0$, we solve for $\hat{H}^2$ from the equation $d\hat{y}/dN = 0$ and plug the obtained solution into the equation $d\hat{H}/dN = 0$. After doing so, we arrive at the following equation:

$$\frac{\hat{H}}{2-3\hat{y}-10c_3\hat{y}^2} = 0, \tag{3.38}$$

whose solution is again $\hat{H} = 0$, signaling that the simple coupling of the scalar field to the Gauss-Bonnet term that we have considered is not able to introduce additional critical points. One can clearly see in Fig. 4 that there are no additional critical points and that the Minkowski solution is the only attractor solution even if we include the additional Gauss-Bonnet term.

F. Shift symmetry breaking term πR

Above, we have shown that even if we include the additional Gauss-Bonnet term in the proxy theory, which also has its origin in the decoupling limit of massive gravity, the only critical point existing in the phase map of the proxy theory is the vacuum Minkowski solution. However, this is not surprising. The problematic term avoiding the existence of de Sitter critical points in the cosmological evolution is the πR term in the action. The original approximation $\pi H \ll \dot{\pi}$ used in [16] actually means that exactly this term is negligible. However, our findings show that such a small term cannot be consistently maintained, and it is the responsible term for the absence of de Sitter solutions in the proxy theory. Thus, a natural modification of the proxy theory that will lead to de Sitter solutions consists in simply dropping the problematic term πR from the action. It is evident that this modified theory will have de Sitter solutions because in that case the approximation used in [16] is exact. In fact, such a term is the only one violating the shift symmetry, so without it, only the derivatives of the scalar field are physically relevant but not the value of the field itself. We can proceed analogously as before to obtain the corresponding autonomous system and look for the critical points. When doing so, one can show that there are de Sitter critical points and that they are stable, since the eigenvalues of the matrix determining the linearized system around the de Sitter critical point are both −3, confirming the results that have been obtained under the approximation $\pi H \ll \dot{\pi}$ in [16]. In

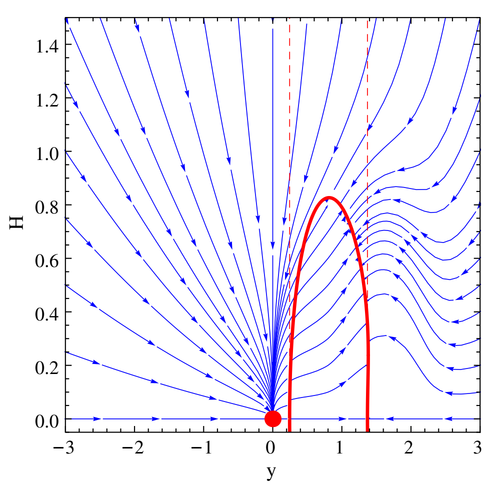

FIG. 4 (color online). In this figure we show an example of the phase map for the proxy theory with the additional Gauss-Bonnet term coming from the covariantization of the decoupling limit of the derivative interactions $\pi\mathcal{L}_{GM}$. One can clearly see that the de Sitter solution is still not an attractor of the cosmological evolution leaving the Minkowski solution as the only existing attractor solution. Thus, the inclusion of this term does not change the cosmological properties of the proxy theory. The red line denotes the separatrix.

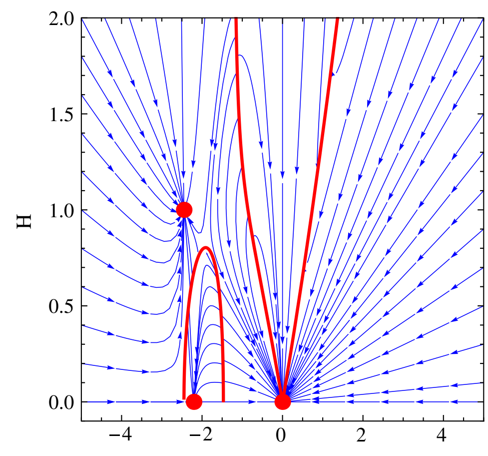

FIG. 5 (color online). In this figure we show an example of the phase map for the proxy theory without the πR term that spoils the existence of de Sitter critical points. We can see that the de Sitter solution is an attractor of the cosmological evolution. The red lines denote the corresponding separatrices.

Fig. 5 we plot an example of the phase map for the case without the πR term in the action, and one can indeed see the existence of the de Sitter attractor. The theory without the πR term can be considered on its own, and it represents an interesting subclass of Horndeski interactions. However, its original motivation from massive gravity would be lost. In the context of massive gravity, putting πR to zero would correspond to putting the kinetic term for the helicity-0 degree of freedom to zero, $h^{\mu\nu}X^{(1)}_{\mu\nu} = 0$. Thus, this would yield strong coupling issues in the original theory. Since we are only interested in the proxy theory related to massive gravity, we do not consider this option any further.

IV. DISCUSSION AND SUMMARY

In this paper, we studied the cosmological dynamics of the proxy theory. For the homogeneous and isotropic universe, there is the de Sitter solution found in [16]; however, we show that this solution can be realized only during the transient regime, and it cannot be an attractor. In order to realize this transient de Sitter regime, we need fine-tuning of the initial conditions of the scalar field; thus, the homogeneous and isotropic universe in the proxy theory cannot be an alternative theory for the dark energy model. Instead, the space-time approaches Minkowski space-time or a type II singularity at the end, depending on the initial conditions.

In the proxy theory, the constant shift symmetry, $\pi \to \pi + c$, is broken by the πR interaction term, while the decoupling limit theory in massive gravity satisfies this symmetry. If the theory satisfies the shift symmetry, then the field equation of the scalar field obeys $\ddot{\phi} + 3H\dot{\phi} = 0$, where $\dot{\phi}$ depends on the models. In this case this equation can be easily solved, giving $\dot{\phi} \propto a^{-3}$. This means that whatever $\dot{\phi}$ is, this variable will be diluted in the future, signaling an attractor solution. Furthermore, thanks to the shift symmetry, $\dot{\phi}$ only depends on $\dot{\pi}$, and π never appears in any equations of motion, which means $\dot{\pi} = \text{const}$ could be an attractor solution with a wide range of initial conditions. This can be applied to the most general second-order scalar-tensor theory which satisfies the shift symmetry. However, it should be noted that this is a sufficient condition to have a de Sitter attractor but not a necessary condition. The shift symmetry breaking example in the Galileon theory can be found in [40,41]; (quasi-) de Sitter attractor solutions exist in these models. In addition, the case of massive scalar fields is an exception.

One should note that there is an exact de Sitter solution in the decoupling limit theory of massive gravity. Since the proxy theory shares the same decoupling limit with massive gravity, there should be an exact de Sitter attractor solution within the patch enclosed by a sphere, whose domain is of order of the current horizon scale H_0^{-1}. This approximate solution should be connected to inhomogeneous or anisotropic solutions in the proxy theory in a similar way as in the case of massive gravity itself. However, this would rely on the successful implementation of the Vainshtein mechanism [7]. It would be interesting to study this kind of inhomogeneous and/or anisotropic solution in a future work.

The original proxy theory was constructed from the decoupling limit on flat space-time. It is a legitimate question to ask whether or not one can find interesting cosmological solutions from proxy theories constructed from different decoupling limits of massive gravity. One can, for instance, construct the decoupling limit on a de Sitter or anti-de Sitter space-time. In the case of a de Sitter reference metric, the decoupling limit has to be taken in such a way that the de Sitter length scale H has to go to zero $H \to 0$ at least at the same rate as the mass of the graviton goes to zero. This is due to the Higuchi bound. The decoupling limit has to be taken in such a way that $H/m \to$ fixed. This gives rise to new nontrivial contributions in the form of Galileon interactions. The constructed proxy theory from this de Sitter decoupling limit would contain the terms which we already considered here, but it would also contain the covariantized Galileon interactions. This would give rise to a different subclass of Horndeski interactions. We anticipate that the cosmological evolution in this new proxy theory would be quite rich, and it would be interesting to explore this in more detail in future works.

ACKNOWLEDGMENTS

L. H. is supported by the Swiss National Science Foundation. R. K. is supported in part by a Grant-in-Aid for JSPS Fellows. The research of K. Y. is supported in part by a Grant-in-Aid for Scientific Research of the Japanese Ministry of Education, Culture, Sports, Science and Technology (No. 21540270 and No. 21244033).

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