PHYSICAL REVIEW D **89**, 085037 (2014)

Langevin description of gauged scalar fields in a thermal bath

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(Received 30 October 2013; published 24 April 2014)

We study the dynamics of the oscillating gauged scalar field in a thermal bath. A Langevin-type equation of motion of the scalar field, which contains both dissipation and fluctuation terms, is derived by using the real-time finite-temperature effective action approach. The existence of the quantum fluctuation-dissipation relation between the nonlocal dissipation term and the Gaussian stochastic noise terms is verified. We find that the noise variables are anticorrelated at equal time. The dissipation rate for each mode is also studied, which turns out to depend on the wave number.

DOI: 10.1103/PhysRevD.89.085037 PACS numbers: 11.10.Wx, 98.80.Cq

I. INTRODUCTION

Recent advancements in observational technology enable us to trace back the history of the Universe. In particular, observations of the cosmic microwave background, including the latest results of the Planck mission [1], provide us with the picture of the Universe at the recombination, the subsequent evolution, and a piece of information on the early Universe. The Universe in a much earlier period, however, is still veiled and many models which are built to explain the physics beyond the energy scale realized in laboratories remain unverified. To select the theory describing our world, we need not only observational developments but also more precise theoretical predictions using fundamental theories of physics.

One of the most interesting phenomena in the early Universe is the phase transition. It has provided mechanisms of inflation [2–7], called "old inflation" [3,4] and "new inflation" [5,6]. In both of the models inflation is driven by the vacuum energy before the end of the phase transition. The thermal inflation [8], also caused by the potential energy of the flaton field, is a relatively short accelerating period after the primordial inflation. Since it changes the expansion history of the Universe, not only are the moduli and gravitinos diluted but the primordial gravitational waves are damped as well [9]. On the other hand, collisions of bubbles generated during a phase transition can produce gravitational waves [10]. Furthermore, depending on the kinds of broken symmetry, various topological defects are expected to be produced. Among them, line-like topological defects known as osmic strings can produce gravitational waves [11] which may be detectable by future experiments [12]. These examples indicate that the phase transition is a key to understanding high energy physics and the early Universe.

A precise description of the dynamics of phase transitions is necessary to compare predictions of each theoretical model with observations. In many models of the early Universe, phase transitions are controlled by the expectation value of the scalar fields. While the effective potential is a useful quantity to derive properties of the phase transitions that happen quasistatically, it often comes up short because of the dynamical nature of the scalar fields. In such cases, we need to directly solve the evolution equations of the scalar fields derived from the effective action. It has been shown that the behavior of a scalar field in a thermal bath can be described by the Langevin equation [13], which includes stochastic noise terms coming from interactions with other fields in the thermal bath. These noise terms may change the types of phase transitions. For example, a previous study [14] indicates that the fermionic noise may lead to the phase mixing, which invalidates the description of a phase transition using the effective potential. In addition, the relevance of thermal fluctuations during inflation in discriminating inflation models has also been pointed out [15].

So far, the effective action and the resultant equation of motion of a scalar field have been studied in models where it has self-interaction and interactions with other fermions and scalar bosons [14,16]. Now, it is an interesting project to extend the previous studies to include interactions with gauge fields. We extend previous analyses to include interactions with gauge fields using the simplest Abelian gauge theory known as scalar quantum electrodynamics. Though the hot scalar QED theory has been studied in Refs. [17,18] to study the dynamics of gauge fields, we

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focus on the scalar field as a system of interest and treat gauge fields as a hot environment.

The organization of this paper is as follows. We briefly review the effective action method and apply it to scalar quantum electrodynamics in Sec. II. Actually, this effective action contains the imaginary part. In Sec. III, we interpret it as stochastic noises and derive a generalized Langevin equation. We consider the meaning of the equation, and explain the validity of this interpretation. We also show the stochastic property of the noise, and compare it with the fermionic and scalar bosonic noises which have been studied in previous studies. The dissipation rate of each mode is also studied. We summarize our study and discuss its applicability in Sec. IV.

II. EFFECTIVE ACTION

As we have mentioned in the Introduction, one of our goals is the precise description of phase transitions in gauge theory which requires the knowledge of the effective action for the scalar field. In this paper, we focus on the derivation of the effective action and investigate the basic properties of the obtained equation of motion for the massive charged scalar field due to interactions with gauge fields. In order to realize the phase transition, we need to add the self-interaction of the scalar field to have a Higgs mechanism. We defer the inclusion of the self-interaction to another study. The simplest approach to describe phase transitions in the hot early Universe is to analyze a finite-temperature effective potential. By using the effective potential, we can explain the symmetry restoration at high temperature or in the early Universe and the subsequent spontaneous symmetry breaking. However, since it is derived under the assumption of a static, homogeneous field configuration, it cannot describe the dynamics of phase transitions accurately. In this section, in order to obtain the equation of motion governing the dynamical phenomenon, we calculate the effective action. Studies so far show that the effective action generally contains an imaginary part, which can be interpreted as the origin of dissipative properties.

A. Settings

To clarify the role of gauge fields, we consider scalar quantum electrodynamics, which is the simplest gauge theory. Its Lagrangian density is given by

$$\begin{aligned}\mathcal{L} &= D_\mu\Phi^\dagger D^\mu\Phi - m^2\Phi^\dagger\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &\quad + ieA_\mu(\Phi^\dagger\partial^\mu\Phi - \Phi\partial^\mu\Phi^\dagger) + e^2A_\mu A^\mu\Phi^\dagger\Phi. \end{aligned} \tag{1}$$

After imposing the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A} = 0$, one can see that the Lagrangian density becomes

$$\begin{aligned}\mathcal{L} &= \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi + \frac{1}{2}\partial_\mu\vec{A}_T\cdot\partial^\mu\vec{A}_T \\ &\quad - ie\vec{A}_T(\Phi\vec{\nabla}\Phi^\dagger - \Phi^\dagger\vec{\nabla}\Phi) - e^2\vec{A}_T\cdot\vec{A}_T\Phi^\dagger\Phi \\ &\quad + \frac{1}{2}(\vec{\nabla}A_0)^2 - ieA_0(\Phi\dot{\Phi}^\dagger - \Phi^\dagger\dot{\Phi}) + e^2A_0^2\Phi^\dagger\Phi. \end{aligned} \tag{2}$$

Here, $\vec{A}_\mathrm{T}$ means the transverse components, which satisfy $\vec{\nabla}\cdot\vec{A}_\mathrm{T} = 0$. Though we use the Coulomb gauge in this study, other choices of gauge, such as axial gauge or Lorenz gauge, should also be possible in principle.[1] Although the field equation may be gauge dependent, physical quantities extracted from it should be gauge invariant. We discuss this issue by showing the gauge independence of the dissipation rate in Appendix A.

In the so-called real-time thermal field theory, we can calculate the thermal average using a path integral.[2] We can choose the time path so that it consists of three paths: (i) A path from $t_i(<0)$ to $-t_i$ on the real axis of t (plus contour), (ii) a path from $-t_i$ to t_i on the real axis of t (minus contour), and (iii) a path from t_i to $t_i - i\beta$, where $\beta = 1/T$ is the inverse of the temperature. We can neglect the contribution from the third contour when taking $T\to\infty$ [20]. We denote field variables on the contour (i) and (ii) by superscripts $+$ and $-$, respectively. Following Boyanovsky *et al.* [18], thermal propagators for the scalar field Φ and gauge field $\vec{A}_\mathrm{T}$ are given as follows.

Propagators for the scalar field:

$$\langle\Phi^{(a)\dagger}(\vec{x},t)\Phi^{(b)}(\vec{x}',t')\rangle = -i\int\frac{d^3k}{(2\pi)^3}G_k^{ab}(t,t')e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{3}$$

$$G_k^{++}(t,t') = G_k^{>}(t,t')\Theta(t-t') + G_k^{<}(t,t')\Theta(t'-t), \tag{4}$$

$$G_k^{--}(t,t') = G_k^{>}(t,t')\Theta(t'-t) + G_k^{<}(t,t')\Theta(t-t'), \tag{5}$$

$$G_k^{+-}(t,t') = -G_k^{<}(t,t'), \tag{6}$$

$$G_k^{-+}(t,t') = -G_k^{>}(t,t'), \tag{7}$$

$$G_k^{>}(t,t') = \frac{i}{2\omega_k}[(1+n_k)e^{-i\omega_k(t-t')} + n_k e^{+i\omega_k(t-t')}], \tag{8}$$

$$G_k^{<}(t,t') = \frac{i}{2\omega_k}[n_k e^{-i\omega_k(t-t')} + (1+n_k)e^{+i\omega_k(t-t')}], \tag{9}$$

[1]In addition to these conditions, we come up with the so-called covariant gauge. While it is often convenient to use it, we should note that it has a different meaning from other gauge conditions. That is to say, we do not specify the gauge-fixing condition in the covariant gauge. In our study the use of this gauge is inappropriate since it is useful only in the problems where physical quantities such as the S-matrix elements can be directly calculated.

[2]One of the introductory textbooks is Ref. [19].

$$\omega_k = \sqrt{|\vec{k}|^2 + m^2}, \qquad n_k = \frac{1}{e^{\beta\omega_k} - 1}. \tag{10}$$

Propagators for the gauge field:

$$\langle A_{Ti}^{(a)}(\vec{x}, t) A_{Tj}^{(b)}(\vec{x}', t')\rangle = -i \int \frac{d^3k}{(2\pi)^3} \mathcal{G}_{kij}^{ab}(t, t') e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{11}$$

$$\mathcal{G}_{kij}^{++}(t, t') = \mathcal{P}_{ij}(\vec{k})[g_k^{>}(t, t')\Theta(t - t') + g_k^{<}(t, t')\Theta(t' - t)], \tag{12}$$

$$\mathcal{G}_{kij}^{--}(t, t') = \mathcal{P}_{ij}(\vec{k})[g_k^{>}(t, t')\Theta(t' - t) + g_k^{<}(t, t')\Theta(t - t')], \tag{13}$$

$$\mathcal{G}_{kij}^{+-}(t, t') = -\mathcal{P}_{ij}(\vec{k}) g_k^{<}(t, t'), \tag{14}$$

$$\mathcal{G}_{kij}^{-+}(t, t') = -\mathcal{P}_{ij}(\vec{k}) g_k^{>}(t, t'), \tag{15}$$

$$g_k^{>}(t, t') = \frac{i}{2k}[(1 + N_k)e^{-ik(t-t')} + N_k e^{+ik(t-t')}], \tag{16}$$

$$g_k^{<}(t, t') = \frac{i}{2k}[N_k e^{-ik(t-t')} + (1 + N_k)e^{+ik(t-t')}], \tag{17}$$

$$k = \sqrt{|\vec{k}|^2}, \qquad N_k = \frac{1}{e^{\beta k} - 1}, \qquad \mathcal{P}_{ij}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{18}$$

The generating functional of the Green's function $Z[J^{(+)}, J^{(-)}]$ is

$$\begin{aligned} Z[J^{(+)}, J^{(-)}] &= \int \mathcal{D}\vec{A}_T \mathcal{D}\Phi \mathcal{D}\Phi^\dagger \exp\left[i(S^{(+)} - S^{(-)}) \right. \\ &\quad \left. + i \int d^4x \{J^{(+)}(x)\Phi^{(+)}(x) - J^{(-)}(x)\Phi^{(-)}(x)\}\right], \end{aligned} \tag{19}$$

where

$$S^{(\pm)} = \int d^4x \mathcal{L}[\vec{A}_T^{(\pm)}, \Phi^{(\pm)}, \Phi^{\dagger(\pm)}]. \tag{20}$$

B. Perturbative expansion

Calculating the effective action corresponds to the summation of one-particle-irreducible (1PI) diagrams. Practically, the effective action can be obtained only by means of a perturbative expansion in terms of the gauge coupling constant e, which we adopt in our study. The lowest nontrivial contributions to the effective action appear at the second order of the coupling constant e. At this order, there are two relevant diagrams, which are shown in Fig. 1. In addition to these 1PI diagrams, we have to rewrite A_0 using its Euler-Lagrange equation as

$$A_0(x) = -\frac{1}{\Delta}\rho(x) + \mathcal{O}(e^2), \tag{21}$$

$$\rho \equiv ie(\Phi\dot{\Phi}^\dagger - \Phi^\dagger\dot{\Phi}), \tag{22}$$

and take the following interaction into account:

$$\begin{aligned} \frac{1}{2}(\vec{\nabla}A_0)^2 &\overset{\text{td}}{=} -\frac{1}{2}A_0\Delta A_0 = \frac{1}{2}\rho\frac{1}{\Delta}\rho = \frac{e^2}{8\pi}\int d^3y \\ &\quad \times \frac{(\Phi\dot{\Phi}^\dagger - \Phi^\dagger\dot{\Phi})(\vec{x}, t)(\Phi\dot{\Phi}^\dagger - \Phi^\dagger\dot{\Phi})(\vec{y}, t)}{|\vec{x} - \vec{y}|}. \end{aligned} \tag{23}$$

Here $\overset{\text{td}}{=}$ means an equality up to a total derivative term. This is nothing but the Coulomb potential. Although the meaning of this interaction is clearly seen in real space, calculations are performed more easily in Fourier space. The action corresponding to this term is written as

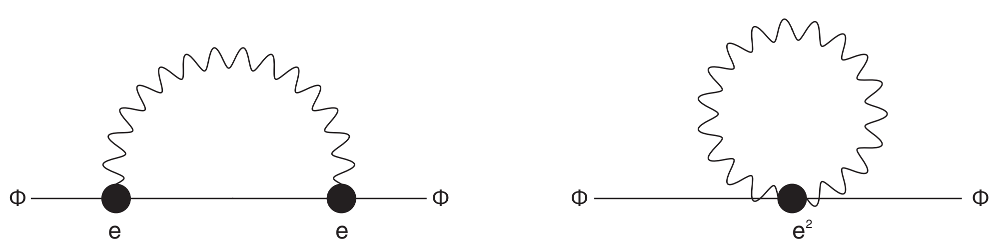

FIG. 1. $\mathcal{O}(e^2)$ 1PI diagrams. The solid/wavy line represents the scalar/photon propagator, respectively. The left diagram produces nonlocal terms in the effective action. The right diagram gives a thermal correction to the mass term, which is proportional to T^2.

$$S \supset \frac{e^2}{2}\int dt \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\vec{k}^2}(\tilde{\Phi}(t,\vec{p})\dot{\tilde{\Phi}}^\dagger(t,\vec{p}-\vec{k}) - \dot{\tilde{\Phi}}(t,\vec{p})\tilde{\Phi}^\dagger(t,\vec{p}-\vec{k}))$$
$$\times(\tilde{\Phi}(t,\vec{q})\dot{\tilde{\Phi}}^\dagger(t,\vec{k}+\vec{q}) - \dot{\tilde{\Phi}}(t,\vec{q})\tilde{\Phi}^\dagger(t,\vec{k}+\vec{q})), \tag{24}$$

where $\tilde{\Phi}(t,\vec{p})$ is the spatial Fourier transformation of $\Phi(t,\vec{x})$, defined as

$$\tilde{\Phi}(t,\vec{p}) \equiv \int d^3x e^{i\vec{p}\cdot\vec{x}}\Phi(t,\vec{x}). \tag{25}$$

The contribution of the left diagram in Fig. 1 to the effective action Γ is

$$\Gamma \supset +4ie^2\int d^4x_1 d^4x_2 \langle A_{Ti}^{(+)}(x_1)A_{Tj}^{(+)}(x_2)\rangle\langle\partial_i\Phi^{\dagger(+)}(x_1)\partial_j\Phi^{(+)}(x_2)\rangle\Phi^{\dagger(+)}(x_2)\Phi^{(+)}(x_1)$$
$$+4ie^2\int d^4x_1 d^4x_2 \langle A_{Ti}^{(-)}(x_1)A_{Tj}^{(-)}(x_2)\rangle\langle\partial_i\Phi^{\dagger(-)}(x_1)\partial_j\Phi^{(-)}(x_2)\rangle\Phi^{\dagger(-)}(x_2)\Phi^{(-)}(x_1)$$
$$-4ie^2\int d^4x_1 d^4x_2 \langle A_{Ti}^{(-)}(x_1)A_{Tj}^{(+)}(x_2)\rangle\langle\partial_i\Phi^{\dagger(-)}(x_1)\partial_j\Phi^{(+)}(x_2)\rangle\Phi^{\dagger(+)}(x_2)\Phi^{(-)}(x_1)$$
$$-4ie^2\int d^4x_1 d^4x_2 \langle A_{Ti}^{(+)}(x_1)A_{Tj}^{(-)}(x_2)\rangle\langle\partial_i\Phi^{\dagger(+)}(x_1)\partial_j\Phi^{(-)}(x_2)\rangle\Phi^{\dagger(-)}(x_2)\Phi^{(+)}(x_1), \tag{26}$$

and the right diagram contributes

$$\Gamma \supset -e^2\int d^4x[\langle A_{Ti}^{(+)}(x)A_{Ti}^{(+)}(x)\rangle\Phi^{\dagger(+)}(x)\Phi^{(+)}(x)$$
$$-\langle A_{Ti}^{(-)}(x)A_{Ti}^{(-)}(x)\rangle\Phi^{\dagger(+)}(x)\Phi^{(-)}(x)]. \tag{27}$$

This local term gives a thermal correction to the mass term,[3]

$$\langle A_{Ti}^{(+)}(x)A_{Ti}^{(+)}(x)\rangle = \langle A_{Ti}^{(-)}(x)A_{Ti}^{(-)}(x)\rangle = \frac{T^2}{6}. \tag{28}$$

For the Coulomb potential term (24), although we do not have diagrammatic correspondence, the calculation procedure is the same as the previous interaction terms. After taking contractions except for two field variables which are going to be external lines, we have

$$\Gamma \supset -ie^2\int dt\int\frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{|\vec{k}-\vec{p}|^2}$$
$$\times[-G_p^{++}(t,t)(\dot{\tilde{\Phi}}^{(+)}\dot{\tilde{\Phi}}^{\dagger(+)} - \dot{\tilde{\Phi}}^{\dagger(-)}\dot{\tilde{\Phi}}^{\dagger(-)})(t,\vec{k})$$
$$+\ddot{G}_p^{++}(t,t)(\tilde{\Phi}^{(+)}\tilde{\Phi}^{\dagger(+)} - \tilde{\Phi}^{\dagger(-)}\tilde{\Phi}^{\dagger(-)})(t,\vec{k})]. \tag{29}$$

It is convenient to replace $\Phi^{(\pm)}$ with new variables,

$$\Phi^{(\pm)} = \phi_c \pm \frac{1}{2}\phi_\Delta. \tag{30}$$

Finally the effective action Γ incorporating these two diagrams and A_0 terms up to the second order in e becomes

$$\Gamma = \int d^4x\left[\phi_\Delta^\dagger(x)\left(-\partial_\mu\partial^\mu - m^2 - e^2\frac{T^2}{6}\right)\phi_c(x) + \phi_\Delta(x)\left(-\partial_\mu\partial^\mu - m^2 - e^2\frac{T^2}{6}\right)\phi_c(x)^\dagger\right] - 4ie^2\int d^4x_1 d^4x_2$$
$$\times\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}e^{-i(\vec{p}_1+\vec{p}_2)\cdot(\vec{x}_1-\vec{x}_2)}\mathcal{P}_{ij}(\vec{p}_1)p_{2i}p_{2j}\Theta(t_2-t_1)[g_{p_1}^<(t_1,t_2)G_{p_2}^<(t_1,t_2) - g_{p_1}^>(t_1,t_2)G_{p_2}^>(t_1,t_2)](\phi_c^\dagger(x_1)\phi_\Delta(x_2)$$
$$+\phi_c(x_1)\phi_\Delta^\dagger(x_2)) - 2ie^2\int d^4x_1 d^4x_2\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}e^{-i(\vec{p}_1+\vec{p}_2)\cdot(\vec{x}_1-\vec{x}_2)}\mathcal{P}_{ij}(\vec{p}_1)p_{2i}p_{2j}[g_{p_1}^<(t_1,t_2)G_{p_2}^<(t_1,t_2)$$
$$+g_{p_1}^>(t_1,t_2)G_{p_2}^>(t_1,t_2)]\phi_\Delta^\dagger(x_1)\phi_\Delta(x_2) + \Gamma_{A_0}, \tag{31}$$

[3]Here we omit the divergent part, which is to be cancelled by a mass counterterm since it exists even at zero temperature.

$$\Gamma_{A_0} = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^3p}{(2\pi)^3} \frac{1}{|\vec{k}-\vec{q}|^2} \cdot \left(\frac{1}{2}+n_p\right) \cdot \left(\frac{\omega^2}{\omega_p}+\omega_p\right) \times (\tilde{\phi}_c(k)\tilde{\phi}_\Delta{}^\dagger(k)+\tilde{\phi}_\Delta(k)\tilde{\phi}_c{}^\dagger(k)) \equiv \int \frac{d^4k}{(2\pi)^4} (\tilde{\phi}_c(k)\tilde{\phi}_\Delta{}^\dagger(k)+\tilde{\phi}_\Delta(k)\tilde{\phi}_c{}^\dagger(k))\tilde{f}_{A_0}(k). \quad (32)$$

Γ_{A_0}, which comes from the Coulomb potential term, gives corrections to the dispersion relation as well as the thermal mass term.

Let us show that the imaginary part of the nonlocal terms which come from diagrams in Fig. 1 is nonzero. First, both of the integrands are invariant under replacements $\vec{p}_1 \to -\vec{p}_1$ and $\vec{p}_2 \to -\vec{p}_2$, respectively. This property allows us to replace $e^{-i(\vec{p}_1+\vec{p}_2)\cdot(\vec{x}_1-\vec{x}_2)}$ with $\cos[(\vec{p}_1+\vec{p}_2)\cdot(\vec{x}_1-\vec{x}_2)]$, which is real. Second, from Eqs. (8), (9), (16), and (17), we note that $g^< G^< - g^> G^>$ is purely imaginary and $g^< G^< + g^> G^>$ is real. Thus, the first nonlocal term is real and the second one is purely imaginary. We will explain how to interpret the imaginary part of the effective action in the next section.

III. LANGEVIN EQUATION AND NOISE PROPERTIES

In the previous section, we have seen that the effective action for the scalar field contains an imaginary part as with the case of pure scalar theory in a thermal environment. It can be written as

$$i\Gamma \supset -\int d^4x_1 d^4x_2 \mathcal{N}(x_1-x_2)(\phi_{\Delta\mathrm{R}}(x_1)\phi_{\Delta\mathrm{R}}(x_2) + \phi_{\Delta\mathrm{I}}(x_1)\phi_{\Delta\mathrm{I}}(x_2)), \quad (33)$$

where

$$\mathcal{N}(x_1-x_2) = -2e^2 \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} e^{-i(\vec{p}_1+\vec{p}_2)\cdot(\vec{x}_1-\vec{x}_2)} \mathcal{P}_{ij}(\vec{p}_1) p_{2i} p_{2j} \times [g^<_{p_1}(t_1,t_2)G^<_{p_2}(t_1,t_2) + g^>_{p_1}(t_1,t_2)G^>_{p_2}(t_1,t_2)]. \quad (34)$$

$\phi_{\Delta\mathrm{R/I}}$ are the real/imaginary parts of ϕ_Δ, respectively. Now we are going to rewrite and interpret it as stochastic noise terms.

A. Mathematical transformation

As in previous studies [13,14,16], we rewrite the imaginary part by using the Gaussian integral formula,

$$\exp\left[-\int d^4x d^4y \varphi(x) M(x,y) \varphi(y)\right] \propto \int \mathcal{D}\xi \exp\left[-\frac{1}{4}\int d^4x d^4y \xi(x) M^{-1}(x,y) \xi(y) + i\int d^4x \xi(x)\varphi(x)\right], \quad (35)$$

and interpret the integration over ξ as an ensemble average, where ξ is regarded as a stochastic Gaussian variable.

This formula is not applicable to arbitrary $M(x,y)$. Just as the one-dimensional Gaussian integral $\int dx \exp[-ax^2]$ requires $a>0$, all of the eigenvalues of M should be positive. After rewriting with Fourier transformations,

$$\int d^4x d^4y \xi(x) M^{-1}(x,y) \xi(y) = \int \frac{d^4k}{(2\pi)^4} \tilde{M}^{-1}(k) |\tilde{\xi}(k)|^2, \quad (36)$$

we notice that $\tilde{M}(k)$ should be positive. The Fourier transformation of $\mathcal{N}$ with respect to $t-t'$ and $\vec{x}-\vec{x}'$ is

$$\tilde{\mathcal{N}}(\omega,\vec{k}) = 2e^2 \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \mathcal{P}_{ij}(\vec{p}_1) p_{2i} p_{2j} (2\pi)^3 \delta(\vec{k}-\vec{p}_1-\vec{p}_2) \times \frac{2\pi}{2p_1 2\omega_{p_2}} [\{(1+N_{p_1})(1+n_{p_2}) + N_{p_1} n_{p_2}\}\delta(\omega - p_1 - \omega_{p_2}) + \{(1+N_{p_1})n_{p_2} + N_{p_1}(1+n_{p_2})\}\delta(\omega - p_1 + \omega_{p_2}) + \{N_{p_1}(1+n_{p_2}) + (1+N_{p_1})n_{p_2}\}\delta(\omega + p_1 - \omega_{p_2}) + \{N_{p_1} n_{p_2} + (1+N_{p_1})(1+n_{p_2})\}\delta(\omega + p_1 + \omega_{p_2})]. \quad (37)$$

Clearly, this is positive for any $(\omega,\vec{k})$, and thus this expression ensures us that we can use the formula (35) and rewrite the effective action with stochastic noise terms. Finally we obtain

$$e^{i\Gamma} = \int \mathcal{D}\xi_a \mathcal{D}\xi_b P[\xi_a] P[\xi_b] \exp\left[i\Gamma_{\mathrm{real}} + i\int d^4x (\xi_a(x)\phi_{\Delta\mathrm{R}}(x) + \xi_b(x)\phi_{\Delta\mathrm{I}}(x))\right] = \int \mathcal{D}\xi_a \mathcal{D}\xi_b P[\xi_a] P[\xi_b] \exp\left[i\Gamma_{\mathrm{real}} + i\int d^4x (\xi^\dagger(x)\phi_\Delta(x) + \xi(x)\phi^\dagger_\Delta(x))\right] \equiv \int \mathcal{D}\xi_a \mathcal{D}\xi_b P[\xi_a] P[\xi_b] \exp[iS_{\mathrm{eff}}], \quad (38)$$

where

$$P[\xi_a] = \exp\left[-\frac{1}{4}\int d^4x d^4y \xi_a(x)\mathcal{N}^{-1}(x-y)\xi_a(y)\right],$$
$$P[\xi_b] = \exp\left[-\frac{1}{4}\int d^4x d^4y \xi_b(x)\mathcal{N}^{-1}(x-y)\xi_b(y)\right],$$
$$\xi = \frac{1}{2}\xi_a + i\frac{1}{2}\xi_b = \xi_{\rm R} + i\xi_{\rm I}. \tag{39}$$

Now we have a real action $S_{\rm eff}$ containing stochastic noise terms. The two-point correlation function of ξ is given by

$$\langle \xi_{\rm R}(x_1)\xi_{\rm R}(x_2)\rangle = \langle \xi_{\rm I}(x_1)\xi_{\rm I}(x_2)\rangle = \frac{1}{2}\mathcal{N}(x_1-x_2), \tag{40}$$

$$\langle \xi_{\rm R}(x_1)\xi_{\rm I}(x_2)\rangle = 0, \tag{41}$$

$$\langle \xi(x_1)\xi^\dagger(x_2)\rangle = \mathcal{N}(x_1-x_2). \tag{42}$$

B. Validity of the interpretation and the fluctuation-dissipation relation

As we saw in the previous section, we obtain a real effective action $S_{\rm eff}$ by introducing noise terms. To derive an equation of motion for the physical variable ϕ_c in the closed time-path formalism we take a variation with respect to ϕ_Δ and set it to zero. The equation of motion of $\phi_c(x)$ is

$$\left.\frac{\delta\Gamma}{\delta\phi_\Delta(x)}\right|_{\phi_\Delta=0} = 0, \tag{43}$$

which is now equivalent to

$$\left.\frac{\delta S_{\rm eff}}{\delta\phi_\Delta(x)}\right|_{\phi_\Delta=0} = 0, \tag{44}$$

supplemented by Eq. (39). The variation of this real action leads to the following Langevin-type equation of motion:

$$\begin{aligned}
&\left(\Box + m^2 + e^2\frac{T^2}{6}\right)\phi_c(x) - \int d^4x' f_{A_0}(x-x')\phi_c(x') \\
&\quad + 4ie^2\int_{-\infty}^t dt'\int d^3x'\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3} \\
&\quad \times e^{-i(\vec{p}_1+\vec{p}_2)\cdot(\vec{x}'-\vec{x})}\mathcal{P}_{ij}(\vec{p}_1)p_{2i}p_{2j}[g^<_{p_1}(t',t)G^<_{p_2}(t',t) \\
&\quad - g^>_{p_1}(t',t)G^>_{p_2}(t',t)]\phi_c(t',\vec{x}') = \xi(x).
\end{aligned} \tag{45}$$

Now let us consider its validity. The right-hand side, ξ, kicks or perturbs the mean field and supplies energy to it from the thermal bath. On the other hand, the last term on the left-hand side represents the friction, which dissipates energy of the mean field ϕ_c into the bath. This nonlocal memory term can be formally written as

$$\int_{-\infty}^t dt'\int d^3x C(x-x')\phi(x'). \tag{46}$$

The equation of motion in Fourier space is

$$\begin{aligned}
&(-\omega^2+k^2+m^2)\tilde{\phi}(\omega,\vec{k}) \\
&\quad + \left(e^2\frac{T^2}{6} - \tilde{f}_{A_0}(\omega,\vec{k}) + \int\frac{d\omega'}{2\pi}\frac{\mathcal{P}}{\omega-\omega'}i\tilde{C}(\omega',\vec{k})\right)\tilde{\phi}(\omega,\vec{k}) \\
&\quad + \frac{1}{2}\tilde{C}(\omega,\vec{k})\tilde{\phi}(\omega,\vec{k}) = \tilde{\xi}(\omega,\vec{k}).
\end{aligned} \tag{47}$$

Note that $\tilde{C}$ is purely imaginary and $\tilde{f}_{A_0}$ is real, so all the coefficients of $\tilde{\phi}$ in the second line are real. We interpret them as corrections to the free part, i.e., the first line. The terms in the third line can be interpreted as dissipation and fluctuation.

The imaginary part of the Fourier transformation of the memory kernel $C(t,\vec{x})$ is

$$\begin{aligned}
&{\rm Im}\tilde{C}(\omega,\vec{k}) \\
&= -4e^2\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\mathcal{P}_{ij}(\vec{p}_1)p_{2i}p_{2j}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{p}_1-\vec{p}_2) \\
&\times\frac{2\pi}{2p_1 2\omega_{p_2}}[\{(1+N_{p_1})(1+n_{p_2})-N_{p_1}n_{p_2}\}\delta(\omega-p_1-\omega_{p_2}) \\
&+\{(1+N_{p_1})n_{p_2}-N_{p_1}(1+n_{p_2})\}\delta(\omega-p_1+\omega_{p_2}) \\
&+\{N_{p_1}(1+n_{p_2})-(1+N_{p_1})n_{p_2}\}\delta(\omega+p_1-\omega_{p_2}) \\
&+\{N_{p_1}n_{p_2}-(1+N_{p_1})(1+n_{p_2})\}\delta(\omega+p_1+\omega_{p_2})].
\end{aligned} \tag{48}$$

Now we have collected all the ingredients necessary for showing the fluctuation-dissipation relation. Expecting that the scalar field and the gauge field reach some equilibrium state, we start our analysis by using finite-temperature propagators. In order for a system to achieve and keep thermal equilibrium, there is a necessary condition between noise terms and the memory term, which is the fluctuation-dissipation relation. Mathematically, it is written as

$$\frac{\tilde{\mathcal{N}}(\omega,\vec{k})}{\frac{-1}{\omega}{\rm Im}\tilde{C}(\omega,\vec{k})} = \frac{\omega}{2}\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \omega\left(\frac{1}{2}+n_\omega\right). \tag{49}$$

It is straightforward to check that this relation indeed holds in our case.[4] This is the quantum fluctuation-dissipation relation [16,21–24]. In light of this fact, we conclude that the introduction of noise terms is not just a mathematical trick but a meaningful transformation to bring out physics.

C. Properties of the stochastic noise

We now show the properties of the noise. From Eq. (34), we see that the spatial noise correlation is expressed as

[4]Owing to delta functions, we can factor out the ratio $\frac{(e^{\beta\omega}+1)}{(e^{\beta\omega}-1)}$ without performing complicated integrals in Eqs. (37) and (48).

$$\begin{aligned}\langle\xi(\vec{x}_1,t)\xi^\dagger(\vec{x}_2,t)\rangle \\ &= \frac{e^2}{2}\int\frac{d^3p_1}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3}e^{-i(\vec{p}_1+\vec{p}_2)\cdot(\vec{x}_1-\vec{x}_2)}\mathcal{P}_{ij}(\vec{p}_1)p_{2i}p_{2j} \\ &\quad\times\frac{1}{p_1\omega_{p_2}}(1+2N_{p_1})(1+2n_{p_2}).\end{aligned} \tag{50}$$

We divide it as

$$\begin{aligned}\langle\xi(\vec{x}_1,t)\xi^\dagger(\vec{x}_2,t)\rangle &= \frac{e^2}{2}(\alpha_{ij}-\beta_{ij})\gamma_{ij} \\ &= \frac{e^2}{2}\left[-\alpha(r)\left(\gamma''(r)+\frac{2}{r}\gamma'(r)\right)\right. \\ &\quad\left.-\frac{2}{r^2}\beta'(r)\gamma'(r)-\beta''(r)\gamma''(r)\right],\end{aligned} \tag{51}$$

where

$$\alpha_{ij} = \int\frac{d^3p_1}{(2\pi)^3}e^{-i\vec{p}_1\cdot\vec{r}}\frac{1}{p_1}\left(1+\frac{2}{e^{\beta p_1}-1}\right)\delta_{ij} \equiv \alpha(r)\delta_{ij}, \tag{52}$$

$$\begin{aligned}\beta_{ij} &= \int\frac{d^3p_1}{(2\pi)^3}e^{-i\vec{p}_1\cdot\vec{r}}\frac{p_{1i}p_{1j}}{p_1^3}\left(1+\frac{2}{e^{\beta p_1}-1}\right) \\ &\equiv -\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\beta(r),\end{aligned} \tag{53}$$

$$\begin{aligned}\gamma_{ij} &= \int\frac{d^3p_2}{(2\pi)^3}e^{-i\vec{p}_2\cdot\vec{r}}\frac{p_{2i}p_{2j}}{\omega_{p_2}}\left(1+\frac{2}{e^{\beta\omega_{p_2}}-1}\right) \\ &\equiv -\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\gamma(r),\end{aligned} \tag{54}$$

and we use $r = |\vec{r}| = |\vec{x}_1-\vec{x}_2|$.

After some calculations, we obtain the following expressions:

$$\alpha(r) = \frac{1}{2\pi r\beta}\coth\left(\frac{r}{\beta}\pi\right), \tag{55}$$

$$\beta'(r) = -\frac{1}{2\pi^2 r}+\sum_{n=1}^{\infty}\frac{-r+n\beta\mathrm{Arccot}(\frac{n\beta}{r})}{\pi^2r^2}, \tag{56}$$

$$\begin{aligned}\beta''(r) &= \frac{1}{\pi^2r^2}-\frac{1}{2\pi r\beta}\coth\left(\frac{r}{\beta}\pi\right) \\ &\quad+\sum_{n=1}^{\infty}\frac{2}{\pi^2r^3}\left[r-n\beta\mathrm{Arccot}\left(\frac{n\beta}{r}\right)\right],\end{aligned} \tag{57}$$

$$\begin{aligned}\gamma'(r) &= -\frac{m^2}{2\pi^2 r}K_2(mr)-\frac{m^2r}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{r^2+n^2\beta^2} \\ &\quad\times K_2\left(m\sqrt{r^2+n^2\beta^2}\right),\end{aligned} \tag{58}$$

$$\begin{aligned}\gamma''(r) &= -\frac{m^2}{2\pi^2}\left[\frac{1}{r^2}K_2(mr)-\frac{m}{r}K_3(mr)\right] \\ &\quad-\frac{m^2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{r^2+n^2\beta^2}K_2\left(m\sqrt{r^2+n^2\beta^2}\right)\right. \\ &\quad\left.-\frac{mr^2}{(r^2+n^2\beta^2)^{3/2}}K_3\left(m\sqrt{r^2+n^2\beta^2}\right)\right].\end{aligned} \tag{59}$$

Here $K_\nu(z)$ is the modified Bessel function of the νth order.

Although the expression of the noise correlation function is quite cumbersome for the general case, and numerical computation is the only feasible way to evaluate it, it reduces to a fairly concise form in some limiting cases. First, in the short-distance limit, we find

$$\langle\xi(\vec{x}_1,t)\xi^\dagger(\vec{x}_2,t)\rangle \simeq -\frac{3e^2}{2\pi^4r^6}\Theta(r)+\frac{e^2}{4\pi^4r^5}\delta(r). \tag{60}$$

For the derivation of this expression, see Appendix B. Here we define the step function as

$$\Theta(x) = \begin{cases}0 & x\le 0, \\ 1 & x>0.\end{cases} \tag{61}$$

On the other hand, we obtain the following behavior in the long-distance limit $r \gg \beta, \frac{1}{m}$:

$$\langle\xi(\vec{x}_1,t)\xi^\dagger(\vec{x}_2,t)\rangle \simeq -\frac{e^2m^2}{8\pi^2\beta^2r^2}e^{-mr}. \tag{62}$$

If the scalar field is massless, we obtain

$$\langle\xi(\vec{x}_1,t)\xi^\dagger(\vec{x}_2,t)\rangle \simeq -\frac{e^2}{8\pi^2\beta^2r^4}. \tag{63}$$

For the derivation of these expressions, see Appendix C.

We show the spatial correlation for various masses in Fig. 2. As the approximate expression (62) shows, the noise correlation is exponentially suppressed at $r \gtrsim \frac{1}{m}$ and monotonically approaches zero. Asymptotically, the noise correlation obtained by numerical evaluation is consistent with the above simple expressions that were obtained analytically. We see that the noise in this model shows anticorrelation, which is different from the previous study [14].

D. Dissipation rate

The Langevin equation provides not only fluctuations to the scalar field but also its dissipation.

According to Refs. [16,24], for the scalar field described by the equation

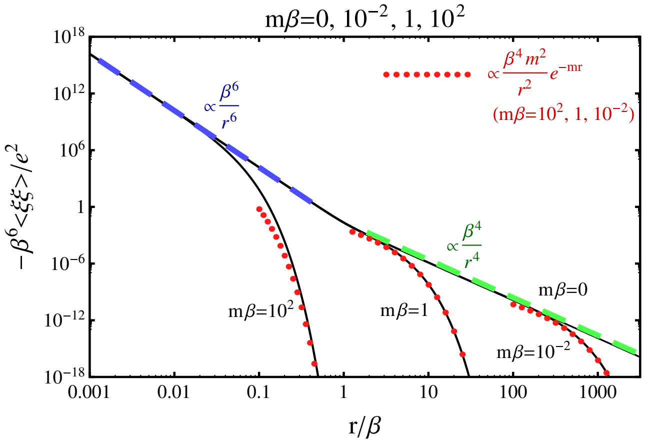

FIG. 2 (color online). Noise spatial correlations for various mass values. The solid black line represents the exact expression (51) with Eqs. (55)–(59). The dashed and dotted lines correspond to the analytical approximations (60), (62), and (63), respectively. For $r < \frac{1}{m}$, correlations obey a power-law decay. They start to decay exponentially when r exceeds $\frac{1}{m}$.

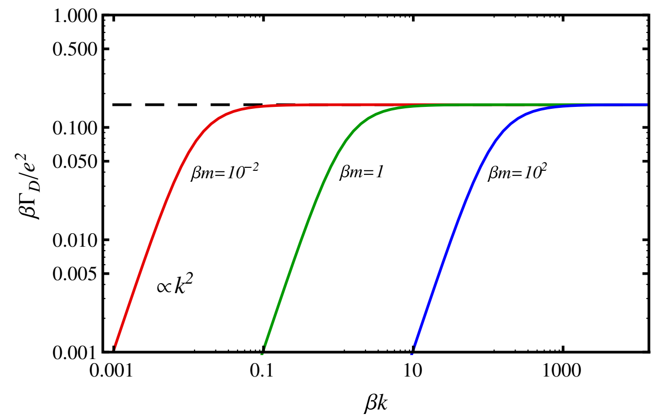

FIG. 3 (color online). The k dependence of the dissipation rate (68). The dashed black line represents $\beta\Gamma_{\mathrm{D}} = e^2/2\pi$. The solid lines represent the dissipation rate for $\beta m = 10^{-2}, 1, 10^2$, respectively. For $k > m$, the dissipation rate is almost independent of k. However, for $k < m$, it is proportional to k^2.

$$(-\omega^2 + M_{k,\omega}^2)\tilde{\phi}(\omega, \vec{k}) + \frac{1}{2}\tilde{C}(\omega, \vec{k})\tilde{\phi}(\omega, \vec{k}) = \tilde{\xi}(\omega, \vec{k}), \tag{64}$$

the dissipation rate of the k-mode oscillation is given by

$$\Gamma_{\mathrm{D}}(\vec{k}) = -\mathrm{Im}\tilde{C}(\vec{k}, M_{k,\omega})/2M_{k,\omega}. \tag{65}$$

This expression is valid if the ω and nontrivial k dependence of $M_{k,\omega}$ is negligibly small, that is

$$M_{k,\omega}^2 = k^2 + M_0^2, \tag{66}$$

where M_0 is a constant. In this study, $M_{k,\omega}$ is given by

$$M_{k,\omega}^2 = k^2 + m^2 + e^2\frac{T^2}{6} - \tilde{f}_{A_0}(\omega, \vec{k}) + \int \frac{d\omega'}{2\pi} \mathrm{P}\frac{1}{\omega - \omega'} i\tilde{C}_{\vec{k}}(\omega'). \tag{67}$$

Both $\tilde{f}_{A_0}$ and the principal value integral are divergent. In Appendix D, we show that this divergence can be removed by renormalizing the scalar field strength. In other words, we can cancel out this divergence with the counterterm which is proportional to the kinetic term of the scalar field. Although the k and ω dependencies of $M_{k,\omega}$ are nontrivial, such corrections are proportional to e^2. If we assume $M_{k,\omega}$ is given by Eq. (66), we find

$$\begin{aligned}\Gamma_{\mathrm{D}}(\vec{k}) = {} & \frac{e^2}{4\pi}\frac{k}{\sqrt{M_0^2 + k^2}} \\ & \times \int_{p_i}^{p_f} dp \left(1 + \frac{1}{e^{\beta p} - 1} - \frac{1}{e^{\beta(\sqrt{M_0^2+k^2} - p)} - 1}\right) \\ & \times \left[-\frac{M_0^2}{k^2} + \frac{(M_0^2 - m^2)\sqrt{M_0^2 + k^2}}{k^2 p} - \frac{(M_0^2 - m^2)^2}{4k^2p^2}\right], \\ p_i = {} & \frac{M_0^2 - m^2}{2(\sqrt{M_0^2 + k^2} + k)}, \qquad p_f = \frac{M_0^2 - m^2}{2(\sqrt{M_0^2 + k^2} - k)}.\end{aligned} \tag{68}$$

Though we have assumed $M_0 > m$ in deriving the above expression, it is finite even in taking $M_0 \to m$. In this limit, we can obtain

$$\Gamma_{\mathrm{D}}(\vec{k}) = \begin{cases} \frac{e^2}{3\pi\beta}\frac{k^2}{m^2} & k \ll m, \\ \frac{e^2}{2\pi\beta} & k \gg m. \end{cases} \tag{69}$$

We show the dissipation rate as a function of k in Fig. 3 for various values of βm.[5]

Since both the dissipation and fluctuation come from the left diagram in Fig. 1, we can see the physical processes related to dissipation and fluctuation by cutting the diagram into two pieces [25]. Considering the fact that a scalar boson cannot decay into a scalar boson of the same species and a massless gauge boson due to energy and momentum conservation, it may be doubtful that Eq. (68) is the physical dissipation rate. Though the dissipation rate shown

[5]Note that we should not take the high-temperature limit ($\beta m \to 0$) for the result in Fig. 3, since in such a case the difference between M_0 and m is not negligible.

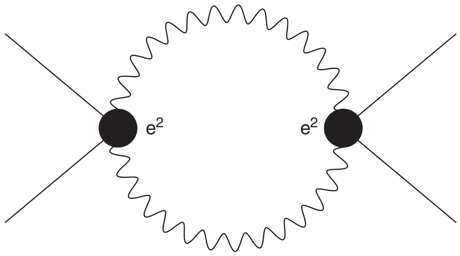

FIG. 4. 1PI diagram relevant to the multiplicative noise. This $\mathcal{O}(e^4)$ diagram leads to the nonzero dissipation rate for the coherently oscillating scalar field.

in Fig. 3 is expressed as an integral over the loop momentum, only the $\vec{p}=\vec{0}$, or a soft-photon loop, contributes to the resultant finite value.[6] From a mathematical point of view, it results from a cancellation between the divergent contribution from the bosonic distribution function and the vanishment of the phase space, like $\int_0 p^2 dp \times \frac{1}{p^2}\delta(p)$. If we include higher-order corrections, for example, by using dressed propagators instead of free ones, gauge fields acquire a plasmon mass. Therefore the bosonic distribution function at zero momentum becomes finite, so that the dissipation rate from this diagram vanishes. In this case Eq. (37) would also vanish, as it should.

The necessity of higher-order corrections to obtain a physical dissipation rate is also shown by a different consideration. Since the contribution of the zero mode seems important, we have considered the same problem in a finite box having a spatial volume V with a periodic boundary condition where momentum is discretized and the zero-mode contribution is isolated. It is found that the zero-mode contribution contains the thermal average of the field value squared which evidently diverges since no particular field value is energetically favored. As a result, the contribution to Eq. (68) scales as Φ_Λ^2/V, where Φ_Λ is a cutoff of the zero-mode field amplitude. Thus, the zero-mode contribution has an ambiguity arising from its dependence on the order of taking the limit $\Phi_\Lambda \to \infty$ and $V \to \infty$. Hence we may not trust the finite value obtained in Eq. (68) which is based on the particular continuum calculation. Indeed Eq. (68) itself would vanish, if we incorporate a plasmon mass into the gauge field using a dressed propagator or simply a mass term generated by a finite value of ϕ. In this case Eq. (37) would also vanish, as it should.

Thus the dissipation arises from diagrams of higher order in e (as shown in Fig. 4) related to the interaction $e^2 A_\mu A^\mu \Phi^\dagger \Phi$. In this case, the noise becomes the multiplicative noise, which appears in the equation of motion of ϕ in a form like $\xi\phi$. The nonlocal memory term in the effective action is

$$\Gamma \supset -4ie^4 \int d^4x_1 d^4x_2 [\phi_{\mathrm{cR}}(x_1)\phi_{\Delta\mathrm{R}}(x_1) + \phi_{\mathrm{cI}}(x_1)\phi_{\Delta\mathrm{I}}(x_1)] \times \left[|\phi_{\mathrm{c}}(x_2)|^2 + \frac{1}{4}|\phi_\Delta(x_2)|^2\right] \times \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{-i(\vec{k}_1+\vec{k}_2)\cdot(\vec{x}_1-\vec{x}_2)} \mathcal{P}_{ij}(\vec{k}_1)\mathcal{P}_{ij}(\vec{k}_2) \times [g^>_{k_1}(t_1,t_2)g^>_{k_2}(t_1,t_2) - g^<_{k_1}(t_1,t_2)g^<_{k_2}(t_1,t_2)]\Theta(t_1-t_2). \tag{70}$$

The dissipation rate corresponding to multiplicative noise cases was also studied in Ref. [16]. Using the quantity

$$C_{\mathrm{m}}(x-x') \equiv 4ie^4 \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{-i(\vec{k}_1+\vec{k}_2)\cdot(\vec{x}-\vec{x}')} \mathcal{P}_{ij}(\vec{k}_1)\mathcal{P}_{ij}(\vec{k}_2) \times [g^>_{k_1}(t,t')g^>_{k_2}(t,t') - g^<_{k_1}(t,t')g^<_{k_2}(t,t')], \tag{71}$$

we can evaluate the dissipation rate for the homogeneous field[7] as

$$\Gamma_{\mathrm{D}} = \frac{\tilde{C}_{\mathrm{m}}(\vec{k}=\vec{0},2M)}{2iM}|\phi(t)|^2 = \frac{e^4|\phi(t)|^2}{4\pi M}(1+2N_M). \tag{72}$$

Here M is the angular frequency of the coherent oscillation and $|\phi(t)|^2$ is a mean square amplitude around the time t. So even the coherent oscillation has nonzero dissipation at this order.

IV. SUMMARY

In this paper, we studied the role of gauge fields in the effective action for the scalar field by considering the scalar QED theory. As can be expected from previous studies, the effective action we obtained contains an imaginary part. We rewrote it by applying the Gaussian functional integral formula, and interpreted the integral over the variable ξ as ensemble averaging. The validity of this arrangement is confirmed by the fluctuation-dissipation relation between the memory term and the introduced noise term. Then we analyzed the spatial correlation of the noise, and found that the noise shows anticorrelation, which is different from the case of scalar and fermionic interactions. The origin of this anticorrelation is due to the existence of derivative interactions between the scalar and gauge fields. We also considered the dissipation rate of the scalar field. Though we obtained a finite dissipation rate, it comes from a soft

[6]We can see this more explicitly by going back to Eq. (48), which is related to the dissipation rate by Eq. (65). After performing the p_2 integral, one can see that only the first and the third delta function can contribute to the p_1 integral at $\vec{p}_1 = 0$ for an on-shell scalar field.

[7]Here we focus on the dissipation rate in the configuration which is essentially equivalent to the single-field dynamics (both the real and imaginary parts are oscillating with the same phase).

photon in the loop. It would vanish if we incorporated a finite mass which may be generated from higher-order loops. Furthermore since the dissipation we have obtained comes from derivative interactions, the dissipation rate for the coherent oscillation vanishes. On the other hand, higher-order diagrams, consisting of a nonderivative interaction as depicted in Fig. 4, gives a nonzero dissipation rate.

Considering that gauge coupling constants are generally larger than Yukawa coupling constants, the absolute value of the noise correlation function for the massless case [Eqs. (60) and (63)] can be larger than that of the fermionic noise studied in Ref. [14]. It would be interesting to study the phase transitions numerically with our results included. Another possible extension is to apply our results to non-Abelian gauge theories in order to treat the realistic phenomena in the early Universe.

ACKNOWLEDGMENTS

We would like to thank Marco Drewes for helpful comments. This work was supported by JSPS Research Fellowships for Young Scientists (Y. M.), the Grant-in-Aid for Scientific Research on Innovative Areas No. 25103505 (T. S.), No. 24103006 (H. M.), No. 21111006 (J. Y.), and No. 25103504 (J. Y.) from The Ministry of Education, Culture, Sports, Science and Technology (MEXT), and JSPS Grant-in-Aid for Scientific Research No. 23340058 (J. Y.).

APPENDIX A: GAUGE INVARIANCE OF THE DISSIPATION RATE

The generating functional $W_f[J]$ is defined by

$$e^{iW_f[J]} = \int \mathcal{D}A_\mu \mathcal{D}\phi^* \mathcal{D}\phi e^{iS[A,\phi]+i\int d^4x(J_A^\mu A_\mu+\phi J+\phi^* J^*)} \times B[f(A;x)]\det(\mathcal{F}_{x,y}), \tag{A1}$$

where $f(A;x)$ is the gauge-fixing condition and A_λ is the transformed gauge field by the gauge transformation specified by λ. The matrix $\mathcal{F}_{x,y}$ is defined by

$$\mathcal{F}_{x,y} = \frac{\delta f(A_\lambda;x)}{\delta\lambda(y)}\bigg|_{\lambda=0}. \tag{A2}$$

In the main text, we have been working in the Coulomb gauge for which we have

$$B[f(A;x)] = \prod_x \delta(f(A;x)), \qquad f(A;x) = \mathrm{div}\vec{A}(x). \tag{A3}$$

The purpose of this appendix is to comment on the gauge dependence of the dissipation rate of ϕ on the gauge-fixing condition. Although all the results in this paper use the in-out formalism for notational simplicity, the same results hold for the case of the in-in formalism.

Under the slight change of the gauge-fixing condition from f to $f+\Delta f$, W_f varies as [26]

$$W_{f+\Delta f}[J] - W_f[J] = \int_{x,y}\langle[\partial^\mu J_{A\mu}(x) + ie(\phi^* J^* - \phi J)(x)]\mathcal{F}^{-1}_{x,y}\Delta f(A;y)\rangle, \tag{A4}$$

where $\langle\mathcal{O}\rangle$ is defined by

$$\langle\mathcal{O}\rangle = e^{-iW_f[J]}\int \mathcal{D}A_\mu \mathcal{D}\phi^* \mathcal{D}\phi e^{iS[A,\phi]+\int_x(J_A^\mu A_\mu+\phi J+\phi^* J^*)} \times \mathcal{O}\det(\mathcal{F}_{x,y}). \tag{A5}$$

Equation (A4) provides the transformation rule of $W_f[J]$ under the change of the gauge-fixing condition.

From the definition of the generating functional, the expectation value of ϕ_f in the gauge f is given by

$$\phi_f(x) = \frac{\delta W_f[J]}{\delta J(x)}. \tag{A6}$$

If we set $J=0$ on the right-hand side of the above equation, ϕ_f constitutes a solution of $\frac{\delta\Gamma_f}{\delta\phi}=0$. From Eq. (A4), we obtain the transformation rule of ϕ_f under the change of the gauge-fixing condition as

$$\phi_{f+\Delta f}(x) - \phi_f(x) = \frac{\delta}{\delta J(x)}\int_{y,z}\langle[\partial^\mu J_{A\mu}(y) + ie(\phi^* J^* - \phi J)(y)]\mathcal{F}^{-1}_{y,z}\Delta f(A;z)\rangle. \tag{A7}$$

In particular, for ϕ_f satisfying $\frac{\delta\Gamma_f}{\delta\phi}=0$, we find

$$\phi_{f+\Delta f}(x) - \phi_f(x) = -\int_y\langle ie\phi(x)\mathcal{F}^{-1}_{x,y}\Delta f(A;y)\rangle = -ie\langle\phi(x)\Lambda(x)\rangle, \tag{A8}$$

where Λ is the gauge transformation connecting two gauges f and $f+\Delta f$ and is related to Δf by

$$\Lambda(x) = \int_y \mathcal{F}^{-1}_{x,y}\Delta f(y). \tag{A9}$$

In a similar way, we find

$$A^\mu_{f+\Delta f}(x) - A^\mu_f(x) = -\partial^\mu\langle\Lambda(x)\rangle. \tag{A10}$$

From Eqs. (A8) and (A10), we find that the gauge transformation of the expectation value of any field is given by the expectation value of the gauge transformation for that field. We also find that $|\phi_f|$ is not gauge invariant in general. On the other hand, $F_f^{\mu\nu} = \partial^\mu A_f^\nu - \partial^\nu A_f^\mu$ is always gauge invariant.

If the path integral to compute the right-hand side of Eq. (A8) is dominated by the field configurations in the close vicinity of ϕ_f, which is the case investigated in the main text, Eq. (A8) approximately becomes

$$\phi_{f+\Delta f}(x) \approx (1 - ie\langle\Lambda(x)\rangle)\phi_f(x), \tag{A11}$$

which means $|\phi_f(x)|^2$ is gauge invariant and hence the dissipation rate is too.

APPENDIX B: SHORT-RANGE NOISE CORRELATION

From Eqs. (55)–(59), we obtain the following asymptotic form as $r \to 0$:

$$\alpha(r) \to \frac{1}{2\pi^2 r^2}, \qquad \beta'(r) \to -\frac{1}{2\pi^2 r}, \qquad \beta''(r) \to \frac{1}{2\pi^2 r^2},$$
$$\gamma'(r) \to -\frac{1}{\pi^2 r^3}, \qquad \gamma''(r) \to \frac{3}{\pi^2 r^4}. \tag{B1}$$

To derive these results, we have used the fact that modified Bessel functions $K_n(x)$ satisfy

$$\lim_{x\to 0} x^n K_n(x) = 2^{-1+n}\Gamma(n). \tag{B2}$$

Using the above expressions, the spatial noise correlation becomes

$$\langle\xi(\vec{x}_1, t)\xi^\dagger(\vec{x}_2, t)\rangle \simeq -\frac{3e^2}{2\pi^4 r^6}, \tag{B3}$$

as r approaches zero.

Since the value at $r = 0$ corresponds to $\langle|\xi(\vec{x}, t)|^2\rangle$, the fact that the correlation function given by Eq. (B3) is negative seems strange. We speculate that the origin of this apparent contradiction lies in the evaluation of γ_{ij}. To see the essence of this, we now consider the case where the scalar field is massless.

The divergence comes from the zero-temperature part,

$$\gamma'_{\rm zero} = \frac{1}{2\pi^2}\int_0^\infty dp\left(\frac{p\cos(pr)}{r} - \frac{\sin(pr)}{r^2}\right), \tag{B4}$$

$$\gamma''_{\rm zero} = \frac{1}{2\pi^2}\int_0^\infty dp \times\left(\frac{-p^2\sin(pr)}{r} - 2\frac{p\cos(pr)}{r^2} + 2\frac{\sin(pr)}{r^3}\right). \tag{B5}$$

These integrals are UV divergent. We regulate them by introducing a cutoff factor $e^{-p/\Lambda}$, getting

$$\gamma'_{\rm zero} = -\frac{1}{2\pi^2}\frac{2r\Lambda^4}{(1 + r^2\Lambda^2)^2}, \tag{B6}$$

$$\gamma''_{\rm zero} = \frac{1}{2\pi^2}\frac{2\Lambda^4(3r^2\Lambda^2 - 1)}{(1 + r^2\Lambda^2)^3}. \tag{B7}$$

If we evaluate the noise correlation with these regulated integrals, the asymptotic form becomes

$$\langle\xi(\vec{x}_1, t)\xi^\dagger(\vec{x}_2, t)\rangle \simeq -\frac{e^2}{2\pi^4}\frac{\Lambda^4(3r^2\Lambda^2 - 1)}{r^2(1 + r^2\Lambda^2)^3}. \tag{B8}$$

When $r > 0$, taking $\Lambda \to \infty$ gives the same result as Eq. (B3). On the other hand, if we keep Λ finite and take $r \to +0$, we see that the spatial noise correlation goes to $+\infty$. At $r = 0$, the dominant part is $\frac{e^2}{2\pi^4}\frac{\Lambda^4}{r^2(1+r^2\Lambda^2)^3}$. If we multiply it by r^5 and integrate from 0 to ∞, we obtain a finite value,

$$\int_0^\infty dr\frac{\Lambda^4}{r^2(1 + r^2\Lambda^2)^3} \times r^5 = \frac{1}{4}. \tag{B9}$$

From this, we can it write as follows:

$$\frac{e^2}{2\pi^4}\frac{\Lambda^4}{r^2(1 + r^2\Lambda^2)^3} = \frac{e^2}{4\pi^4 r^5}\delta(r). \tag{B10}$$

APPENDIX C: LONG-RANGE NOISE CORRELATION

We briefly show the long-range ($r \gg \beta, \frac{1}{m}$) behavior. In this limit, we obtain

$$\alpha(r) \to \frac{1}{2\pi\beta r}, \qquad \beta'(r) \to -\frac{1}{4\pi\beta}, \qquad \beta''(r) \to \frac{\beta}{12\pi r^3},$$
$$\gamma'(r) \to \begin{cases} -\frac{1}{2\pi\beta r^2} & m = 0, \\ -\frac{m}{2\pi r\beta}e^{-mr} & m \neq 0, \end{cases}$$
$$\gamma''(r) \to \begin{cases} \frac{1}{\pi\beta r^3} & m = 0, \\ \frac{m^2}{2\pi r\beta}e^{-mr} & m \neq 0. \end{cases} \tag{C1}$$

In the evaluation of γ' and γ'', we used the asymptotic form for modified Bessel functions $K_n(x)$,

$$K_n(x) \to \sqrt{\frac{\pi}{2}}x^{-1/2}e^{-x} \qquad \text{as } x \to \infty. \tag{C2}$$

APPENDIX D: SCALAR FIELD STRENGTH RENORMALIZATION

We show that the divergent part of

$$-\tilde{f}_{A_0}(\omega, \vec{k}) + \int\frac{d\omega'}{2\pi}\mathrm{P}\frac{1}{\omega - \omega'}i\tilde{C}_{\vec{k}}(\omega') \tag{D1}$$

can be removed by renormalizing the field strength of the scalar field.

First, $\tilde{f}_{A_0}$ can be expressed as

$$-\tilde{f}_{A_0}(\omega,\vec{k}) = \frac{e^2}{2}\int\frac{d^3q}{(2\pi)^3}\frac{\omega^2+\omega_q^2}{\omega_q|\vec{k}-\vec{q}|^2}(1+2n_q). \quad \text{(D2)}$$

This is a UV-divergent integral, whose divergence comes from the zero-temperature part. Sticking to the massless case which does not cause a loss of generality of the analysis in this section, we find

$$-\tilde{f}_{A_0}(\omega,\vec{k}) \to \frac{e^2}{16\pi^2 k}\int_0^\infty dq(\omega^2+q^2)\ln\frac{(k+q)^2}{(k-q)^2}. \quad \text{(D3)}$$

Now we use the dimensional regularization method. Changing the dimension from 3 to $3+\varepsilon$ in Eq. (D2) enables us to extract the divergence as follows:

$$-\tilde{f}_{A_0}(\omega,\vec{k}) = \frac{e^2}{12\pi^2}(3\omega^2+k^2)\frac{1}{\varepsilon} + (\text{regular terms}). \quad \text{(D4)}$$

Second, it is convenient to use another expression for the principal integral term,

$$\int\frac{d\omega'}{2\pi}\mathrm{P}\frac{1}{\omega-\omega'}i\tilde{C}_{\vec{k}}(\omega') = -\frac{e^2}{2}\int\frac{d^3p}{(2\pi)^3}\mathcal{P}_{ij}k_ik_j\frac{1}{p\omega_{k+p}} \times\Bigg[(1+2N_p)\left(\frac{\mathrm{P}}{\omega+p+\omega_{k+p}}-\frac{\mathrm{P}}{\omega-p-\omega_{k+p}}-\frac{\mathrm{P}}{\omega+p-\omega_{k+p}}+\frac{\mathrm{P}}{\omega-p+\omega_{k+p}}\right) + (1+2n_{k+p})\left(\frac{\mathrm{P}}{\omega+p+\omega_{k+p}}-\frac{\mathrm{P}}{\omega-p-\omega_{k+p}}-\frac{\mathrm{P}}{\omega-p+\omega_{k+p}}+\frac{\mathrm{P}}{\omega+p-\omega_{k+p}}\right)\Bigg]. \quad \text{(D5)}$$

This is also UV divergent and we use the dimensional regularization method once more. The above integral at large p is simplified to

$$-\frac{e^2k^2}{3\pi}\int^\infty dp\,p^{-1+\epsilon} \to -\frac{e^2k^2}{3\pi^2}\frac{1}{\epsilon}. \quad \text{(D6)}$$

Finally we find that Eq. (D1) diverges like $\frac{e^2}{4\pi^2}(\omega^2-k^2)\frac{1}{\epsilon}$. This combination of (ω^2-k^2) ensures that we can remove this divergence by the renormalization of the scalar field strength.

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