PHYSICAL REVIEW D **89,** 087502 (2014)

Exact and unique metric for Kerr-Newman black hole in harmonic coordinates

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(Received 3 February 2014; published 8 April 2014)

We present an exact and close-form harmonic metric for Kerr-Newman black hole, and demonstrate it is unique in the harmonic coordinates.

DOI: 10.1103/PhysRevD.89.087502 PACS numbers: 04.20.Cv, 04.20.Jb, 04.25.Nx, 04.70.Bw

Einstein field equations describe the fundamental gravitation interaction between spacetime geometry and matter as well as energy [1]. However, they are redundant due to Bianchi identity of the Riemann curvature tensor. Therefore, to solve the equations we must fix a specific coordinate system (also called gauge fixing). A particularly useful coordinate system is harmonic coordinates, which were first employed by de Donder [2] and Lanczos [3] in dealing with Einstein's field equations. In 1939, Fock [4] obtained the equations of motion and the explicit gravitational potentials for an isolated mass system in harmonic coordinates. Fock stated that one cannot clarify the physical significance of various expressions in general relativity without a harmonic coordinate system and believed that the harmonic coordinates are the preferred system in physical nature [5]. Although this viewpoint is not accepted by the relativity community, the harmonic coordinates play a substantial role in deriving post-Newtonian dynamics and gravitational radiation [6]. The harmonic coordinates are also often used in numerical relativity, e.g., [7].

The exact harmonic metric for the Schwarzschild black hole has been well known [6]. However, it is very difficult to obtain the explicit harmonic metric for other black holes [8–13]. Recently, we have obtained the harmonic metric for the Kerr black hole [14]. In this paper we present an exact and close-form harmonic metric for a spherically symmetric black hole with rotation and electric charge, which is generally known as the Kerr-Newman black hole. Moreover, we also demonstrate that it is unique in the harmonic coordinates.

We start with the metric of the Kerr-Newman black hole in the Boyer-Lindquist coordinate system, which reads [15]

$$ds^2 = -\left(1-\frac{2m\bar{r}-Q^2}{\bar{\rho}^2}\right)d\bar{t}^2 + \frac{\bar{\rho}^2}{\bar{\Delta}}d\bar{r}^2 + \bar{\rho}^2 d\bar{\theta}^2 + \frac{(\bar{r}^2+a^2)^2-\bar{\Delta}a^2\sin^2\bar{\theta}}{\bar{\rho}^2}\sin^2\bar{\theta}d\bar{\varphi}^2 - 2\frac{(2m\bar{r}-Q^2)a\sin^2\bar{\theta}}{\bar{\rho}^2}d\bar{t}d\bar{\varphi}, \tag{1}$$

where $\bar{\rho}^2 \equiv \bar{r}^2 + a^2\cos^2\bar{\theta}$, $\bar{\Delta} \equiv \bar{r}^2 + a^2 - 2m\bar{r} + Q^2$. m, a, and Q denote the mass, angular momentum per unit mass, and electric charge of the Kerr-Newman black hole, respectively. The charge and angular momentum are restricted by the relation $m^2 \geq a^2 + Q^2$ to ensure there is no naked singularity for the black hole. Throughout this paper, we use the geometrized units, in which the speed of light in vacuum and the gravitational constant are set equal to unity.

Applying the following transform,

$$t = \bar{t} + \int f_0(\bar{r})d\bar{r}, \qquad r = \bar{r}, \qquad \theta = \bar{\theta}, \qquad \varphi = \bar{\varphi} + \int g(\bar{r})d\bar{r}, \tag{2}$$

to Boyer-Lindquist formulation, we have

$$ds^2 = Adt^2 + 2Bdtd\varphi + Cd\varphi^2 + Dd\theta^2 + Edr^2 + 2Fdtdr + 2Gdrd\varphi, \tag{3}$$

where

$$A = -1 + (2mr - Q^2)/\rho^2, \qquad B = (Q^2 - 2mr)a\sin^2\theta/\rho^2,$$
$$C = (r^2 + a^2 - aB)\sin^2\theta, \qquad D = \rho^2,$$
$$E = \rho^2/\Delta + f_0^2 A + 2f_0 gB + g^2 C, \qquad F = -f_0 A - gB,$$
$$G = -f_0 B - gC, \tag{4}$$

with $\Delta \equiv r^2 + a^2 - 2mr + Q^2$, and $\rho^2 \equiv r^2 + a^2\cos^2\theta$.

The harmonic coordinates X_μ can be constructed as follows:

$$X_0 = t, \qquad X_1 = f_1(r,\varphi)\sin\theta, \qquad X_2 = f_2(r,\varphi)\sin\theta, \qquad X_3 = f_3(r,\varphi)\cos\theta, \tag{5}$$

with f_1, f_2, f_3 being unknown functions determined by harmonic-coordinate conditions [5,6]

$$\Box^2 X_\mu \equiv g^{\lambda\rho}\frac{\partial^2 X_\mu}{\partial x^\lambda \partial x^\rho} - g^{\lambda\rho}\Gamma^\kappa_{\lambda\rho}\frac{\partial X_\mu}{\partial x^\kappa} = 0, \tag{6}$$

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where $\square^2$ is the invariant d'Alembertian operation, and $\Gamma^{\kappa}_{\lambda\rho}$ is the affine connection.

Substituting Eqs. (3)–(5) into Eq. (6), we can obtain

$$\frac{(f_0\Delta)'}{\rho^2}=0, \tag{7}$$

$$\frac{\sin\theta}{\rho^2}\left[\Delta\frac{\partial^2}{\partial r^2}-2+\left(g^2\Delta-\frac{a^2}{\Delta}\right)\frac{\partial^2}{\partial\varphi^2}+2g\Delta\frac{\partial^2}{\partial r\partial\varphi}+\Delta'\frac{\partial}{\partial r}+(g\Delta)'\frac{\partial}{\partial\varphi}+\frac{(\frac{\partial^2}{\partial\varphi^2}+1)}{\sin^2\theta}\right]f_1=0, \tag{8}$$

$$\frac{\sin\theta}{\rho^2}\left[\Delta\frac{\partial^2}{\partial r^2}-2+\left(g^2\Delta-\frac{a^2}{\Delta}\right)\frac{\partial^2}{\partial\varphi^2}+2g\Delta\frac{\partial^2}{\partial r\partial\varphi}+\Delta'\frac{\partial}{\partial r}+(g\Delta)'\frac{\partial}{\partial\varphi}+\frac{(\frac{\partial^2}{\partial\varphi^2}+1)}{\sin^2\theta}\right]f_2=0, \tag{9}$$

$$\frac{\cos\theta}{\rho^2}\left[\Delta\frac{\partial^2}{\partial r^2}-2+\left(g^2\Delta-\frac{a^2}{\Delta}\right)\frac{\partial^2}{\partial\varphi^2}+2g\Delta\frac{\partial^2}{\partial r\partial\varphi}+\Delta'\frac{\partial}{\partial r}+(g\Delta)'\frac{\partial}{\partial\varphi}+\frac{\frac{\partial^2}{\partial\varphi^2}}{\sin^2\theta}\right]f_3=0, \tag{10}$$

where the prime denotes the derivative with respect to r.

It follows from Eq. (7) that

$$f_0=\frac{c}{\Delta}, \tag{11}$$

with c being a constant.

Similar to the paper [14], we can find a particular solution to Eqs. (8)–(10) as follows:

$$g=\frac{a}{\Delta}, \tag{12}$$

$$f_1=(r-m)\cos\varphi-a\sin\varphi, \tag{13}$$

$$f_2=a\cos\varphi+(r-m)\sin\varphi, \tag{14}$$

$$f_3=r-m. \tag{15}$$

Thus, the harmonic coordinates, Eq. (5), can be rewritten as

$$X_0=t,\qquad X_1=\sqrt{R^2+a^2}\cos\Phi\sin\theta,$$
$$X_2=\sqrt{R^2+a^2}\sin\Phi\sin\theta,\qquad X_3=R\cos\theta, \tag{16}$$

where $R\equiv r-m$, and $\Phi\equiv\varphi+\arctan\frac{a}{R}$.

Applying the ellipsoidal coordinate transformation, Eq. (16), to the metric given in Eqs. (3) and (4), and taking into account Eqs. (11) and (12), we can formulate the metric of Kerr-Newman black hole in the harmonic coordinates as

$$\begin{aligned}ds^2=&\frac{R^2(R+m)^2+a^2X_3^2}{(R^2+\frac{a^2}{R^2}X_3^2)^2}\left[\frac{(\boldsymbol{X}\cdot d\boldsymbol{X}+\frac{a^2}{R^2}X_3dX_3)^2}{R^2+a^2-m^2+Q^2}+\frac{X_3^2}{R^2}\frac{(\boldsymbol{X}\cdot d\boldsymbol{X}-\frac{R^2}{X_3}dX_3)^2}{R^2-X_3^2}\right]\\&+\frac{(R+m)^2+a^2}{R^2-X_3^2}\left[\frac{aR^2(m^2-Q^2)(R^2-X_3^2)(\boldsymbol{X}\cdot d\boldsymbol{X}+\frac{a^2}{R^2}X_3dX_3)}{(R^2+a^2-m^2+Q^2)(R^2+a^2)(R^4+a^2X_3^2)}+\frac{R(X_2dX_1-X_1dX_2)}{R^2+a^2}\right]^2\\&+\frac{2m(R+m)-Q^2}{(R+m)^2+\frac{a^2}{R^2}X_3^2}\left[\frac{a^2R(m^2-Q^2)(R^2-X_3^2)(\boldsymbol{X}\cdot d\boldsymbol{X}+\frac{a^2}{R^2}X_3dX_3)}{(R^2+a^2-m^2+Q^2)(R^2+a^2)(R^4+a^2X_3^2)}+\frac{a(X_2dX_1-X_1dX_2)}{R^2+a^2}\right.\\&\left.-\frac{cR^3(\boldsymbol{X}\cdot d\boldsymbol{X}+\frac{a^2}{R^2}X_3dX_3)}{(R^4+a^2X_3^2)(R^2+a^2-m^2+Q^2)}+dX_0\right]^2-\left[dX_0-\frac{cR^3(\boldsymbol{X}\cdot d\boldsymbol{X}+\frac{a^2}{R^2}X_3dX_3)}{(R^4+a^2X_3^2)(R^2+a^2-m^2+Q^2)}\right]^2,\end{aligned} \tag{17}$$

where $\boldsymbol{X}\equiv(X_1,X_2,X_3)$, and $\boldsymbol{X}\cdot d\boldsymbol{X}\equiv X_1dX_1+X_2dX_2+X_3dX_3$. The relation between R and X_i can be written as $\frac{X_1^2+X_2^2}{R^2+a^2}+\frac{X_3^2}{R^2}=1$. It is worth mentioning that Cook and Scheel [16] presented a $3+1$ form of the harmonic solution to the Kerr-Newman black hole, and their solution is corresponding to Eq. (17) with $c=2m[m+(m^2-a^2-Q^2)^{\frac{1}{2}}]-Q^2$.

At first glance, it seems that Eq. (17) with any constant c can correspond to the metric of the Kerr-Newman black hole in the harmonic coordinates, however, this is not the case. We know Einstein's equations are redundant, but this redundancy will be removed once four coordinate conditions are imposed. Therefore, there is *one and only one* metric corresponding to the Kerr-Newman black hole in the harmonic coordinates, since four harmonic conditions have been used [see Eq. (6)].

In fact, we can determine the constant c via the post-Newtonian approximation, which does not rely on any assumption except that the energy-momentum tensor and the corresponding metric can be expanded with some small parameters. Let $\bar{M}$, $\bar{v}$, and $\bar{R}$ denote the typical values of mass, nonrelativistic velocity, and distance in a system. In the post-Newtonian approximation, the metric is expanded in the powers of $\bar{v}^2$, which is assumed to be roughly of the same order of the typical potential $\bar{\phi} = -\bar{M}/\bar{R}$, as follows [6]:

$$\begin{aligned} g_{00} &= -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \cdots \\ g_{ij} &= \delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \cdots \\ g_{0i} &= \overset{3}{g}_{0i} + \overset{5}{g}_{0i} + \cdots, \end{aligned} \tag{18}$$

where δ_{ij} is Kronecker's delta, and the symbol $\overset{N}{g}_{\mu\nu}$ denotes the terms in $g_{\mu\nu}$ of order $\bar{v}^N$. The corresponding energy-momentum tensor is expanded as

$$\begin{aligned} T^{00} &= \overset{0}{T}{}^{00} + \overset{2}{T}{}^{00} + \cdots \\ T^{ij} &= \overset{2}{T}{}^{ij} + \overset{4}{T}{}^{ij} + \cdots \\ T^{0i} &= \overset{1}{T}{}^{0i} + \overset{3}{T}{}^{0i} + \cdots, \end{aligned} \tag{19}$$

where the symbol $\overset{N}{T}{}^{\mu\nu}$ denotes the terms in $T^{\mu\nu}$ of order $(\bar{M}/\bar{R}^3)\bar{v}^N$. With the harmonic conditions, the leading terms of the time-spatial component of metric are related to the leading time-spatial terms of a general energy-momentum tensor by [6]

$$\overset{3}{g}_{0i}(\boldsymbol{X}, t) = -4 \int \frac{\overset{1}{T}{}^{0i}(\boldsymbol{X}', t)}{|\boldsymbol{X} - \boldsymbol{X}'|} d^3X', \quad i = 1, 2, 3. \tag{20}$$

It follows from Eq. (20) that $\overset{3}{g}_{0i}$ is uniquely determined once T^{0i} is given, and any nonzero g_{0i} implies the time-spatial component T^{0i} of the corresponding energy-momentum tensor is nonzero.

Now we apply the post-Newtonian approximation to the far field of the Kerr-Newman black hole with $Q = 0$ and $a = 0$ (i.e., Schwarzschild black hole). Setting $Q = 0$ and $a = 0$ in Eq. (17) and expanding the metric in the powers of $1/R$, we obtain that the leading term in the time-spatial component of metric is $\frac{cX_i}{R^3}$. Therefore, any nonzero constant c will lead to a nonphysical result that the time-spatial component T^{0i} of energy-momentum of the Schwarzschild black hole is nonzero, i.e., $c = 0$ is the direct consequence of the post-Newtonian approximation, which does not rely on any assumed symmetry of system.

Substituting $c = 0$ into Eq. (17), we can obtain the unique metric of the Kerr-Newman black hole in the harmonic coordinates as follows:

$$\begin{aligned} ds^2 = {} & -dX_0^2 + \frac{R^2(R+m)^2 + a^2X_3^2}{(R^2 + \frac{a^2}{R^2}X_3^2)^2}\left[\frac{(\boldsymbol{X}\cdot d\boldsymbol{X} + \frac{a^2}{R^2}X_3dX_3)^2}{R^2 + a^2 - m^2 + Q^2} + \frac{X_3^2}{R^2}\frac{(\boldsymbol{X}\cdot d\boldsymbol{X} - \frac{R^2}{X_3}dX_3)^2}{R^2 - X_3^2}\right] \\ & + \frac{(R+m)^2 + a^2}{R^2 - X_3^2}\left[\frac{aR^2(m^2 - Q^2)(R^2 - X_3^2)(\boldsymbol{X}\cdot d\boldsymbol{X} + \frac{a^2}{R^2}X_3dX_3)}{(R^2 + a^2 - m^2 + Q^2)(R^2 + a^2)(R^4 + a^2X_3^2)} + \frac{R(X_2dX_1 - X_1dX_2)}{R^2 + a^2}\right]^2 \\ & + \frac{2m(R+m) - Q^2}{(R+m)^2 + \frac{a^2}{R^2}X_3^2}\left[\frac{a^2R(m^2 - Q^2)(R^2 - X_3^2)(\boldsymbol{X}\cdot d\boldsymbol{X} + \frac{a^2}{R^2}X_3dX_3)}{(R^2 + a^2 - m^2 + Q^2)(R^2 + a^2)(R^4 + a^2X_3^2)} + \frac{a(X_2dX_1 - X_1dX_2)}{R^2 + a^2} + dX_0\right]^2. \end{aligned} \tag{21}$$

This equation reduces to the harmonic metric of the Kerr black hole for $Q = 0$ [14] and that of the Schwarzschild black hole when both $Q = 0$ and $a = 0$ [6]. In the work by Hergt and Schäfer [17], the leading terms to order $1/R^4$ for the harmonic metric of the Kerr black hole are obtained and formulated in the spherical coordinates. We have verified that our solution reduces to their results when we set $Q = 0$ and expand Eq. (21) in the powers of $1/R$ to the same order.

In summary, we have derived an exact and unique harmonic metric for the spherically symmetric black hole with rotation and charge, basing on the Kerr-Newman metric in the Boyer-Lindquist coordinates. There may exist other black holes, e.g., the one whose external field is described by Tomimatsu and Sato's solutions [18], which can characterize not only the Kerr metric but also the axially symmetric metric for a rotating deformed mass. The harmonic metrics for these kinds of black holes will be pursued in future work.

ACKNOWLEDGMENTS

The authors would like to thank the referee for providing valuable comments on this paper. This work was supported in part by the Program for New Century Excellent Talents in University (No. NCET-10-0702), the National Basic Research Program of China (973 Program) Grant No. 2013CB328904, and the Ph.D. Programs Foundation of Ministry of Education of China (No. 20110184110016).

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