PHYSICAL REVIEW D **89,** 086003 (2014)

Chern-Simons diffusion rate in anisotropic plasma at strong coupling

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(Received 8 November 2013; published 3 April 2014)

In this work we probe the anisotropy of gauge theory plasma at strong coupling regime within the framework of holographic AdS/CFT correspondence. To be specific, we compute the Chern-Simons diffusion rate in an anisotropic, strongly coupled $\mathcal{N} = 4$ super-Yang-Mills plasma by working with its supergravity dual in an anisotropic spacetime. The Chern-Simons diffusion rate is an important observable in CP-odd phenomena, which may happen in the quark-gluon plasma. We worked out a generically analytic formula for the Chern-Simons diffusion rate using both linear hydrodynamic expansion and holographic renormalization group flow equation methods. In the high temperature phase (compared to the spatial anisotropy), we found that the anisotropy will decrease the Chern-Simons diffusion rate. We also used the generic formula to extract the stringy/higher order gravity corrections to the Chern-Simons diffusion rate.

DOI: 10.1103/PhysRevD.89.086003 PACS numbers: 11.25.Tq, 12.38.Mh

I. INTRODUCTION

In the past few years, a lot of effort has been made to understand the heavy ion collisions and the properties of the quark-gluon plasma (QGP). However, the theoretical analysis based on the experimental discoveries made at the Relativistic Heavy Ion Collider (RHIC) experiments points out that the QGP is most probably a strongly coupled fluid [1,2], which should be quite different from the weakly coupled quasiparticle gas. Furthermore, the elliptic flow observed in RHIC experiments can be effectively described by hydrodynamical models, which further favors small values of the ratio of the shear viscosity over entropy density η/s of the QGP [1–4]. These indications pose one significant challenge for the methods used to describe the QGP: conventional calculations based on perturbative QCD are in general not appropriate. On the other hand, the lattice simulation can be useful in computing some quantities such as hadron mass spectrum and thermodynamical behaviors, but is still impotent in real-time dynamics due to a formidable problem of analytic continuation.

The gauge/gravity duality [5] conjectures that a strongly coupled quantum field theory living in a flat spacetime is equivalent to a weakly interacting (classical) gravity propagating in an asymptotically anti-de Sitter (AdS) spacetime but with one more dimension. Moreover, gauge/gravity duality is a holographic correspondence: the quantum field theory lives at the boundary of the asymptotically AdS spacetime. Quite shortly after its discovery, a promising approach within gauge/gravity duality was developed in order to study the phenomena in the strongly coupled QGP, an updated progress on this topic can be found in one recent review [6]. Attractively, gauge/gravity duality provides us with an analytical treatment of non-Abelian gauge theories in the strong coupling regime.

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Although an exact gravity dual of the QCD is not known yet, the calculations based on gauge/gravity duality have predicted several important results that appear to have some kind of universality among the different theories. In particular, some computations based on large N_c gauge theory (like $\mathcal{N} = 4$ super-Yang-Mills theory) can match quite well with some QCD phenomena. In the limit of infinite 't Hooft coupling and infinite N_c, the ratio η/s for $\mathcal{N} = 4$ super-Yang-Mills plasma has been found to be $1/4\pi$ [7]. The smallness of η/s obtained from gauge/gravity duality is quite close to the value extracted from RHIC experiments. What is more, this small value for η/s has been further proven to be universal within a large number of examples of gauge/gravity duality. Therefore, gauge/gravity duality should be a very useful tool in computing field theory predictions at strong coupling regime and even revealing some universal features of strongly coupled dynamics.

The relativistic hydrodynamics, which can quite accurately describe some problems of the QGP, usually assumes an isotropic pressure. While just after the collision the energy momentum tensor is definitely anisotropic. In other words, the QGP system is locally anisotropic for a very short time just after the collision and soon experiences a quick isotropization process toward locally isotropic. The work [8] proposed a formalism of intrinsically anisotropic hydrodynamics to study the very early stage after the collisions. A recent review about aspects of anisotropic QGP can be found in Ref. [9]. However, due to the mixed weak/strong coupling physics in the early stage of the QGP, we are still lacking a definite understanding of its essential aspects. On the other hand, some attempts within gauge/gravity duality have been made to construct a gravity dual of an anisotropic gauge theory plasma at strong coupling regime [10–12]. The anisotropic version of $\mathcal{N} = 4$ super-Yang-Mills plasma proposed in Refs. [11,12] was soon used to probe some useful properties of anisotropic QGP in Refs. [13–23].

In this work we probe the anisotropic feature of the model [11,12] by computing the Chern-Simons diffusion rate $\Gamma_{\rm CS}$, which is determined by the zero momentum, zero frequency limit of the retarded Green's function of the CP-odd operator $\mathcal{O}(x) = \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$ where $F^a_{\mu\nu}(x)$ is the Yang-Mills field strength. Previous calculations for this quantity within the gauge/gravity duality framework can be found in Refs. [24–27]. In particular, the computations of [25] show that, due to the strong interactions between the charged fields and non-Abelian gauge fields, the external magnetic field has the effect of increasing the diffusion rate, regardless of its strength. However, we found that the anisotropy introduced in the Chern-Simons deformed gauge theory [11,12] will decrease the diffusion rate, at least to quadratic order in the anisotropic parameter a. Apparently, this is due to the nonminimal coupling for the axion field in the bulk side, which is the gravity dual of the CP-odd operator $\mathcal{O}(x)$.

In the next section, we will review the gravity dual of an anisotropic version of $\mathcal{N} = 4$ super-Yang-Mills plasma constructed by Mateos and Trancanelli in [11,12] and classify our conventions. Section III is the main part of this work, where we use two different approaches to compute the diffusion rate. In Sec. III B we solve the holographic renormalization group (RG) flow equations taken from Ref. [28] of the retarded Green's function for the CP-odd operator $\mathcal{O}(x)$. In Sec. III C we directly solve the bulk equation of motion in the conventional linear hydrodynamic regime following the trick of Ref. [29]. With the two methods, we can obtain one very generic analytic formula for $\Gamma_{\rm CS}$. Then we briefly compare our results with those in Refs. [24–27]. Section IV is devoted to discussions of confronting our formula for the Chern-Simons diffusion rate with corrected gravity dual of the $\mathcal{N} = 4$ super-Yang-Mills plasma.

II. THE MODEL: MATEOS-TRANCANELLI GEOMETRY

The type IIB supergravity solution dual to an anisotropic $\mathcal{N} = 4$ super-Yang-Mills plasma at strong coupling constructed by Mateos and Trancanelli is a finite temperature generalization of Ref. [30]. More precisely, on the field theory side the anisotropic version of $\mathcal{N} = 4$ $SU(N_c)$ super-Yang-Mills plasma is given by deforming its isotropic counterpart by the Chern-Simons term,

$$\delta S = \frac{1}{8\pi^2} \int \theta(z)\, \mathrm{Tr} F \wedge F, \tag{1}$$

where the theta angle $\theta(z) = 2\pi a z$ linearly depends on one spatial coordinate z. In other words, the total gauge theory action for an anisotropic $\mathcal{N} = 4$ super-Yang-Mills plasma contains two parts

$$S_{\rm total} = S_{\mathcal{N}=4} + \delta S. \tag{2}$$

Holographically, the anisotropic parameter a can be thought of as the density of D7-brane homogeneously distributed along the anisotropic direction z. As argued in Refs. [11,12], in the dual gravity side it suffices to consider a five-dimensional gravity-dilaton-axion action, which takes the form (in the Einstein frame)

$$S_{\rm bulk} = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left(R + 12 - \frac{(\partial\phi)^2}{2} - e^{2\phi} \frac{(\partial\chi)^2}{2} \right) + S_{\rm bdry}, \tag{3}$$

where the gravitational constant κ^2 is related to the degree of freedom on the dual gauge theory side by $\kappa^2 = 4\pi^2/N_c^2$. The boundary term $S_{\rm bdry}$ is added to make the variational problem well defined and does not modify the equations of motion.

The metric that solves the action (3) is given by an ansatz of the form

$$\begin{aligned} ds^2 &= g_{MN} dx^M dx^N = g_{tt} dt^2 + g_{xx} dx^2 + g_{yy} dy^2 + g_{zz} dz^2 + g_{uu} du^2 \\ &= \frac{e^{-\phi(u)/2}}{u^2} \left(-\mathcal{F}(u)\mathcal{B}(u) dt^2 + \frac{du^2}{\mathcal{F}(u)} + dx^2 + dy^2 + \mathcal{H}(u) dz^2 \right), \end{aligned} \tag{4}$$

and the axion $\chi = az$. For generic anisotropy parameter a, one can numerically construct the metric. In this work, we are interested in the high temperature limit (compared to the parameter a), which allows one to obtain one analytical solution for these coefficients in the metric (4). In other words, if $a \ll T$, we have

$$\begin{aligned} \phi(u) &= -\frac{a^2 u_h^2}{4} \log\left(1 + \frac{u^2}{u_h^2}\right) + \mathcal{O}(a^4), \\ \mathcal{H}(u) &= e^{-\phi(u)}, \mathcal{F}(u) = 1 - \frac{u^4}{u_h^4} + \frac{a^2}{24 u_h^2} \left[8u^2(u_h^2 - u^2) - 10u^4 \log 2 + (3u_h^4 + 7u^4) \log\left(1 + \frac{u^2}{u_h^2}\right) \right] + \mathcal{O}(a^4), \\ \mathcal{B}(u) &= 1 - \frac{a^2 u_h^2}{24} \left[\frac{10u^2}{u_h^2 + u^2} + \log\left(1 + \frac{u^2}{u_h^2}\right) \right] + \mathcal{O}(a^4). \end{aligned} \tag{5}$$

Explicitly, the above metric has a regular, nondegenerate horizon at $u = u_h$, which is related to the Hawking temperature of the black hole by

$$u_h = \frac{1}{\pi T} + \frac{5\log 2 - 2}{48\pi^2 T^3} a^2 + \mathcal{O}(a^4). \tag{6}$$

The thermodynamics and instability analysis of the above gravity-dilaton-axion system has been completely studied in Refs. [11,12]. However, for our purpose of computing the Chern-Simons diffusion rate, we will focus on the axion part in the bulk action.

III. CALCULATION OF CHERN-SIMONS DIFFUSION RATE

A. Preliminary

Let us proceed by explaining the physics of Chern-Simons diffusion rate from the field theory point of view. Our discussions below mainly follow the associated descriptions in Refs. [24,25].

The vacuum of the non-Abelian gauge theories contains an infinite number of degenerate states which are characterized by an integer—the Chern-Simons number $N_{\rm CS}$. Furthermore, the integer $N_{\rm CS}$ is a topological quantity and is determined by the global structure of non-Abelian gauge theories. At zero temperature, the quantum tunneling effect allows the transition between the states with different $N_{\rm CS}$. Consider non-Abelian gauge theories at finite temperature. The thermal effect can also activate the change between the states characterized by a different Chern-Simons number $N_{\rm CS}$. In particular, as opposed to the tunneling effect at zero temperature, the change rate due to thermal activation is not necessarily exponentially suppressed. Quantitatively, the change of the Chern-Simons number in such a process due to thermal effect is given by a formula

$$\Delta N_{\rm CS} = \frac{g^2}{32\pi^2}\int d^4x \mathcal{O}(x), \qquad \mathcal{O}(x) \equiv \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}, \tag{7}$$

where g is the Yang-Mills coupling constant. Then the Chern-Simons diffusion rate $\Gamma_{\rm CS}$, defined by the rate of change of the Chern-Simons number, is nothing but simply the probability of the Chern-Simons number changing process to occur per unit volume and unit time,

$$\Gamma_{\rm CS} = \frac{\langle \Delta N_{\rm CS}^2\rangle}{Vt} = \int d^4x \left\langle \frac{g^2}{32\pi^2}\mathcal{O}(x)\frac{g^2}{32\pi^2}\mathcal{O}(0)\right\rangle. \tag{8}$$

The diffusion rate $\Gamma_{\rm CS}$ is by itself important in CP-odd phenomena, like the chiral magnetic effect, which may happen in heavy ion collisions. A simple formula relating the diffusion rate $\Gamma_{\rm CS}$ with the generation of chiral charge $N_5 = \langle J_5^0\rangle$ is given by

$$\frac{dN_5}{dt} = -CN_5\frac{\Gamma_{\rm CS}}{T^3}, \tag{9}$$

where the constant C is determined by the specific theory considered.

Transferred to the momentum space, the diffusion rate defined in Eq. (8) is given by [24]

$$\Gamma_{\rm CS} = -\left(\frac{g^2}{8\pi^2}\right)^2 \lim_{\omega\to 0}\frac{2T}{\omega}[G^R(\omega, \vec{k}=0)], \tag{10}$$

where the retarded Green's function $G^R(\omega,\vec{k})$ is defined by

$$G^R(\omega,\vec{k}) \equiv -i\int d^4x e^{-ikx}\theta(t)\left\langle\left[\frac{1}{4}\mathcal{O}(x), \frac{1}{4}\mathcal{O}(0)\right]\right\rangle. \tag{11}$$

Gauge/gravity duality relates the operator $\frac{1}{4}\mathcal{O}$ with the Ramond-Ramond scalar C_0, which is exactly the axion field χ in type IIB supergravity theory. In particular, as shown in Ref. [25], the axion field behaves the same as a free massless scalar in the consistent Kaluza-Klein reduction of the ten-dimensional type IIB supergravity to the five-dimensional bulk theory.

However, there is a slight difference in our case due to the nontrivial profile of the dilaton mode. The equation of motion for the axion χ comes from the variation of the action S,

$$\frac{\delta S}{\delta\chi} = 0 \Rightarrow \partial_M(\sqrt{-g}e^{2\phi}g^{MN}\partial_N\chi) = 0. \tag{12}$$

In order to calculate the retarded Green's function $G^R(\omega,\vec{k}=0)$ using gauge/gravity duality, one needs to do perturbations of the gravitational system,

$$g_{MN} \to g_{MN} + \delta g_{MN}, \qquad \phi\to\phi+\delta\phi, \qquad \chi\to\chi+\delta\chi. \tag{13}$$

Linearizing the equation of motion for χ results in

$$\begin{aligned}0 &= \partial_M(\delta\sqrt{-g}e^{2\phi}g^{MN}\partial_N\chi) + \partial_M(\sqrt{-g}\delta e^{2\phi}g^{MN}\partial_N\chi)\\ &\quad + \partial_M(\sqrt{-g}e^{2\phi}\delta g^{MN}\partial_N\chi) + \partial_M(\sqrt{-g}e^{2\phi}g^{MN}\partial_N\delta\chi).\end{aligned} \tag{14}$$

At first glance, the different perturbations $\{\delta g_{MN}, \delta\phi, \delta\chi\}$ will couple together, which makes the problem quite sophisticated. However, recall that we only need to compute $G^R(\omega,\vec{k}=0)$, which means that we can set all spatial momenta to zero,

$$\delta\chi(u, x_\mu) = \int\frac{dt}{2\pi}e^{-i\omega t}\delta\chi(u,\omega), \tag{15}$$

and similar Fourier expansions can be done for δg_{MN} and $\delta\phi$. Combining the Fourier ansatz (15) with the profiles of

$\{g_{MN}, \phi, \chi\}$, it is not difficult to see that the first two terms in the linearized equation (14) exactly vanish. However, the fluctuations of the metric and the axion still couple together. Explicitly, Eq. (14) takes the following form,

$$0 = i\omega\sqrt{-g}e^{2\phi}g^{tt}g^{zz}\delta g_{tz} \times a + \partial_M(\sqrt{-g}e^{2\phi}g^{MN}\partial_N\delta\chi). \tag{16}$$

Since we will work under the condition $a \ll T$, it is reasonable to ignore the first term in the above equation if one further assumes that the fluctuations and the anisotropy parameter are of the same order, $\delta g_{tz}, \delta\chi \sim a$. In this work we will take this assumption to simplify the computations and leave the complete treatments about the coupling between δg_{tz} and $\delta\chi$ for further work. In other words, we will switch off all the fluctuations except the axion part,

$$\delta g_{MN} = \delta\phi = 0.$$

The approximation adopted here is quite similar to the probe limit widely used in the literature.

Hence, in the Fourier space, the linearized equation of motion (14) is simply

$$\partial_M(\sqrt{-g}e^{2\phi}g^{MN}\partial_N\delta\chi) = 0 \Rightarrow$$
$$\partial_u^2\delta\chi + \partial_u \ln(\sqrt{-g}e^{2\phi}g^{uu})\partial_u\delta\chi - \omega^2\frac{g^{tt}}{g^{uu}}\delta\chi = 0. \tag{17}$$

To quadratic order in the perturbations, the action relevant for the axion part is

$$\begin{aligned} S_\chi &= -\frac{1}{4\kappa^2}\int d^5x\sqrt{-g}e^{2\phi}g^{MN}\partial_M\delta\chi\partial_N\delta\chi \\ &= -\frac{1}{4\kappa^2}\int\frac{d\omega}{2\pi}\sqrt{-g}e^{2\phi}g^{uu}\delta\chi(u,-\omega)\partial_u\delta\chi(u,\omega)|_{u=0}^{u=u_h}, \end{aligned} \tag{18}$$

where in the second line of the above action we have reduced it to the surface term by making use of the equation of motion and Fourier ansatz.

The remaining task is to solve the equation of motion (17) and extract the retarded Green's function from the on-shell action (18) using the prescription proposed in Ref. [24]. However, to make our derivations as general as possible, in what follows we will not substitute for the explicit form of the metric and dilaton profile until the final step toward obtaining the diffusion rate. Doing so allows us to derive one generic formula for Γ_{CS} as alluded at the beginning.

B. Holographic RG flow equation

In this section, using the equation of motion for the axion field, we derive the holographic RG flow equation for the retarded Green's function. The canonical momentum conjugate to $\delta\chi$ with respect to a foliation in the holographic direction u is given by

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_u\delta\chi)} = -\sqrt{-g}e^{2\phi}g^{uu}\partial_u\delta\chi, \tag{19}$$

where $\mathcal{L}$ is the Lagrangian density of the action S_χ given in Eq. (18). We have thrown away the prefactor in the final expression for Π to make the equation more concise. The Hamilton equation for Π can be worked out with help of the equation for $\delta\chi$ and is given by

$$\partial_u\Pi = -\sqrt{-g}e^{2\phi}\omega^2\delta\chi. \tag{20}$$

As in the Ref. [28], we define a generalized response function $\xi(\omega,u)$ by

$$\xi(\omega,u) \equiv \frac{\Pi}{i\omega\delta\chi}, \quad \text{with} \quad G^R(\omega,\vec{k}=0) = \frac{\Pi}{\delta\chi}\bigg|_{u=0}. \tag{21}$$

Using Eqs. (19) and (20), one can show that the response function satisfies a first order nonlinear differential equation

$$\begin{aligned} \partial_u\xi &= i\omega\sqrt{-\frac{g_{uu}}{g_{tt}}}\left[\frac{\xi^2}{\Sigma(u)} - \Sigma(u)\right], \\ \Sigma(u) &= e^{2\phi}\sqrt{\frac{-g}{g_{uu}g_{tt}}}, \end{aligned} \tag{22}$$

where we have factorized out the singular term in the above equation.

In the hydrodynamic limit, we can perturbatively solve the RG flow equation (22) by expanding the response function in terms of ω

$$\xi(\omega,u) = \xi_0(u) + \omega\xi_1(u) + \mathcal{O}(\omega^2). \tag{23}$$

At the zeroth order, there is no holographic RG flow for $\xi_0(u)$ as

$$\partial_u\xi_0(u) = 0 \Rightarrow \xi_0(u) = \text{Constant}. \tag{24}$$

However, due to the appearance of a singularity for the prefactor $\sqrt{-\frac{g_{uu}}{g_{tt}}}$ at the horizon $u = u_h$, the expression in the parentheses of Eq. (22) must vanish at the horizon,

$$\begin{aligned} \xi_0(u=u_h) = \Sigma(u=u_h) &= e^{2\phi}\sqrt{\frac{-g}{g_{uu}g_{tt}}}\bigg|_{u=u_h} \\ &= e^{2\phi}\sqrt{g_{xx}g_{yy}g_{zz}}|_{u=u_h}. \end{aligned} \tag{25}$$

Combining Eqs. (24) and (25) tells us that, to lowest order in ω, the retarded Green's function is given by

$$G^R(\omega,\vec{k}=0) = -\frac{1}{2\kappa^2}\times i\omega\sqrt{g_{xx}g_{yy}g_{zz}}e^{2\phi}|_{u=u_h} + \mathcal{O}(\omega^2). \tag{26}$$

In the next section, we will derive this formula by direct linear hydrodynamic expansion method, which is a little slower than the method presented here.

C. Linear hydrodynamic expansion

In this section we will use the conventional linear hydrodynamic expansion method to solve the equation of motion for $\delta\chi$. We will also use the trick of Ref. [29] to make the derivation more general, and concise as well.

As stated by the authors of Ref. [29], the equations of motion for $\delta\chi$ can be solved as follows. First, one should consider the equation of motion near the black hole horizon and impose the ingoing wave boundary condition there (for the purpose of retarded Green's function). Then, in the hydrodynamic limit, one considers the whole region in the bulk and solves the equation of motion. After doing these, one can match the solutions and extract the retarded Green's function.

Let us proceed by studying the behavior of Eq. (17) near the horizon $u=u_h$. As alluded above, there is a nondegenerate horizon at $u=u_h$, which means that the metric components have the following behavior near the horizon

$$g_{tt}\to c_t(u_h-u),\qquad g_{uu}\to c_u(u_h-u)^{-1},\qquad g_{xx,yy,zz}\to g_{xx,yy,zz}(u_h). \tag{27}$$

In particular, the Hawking temperature T is given by

$$T=\frac{1}{4\pi}\sqrt{-\frac{c_t}{c_u}}. \tag{28}$$

Equation (17) takes the form as approaching the horizon

$$\partial_u^2\delta\chi - \frac{1}{(u_h-u)}\partial_u\delta\chi + \frac{\tilde{\omega}^2}{(u_h-u)^2}\delta\chi = 0,\qquad \tilde{\omega}\equiv\frac{\omega}{4\pi T}. \tag{29}$$

The Frobenius analysis tells us that

$$\delta\chi\approx(u_h-u)^{-i\tilde{\omega}},\qquad u\to u_h, \tag{30}$$

where we have kept only the ingoing mode in the above solution. In the hydrodynamic limit, the above ingoing wave can be further expanded in terms of small parameter $\tilde{\omega}\ll 1$,

$$\delta\chi\approx 1-i\tilde{\omega}\log(u_h-u)+\mathcal{O}(\tilde{\omega}^2),\qquad u\to u_h,\qquad \tilde{\omega}\ll 1. \tag{31}$$

To solve the equation over the whole region in the bulk, the hydrodynamic limit greatly facilitates the computations. Expand the solution over the whole region by orders of $\tilde{\omega}$,

$$\delta\chi(u,\tilde{\omega}) = \delta\chi_{(0)}(u)+\tilde{\omega}\delta\chi_{(1)}(u)+\mathcal{O}(\tilde{\omega}^2). \tag{32}$$

To each order, $\delta\chi_{(i)}(u)$ are determined by

$$\partial_u^2\delta\chi_{(i)} + \partial_u\ln(\sqrt{-g}g^{uu}e^{2\phi})\partial_u\delta\chi_{(i)} = 0,\qquad i=0,1. \tag{33}$$

The general solutions to $\delta\chi_{(i)}$ can be found by direct integration and given by

$$\delta\chi_{(i)}(u) = C_1^{(i)} + C_2^{(i)}\int_0^u\frac{du}{\sqrt{-g}g^{uu}e^{2\phi}},\qquad i=0,1, \tag{34}$$

where the integration constants $C_{1,2}^{(i)}$ can be partially fixed by the Dirichlet boundary conditions at the conformal boundary $u=0$ (we set the total scale of $\delta\chi$ at the conformal boundary to unit for convenience)

$$C_1^{(0)}=1,\qquad C_1^{(1)}=0. \tag{35}$$

The final step is to match the solutions. For this purpose, we need to take a close look at the solutions given by (34) near the horizon. When $u\longrightarrow u_h$, the integrand behaves as follows

$$\frac{1}{\sqrt{-g}g^{uu}e^{2\phi}} = \frac{1}{\sqrt{g_{xx}g_{yy}g_{zz}}e^{2\phi}}\sqrt{-\frac{g_{uu}}{g_{tt}}} \to \frac{1}{\sqrt{g_{xx}g_{yy}g_{zz}}e^{2\phi}}\bigg|_{u=u_h}\sqrt{-\frac{c_u}{c_t}}\frac{1}{(u_h-u)}. \tag{36}$$

Then, the horizon behavior for the general solutions is

$$\delta\chi_{(i)}(u\to u_h) = C_1^{(i)} - C_2^{(i)}\frac{1}{4\pi T}\frac{1}{\vartheta(u_h)}\log(u_h-u),\qquad \vartheta(u_h)\equiv\sqrt{g_{xx}g_{yy}g_{zz}}e^{2\phi}|_{u=u_h}. \tag{37}$$

Combining Eqs. (37) and (32), then matching it with the solution around the horizon (31) determines the remained integration constants

$$C_2^{(0)}=0,\qquad C_2^{(1)}=i\vartheta(u_h)\times 4\pi T. \tag{38}$$

In other words, the solution over the whole region in the bulk is given by (to first order in $\tilde{\omega}$)

$$\delta\chi(u,\omega) = 1-i\omega\vartheta(u_h)\int_0^u\frac{du}{\sqrt{-g}g^{uu}e^{2\phi}}+\mathcal{O}(\omega^2). \tag{39}$$

Using the Minkowskian prescription of Ref. [24] gives the retarded Green's function

$$
\begin{aligned}
G^R(\omega, \vec{k} = 0) \\
&= -2 \times \left(-\frac{1}{4\kappa^2}\right) \sqrt{-g} g^{uu} e^{2\phi} \delta\chi(u,\omega) \partial_u \delta\chi(u,-\omega)|_{u=0} \\
&= -\frac{1}{2\kappa^2} \times i\omega\vartheta(u_h) + \mathcal{O}(\omega^2),
\end{aligned}
\tag{40}
$$

which is exactly the same as the solution obtained from holographic RG flow equation method once substituting for the expression of the $\vartheta(u_h)$ function defined in Eq. (37).

The Chern-Simons diffusion rate Γ_{CS} can now be easily obtained using the formula given in Sec. III A,

$$
\begin{aligned}
\Gamma_{\mathrm{CS}} &= \left(\frac{g^2}{8\pi^2}\right)^2 \times \frac{N_c^2}{8\pi^2} \times 2T \times \frac{\exp[3\phi(u_h)/4]}{u_h} \\
&= \frac{(g^2 N_c)^2}{256\pi^3} T^4 (1 - \gamma \tilde{a}^2), \qquad \tilde{a} \equiv \frac{a}{\pi T},
\end{aligned}
\tag{41}
$$

where we have used the relation between the horizon radius and the Hawking temperature T given in Eq. (6). Notice that the constant γ is a positive number

$$
\gamma = \frac{4\log 2 - 1}{8} > 0,
\tag{42}
$$

which implies that the anisotropy introduced by nonzero a will decrease the Chern-Simons diffusion rate Γ_{CS}. This should be compared with the result of Ref. [25] where the $U(1)_R$ magnetic field also makes the spatial directions anisotropic but increases the Γ_{CS}.

In the next section, we will use the formula obtained here to probe the effect of some corrections in the bulk gravity (higher gravity corrections and stringy ones). In particular, we will find that the stringy corrections also decrease the Chern-Simons diffusion rate while the minimal Gauss-Bonnet corrections will enhance the diffusion rate.

IV. DISCUSSION

Using the formula (26), one can easily produce the Chern-Simons diffusion rate once knowing the background metric. Here, we will consider the corrections in the bulk gravity and probe their effect by computing the diffusion rate Γ_{CS}. Two examples worthy of being considered are the α' corrected black D3-brane geometry [31,32] and the Gauss-Bonnet corrected Schwarzschild-AdS$_5$ black hole [33,34]. In particular, it was found that the conjectured shear viscosity bound $\eta/s \geq 1/4\pi$ can be violated by higher order gravity corrections [35]. However, the computations of Ref. [36] show that the stringy α' corrections do not violate the viscosity bound.

After a five-dimensional reduction over the compact space S^5, the α' corrected metric for a black D3-brane is given by (with the notations of Ref. [36])

$$
\begin{aligned}
ds_5^2 &= \frac{r_0^2}{u} e^{c(u)} (-f e^{a(u)} dt^2 + dx^2 + dy^2 + dz^2) + \frac{du^2}{4u^2 f} e^{b(u)}, \\
f &= 1 - u^2.
\end{aligned}
\tag{43}
$$

To leading order in $\beta = \zeta(3)\alpha'^3$, the functions $a(u), b(u), c(u)$ are given by

$$
\begin{aligned}
a(u) &= 15\beta(5u^2 + 5u^4 - 3u^6), \\
b(u) &= 15\beta(5u^2 + 5u^4 - 19u^6), \\
c(u) &= 0.
\end{aligned}
\tag{44}
$$

The parameter r_0 gives the Hawking temperature T as

$$
T = \frac{r_0}{\pi}(1 + 15\beta).
\tag{45}
$$

On the other hand, adding one specific combination of four derivative gravity corrections to the Einstein-Hilbert action, one is led to the Gauss-Bonnet gravity action

$$
\begin{aligned}
S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \\
\times \left[R + 12 + \frac{\alpha}{2}(R^{MNPQ}R_{MNPQ} - 4R^{MN}R_{MN} + R^2)\right].
\end{aligned}
\tag{46}
$$

Fortunately, the following analytic metric solves the above action

$$
\begin{aligned}
ds^2 &= -\frac{f(r)}{f(\infty)} dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2 + dz^2), \\
f(r) &= \frac{r^2}{2\alpha}\left[1 - \sqrt{1 - 4\alpha(1 - r_h^4/r^4)}\right], \\
f(\infty) &= \lim_{r\to\infty} \frac{f(r)}{r^2} = \frac{1 - \sqrt{1 - 4\alpha}}{2\alpha},
\end{aligned}
\tag{47}
$$

and the Hawking temperature is simply given by

$$
T = \frac{r_h}{\pi\sqrt{f(\infty)}}.
\tag{48}
$$

With the generic formula for the retarded Green's function dual to the axion field χ obtained in Sec. III, the Chern-Simons diffusion rate Γ_{CS} can be quickly extracted

$$\Gamma_{\mathrm{CS}} = \frac{(g^2 N_c)^2}{256\pi^3} T^4 (1 - 45\beta), \qquad \alpha' - \text{corrected D3 black hole},$$
$$\Gamma_{\mathrm{CS}} = \frac{(g^2 N_c)^2}{256\pi^3} T^4 \left(\frac{1-\sqrt{1-4\alpha}}{2\alpha}\right)^{3/2}, \qquad \text{Gauss-Bonnet corrected Schwarzschild-AdS}_5. \tag{49}$$

Clearly, the stringy α' correction has the same effect of decreasing the Chern-Simons diffusion rate as in the anisotropic $\mathcal{N} = 4$ super-Yang-Mills plasma at strong coupling, while the Gauss-Bonnet corrections will enhance the Γ_{CS}, especially when α is quite close to its upper bound $1/4$.

ACKNOWLEDGMENTS

The author would like to thank the Max Planck Institute for Physics, Munich for the hospitality during the initial stage of this research. This work was supported by the Israeli Science Foundation Grant No. 87277111 and the Council for Higher Education of Israel under the PBC Program of Fellowships for Outstanding Post-doctoral Researchers from China and India (2013–2014).

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