

# Correlation between Polyakov loops oriented in two different directions in SU(N) gauge theory on a two-dimensional torus

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We consider SU(N) gauge theories on a two-dimensional torus with finite area,  $A$ . Let  $T_\mu(A)$  denote the Polyakov loop operator in the  $\mu$  direction. Starting from the lattice gauge theory on the torus, we derive a formula for the continuum limit of  $\langle g_1(T_1(A))g_2(T_2(A)) \rangle$  as a function of the area of the torus where  $g_1$  and  $g_2$  are class functions. We show that there exists a class function  $\xi_0$  for SU(2) such that  $\langle \xi_0(T_1(A))\xi_0(T_2(A)) \rangle > 1$  for all finite area of the torus with the limit being unity as the area of the torus goes to infinity. Only the trivial representation contributes to  $\xi_0$  as  $A \rightarrow \infty$  whereas all representations become equally important as  $A \rightarrow 0$ .

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## I. INTRODUCTION WITH AN OVERVIEW OF THE MAIN RESULT

Two-dimensional non-Abelian gauge theories are particularly simple to study but reveal a wealth of physics insights. Migdal [1] studied this theory in the context of recursion equations since these equations become exact in two dimensions. Gross and Taylor [2] showed that the partition function of two-dimensional QCD is a string theory. Gross and Witten [3] started from the lattice theory with the standard Wilson action on an infinite lattice and showed factorization to independent plaquettes prompting a possible connection between infinite volume gauge theories and matrix models in a certain limit [4].

In this paper, we start with an SU(N) lattice gauge theory on a torus and rewrite the theory in terms of plaquette degrees of freedom and two additional toron degrees of freedom. There is a constraint imposed by the theory being defined on a two-dimensional torus and the resulting partition function is

$$Z(\beta; L_1 L_2) = \sum_r d_r \prod_{n_1=0}^{L_1-1} \prod_{n_2=0}^{L_2-1} \int [dU_p(n_1, n_2)] f_p[U_p(n_1, n_2); \beta] \times \int dT_1 dT_2 \chi_r[W(L_1 L_2) T_2 T_1 T_2^\dagger T_1^\dagger], \quad (1)$$

where:

- (i)  $\beta$  is the dimensionless lattice coupling related to the continuum coupling,  $g$ , by  $\beta = \frac{1}{g^2 a^2}$  where  $a$  is the lattice spacing.
- (ii) The single plaquette action,  $f_p$ , is a coupling dependent class function of the plaquette variable,  $U_p(n_1, n_2)$ .
- (iii)  $\chi_r$  is the character in the representation labeled by  $r$  and  $d_r$  is the dimension of that representation. The fundamental representation is labelled by  $f$  and the trivial representation by 0.
- (iv) The  $L_1 \times L_2$  periodic lattice has  $L_1 L_2$  plaquettes with  $U_p(n_1, n_2)$ ;  $0 \leq n_1 < L_1$  and  $0 \leq n_2 < L_2$  being the corresponding plaquette variables.
- (v)  $T_1$  and  $T_2$  are toron variables that arise from the presence of noncontractable loops on the torus.
- (vi) The largest Wilson loop on the torus is the ordered product

$$W(L_1 L_2) = \prod_{n_2=0}^{L_2-1} \left[ \prod_{n_1=L_1-1}^0 U_p(n_1, n_2) \right]. \quad (2)$$

The above results on the lattice can be used to study observables in the continuum limit on a torus of area  $A$ . Ignoring a possible overall factor that does not affect computation of observables, the continuum partition function upon integration of all variables is of the form

$$Z(A) = \sum_r e^{-\frac{1}{N} C_r^{(2)} A}, \quad (3)$$

where  $A$  is the dimensionless area of the torus and  $C_r^{(2)}$  is the quadratic Casimir in the  $r$  representation. This is the starting point in [2] for the case of a torus.

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Starting from (1), we show that the continuum limit of the expectation value of a Wilson loop of area  $0 \leq X \leq A$  in representation,  $r$ , is given by

$$\frac{1}{d_r} \langle \chi_r(W(X, A)) \rangle = \frac{1}{Z(A)} \sum_{r', r''} \frac{n(r, r'; r'') d_{r''}}{d_r d_{r'}} \times e^{-\frac{1}{N} C_{r'}^{(2)}(A-X) - \frac{1}{N} C_{r''}^{(2)} X}, \quad (4)$$

where  $n(r, r'; r'')$  is the number of times the representation  $r''$  appears in the tensor product,  $r \otimes r'$ . This coincides with the formula derived in [5] where the techniques used for the calculation are close to the one used in this paper.

Polyakov loop expectation values have been considered in the past [6–8] but the focus has been mainly on Polyakov loop correlators in order to see the confinement behavior. In this paper, we consider correlators of two Polyakov loops oriented in the two different directions. The result only depends on the area of the torus and the representations of the Polyakov loops and we find

$$\begin{aligned} M_{r_1 r_2}(A) &= \langle \chi_{r_1}(T_1(A)) \chi_{r_2}(T_2(A)) \rangle \\ &= \langle \chi_{r_2}(T_1(A)) \chi_{r_1}(T_2(A)) \rangle \\ &= \frac{1}{Z(A)} \sum_r a(r_1 r_2; r) e^{-\frac{1}{N} C_r^{(2)} A}, \end{aligned} \quad (5)$$

where

$$a(r_1, r_2; r) = d_r \int dT_1 dT_2 \chi_r [T_2 T_1 T_2^\dagger T_1^\dagger] \chi_{r_1}(T_1) \chi_{r_2}(T_2). \quad (6)$$

By diagonalizing the infinite-dimensional matrix,  $M(A)$ , at each  $A$ , we can find a new set of  $A$  dependent orthonormal basis of class functions,

$$\xi_i(T(A)) = \sum_r b_i^r(A) \chi_r(T(A)),$$

$$\int dT(A) \xi_i^*(T(A)) \xi_j(T(A)) = \delta_{ij}; \quad i = 0, 1, \dots \quad (7)$$

such that

$$\langle \xi_i(T_1(A)) \xi_j(T_2(A)) \rangle = \lambda_i(A) \delta_{ij}; \quad \lambda_i(A) > \lambda_{i+1}(A) \forall i. \quad (8)$$

An explicit computation for the case of  $SU(2)$  results in only one eigenvalue,  $\lambda_0(A)$ , satisfying the condition  $\lambda_0(A) > 1$  for all finite  $A$  with  $\lambda_0(\infty) \rightarrow 1$   $A \rightarrow \infty$  and  $\lambda_0(A) \rightarrow \infty$  as  $A \rightarrow 0$ .

## II. GAUGE ACTION IN TERMS OF PLAQUETTE VARIABLES

Consider a  $L_1 \times L_2$  periodic lattice. We wish to study a gauge invariant non-Abelian gauge theory on this lattice. The  $(2L_1 L_2)$   $SU(N)$  link variables are denoted by  $U_\mu^g(n_1, n_2)$  for  $0 \leq n_1 < L_1$ ;  $0 \leq n_2 < L_2$  and  $\mu = 1, 2$ . They obey periodic boundary conditions:

$$\begin{aligned} U_1^g(n_1, L_2) &= U_1^g(n_1, 0); & U_2^g(L_1, n_2) &= U_2^g(0, n_2); \\ 0 \leq n_1 < L_1; & & 0 \leq n_2 < L_2. \end{aligned} \quad (9)$$

Given a representative gauge field configuration,  $U_\mu(n_1, n_2)$ , all configurations on this gauge orbit are given by

$$\begin{aligned} U_1^g(n_1, n_2) &= g(n_1, n_2) U_1(n_1, n_2) g(n_1 + 1, n_2); \\ U_2^g(n_1, n_2) &= g^\dagger(n_1, n_2) U_2(n_1, n_2) g(n_1, n_2 + 1); \end{aligned} \quad (10)$$

where  $g(n_1, n_2)$  is a periodic function defined on the lattice sites.

We start with the following representative gauge field configuration (see Fig. 1):

- (i)  $U_1(n_1, n_2) = 1$  for  $0 \leq n_1 < L_1 - 1$  and  $0 \leq n_2 < L_2$ .
- (ii)  $U_2(0, n_2) = 1$  for  $0 \leq n_2 < L_2 - 1$ .
- (iii)  $U_1(L_1 - 1, 0) = T_1$ .
- (iv)  $U_2(0, L_2 - 1) = T_2$ .
- (v)  $U_2(n_1 + 1, n_2) = U_p(n_1, n_2) U_2(n_1, n_2)$  for  $0 \leq n_1 < L_1 - 1$ ; and  $0 \leq n_2 < L_2$ .
- (vi)  $U_1(L_1 - 1, n_2 + 1) = U_2^\dagger(L_1 - 1, n_2) U_p^\dagger(L_1 - 1, n_2) \times U_1(L_1 - 1, n_2)$  for  $0 \leq n_2 < L_2 - 1$ .

This configuration still has a global gauge symmetry given by  $g(n_1, n_2) = g$ . The integration over all  $2(L_1 L_2)$   $U_\mu^g(n_1, n_2)$  variables can be split into

- (i)  $((L_1 L_2) - 1)$   $U_p(n_1, n_2)$  variables for all  $0 \leq n_1 < L_1$ ;  $0 \leq n_2 < L_2$  except  $(n_1, n_2) = (L_1 - 1, L_2 - 1)$ ;
- (ii)  $((L_1 L_2) - 1)$   $g(n_1, n_2)$  variables for all  $0 \leq n_1 < L_1$ ;  $0 \leq n_2 < L_2$  except  $(n_1, n_2) = (0, 0)$ ;
- (iii)  $T_1$  and  $T_2$ .

Note that  $U_p(n_1, n_2)$  is nothing but the plaquette variable associated with the plaquette with  $(n_1, n_2)$  as the bottom-left corner site and taken in the counterclockwise direction (see Fig. 1). The plaquette variable,  $U_p(L_1 - 1, L_2 - 1)$ , is constrained by

$$T_1 T_2 T_1^\dagger T_2^\dagger = W(L_1 L_2), \quad (11)$$

where

$$W(L_1 L_2) = \prod_{n_2=0}^{L_2-1} \left[ \prod_{n_1=L_1-1}^0 U_p(n_1, n_2) \right] \quad (12)$$

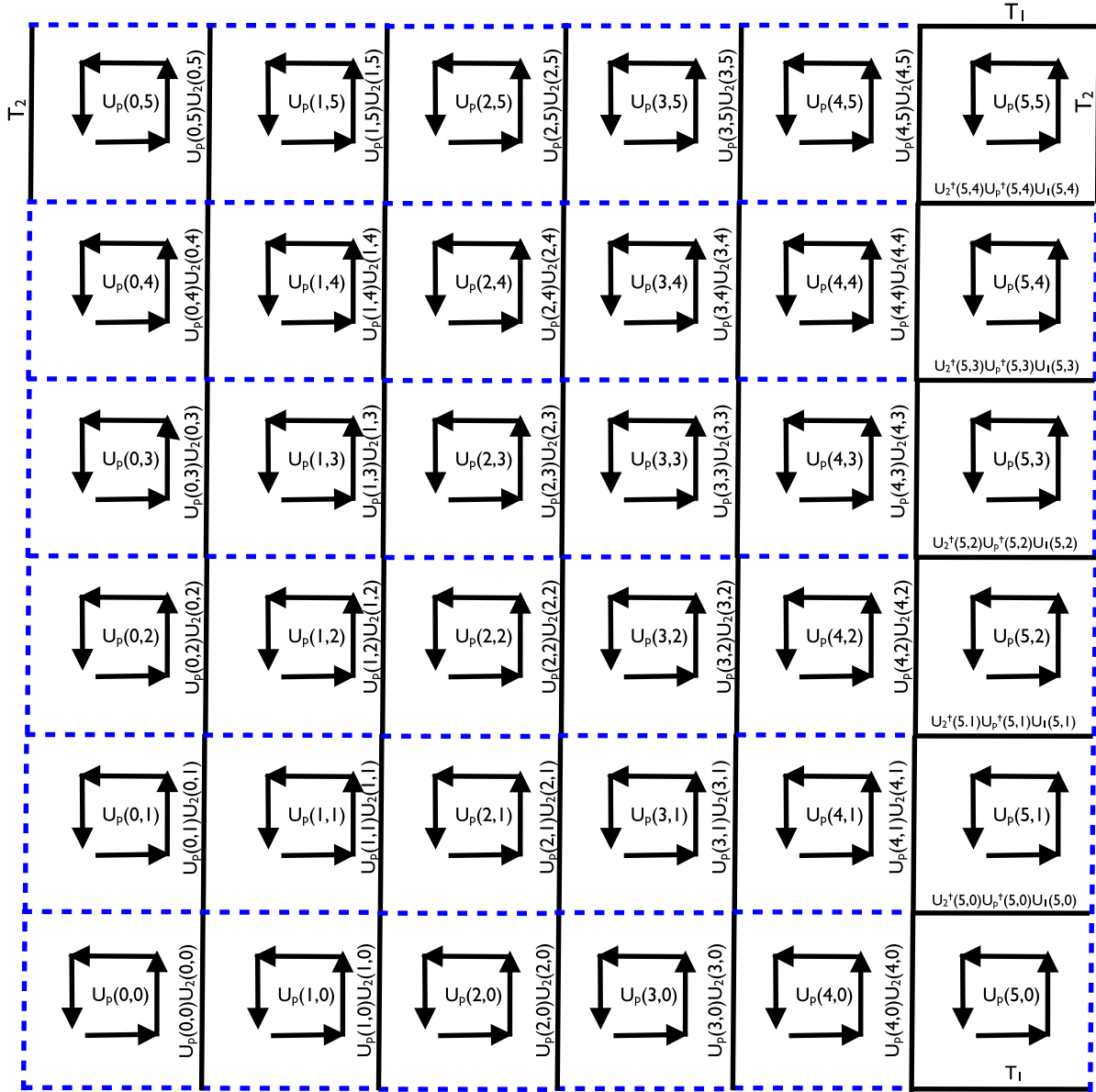


FIG. 1 (color online). A pictorial representation of the representative gauge field configuration on a  $6 \times 6$  lattice. The dashed links are set to unity. The horizontal links are in the 1 direction and are oriented from left to right. The vertical links are in the 2 direction and are oriented from bottom to top.

is the largest Wilson loop on the torus and the product is path ordered.

The next step in the definition of the model is the partition function. We assume a single plaquette action of the form

$$e^{S_g} = \prod_{n_1=0}^{L_1-1} \prod_{n_2=0}^{L_2-1} f_p[U_p(n_1, n_2); \beta], \quad (13)$$

where  $\beta$  is the dimensionless coupling constant on the lattice and  $f_p$  is a coupling dependent class function which can be expanded in terms of characters in the form

$$f_p[U; \beta] = \sum_r \tilde{\beta}_r(\beta) \chi_r(U); \quad (14)$$

with  $\tilde{\beta}_r(\beta) = \tilde{\beta}_r(\beta)$  and real. The continuum limit at a fixed physical coupling,  $g^2$ , is obtained by setting  $\beta = \frac{1}{g^2 a^2}$  and taking the lattice spacing,  $a \rightarrow 0$ . We will keep the size of the torus fixed as we take the continuum limit by setting the dimensionless area

$$A = \frac{L_1 L_2}{\beta} = (a L_1)(a L_2) g^2 \quad (15)$$

fixed as we take  $a \rightarrow 0$  and  $(L_1 L_2) \rightarrow \infty$ .

We will use all  $(L_1 L_2)$  plaquette variables in our definition of the partition function and use (11) to restrict the integral over  $T_1$  and  $T_2$ . The finite volume partition function is defined as

$$Z(\beta; L_1 L_2) = \prod_{n_1=0}^{L_1-1} \prod_{n_2=0}^{L_2-1} \int [dU_p(n_1, n_2)] f_p[U_p(n_1, n_2; \beta)] \times \int dT_1 dT_2 \delta[W(L_1 L_2), T_1 T_2 T_1^\dagger T_2^\dagger], \quad (16)$$

where [9]

$$\delta(U, V) = \sum_r d_r \chi_r(UV^\dagger) \quad (17)$$

is the delta function defined on the group with  $U, V \in SU(N)$  and the sum running over all representations,  $r$ , with  $\chi_r$  being the character and  $d_r$  being the dimension of that representation. Insertion of (17) in (16) yields the form of the partition function, (1), stated in Sec. I.

Given that  $D_{\alpha\beta}^r(U)$  is the representation of  $U$  labeled  $r$ , we have the following orthogonality relation [9]:

$$\int DUD_{\alpha\beta}^r(U) D_{\gamma\delta}^s(U^\dagger) = \delta_{rs} \frac{\delta_{\alpha\delta} \delta_{\beta\gamma}}{d_r}. \quad (18)$$

Using (17) and (18), we can perform the integrals over  $T_1$  and  $T_2$  in (16) to arrive at

$$Z(\beta; L_1 L_2) = \prod_{n_1=0}^{L_1-1} \prod_{n_2=0}^{L_2-1} \int [dU_p(n_1, n_2)] f_p[U_p(n_1, n_2; \beta)] \times \left[ \sum_r \frac{1}{d_r} \chi_r(W(L_1 L_2)) \right]. \quad (19)$$

Using the identity that follows from (18),

$$\int DUX_s(U) \chi_r(VU^\dagger W) = \delta_{sr} \frac{\chi_r(VW)}{d_r}, \quad (20)$$

we can integrate out all  $U_p(n_1, n_2)$ , one after the other, to obtain

$$Z(\beta; L_1 L_2) = \sum_r \left[ \frac{\tilde{\beta}_r(\beta)}{d_r} \right]^{L_1 L_2} = [\tilde{\beta}_0(\beta)]^{L_1 L_2} \sum_r \left[ \frac{\tilde{\beta}_r(\beta)}{d_r \tilde{\beta}_0(\beta)} \right]^{L_1 L_2}. \quad (21)$$

### III. WILSON LOOPS

Consider a  $K_1 K_2$  rectangular loop with corners at  $(0,0)$ ,  $(K_1 - 1, 0)$ ,  $(0, K_2 - 1)$ , and  $(K_1 - 1, K_2 - 1)$  and with  $0 < K_1 \leq L_1 - 1$  and  $0 < K_2 \leq L_2 - 1$ . As in the case of

the physical size of the torus defined in (15), we will keep the size of the loop fixed as we take the continuum limit by setting the dimensionless area of the loop

$$X = \frac{K_1 K_2}{\beta} = (aK_1)(aK_2)g^2 \quad (22)$$

fixed as we take  $a \rightarrow 0$  and  $(L_1 L_2) \rightarrow \infty$ .

The operator is given by (see Fig. 1)

$$W(K_1 K_2) = \prod_{i_2=0}^{K_2-1} \left[ \prod_{i_1=K_1-1}^0 U_p(i_1, i_2) \right]. \quad (23)$$

Starting from (19), we have

$$Z(\beta, L_1 L_2) \frac{1}{d_r} \langle \chi_r(W(K_1 K_2)) \rangle = \prod_{n_1=0}^{L_1-1} \prod_{n_2=0}^{L_2-1} \int [dU_p(n_1, n_2)] f_p[U_p(n_1, n_2; \beta)] \times \left[ \sum_{r'} \frac{1}{d_{r'}} \chi_{r'}(W(L_1 L_2)) \right] \frac{1}{d_r} \chi_r(W(K_1 K_2)). \quad (24)$$

As in the case of the partition function, we can use (14) and (20) and integrate out all  $U_p(n_1, n_2)$  that does not appear in  $W(K_1 K_2)$  to obtain

$$Z(\beta, L_1 L_2) \frac{1}{d_r} \langle \chi_r(W(K_1 K_2)) \rangle = \sum_{r'} \left[ \frac{\tilde{\beta}_{r'}(\beta)}{d_{r'}} \right]^{L_1 L_2 - K_1 K_2} \prod_{n_1=0}^{K_1-1} \prod_{n_2=0}^{K_2-1} \int [dU_p(n_1, n_2)] \times f_p[U_p(n_1, n_2; \beta)] \frac{1}{d_{r'}} \chi_{r'}(W(K_1 K_2)) \frac{1}{d_r} \chi_r(W(K_1 K_2)). \quad (25)$$

Using the Clebsch-Gordan series [10], namely,

$$\chi_r(U) \chi_{r'}(U) = \sum_{r''} n(r, r'; r'') \chi_{r''}(U), \quad (26)$$

where  $n(r, r'; r'')$  is the number of times the representation,  $r''$ , appears in the tensor product  $r \otimes r'$ , we can perform the rest of the integrals to obtain

$$Z(\beta, L_1 L_2) \frac{1}{d_r} \langle \chi_r(W(K_1 K_2)) \rangle = \sum_{r'} \left[ \frac{\tilde{\beta}_{r'}(\beta)}{d_{r'}} \right]^{L_1 L_2 - K_1 K_2} \sum_{r''} \frac{n(r, r'; r'') d_{r''}}{d_r d_{r'}} \left[ \frac{\tilde{\beta}_{r''}(\beta)}{d_{r''}} \right]^{K_1 K_2}. \quad (27)$$

Using (21), we can write the result in the form

$$\begin{aligned} & \frac{1}{d_r} \langle \chi_r(W(K_1 K_2)) \rangle \\ &= \frac{\sum_{r'} \left[ \frac{\tilde{\beta}_{r'}(\beta)}{d_r \tilde{\beta}_0(\beta)} \right]^{L_1 L_2 - K_1 K_2} \sum_{r''} \frac{n(r, r'; r'') d_{r''}}{d_r d_{r'}} \left[ \frac{\tilde{\beta}_{r''}(\beta)}{d_{r''} \tilde{\beta}_0(\beta)} \right]^{K_1 K_2}}{\sum_{r'} \left[ \frac{\tilde{\beta}_{r'}(\beta)}{d_r \tilde{\beta}_0(\beta)} \right]^{L_1 L_2}}. \end{aligned} \quad (28)$$

One can proceed further and compute the correlations of multiple Wilson loops where no two loops have a single plaquette in common and show that the correlations do not depend on the separation. This is a consequence of the form of the partition function in (1) where all plaquettes are independent except for a global constraint that only depends on the area.

#### IV. POLYAKOV LOOPS

In order to consider the correlation between Polyakov loops oriented in different directions, we start from (16) and (17) and consider expectation values of the form

$$\begin{aligned} & Z(\beta, L_1 L_2) \langle \chi_{r_1}(T_1) \chi_{r_2}(T_2) \rangle \\ &= \prod_{n_1=0}^{L_1-1} \prod_{n_2=0}^{L_2-1} \int [dU_p(n_1, n_2)] f_p[U_p(n_1, n_2); \beta] \\ & \times \int dT_1 dT_2 \left\{ \sum_r d_r \chi_r[W(L_1 L_2) T_2 T_1 T_2^\dagger T_1^\dagger] \right\} \\ & \times \chi_{r_1}(T_1) \chi_{r_2}(T_2). \end{aligned} \quad (29)$$

We can use (14) and (20) and integrate out all  $U_p(n_1, n_2)$  to obtain

$$Z(\beta, L_1 L_2) \langle \chi_{r_1}(T_1) \chi_{r_2}(T_2) \rangle = \sum_r a(r_1, r_2; r) \left[ \frac{\tilde{\beta}_r(\beta)}{d_r} \right]^{L_1 L_2}, \quad (30)$$

where

$$a(r_1, r_2; r) = d_r \int dT_1 dT_2 \chi_r[T_2 T_1 T_2^\dagger T_1^\dagger] \chi_{r_1}(T_1) \chi_{r_2}(T_2) \quad (31)$$

are real coefficients. Using (21), we can write the expectation value of the Polyakov loops in the form

$$M_{r_1 r_2}(A) = \langle \chi_{r_1}(T_1) \chi_{r_2}(T_2) \rangle = \frac{\sum_r a(r_1, r_2; r) \left[ \frac{\tilde{\beta}_r(\beta)}{d_r \tilde{\beta}_0(\beta)} \right]^{L_1 L_2}}{\sum_r \left[ \frac{\tilde{\beta}_r(\beta)}{d_r \tilde{\beta}_0(\beta)} \right]^{L_1 L_2}}. \quad (32)$$

Since

$$a(r_2, r_1; r) = a(r_1, r_2; \bar{r}), \quad (33)$$

it follows that  $M(A)$  is a real symmetric matrix.

#### V. CONTINUUM LIMIT

In order to take the continuum limit, we need to take a specific lattice action. Since the continuum limit will not depend on the specific choice as long as it satisfies some essential properties, the simplest choice is the heat kernel action given by [9]

$$\tilde{\beta}_r(\beta) = d_r e^{-\frac{C_r^{(2)}}{N\beta}}, \quad (34)$$

where  $C_r^{(2)}$  is the quadratic Casimir in the  $r$  representation. In this case,  $\tilde{\beta}_0(\beta) = 1$  and

$$\lim_{a \rightarrow 0} \left[ \frac{\tilde{\beta}_r(\frac{1}{g^2 a^2})}{d_r \tilde{\beta}_0(\frac{1}{g^2 a^2})} \right]^{\frac{Y}{g^2 a^2}} = e^{-\frac{1}{N} C_r^{(2)} Y}. \quad (35)$$

- (1) The continuum limit of the partition function, (21), is

$$Z(A) = \sum_r e^{-\frac{1}{N} C_r^{(2)} A}, \quad (36)$$

as stated in Sec. I.

- (2) The continuum limit of the expectation value of the Wilson loop, (28), is

$$\begin{aligned} & \frac{1}{d_r} \langle \chi_r(W(X, A)) \rangle \\ &= \frac{\sum_{r', r''} \frac{n(r, r'; r'') d_{r''}}{d_r d_{r'}} e^{-\frac{1}{N} C_{r'}^{(2)} (A-X) - \frac{1}{N} C_{r''}^{(2)} X}}{\sum_r e^{-\frac{1}{N} C_r^{(2)} A}}, \end{aligned} \quad (37)$$

as stated in Sec. I.

- (a) Since all  $C_r^{(2)} > 0$  for  $r \neq 0$ , it follows that

$$\frac{1}{d_r} \langle \chi_r(W(X, \infty)) \rangle = e^{-\frac{1}{N} C_r^{(2)} X}, \quad (38)$$

which shows Casimir scaling of the string tension in the infinite area limit.

- (b) Since

$$\sum_{r''} n(r, r'; r'') d_{r''} = d_r d_{r'}, \quad (39)$$

it follows that

$$\frac{1}{d_r} \langle \chi_r(W(0, A)) \rangle = 1. \quad (40)$$

(c) For the special case of  $X = A$ , we have

$$\frac{1}{d_r} \langle \chi_r(W(A, A)) \rangle = \frac{\sum_{r', r''} \frac{n(r, r'; r'') d_{r''}}{d_r d_{r'}} e^{-\frac{1}{N} C_{r''}^{(2)} A}}{\sum_{r'} e^{-\frac{1}{N} C_{r'}^{(2)} A}}. \quad (41)$$

In the limit of  $A \rightarrow \infty$ , only  $r' = \bar{r}$  contributes to the numerator and we have

$$\frac{1}{d_r} \langle \chi_r(W(\infty, \infty)) \rangle = \frac{1}{d_r^2}. \quad (42)$$

(3) The continuum limit of the correlation of Polyakov loops oriented in two different directions, (32), is

$$\begin{aligned} M_{r_1 r_2}(A) &= \langle \chi_{r_1}(T_1(A)) \chi_{r_2}(T_2(A)) \rangle \\ &= \frac{\sum_r a(r_1, r_2; r) e^{-\frac{1}{N} C_r^{(2)} A}}{\sum_r e^{-\frac{1}{N} C_r^{(2)} A}}, \end{aligned} \quad (43)$$

as stated in Sec. I.

(d) In the limit of  $A \rightarrow \infty$ , only  $r = 0$  contributes to the sum in the numerator and denominator. For this special case, it follows from (6) that  $a(r_1, r_2; 0) = \delta_{r_1 0} \delta_{r_2 0}$ . Therefore,

$$M_{r_1 r_2}(\infty) = \delta_{r_1 0} \delta_{r_2 0}. \quad (44)$$

It has one eigenvalue equal to unity and all other eigenvalues are zero.

(e) For the special case of  $r_2 = 0$  (or  $r_1 = 0$ ), we have

$$M_{r_1 0}(A) = \frac{\sum_r n(r, r_1; r) e^{-\frac{1}{N} C_r^{(2)} A}}{\sum_r e^{-\frac{1}{N} C_r^{(2)} A}}. \quad (45)$$

## VI. EIGENVALUES OF $M$ FOR THE CASE OF $SU(2)$

The representations of  $SU(2)$  are labeled by  $s \geq 0$  with  $s$  being an integer or a half-integer. The matrix elements obtained in (43) become

$$M_{s_1 s_2}(A) = \frac{\sum_s a(s_1, s_2; s) e^{-\frac{s(s+1)}{2} A}}{\sum_s e^{-\frac{s(s+1)}{2} A}}. \quad (46)$$

The selection rules for  $a(s_1, s_2; s)$  defined in (31) imply that  $s_1$  and  $s_2$  have to be integers. Furthermore, for a given  $s$ ,  $a(s_1, s_2; s)$  can be nonzero only if  $0 \leq s_1, s_2 \leq 2s$ . Therefore, if we restrict the sum in the numerator of

(46) to  $s \leq S$ , then we have a finite-dimensional matrix of size  $(2S + 1) \times (2S + 1)$ . The integral involved in the evaluation of  $a(s_1, s_2; s)$  defined in (31) can be computed using Clebsch-Gordan coefficients but we found it easier to perform a numerical integration by explicitly writing out  $T_1$  and  $T_2$  in a fixed choice of coordinates. We can work in a gauge where  $T_1$  is diagonal. Working in the fundamental representation, we have

$$\begin{aligned} T_1 &= \begin{pmatrix} e^{i\eta_1} & 0 \\ 0 & e^{-i\eta_1} \end{pmatrix} \quad \eta_1 \in [0, \pi]; \\ T_2 &= \begin{pmatrix} \cos \theta_2 e^{i\alpha_2} & \sin \theta_2 e^{i\beta_2} \\ -\sin \theta_2 e^{-i\beta_2} & \cos \theta_2 e^{-i\alpha_2} \end{pmatrix} \quad \theta_2 \in \left[0, \frac{\pi}{2}\right]; \\ \alpha_2, \beta_2 &\in [0, 2\pi]. \end{aligned} \quad (47)$$

The eigenvalues of  $T_2$  are

$$e^{\pm i\eta_2}; \quad \cos \eta_2 = \cos \theta_2 \cos \alpha_2; \quad \eta_2 \in [0, \pi]. \quad (48)$$

The eigenvalues of  $(T_2 T_1 T_2^\dagger T_1^\dagger)$  are

$$e^{\pm i\eta}; \quad \cos \eta = 1 - 2\sin^2 \theta_2 \sin^2 \eta_1; \quad \eta \in [0, \pi]. \quad (49)$$

The explicit result for (31) is

$$\begin{aligned} a(s_1, s_2; s) &= \frac{2(2s+1)}{\pi} \int_0^\pi d\eta_1 \sin \eta_1 \sin(2s_1 + 1)\eta_1 \\ &\quad \times \int_0^{\frac{\pi}{2}} d\theta_2 \sin 2\theta_2 \frac{\sin(2s+1)\eta}{\sin \eta} \\ &\quad \times \int_0^{2\pi} \frac{d\alpha_2}{2\pi} \frac{\sin(2s_2 + 1)\eta_2}{\sin \eta_2}. \end{aligned} \quad (50)$$

Numerical results show that  $|a(s_1, s_2; s)| \leq 1$  and therefore it follows that every entry in the matrix,  $M(A)$ , is in the range  $[-1, 1]$ . If we restrict the sum in the numerator of (46) to  $s \leq S$ , then we have a finite-dimensional matrix which we can diagonalize and compute all the eigenvalues. These eigenvalues will converge to the correct results and the convergence will be slower for smaller  $A$ . The converged results in the range of  $A \geq 10^{-3}$  are plotted Fig. 2. The eigenvalues diverge as  $A \rightarrow 0$ .

## VII. DISCUSSION

In this paper, we have studied two-dimensional non-Abelian gauge theories on a torus. There is a global constraint on the plaquette variables induced by the geometry of the torus and it only depends on the area of the torus. We explored the area dependence on physical observables. After showing consistency with previously known results, we studied the correlation of two Polyakov loops oriented in two different directions on a finite torus.

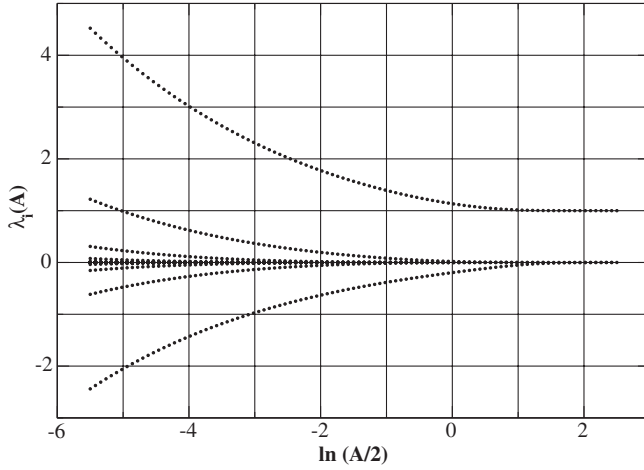


FIG. 2. A plot of  $\lambda_i(A)$  as a function of  $A$ .

This quantity also only depends on the dimensionless area,  $A$ . Correlations of Polyakov loops in representations  $r_1$  and  $r_2$ ,  $M_{r_1 r_2}(A)$ , is a real symmetric matrix. In the large area limit,  $M_{00}(\infty) = 1$  and all others are zero. This says that insertion of Polyakov loops in any nontrivial representation costs infinite amount of energy. The matrix,  $M(A)$ , for  $SU(2)$  at finite  $A$  has every entry in the range  $[-1, 1]$ . Upon diagonalization at a fixed  $A$ , we have new normalized eigenvectors of the form

$$\xi_i(T(A)) = \sum_s b_i^s(A) \chi_s(T(A)), \quad i = 0, 1, \dots \quad (51)$$

with corresponding eigenvalues,  $\lambda_i(A)$ , satisfying  $\lambda_i(A) > \lambda_{i+1}(A)$ . Each eigenvector,  $\xi_i(\theta; A)$ , is an even function of  $\theta \in [-\pi, \pi]$  where  $e^{\pm i\theta}$  are the eigenvalues of  $T(A)$  in the fundamental representation. The eigenvectors are normalized according to

$$\frac{2}{\pi} \int_0^\pi d\theta \sin^2 \theta \xi_i(\theta; A) \xi_j(\theta; A) = \delta_{ij}. \quad (52)$$

Only integer valued  $s$  contribute to the sum and therefore,  $\xi_j(\theta; A) = \xi_j(\pi - \theta; A)$ .

The plot of the eigenvalues  $\lambda_i(A)$  shown in Fig. 2 has two main features:

- (i) There is one eigenvalue,  $\lambda_0(A) > 1$ , for all finite  $A$  and it approaches unity as  $A \rightarrow \infty$ .
- (ii) All other eigenvalues are less than  $\lambda_0(A)$  in magnitude and approach zero as  $A \rightarrow \infty$ .

Since the expectation value of  $\xi_0(T_1(A))\xi_0(T_2(A))$  is greater than unity, the true vacuum of the theory contains the insertion of this operator. Viewed as a function of  $\theta$ ,  $\xi_0(\theta; A)$  will develop a peak at  $\theta = 0$  as we decrease  $A$  from infinity.

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