PHYSICAL REVIEW D **89,** 086002 (2014)

Note on high energy scattering of open superstrings

Francisco Rojas*

Department of Physics, Institute for Fundamental Theory, University of Florida, Gainesville, Florida 32611, USA

(Received 16 October 2013; published 3 April 2014)

We study the Regge and hard-scattering limits of the one-loop amplitude for massless open string states in the type I theory in flat space. For hard scattering we find the exact kinematic dependence in terms of the scattering angle of the factor multiplying the known exponential falloff, without relying on a saddle point approximation for the integration over the cross ratio. This bypasses the issues of estimating the contributions from flat directions as well as those that arise from fluctuations of the Gaussian integration about the saddle point. This result allows for a straightforward computation of the small-angle behavior of the hard-scattering regime and we find complete agreement with the Regge limit at high momentum transfer, as expected.

DOI: 10.1103/PhysRevD.89.086002 PACS numbers: 11.25.-w, 11.25.Db

I. INTRODUCTION

Open string amplitudes, including one-loop corrections in the high energy regime, have been studied since the very early days of string theory [1–3], and subsequently the subject was taken up again by many other authors in the 1980s [4,5]. In [5], the authors analyzed the $(2+2)$ nonplanar amplitude at one loop, i.e., the annulus diagram with two external states attached to each boundary, and found the (now well-known) characteristic exponential falloff of stringy amplitudes in this regime. The coefficient multiplying this exponential behavior, which contains dependence on the scattering angle, involves the typical problem of inversion in the theory of elliptic modular functions. As a consequence of this, the angular dependence in the aforementioned coefficient could only be expressed in terms of an infinite series.

For the case of the planar and nonorientable amplitudes, Gross and Mañes [5] studied this high energy regime at a fixed scattering angle and found that, contrary to the $(2+2)$ nonplanar case, they do not possess a dominant saddle point in the interior of the integration region. Moreover, they were able to show that the dominant contributions come from the boundaries of this region, the one where the annulus shrinks to a point being the dominant boundary in this case.

The study of the fixed-angle limit of the one-loop amplitude in different situations has been carried out by many authors [5–8], but as far as we aware of, we believe that the exact dependence on the scattering angle for the amplitude we study here has not been worked out in the literature in a closed form.

We organize this short paper as follows: in Sec. II we review the calculation of the Regge limit of the sum of the planar and nonorientable diagrams of the type I theory. We also compute its large momentum transfer limit ($\alpha'|t| \to \infty$) in order to make a comparison with the small-angle behavior of the hard-scattering limit which we also review in this section. In Sec. III, by making use of an identity originally used in [9], although in a different context, we compute the exact form of the coefficient that multiplies the exponential falloff of the amplitude in the high energy regime at a fixed scattering angle. This permits a straightforward evaluation of the hard-scattering amplitude in the limit where $t \ll s$, which indeed matches with the Regge behavior at high momentum transfer computed in Sec. II.

II. HIGH ENERGY SCATTERING OF THE TYPE I OPEN SUPERSTRINGS

A. Regge behavior at one loop

We begin by computing the Regge limit; i.e., we take large $\alpha'|s|$ (holding $\alpha' t$ fixed) of the one-loop amplitude for type I open superstrings. With the metric signature $\{- + + \cdots\}$, the Mandelstam variables are conventionally defined as $s = -(k_1 + k_2)^2$, $t = -(k_2 + k_3)^2$, and $u = -(k_2 + k_4)^2$. The details of the calculation are basically the same as the ones computed for the type 0 string in [10], with the only difference being the nature of the cancellation of divergences due to the propagation of closed string tachyons and dilatons. In [10], the remnants of closed string tachyon divergences were canceled by the inclusion of a counterterm which, after analytic continuation using a momentum conservation regulator, turned out to be zero in the Regge limit. The "would-be" subleading divergences due to closed string dilatons were simply absent with the inclusion of Dp-branes as long as $p < 7$.

The amplitude for four massless vector states is much simpler in the superstring compared to the type 0 theory, because in the former the full polarization structure can be factored out of the integration over the moduli, whereas in the latter each combination of polarization vectors must be

*frojasf@phys.ufl.edu

worked out separately. For the $SO(32)$ gauge group, the planar and nonorientable one-loop diagrams combine to give a finite expression [11] and we focus our attention on this case in this article. The amplitude for each diagram (planar and nonorientable) was computed a long time ago (see for instance [12]), and for the $SO(32)$ gauge group they can be combined as

$$A_P + A_N = 16\pi^3 g^4 G_P K \int_0^1 \frac{dq}{q}[F(q^2) - F(-q^2)], \quad (1)$$

with

$$F(q^2) = \int_R \prod_{i=1}^{3} d\theta_i \prod_{i<j} \psi(\theta_{ji})^{2\alpha' k_i \cdot k_j}$$
$$\psi(q,\theta) = \sin\theta \prod_{n=1}^{\infty} \frac{1 - 2q^{2n}\cos 2\theta + q^{4n}}{(1-q^{2n})^2}, \quad (2)$$

and K is the kinematic factor which can be found, for example, in [12]. The region of integration R is given by $0 < \theta_2 < \theta_3 < \theta_4 < \pi$, $\theta_{ji} \equiv \theta_j - \theta_i$, and G_P is the group theory factor $G_P = \mathrm{Tr}(\lambda_1\lambda_2\lambda_3\lambda_4)$. It is convenient to define the expression

$$V_s \equiv \ln\left[\frac{\psi(\theta_{43})\psi(\theta_2)}{\psi(\theta_{42})\psi(\theta_3)}\right]; \quad (3)$$

therefore,

$$\prod_{i<j} \psi(\theta_{ji})^{2\alpha' k_i \cdot k_j} = \left[\frac{\psi(\theta_{43})\psi(\theta_2)}{\psi(\theta_{42})\psi(\theta_3)}\right]^{-\alpha' s}\left[\frac{\psi(\theta_4)\psi(\theta_{32})}{\psi(\theta_{42})\psi(\theta_3)}\right]^{-\alpha' t} \quad (4)$$

$$= \exp\left\{-\alpha' s V_s\right\}\left[\frac{\psi(\theta_4)\psi(\theta_{32})}{\psi(\theta_{42})\psi(\theta_3)}\right]^{-\alpha' t}. \quad (5)$$

Recall that the amplitude above has physical resonances in both the s and t channels, i.e., the integral representation (1) has open-string poles whenever $\alpha' s = 0, 1, 2, \ldots$ (and also when $\alpha' t = 0, 1, 2, \ldots$). Thus, in order to avoid these poles for computing the Regge limit, we take $\alpha' s \to -\infty$. Note that a similar situation also appears at tree level string scattering. For example, if we take the Regge limit in the Veneziano amplitude

$$A(s,t) = \frac{\Gamma(-\alpha' s - 1)\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha' s - \alpha' t - 2)}, \quad (6)$$

we would also "hit" all the poles at $\alpha' s = n$ for large values of the positive integer n as we take $\alpha' s \to \infty$. Note also that, although the usual integral representation of the Veneziano amplitude

$$A(s,t) = \int_0^1 x^{-\alpha' s - 2}(1-x)^{-\alpha' t - 2}\mathrm{d}x \quad (7)$$

only converges for $\mathrm{Re}(\alpha' s) < -1$ [and $\mathrm{Re}(\alpha' t) < -1$], the Regge limit obtained by evaluating this integral for $\alpha' s \to -\infty$ with $\alpha' t$ fixed gives the same answer as the one computed from (6) which defines the analytic continuation of (7) to the full complex plane.

We now go ahead and compute the Regge limit of the amplitude (1). The dominant contributions come from the integration regions where $\alpha' s V_s$ is finite at large energies. It is also important to note that, due to the fact that the angular variables are ordered $0 < \theta_2 < \theta_3 < \theta_4 < \pi$ (contrary to the nonplanar case), V_s is negative definite in the full integration region [2]. Thus, in the $\alpha' s \to -\infty$ limit, the amplitude is dominated by the regions where V_s vanishes. The dominant region corresponds to the simultaneous limits $\theta_2 \sim \theta_3$ and $\theta_4 \sim \pi$. It is worth noting that the single limits, either $\theta_2 \sim \theta_3$ or $\theta_4 \sim \pi$ separately, also produce contributions that are not exponentially suppressed and, in principle, could also contribute to the Regge limit. However, as we will show later on, these contributions are subleading with respect to the double-limit one.

In the double limit $\theta_2 \sim \theta_3$ and $\theta_4 \sim \pi$, we will need the following approximations:

$$\left[\frac{\psi(\theta_{43})\psi(\theta_2)}{\psi(\theta_{42})\psi(\theta_3)}\right]^{-\alpha' s} \sim \exp\{-\alpha' s \theta_{32}(\pi - \theta_4)(\ln\psi)''\}$$
$$\left[\frac{\psi(\theta_{41})\psi(\theta_{32})}{\psi(\theta_{42})\psi(\theta_3)}\right]^{-\alpha' t} \sim \left(\frac{\theta_{32}(\pi - \theta_4)}{\psi^2(\theta_3)}\right)^{-\alpha' t}. \quad (8)$$

The dominant term in K for this limit is

$$K \sim \frac{1}{4}\epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot \epsilon_4 s^2. \quad (9)$$

Using the approximations above, we see that we need to compute the integral

$$I \equiv \int_0^\epsilon dx \int_0^\epsilon dy (xy)^a e^{-xyk} \quad (10)$$

in the limit when $k \to \infty$. After some algebra, this becomes

$$I = k^{-a-1}\left[\ln(\epsilon^2)\int_0^{\epsilon^2 k} dz z^a e^{-z} + \ln k \int_0^{\epsilon^2 k} dz z^a e^{-z} - \int_0^{\epsilon^2 k} dz z^a e^{-z}\ln z\right]$$
$$\sim k^{-a-1}[\Gamma(1+a)\ln k - \ln(\epsilon^2)\Gamma(1+a) - \Gamma'(1+a)] + \mathcal{O}(k^{-a-2}\ln k)$$
$$\sim k^{-a-1}\Gamma(1+a)\ln k. \quad (11)$$

Using this formula for $k = (-\alpha' s)(-\ln\psi)''$ and $a = -\alpha' t$, we obtain the Regge limit of the amplitude, which becomes

$$A_P + A_M \sim g^4(-\alpha' s)^{1+\alpha' t}\Gamma(-\alpha' t)\ln(-\alpha' s)\Sigma(t), \qquad (12)$$

where

$$\Sigma(t) \equiv \alpha' t \int_0^1 \frac{dq}{q}\int_0^\pi d\theta(\psi^{2\alpha' t}[-\ln\psi'']^{\alpha' t-1} - \psi_N^{2\alpha' t}[-\ln\psi_N'']^{\alpha' t-1}) \qquad (13)$$

and $\psi_N(\theta, q^2) = \psi(\theta, -q^2)$. This completes the calculation of the asymptotic behavior of the amplitude in the Regge limit. Notice also that the function $\Sigma(t)$ gives the one-loop correction to the open string Regge trajectory. This can be easily seen as follows. At tree level, Regge behavior implies that the amplitude is of the form $A \sim \beta(t)s^{\alpha(t)}$, with $\alpha(t) = 1 + \alpha' t$. Including one-loop corrections modifies both the Regge trajectory $\alpha(t)$ and the residue $\beta(t)$ by small corrections, say, $\delta\alpha$ and $\delta\beta$, respectively, i.e.,

$$(\beta(t)+\delta\beta)s^{\alpha(t)+\delta\alpha} \sim \beta s^{\alpha(t)} + \beta s^{\alpha(t)}\delta\alpha\log s + \delta\beta s^{\alpha(t)}. \qquad (14)$$

Thus, the new trajectory $\alpha(t)_{\mathrm{new}} = 1 + \alpha' t + \delta\alpha$ is captured by the term containing the $\log s$ factor above.

Before continuing, and as mentioned previously, we should also evaluate the contributions from the regions represented by the single limits $\theta_2 \sim \theta_3$ or $\theta_4 \sim \pi$ *separately*, since they can also render the exponent $\alpha' s V_s$ finite at large $|s|$. Consider first the single limit region where $\theta_2 \sim \theta_3$. Notice that in this case the integral we need to estimate now is

$$I' \equiv \int_0^\epsilon dx x^a e^{-xk} \qquad (15)$$

in the limit $k \to \infty$. Making the change $x \to kx$ yields

$$I' = k^{-1-a}\int_0^{\epsilon k} dx x^a e^{-x} \sim k^{-1-a}\Gamma(1+a) \qquad (16)$$

from where we see that, comparing it with (11), it is suppressed by a factor of $\ln k$. In other words, the joint region $(\theta_2 \sim \theta_3)\cap(\theta_4 \sim \pi)$ has an *enhanced* contribution to the amplitude over the single region $\theta_2 \sim \theta_3$ by a factor of $\ln(-\alpha' s)$. One also arrives at the same conclusion by analyzing the other single limit $\theta_4 \sim \pi$, since the integrand takes the same form as in (15) in the appropriate variables.

Given that we are also interested in recovering the Regge behavior from the hard-scattering limit, we need to extract the large t limit of $\Sigma(t)$. In order to do so, we rewrite the integral (13) as

$$\Sigma(t) = \alpha' t \int_0^1 \frac{dq}{q}\int_0^\pi d\theta(e^{\alpha' t\ln(-\psi^2[\ln\psi]'')}[-\ln\psi]''^{-1} - e^{\alpha' t\ln(-\psi_N^2[\ln\psi_N]'')}[-\ln\psi_N]''^{-1}), \qquad (17)$$

from which we see that its leading behavior at large t is given by the critical points of $\ln(-\psi^2[\ln\psi]'')$ and $\ln(-\psi_N^2[\ln\psi_N]'')$. Notice that now we only have a two-dimensional integration region, for which the critical points should be easier to analyze in principle.

The leading contribution comes from the $q \sim 0$ region (the closed string channel). Notice that the open string channel ($q \to 1$) is exponentially suppressed at high t. This can be seen as follows: performing the transformation $w = \exp\{2\pi^2/\log q\}$, which maps $q \to 1$ to $w \to 0$, yields (for fixed θ)

$$\begin{aligned}\ln\psi &= \ln\left(\frac{\pi}{-\ln w}\right) - \frac{\theta(\pi-\theta)\ln w}{2\pi^2} + \ln(1-w^{\theta/\pi}) \\ &\quad + \sum_{n=1}^\infty \ln\frac{(1-w^{n+\theta/\pi})(1-w^{n-\theta/\pi})}{(1-w^n)^2} \\ &= \ln\left(\frac{\pi}{-\ln w}\right) - \frac{\theta(\pi-\theta)\ln w}{2\pi^2} + \mathcal{O}(w). \end{aligned} \qquad (18)$$

Thus, for $w \sim 0$, the factor multiplying $\alpha' t$ in the exponent in the integrand of (17) becomes

$$\ln(-\psi^2[\ln\psi]'') \simeq -\frac{\theta(\pi-\theta)}{\pi^2}\ln w, \qquad (19)$$

from which we see that the exponent $\alpha' t\ln(-\psi^2[\ln\psi]'')$ is not finite in the large t limit (for fixed θ held a finite amount away from 0 or π), making the $w \sim 0$ contribution exponentially suppressed. On the contrary, in the $q \sim 0$ region we have

$$\begin{aligned}\ln(-\psi^2[\ln\psi]'') &\sim -\ln(-\psi_N^2[\ln\psi_N]'') \\ &\sim 16q^2\sin^4\theta + \mathcal{O}(q^4),\end{aligned} \qquad (20)$$

implying that the $q \sim 0$ region will become important as $\alpha' t$ increases.

Notice that the regions $\theta \sim 0$, π could also produce important contributions to the integral for large t and need to be analyzed separately. For this purpose, we would need the corresponding asymptotic expressions for the functions ψ and ψ_N near these points and to integrate over the full range $0 < q < 1$ (Fig. 1 shows the diagram corresponding to the planar amplitude at fixed q). We will come back to this point at the end of this section and we will find that these regions produce subleading behavior with respect to the contribution coming from $q \sim 0$.

For small q, and also using $[-\ln\psi]'' \sim [-\ln\psi_N]'' \sim \csc^2\theta$, the Regge trajectory $\Sigma(t)$ becomes

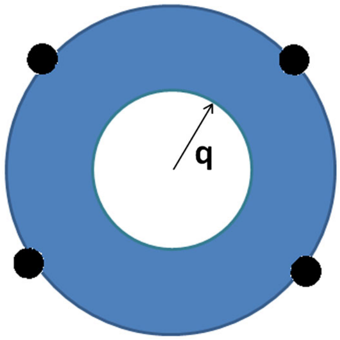

FIG. 1 (color online). For the planar one-loop amplitude, all the external states lie at only one of the two boundaries, and the integration over q is represented as a radial variable. The region $q \sim 0$ corresponds to highly energetic open strings, and it gives the dominant contribution in the hard-scattering regime.

$$\Sigma(t) \sim i\alpha' t \int_0^{\epsilon} \frac{dq}{q} \int_0^{\pi} d\theta \sin^2\theta (e^{i16q^2\sin^4\theta\alpha' t} - e^{-i16q^2\sin^4\theta\alpha' t}). \tag{21}$$

Note that we have also defined the integral above by analytical continuation ($t \to it$) as in [5]. Therefore, we wish to obtain the large $|t|$ behavior of the expression

$$\Sigma(t) \sim i\alpha' t \int_{\delta}^{\pi-\delta} d\theta \sin^2\theta \int_0^{\epsilon} \frac{dq}{q} (e^{itaq^2} - e^{-itaq^2}) \tag{22}$$

for fixed ϵ with $a = 16\sin^4\theta$. We have also introduced the cutoff δ to stress the fact that we need to examine the contributions from the regions where $\theta \sim 0$, π separately. Performing the change $atq^2 \equiv u$, we have

$$\Sigma(t) \sim i\alpha' t \int_{\delta}^{\pi-\delta} \sin^2\theta d\theta i \int_0^{\epsilon^2 ta} \frac{du}{u} \sin u. \tag{23}$$

Since ϵ is small but fixed we can take the upper limit of the u integral to be ∞ in the $|t| \to \infty$ limit. Also, in this limit, the θ dependence in the integration over u disappears, which allows us to send δ to zero; thus,

$$\Sigma(t) \sim -\alpha' t \int_0^{\pi} \sin^2\theta d\theta \int_0^{\infty} \frac{du}{u} \sin u = -\alpha' t \frac{\pi^2}{4}. \tag{24}$$

Therefore, continuing back to $t \to -it$, we have

$$\Sigma(t) \sim i\alpha' t \quad \text{as} \quad t \to -\infty. \tag{25}$$

Finally, as $t \to -\infty$, combining Eqs. (12) and (25) yields

$$\begin{aligned} A_P + A_M &\sim i(-\alpha' s)^{1+\alpha' t} \Gamma(-\alpha' t) \ln(-\alpha' s) \alpha' t \\ &= i(-\alpha' s)^{1+\alpha' t} \Gamma(1-\alpha' t) \ln(-\alpha' s). \end{aligned} \tag{26}$$

We could use Stirling's approximation $\Gamma(1-\alpha' t) \sim \sqrt{2\pi}(-\alpha' t)^{1/2-\alpha' t} e^{\alpha' t}$ valid for $-\alpha' t \gg 1$, which yields

$$A_P + A_M \sim i(-\alpha' s)^{1+\alpha' t} (-\alpha' t)^{1/2-\alpha' t} e^{\alpha' t} \ln(-\alpha' s). \tag{27}$$

To conclude, we take a moment to analyze the regions where $\theta \sim 0, \pi$, which are also important as $|t|$ becomes large. Using the following expression for the logarithm of ψ,

$$\ln\psi(\theta) = \ln\sin\theta + 2\sum_{m=1}^{\infty} \frac{1}{m} \frac{q^{2m}}{1-q^{2m}} (1 - \cos 2m\theta), \tag{28}$$

one can see that

$$\ln(-\psi^2[\ln\psi]'') \sim -\ln(-\psi_N^2[\ln\psi_N]'') \sim \mathcal{O}(\theta^4). \tag{29}$$

Thus, the main contribution at large t comes from the region where θ is of the order of $\sim(-\alpha' t)^{-1/4}$. A rough estimation from these regions gives $\Sigma(t) \sim (-\alpha' t)^{-3/4}$, which is subleading with respect to the $q \sim 0$ contribution given in (25).

B. Hard scattering at one loop

The high energy limit at fixed scattering angle for the one-loop amplitude was first computed in [2] in the early days of string theory. There, the computation was done for the nonplanar amplitude which had a dominant saddle point in the interior of the integration region. In [5], Gross and Mañes showed that only the $(2+2)$ nonplanar amplitude (i.e., the amplitude with two particles on each boundary of the annulus) has a saddle point in the interior of the region of integration. The planar, nonorientable and the $(3+1)$ nonplanar amplitudes do not possess a dominant saddle point in the interior, but points in the boundary of the region do give subdominant contributions [with respect to the $(2+2)$ nonplanar]. They also showed that the leading contribution for the sum of the planar and nonorientable diagrams (1) comes from the region where $q \sim 0$, $x \equiv \frac{\sin\theta_2 \sin\theta_{43}}{\sin\theta_{42} \sin\theta_3} \sim (1+t/s)^{-1}$. We begin this section by recalculating the leading behavior known in the literature using the saddle point approximation for the cross ratio, although we use a different set of integration variables [13] where $\theta_2 \to x$, $\theta_3 \to r \equiv \sin\theta_{43}/\sin\theta_3$. Starting from Eqs. (1) and (2), the relevant factor in the integrand when all kinematic invariants $\alpha' k_i \cdot k_j$ are large is

$$\prod_{i<j} \psi^{2\alpha' k_i \cdot k_j} = \exp\{-\alpha' s V_\lambda\}, \tag{30}$$

where

$$V_\lambda \equiv \ln x - \lambda \ln(1-x) + 2\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{2n}}{1-q^{2n}} (S_n - \lambda T_n), \tag{31}$$

with

$$x \equiv \frac{\sin\theta_{43}\sin\theta_2}{\sin\theta_{42}\sin\theta_3} \tag{32}$$

$$\begin{aligned} S_n &\equiv 2\cos n(\theta_2-\theta_{43})[\cos n(\theta_{42}+\theta_3)-\cos n(\theta_2+\theta_{43})] \\ T_n &\equiv 2\cos n(\theta_{42}+\theta_3)[\cos n(\theta_2-\theta_{43})-\cos n(\theta_2+\theta_{43})] \end{aligned} \tag{33}$$

and $\lambda = -t/s < 0$. Expanding the function V_λ about the critical region mentioned above yields

$$e^{-\alpha' s V_\lambda} \approx e^{-\mathcal{E}_0} e^{-\alpha' s[\frac{(1-\lambda)^3}{2\lambda}(x-x_c)^2 \pm 2q^2(S_1-\lambda T_1)]}, \tag{34}$$

where

$$\begin{aligned} \mathcal{E}_0 &\equiv \alpha'|s|[\lambda\ln(-\lambda)+(1-\lambda)\ln(1-\lambda)] \\ &= \alpha' s\ln(-\alpha' s)+\alpha' t\ln(-\alpha' t)+\alpha' u\ln(\alpha' u). \end{aligned} \tag{35}$$

In the $\alpha' s \to -\infty$ limit, the integration over x can be approximated by a Gaussian, giving

$$\int_{-\infty}^{\infty} dx e^{-\alpha' s\frac{(1-\lambda)^3}{2\lambda}(x-x_c)^2} \sim \sqrt{\frac{-2\pi\lambda}{(1-\lambda)^3}}(-\alpha' s)^{-1/2}. \tag{36}$$

The integral over q is dominated by the small q region which, after analytic continuation to $s \to is$, behaves as

$$\int_0^\epsilon \frac{dq}{q}\left(e^{2i\alpha' sq^2(S_1-\lambda T_1)} - e^{-2i\alpha' sq^2(S_1-\lambda T_1)}\right) \sim \frac{i\pi}{2}, \tag{37}$$

a result which we already encountered in (21). All in all, for the coefficient of $\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3$, we obtain

$$\begin{aligned} A_P + A_M &\sim sue^{-\mathcal{E}_0}\sqrt{\frac{-2\pi\lambda}{(1-\lambda)^3}}(-\alpha' s)^{-1/2}F(\lambda) \\ &\sim s^2(1+t/s)e^{-\mathcal{E}_0}(-\alpha' t)^{1/2}(-\alpha' s)^{-1/2} \\ &\quad \times (1+t/s)^{-3/2}(-\alpha' s)^{-1/2}F(\lambda) \\ &\sim (-\alpha' s)^{3/2}e^{-\mathcal{E}_0}(-\lambda)^{1/2}(1-\lambda)^{-1/2}F(\lambda), \end{aligned} \tag{38}$$

which shows the usual exponential suppression $e^{-\mathcal{E}_0}$ factor, and the function $F(\lambda)$ is given by

$$F(\lambda) = \int_0^\infty dr\int_0^\pi d\theta \frac{r\sin^2\theta(r^2+2r\cos\theta+1)^{-1}}{r^2(1-\lambda)^2+2r(1-\lambda)\cos\theta+1}. \tag{39}$$

Writing the exponential factor as

$$e^{-\mathcal{E}_0} = (-\alpha' s)^{\alpha' t}(-\alpha' t)^{-\alpha' t}(1+t/s)^{\alpha' s+\alpha' t}, \tag{40}$$

we have

$$A_P + A_M \sim i(-\alpha' s)^{1+\alpha' t}(-\alpha' t)^{1/2-\alpha' t}(1+t/s)^{\alpha' s+\alpha' t-1/2}F(\lambda) \tag{41}$$

which completes the hard-scattering limit of the one-loop amplitude.

Before closing this section, a few remarks are important to note about (39). First, the integral is divergent for $\lambda = 0$. Second, since $\lambda = -t/s$ and s and t are both negative, it is convergent in the entire range of interest $-\infty < \lambda < 0$. If we would like to study the $\lambda \sim 0$ behavior of $F(\lambda)$, we need to examine the divergent regions of the integral for $\lambda = 0$.

In the next section, we will study the $s \gg t$ limit (i.e., $\lambda \sim 0$) of the hard-scattering amplitude and we will have at our disposal a full analytic result (no integrals are left to be evaluated) for the leading behavior of the amplitude. However, this is not the case in the study of other theories, such as the type 0 string studied in [10]. Thus, in those cases, one only has at one's disposal the hard-scattering limit in terms of integral representations similar to (39), and an estimation of $F(\lambda)$ for $\lambda \sim 0$ is indeed necessary. We will come back to these remarks at the end of Sec. III, but for now we will go ahead and conclude this section by estimating the small λ behavior of $F(\lambda)$.

By simple inspection we see that, as $\lambda \to 0$, the only singular integration region is the joint region $\theta \sim \pi$ and $r \sim 1$. It is straightforward to see this by recalling that, in terms of the cross ratio x, the dominant saddle point is given by $x_c = (1-\lambda)^{-1}$. This perfectly matches with the fact that the Regge behavior of the amplitude is obtained from the region $\theta_2 \sim \theta_3$, $\theta_4 \sim \pi$, since $x \sim \theta_{32}(\pi - \theta_4)$, which gives the leading behavior [3,10]. Thus, the Regge limit occurs when $x \to 1$. Therefore, in the small scattering angle limit (i.e., $\lambda \sim 0$), the integral above is singular where $\theta \sim \pi$, $r \sim x$; thus

$$\begin{aligned} F(\lambda) &\sim \int_{x-\delta}^{x+\delta} dr\int_{\pi-\epsilon}^{\pi} d\theta \\ &\qquad \times \frac{(\pi-\theta)^2}{((x-1)^2+x(\pi-\theta)^2)((r/x-1)^2+(\pi-\theta)^2)} \\ &\sim 2\int_0^\epsilon \frac{\theta}{(x-1)^2+x\theta^2} \\ &= -2\ln|1-x| + \ln((1-x)^2+\epsilon^2). \end{aligned} \tag{42}$$

Therefore, as $\lambda \to 0$ for fixed ϵ, we have

$$F(\lambda) \sim -2\ln(-\lambda) + 2\ln(1-\lambda). \tag{43}$$

III. RECOVERY OF THE REGGE LIMIT

The high energy behavior at fixed angle given in Eq. (41) uses a Gaussian approximation around the dominant saddle point given by $x_c = (1-\lambda)^{-1}$. We will now calculate this limit using a different method which does not require the

Gaussian approximation, but instead we will compute the integral over the x variable in an exact closed form. However, we still need to approximate the exponent for small q, but this is not too serious since this is the only place in the q integration where there is a dominant critical point [5]. One could regard the calculation we perform in this section as a computation of the Gaussian approximation, including all the possible fluctuations around the saddle. This allows us to bypass the issue of computing the contributions coming from any other region in the θ_k integrations, since we will be computing this triple integral in exact form. Starting from (1), for small q we obtain

$$\prod_{i<j}\psi(\theta_{ji})^{2\alpha' k_i\cdot k_j} = e^{-\alpha' s V_\lambda} \approx e^{-\alpha' s[\ln x - \lambda\ln(1-x)+2q^2(S_1-\lambda T_1)]} \approx x^{-\alpha' s}(1-x)^{-\alpha' t}e^{-2\alpha' s q^2(S_1-\lambda T_1)}. \tag{44}$$

Notice that this time we are not expanding the function $\ln x - \lambda\ln(1-x)$ about the saddle point x_c. The small q contribution to the total amplitude can be written as

$$A_P + A_N \sim \alpha'^2 su\int\prod_k d\theta_k x^{-\alpha' s}(1-x)^{-\alpha' t} \times\int_0^\epsilon\frac{dq}{q}[e^{-2\alpha' s q^2(S_1-\lambda T_1)} - e^{2\alpha' s q^2(S_1-\lambda T_1)}], \tag{45}$$

where we have included the overall $\alpha'^2 su$ coefficient coming from the coefficient of $\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3$. We have already encountered the expression for the q integral above, with the very satisfying result that it does not depend on the coefficient of q^2 in the exponent; therefore, it does not bring an angular dependence from the combination $S_1 - \lambda T_1$ which will allow us to perform an exact evaluation of the integration over the θ_k variables. The integral

$$I \equiv \int_0^\pi d\theta_4\int_0^{\theta_4}d\theta_3\int_0^{\theta_3}d\theta_2 x^{-\alpha' s}(1-x)^{-\alpha' t} \tag{46}$$

was evaluated long ago by Green and Schwarz in [9], where they proved that dilaton tadpole divergences could be absorbed in a renormalization of the Regge slope α'. This was realized before it was recognized that this divergence is absent for the $SO(32)$ gauge group. We simply quote the answer here:

$$I = \int\prod_k d\theta_k x^{-\alpha' s}(1-x)^{-\alpha' t} = \gamma\frac{1}{\alpha'}\frac{\partial}{\partial\alpha'}\left[\alpha'\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)}\right], \tag{47}$$

where γ is a numerical constant. Using this and the result for the integral over q given in Eq. (37), we have

$$A_P + A_N \sim i\alpha'^2 su\frac{1}{\alpha'}\frac{\partial}{\partial\alpha'}\left[\alpha'^2\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)}\right], \tag{48}$$

where we have omitted the numerical coefficient γ for simplicity. We can now take the limit $s, t \to -\infty$, holding t/s fixed directly inside the brackets, to obtain

$$A_P + A_N \sim i\alpha'^2 su\frac{1}{\alpha'}\frac{\partial}{\partial\alpha'}[\alpha'^2(-\alpha' s)^{-1+\alpha' t}(-\alpha' t)^{-1/2-\alpha' t}(1+t/s)^{-1/2+\alpha' s+\alpha' t}] \sim i(-\alpha' s)^{1/2}(-\lambda)^{-1/2-\alpha' t}(1-\lambda)^{1/2+\alpha' s+\alpha' t}[1+2\alpha' s(\lambda\ln(-\lambda)+(1-\lambda)\ln(1-\lambda))]. \tag{49}$$

Taking again $-\alpha' s \gg 1$, we end up with

$$A_P + A_N \sim i(-\alpha' s)^{3/2}(-\lambda)^{-1/2}(1-\lambda)^{1/2}e^{\alpha' s[\lambda\ln(-\lambda)+(1-\lambda)\ln(1-\lambda)]}[\lambda\ln(-\lambda)+(1-\lambda)\ln(1-\lambda)].$$

To recover the Regge behavior, we take $s \gg t$ above. The exponential becomes

$$e^{\alpha' s[\lambda\ln(-\lambda)+(1-\lambda)\ln(1-\lambda)]} = (-\lambda)^{-\alpha' t}(1-\lambda)^{\alpha' s+\alpha' t} \sim (-\alpha' s)^{\alpha' t}(-\alpha' t)^{-\alpha' t}e^{\alpha' t}, \tag{50}$$

and the last factor becomes

$$[\lambda\ln(-\lambda)+(1-\lambda)\ln(1-\lambda)] \sim \lambda\ln(-\lambda) = -t/s[\ln(-\alpha' t)-\ln(-\alpha' s)] \sim (-\alpha' t)(-\alpha' s)^{-1}\ln(-\alpha' s). \tag{51}$$

Therefore, the Regge limit at high t is

$$A_P + A_N \sim i(-\alpha' s)^{3/2}(-\lambda)^{-1/2}(-\alpha' s)^{\alpha' t}(-\alpha' t)^{-\alpha' t}e^{\alpha' t}(-\alpha' t)(-\alpha' s)^{-1}\ln(-\alpha' s) \sim i(-\alpha' s)^{1+\alpha' t}(-\alpha' t)^{1/2-\alpha' t}e^{\alpha' t}\ln(-\alpha' s), \tag{52}$$

which is exactly the result we found in (27).

As anticipated at the end of Sec. II B, we finish this section by showing that the result (52) can also be obtained from the approximate expression in (41) by using the small λ behavior of $F(\lambda)$ defined in (39). We believe it is instructive to do this because we are also interested in the small λ behavior of the hard-scattering limit of the type 0 model in the context of [10,14,15], where we cannot afford the luxury of having an exact expression for the coefficient of the exponential falloff. From (43) we have

$$F(\lambda) \sim -2\ln(-\lambda) + 2\ln(1-\lambda) \sim 2\ln(-\alpha' s) \quad \text{for } s \gg t, \tag{53}$$

which provides the logarithm that appears in the Regge limit of the amplitude in (27). Thus, putting this into (41) one immediately recovers (52).

IV. CONCLUSIONS

By studying the hard-scattering limit of the sum of the one-loop planar and nonorientable diagrams of type I superstrings in flat spacetime, we found the exact dependence in the scattering angle that multiplies the known exponential suppression at high energies. This avoids the issue of having to estimate the contributions from flat directions in the angular integrals and the fluctuations around the saddle point, since we have at our disposal an exact result for the triple integral over the angular variables in a closed form (i.e., the integral over the moduli representing the positions of the vertex operators). This allowed us to compare both the hard-scattering and Regge regimes of the amplitude, since they should coincide in the limit of high-momentum transfer (high t) of the latter regime. We indeed confirmed that this matching occurs by making use of the closed form of the angular integrals given in (47). As a check, we were also able to obtain this result from the approximate expression (41) by analyzing the behavior of the integral $F(\lambda)$ in (39) as $\lambda \to 0$.

It would be interesting to see if such a smooth connection between the hard-scattering and Regge regimes still holds for the one-loop nonplanar diagram. In that case, the amplitude in the hard limit is dominated by a saddle in the interior of the moduli space, i.e., away from the boundaries at $q = 0$ and $q = 1$, given in terms of the Jacobi theta functions as $\theta_4(0,q)/\theta_4(\pi,q) = (1-\lambda)^{1/4}$. However, as $\lambda \to 0$, the only solution for the saddle equation above is $q = 0$, thus moving the saddle to the boundary. On the other hand, the high energy ($\alpha'|t| \gg 1$) limit of the Regge regime is again dominated by the $q \sim 0$ region. Therefore, a smooth transition between the hard and Regge behaviors is also expected for the one-loop nonplanar diagram, although it would be nice to obtain this explicitly.

Another immediate extension of this work would be to allow the open strings to be attached to smaller dimensional coincident Dp-branes (here we considered the case of a space-filling D-brane) where the small q behavior can be analyzed separately for the planar and nonorientable diagrams, since the amplitudes are finite as long as $p < 8$. It would also be interesting to check if our methods can be applied to the situation studied in [6], where the authors analyzed the case where of open strings living on different D-branes separated by a fixed distance.

ACKNOWLEDGEMENTS

I wish to thank professor Charles Thorn for many valuable discussions and constant encouragement. I also thank Juan I. Jottar for comments on the manuscript, André Neveu, and Juan L. Mañes for correspondence, and to Chrysostomos Kalousios, Jay Perez, Christoph Sachse, and Heywood Tam for useful discussions and support. This research was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-97ER-41029.

[1] A. Neveu and J. Scherk, Phys. Rev. D **1**, 2355 (1970).
[2] V. Alessandrini, D. Amati, and B. Morel, Nuovo Cimento Soc. Ital. Fis. **7A**, 797 (1972).
[3] H. J. Otto, V. N. Pervushin, and D. Ebert, Teor. Mat. Fiz. **35**, 48 (1978); [Theor. Math. Phys. **35**, 308 (1978)].
[4] L. Clavelli, P. Coulter, S. T. Jones, and Z. H. Lin, Phys. Rev. D **35**, 2462 (1987).
[5] D. J. Gross and J. L. Manes, Nucl. Phys. **B326**, 73 (1989).
[6] C. Bachas and B. Pioline, J. High Energy Phys. 12 (1999) 004.
[7] J. L. F. Barbon, Phys. Lett. B **382**, 60 (1996).
[8] G. D'Appollonio, P. Di Vecchia, R. Russo, and G. Veneziano, J. High Energy Phys. 11 (2010) 100.
[9] M. B. Green and J. H. Schwarz, Nucl. Phys. **B198**, 441 (1982).
[10] F. Rojas and C. B. Thorn, Phys. Rev. D **84**, 026006 (2011).
[11] M. B. Green and J. H. Schwarz, Phys. Lett. B **151**, 21 (1985).
[12] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory: Loop Amplitudes, Anomalies and Phenomenology*, Cambridge Monographs on Mathematical Physics Vol. 2 (Cambridge University Press, Cambridge, U.K., 1987), p. 596.
[13] A. Neveu and J. Scherk, Nucl. Phys. **B36**, 317 (1972).
[14] C. B. Thorn, Phys. Rev. D **80**, 086010 (2009).
[15] C. B. Thorn, Phys. Rev. D **82**, 065009 (2010).