Three-dimensional $\mathcal{N} = 2$ supergravity theories: From superspace to components

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For general off-shell $\mathcal{N} = 2$ supergravity-matter systems in three spacetime dimensions, a formalism is developed to reduce the corresponding actions from superspace to components. The component actions are explicitly computed in the cases of type I and type II minimal supergravity formulations. We describe the models for topologically massive supergravity which correspond to all the known off-shell formulations for three-dimensional $\mathcal{N} = 2$ supergravity. We also present a universal setting to construct supersymmetric backgrounds associated with these off-shell supergravities.

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I. INTRODUCTION

The simplest way to construct $\mathcal{N} = 2$ locally supersymmetric systems in three spacetime dimensions (3D) is perhaps through dimensional reduction from 4D $\mathcal{N} = 1$ theories (see [1–3] for reviews). However, not all 3D theories with four supercharges can be obtained in this way. For instance, $\mathcal{N} = 2$ conformal supergravity [4] and (2,0) anti-de Sitter (AdS) supergravity[1] [5] cannot be so constructed. A more systematic approach to generate 3D $\mathcal{N} = 2$ supergravity-matter systems is clearly desirable.

Matter couplings in three-dimensional $\mathcal{N} = 2$ supergravity were thoroughly studied in the 1990s using on-shell component approaches [6–8] (see also [9]). More recently, off-shell formulations for general $\mathcal{N} = 2$ supergravity-matter systems have systematically been developed [10,11] purely within the superspace framework, extending earlier off-shell constructions [12–14]. One of the main goals of this paper is to work out techniques to reduce any manifestly $\mathcal{N} = 2$ locally supersymmetric theory presented in [10,11] to components. Upon elimination of the auxiliary fields, one naturally reproduces the partial component results obtained earlier in [6–8].

The prepotential formulation for 3D $\mathcal{N} = 2$ conformal supergravity was constructed in [15]. In principle, this prepotential solution could be obtained by off-shell dimensional reduction from 4D $\mathcal{N} = 1$ conformal supergravity following the procedure sketched in Sec. 7.2 of *Superspace* [2]. In practice, however, it is more advantageous to follow a manifestly covariant approach and derive the solution from scratch. In this sense the 3D story is similar to that of $\mathcal{N} = (2, 2)$ supergravity in two dimensions [16,17].

Similarly to $\mathcal{N} = 1$ supergravity in four dimensions (see [2,3,18,19] for more details), different off-shell formulations for 3D $\mathcal{N} = 2$ Poincaré and AdS supergravity theories in superspace can be obtained by coupling conformal supergravity to different conformal compensators [10,11]. There are three inequivalent types of conformal compensator: (i) a chiral scalar; (ii) a real linear scalar; and (iii) a (deformed) complex linear scalar.

Choosing the chiral compensator leads to the type I minimal supergravity [11] which is the 3D analogue of the old minimal formulation for 4D $\mathcal{N} = 1$ supergravity [20]. As in four dimensions, this formulation can be used to realize both Poincaré and AdS supergravity theories; the latter actually describes the so-called (1,1) AdS supergravity, following the terminology of [5].

Choosing the real linear compensator leads to the type II minimal supergravity [11] which is a natural extension of the new minimal formulation for 4D $\mathcal{N} = 1$ supergravity [21]. Unlike the four-dimensional case, the type II formulation is suitable to realize both Poincaré and AdS supergravity theories (the new minimal formulation cannot be used to describe 4D $\mathcal{N} = 1$ AdS supergravity). The point is that in three dimensions the real linear superfield is the field strength of an Abelian vector multiplet, and the corresponding Chern-Simons terms may be interpreted as a

[1]In three dimensions, $\mathcal{N}$-extended AdS supergravity exists in $[\mathcal{N}/2] + 1$ different versions [5], with $[\mathcal{N}/2]$ the integer part of $\mathcal{N}/2$. These were called the (p, q) AdS supergravity theories where the non-negative integers $p \geq q$ are such that $\mathcal{N} = p + q$. These theories are naturally associated with the 3D AdS supergroups $\mathrm{OSp}(p|2;\mathbb{R}) \times \mathrm{OSp}(q|2;\mathbb{R})$.

cosmological term [14]. Adding such a Chern-Simons term to the supergravity action results in the action for (2,0) AdS supergravity.

Finally, choosing the complex linear compensator leads to the nonminimal supergravity presented in [11]. It is analogous to the nonminimal formulation for 4D $\mathcal{N}=1$ supergravity [22,23], the oldest off-shell locally supersymmetric theory. Both in three and four dimensions, this formulation exists in several versions labeled by a real parameter $n \neq -1/3$, 0 in the 4D case [23] or, more conveniently, by $w = (1-n)/(3n+1)$ in the 3D case [11]. The reason for such a freedom is that the super-Weyl transformation of the complex linear compensator is not fixed uniquely [10]. With the standard constraint

$$(\bar{\mathcal{D}}^2 - 4R)\Sigma = 0 \tag{1.1}$$

obeyed by the complex linear compensator Σ, the 4D $\mathcal{N}=1$ nonminimal formulation is only suitable, for any value of n, to describe Poincaré supergravity [2]. The situation in the 3D case is completely similar [11]. However, it was shown in [24] that $n=-1$ nonminimal supergravity can be used to describe 4D $\mathcal{N}=1$ AdS supergravity provided the constraint (1.1) is replaced with a deformed one,[2]

$$(\bar{\mathcal{D}}^2 - 4R)\Gamma = -4\mu \neq 0, \qquad \mu = \text{const.} \tag{1.2}$$

Applying the same ideas in the 3D case gives us the nonminimal formulation for (1,1) AdS supergravity [11].

All supergravity-matter actions introduced in [10,11] are realized as integrals over the full superspace or over its chiral subspace. The most economical way to reduce such an action to components consists in recasting it as an integral of a closed super three-form over spacetime (that is, the bosonic body of the full superspace), in the spirit of the superform approach[3] to the construction of supersymmetric invariants [25–28]. The required superform construction is given in Sec. III.

In this paper, we work out the component supergravity-matter actions in the cases of type I and type II minimal supergravity formulations.[4] The case of nonminimal supergravity can be treated in a similar way. As an application, we describe off-shell models for topologically massive $\mathcal{N}=2$ supergravity[5] which correspond to all the known off-shell formulations for three-dimensional $\mathcal{N}=2$ supergravity. However, the component actions for topologically massive supergravity are given only for the type I and type II minimal formulations.

Recently, supersymmetric backgrounds in the type II supergravity have been studied within the component approach, both in the Euclidean [34] and Lorentzian [35] signatures, building on the earlier results in four and five dimensions; see [36–49] and references therein. Since the authors of [34,35] did not have access to the complete off-shell component actions for type II supergravity and its matter couplings, their analysis was based either on the considerations of linearized supergravity [34] or on the dimensional reduction $4D \to 3D$ of the new minimal supergravity [35]. Here we present a universal setting to construct supersymmetric backgrounds associated with all the known off-shell formulations for 3D $\mathcal{N}=2$ supergravity, that is the type I and type II minimal and the nonminimal supergravity theories.[6] Our approach will be an extension of the 4D $\mathcal{N}=1$ formalism to determine (conformal) isometries of curved superspaces which was originally developed almost 20 years ago in [3] and further elaborated in [51].[7]

This paper is organized as follows. In Sec. II we review the superspace formulation for the Weyl multiplet of $\mathcal{N}=2$ conformal supergravity, following [10,11,14].[8] In Sec. III we present the locally supersymmetric and super-Weyl invariant action principle which is based on a closed super three-form. The formalism for component reduction, including the important Weyl multiplet gauge, is worked out in Sec. IV. The component actions for type I and type II supergravity-matter systems are derived in Secs. V and VI respectively. In Sec. VII we study the off-shell formulations for topologically massive $\mathcal{N}=2$ supergravity. Section VIII is devoted to the construction of supersymmetric backgrounds in all the known off-shell formulations for $\mathcal{N}=2$ supergravity.

The main body of the paper is accompanied by four appendixes. In Appendix A we give a summary of the notation and conventions used as well as include some technical relations. In Appendix B we give an alternative form for the component action of the most general off-shell nonlinear σ-model in type I supergravity. Appendix C contains the component Lagrangian for the model of an Abelian vector multiplet in conformal supergravity. Appendix D is devoted to the superspace action for $\mathcal{N}=2$ conformal supergravity; at the component level,

[2]The constraint (1.2) is super-Weyl invariant if and only if $n=-1$.

[3]It is also known as the rheonomic approach [25] or the ectoplasm formalism [26,27].

[4]Various aspects of the component reduction in 4D $\mathcal{N}=1$ supergravity theories were studied in the late 1970s [23,29–31]. More complete presentations were given in the textbooks [1–3].

[5]Topologically massive $\mathcal{N}=1$ supergravity was introduced in [32,33]. Its $\mathcal{N}=2$ extended version was discussed in [4].

[6]After our work was completed, there appeared a new paper in the hep-th archive [50] which also studied supersymmetric backgrounds in type I supergravity.

[7]This approach has been used to construct rigid supersymmetric field theories in 5D $\mathcal{N}=1$ [52], 4D $\mathcal{N}=2$ [53,54], and 3D (p,q) anti-de Sitter [11,55,56] superspaces.

[8]There exists a more general off-shell formulation for $\mathcal{N}=2$ conformal supergravity [57]. It will be briefly reviewed in Appendix D.

this action reduces to that constructed many years ago by Roček and van Nieuwenhuizen [4].

II. THE WEYL MULTIPLET IN U(1) SUPERSPACE

In this section we recall the superspace description of $\mathcal{N}=2$ conformal supergravity. The results given here are essential for the rest of the paper.

A. U(1) superspace geometry

We consider a curved superspace in three spacetime dimensions, $\mathcal{M}^{3|4}$, parametrized by local bosonic (x^m) and fermionic $(\theta^\mu, \bar\theta_\mu)$ coordinates $z^M = (x^m, \theta^\mu, \bar\theta_\mu)$, where $m = 0, 1, 2$ and $\mu = 1, 2$. The Grassmann variables θ^μ and $\bar\theta_\mu$ are related to each other by complex conjugation: $\overline{\theta^\mu} = \bar\theta^\mu$. The superspace structure group is chosen to be $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)_R$, and the covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha)$ have the form

$$\mathcal{D}_A = E_A + \Omega_A + \mathrm{i}\Phi_A \mathcal{J}. \tag{2.1}$$

Here $E_A = (E_a, E_\alpha, \bar E^\alpha) = E_A{}^M(z)\partial/\partial z^M$ is the inverse superspace vielbein,

$$\Omega_A = \frac{1}{2}\Omega_A{}^{bc}\mathcal{M}_{bc} = \frac{1}{2}\Omega_A{}^{\beta\gamma}\mathcal{M}_{\beta\gamma} = -\Omega_A{}^c\mathcal{M}_c \tag{2.2}$$

is the Lorentz connection, and Φ_A is the $\mathrm{U}(1)_R$ connection. The Lorentz generators with two vector indices ($\mathcal{M}_{ab} = -\mathcal{M}_{ba}$), one vector index ($\mathcal{M}_a$), and two spinor indices ($\mathcal{M}_{\alpha\beta} = \mathcal{M}_{\beta\alpha}$) are related to each other as follows:

$$\mathcal{M}_a = \frac{1}{2}\varepsilon_{abc}\mathcal{M}^{bc}, \qquad \mathcal{M}_{ab} = -\varepsilon_{abc}\mathcal{M}^c,$$
$$\mathcal{M}_{\alpha\beta} = (\gamma^a)_{\alpha\beta}\mathcal{M}_a, \qquad \mathcal{M}_a = -\frac{1}{2}(\gamma_a)^{\alpha\beta}\mathcal{M}_{\alpha\beta}.$$

The Levi-Civita tensor ε_{abc} and the gamma matrices $(\gamma_a)_{\alpha\beta}$ are defined in Appendix A. The generators of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)_R$ act on the covariant derivatives as follows:

$$[\mathcal{J}, \mathcal{D}_\alpha] = \mathcal{D}_\alpha, \qquad [\mathcal{J}, \bar{\mathcal{D}}^\alpha] = -\bar{\mathcal{D}}^\alpha, \qquad [\mathcal{J}, \mathcal{D}_a] = 0,$$
$$[\mathcal{M}_{\alpha\beta}, \mathcal{D}_\gamma] = \varepsilon_{\gamma(\alpha}\mathcal{D}_{\beta)}, \qquad [\mathcal{M}_{\alpha\beta}, \bar{\mathcal{D}}_\gamma] = \varepsilon_{\gamma(\alpha}\bar{\mathcal{D}}_{\beta)},$$
$$[\mathcal{M}_{ab}, \mathcal{D}_c] = 2\eta_{c[a}\mathcal{D}_{b]}. \tag{2.3}$$

The supergravity gauge group includes local $\mathcal{K}$ transformations of the form

$$\delta_\mathcal{K}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A], \qquad \mathcal{K} = \xi^C\mathcal{D}_C + \frac{1}{2}K^{cd}\mathcal{M}_{cd} + \mathrm{i}\tau\mathcal{J}, \tag{2.4}$$

where the gauge parameters obey natural reality conditions, but are otherwise arbitrary. Given a tensor superfield $U(z)$, with its indices suppressed, it transforms as follows:

$$\delta_\mathcal{K} U = \mathcal{K}U. \tag{2.5}$$

The covariant derivatives obey (anti-)commutation relations of the form

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C\mathcal{D}_C + \frac{1}{2}R_{AB}{}^{cd}\mathcal{M}_{cd} + \mathrm{i}R_{AB}\mathcal{J}, \tag{2.6}$$

where $T_{AB}{}^C$ is the torsion, and $R_{AB}{}^{cd}$ and R_{AB} constitute the curvature tensors.

Unlike the 4D case, the spinor covariant derivatives $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ transform in the same representation of the Lorentz group, and this may lead to misunderstandings. If there is a risk for confusion, we will underline the spinor indices associated with the covariant derivatives $\bar{\mathcal{D}}$. For instance, when the index C of the torsion $T_{AB}{}^C$ takes spinor values, we will write the corresponding components as $T_{AB}{}^\gamma$ and $T_{AB\underline{\gamma}}$.

In order to describe $\mathcal{N}=2$ conformal supergravity, the torsion has to obey the covariant constraints given in [14]. The resulting algebra of covariant derivatives is [10,11]

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar R\mathcal{M}_{\alpha\beta}, \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 4R\mathcal{M}_{\alpha\beta}, \tag{2.7a}$$

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = -2\mathrm{i}(\gamma^c)_{\alpha\beta}\mathcal{D}_c - 2\mathcal{C}_{\alpha\beta}\mathcal{J} - 4\mathrm{i}\varepsilon_{\alpha\beta}\mathcal{S}\mathcal{J} + 4\mathrm{i}\mathcal{S}\mathcal{M}_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta}, \tag{2.7b}$$

$$\begin{aligned}[\mathcal{D}_a, \mathcal{D}_\beta] ={}& \mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma\mathcal{C}^c\mathcal{D}_\gamma + (\gamma_a)_\beta{}^\gamma\mathcal{S}\mathcal{D}_\gamma - \mathrm{i}(\gamma_a)_{\beta\gamma}\bar R\bar{\mathcal{D}}^\gamma - (\gamma_a)_\beta{}^\gamma\boldsymbol{C}_{\gamma\delta\rho}\mathcal{M}^{\delta\rho} - \frac{1}{3}(2\mathcal{D}_\beta\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_\beta\bar R)\mathcal{M}_a\\ &- \frac{2}{3}\varepsilon_{abc}(\gamma^b)_\beta{}^\alpha(2\mathcal{D}_\alpha\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_\alpha\bar R)\mathcal{M}^c - \frac{1}{2}\left((\gamma_a)^{\alpha\gamma}\boldsymbol{C}_{\alpha\beta\gamma} + \frac{1}{3}(\gamma_a)_\beta{}^\gamma(8\mathcal{D}_\gamma\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_\gamma\bar R)\right)\mathcal{J},\end{aligned} \tag{2.7c}$$

$$\begin{aligned}[\mathcal{D}_a, \bar{\mathcal{D}}_\beta] ={}& -\mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma\mathcal{C}^c\bar{\mathcal{D}}_\gamma + (\gamma_a)_\beta{}^\gamma\mathcal{S}\bar{\mathcal{D}}_\gamma - \mathrm{i}(\gamma_a)_\beta{}^\gamma R\mathcal{D}_\gamma - (\gamma_a)_\beta{}^\gamma\bar{\boldsymbol{C}}_{\gamma\delta\rho}\mathcal{M}^{\delta\rho} - \frac{1}{3}(2\bar{\mathcal{D}}_\beta\mathcal{S} - \mathrm{i}\mathcal{D}_\beta R)\mathcal{M}_a\\ &- \frac{2}{3}\varepsilon_{abc}(\gamma^b)_\beta{}^\alpha(2\bar{\mathcal{D}}_\alpha\mathcal{S} - \mathrm{i}\mathcal{D}_\alpha R)\mathcal{M}^c + \frac{1}{2}\left((\gamma_a)^{\alpha\gamma}\bar{\boldsymbol{C}}_{\alpha\beta\gamma} + \frac{1}{3}(\gamma_a)_\beta{}^\gamma(8\bar{\mathcal{D}}_\gamma\mathcal{S} - \mathrm{i}\mathcal{D}_\gamma R)\right)\mathcal{J},\end{aligned} \tag{2.7d}$$

$$[\mathcal{D}_a,\mathcal{D}_b]=\frac{1}{2}\varepsilon_{abc}(\gamma^c)^{\alpha\beta}\varepsilon^{\gamma\delta}\left(-\mathrm{i}\bar{\boldsymbol{C}}_{\alpha\beta\delta}+\frac{4\mathrm{i}}{3}\varepsilon_{\delta(\alpha}\bar{\mathcal{D}}_{\beta)}\mathcal{S}+\frac{2}{3}\varepsilon_{\delta(\alpha}\mathcal{D}_{\beta)}R\right)\mathcal{D}_\gamma+\frac{1}{2}\varepsilon_{abc}(\gamma^c)^{\alpha\beta}\varepsilon^{\gamma\delta}\left(-\mathrm{i}\boldsymbol{C}_{\alpha\beta\delta}+\frac{4\mathrm{i}}{3}\varepsilon_{\delta(\alpha}\mathcal{D}_{\beta)}\mathcal{S}-\frac{2}{3}\varepsilon_{\delta(\alpha}\bar{\mathcal{D}}_{\beta)}\bar{R}\right)\bar{\mathcal{D}}_\gamma$$
$$-\varepsilon_{abc}\left(\frac{1}{4}(\gamma^c)^{\alpha\beta}(\gamma_d)^{\tau\delta}(\mathrm{i}\mathcal{D}_{(\tau}\bar{\boldsymbol{C}}_{\delta\alpha\beta)}+\mathrm{i}\bar{\mathcal{D}}_{(\tau}\boldsymbol{C}_{\delta\alpha\beta)})+\frac{1}{6}(\mathcal{D}^2R+\bar{\mathcal{D}}^2\bar{R})+\frac{2}{3}\mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathcal{S}-4\mathcal{C}^c\mathcal{C}_d-4\mathcal{S}^2-4\bar{R}R\right)\mathcal{M}^d$$
$$+\mathrm{i}\varepsilon_{abc}\left(\frac{1}{2}(\gamma^c)^{\alpha\beta}[\mathcal{D}_\alpha,\bar{\mathcal{D}}_\beta]\mathcal{S}-\varepsilon^{cef}\mathcal{D}_e\mathcal{C}_f-4\mathcal{S}\mathcal{C}^c\right)\mathcal{J}, \qquad (2.7e)$$

with $\boldsymbol{C}_{\alpha\beta\gamma}$ defined by $\boldsymbol{C}_{\alpha\beta\gamma}=-\mathrm{i}\mathcal{D}_{(\alpha}\mathcal{C}_{\beta\gamma)}$. The algebra involves three dimension-1 torsion superfields: a real scalar $\mathcal{S}$, a complex scalar R and its conjugate $\bar{R}$, and a real vector $\mathcal{C}_a$; the $\mathrm{U}(1)_R$ charge of R is -2. The torsion superfields obey differential constraints implied by the Bianchi identities. The constraints are

$$\bar{\mathcal{D}}_\alpha R=0, \qquad (2.8a)$$

$$(\bar{\mathcal{D}}^2-4R)\mathcal{S}=0, \qquad (2.8b)$$

$$\mathcal{D}_\alpha\mathcal{C}_{\beta\gamma}=\mathrm{i}\boldsymbol{C}_{\alpha\beta\gamma}-\frac{1}{3}\varepsilon_{\alpha(\beta}(\bar{\mathcal{D}}_{\gamma)}\bar{R}+4\mathrm{i}\mathcal{D}_{\gamma)}\mathcal{S}). \qquad (2.8c)$$

Equation (2.8b) means that $\mathcal{S}$ is a covariantly linear superfield. When doing explicit calculations, it is useful to deal with equivalent forms of the relations (2.7c) and (2.7d) in which the vector index of $\mathcal{D}_a$ is replaced by a pair of spinor indices. Such identities are given in Appendix A.

As an immediate application of the (anti-)commutation relations (2.7), we compute a covariantly chiral d'Alembertian. Let χ be a covariantly chiral scalar, $\bar{\mathcal{D}}_\alpha\chi=0$, of $\mathrm{U}(1)_R$ charge $-w$, that is $\mathcal{J}\chi=-w\chi$.[9] The covariantly chiral d'Alembertian $\Box_c$ is defined by

$$\Box_c\chi\coloneqq\frac{1}{16}(\bar{\mathcal{D}}^2-4R)\mathcal{D}^2\chi. \qquad (2.9)$$

By construction, the scalar $\Box_c\chi$ is covariantly chiral and has $\mathrm{U}(1)_R$ charge $-w$. It is an instructive exercise to evaluate the explicit form of $\Box_c\chi$ using the chirality of χ and the relations (2.7). The result is

$$\Box_c\chi=\Big\{\mathcal{D}^a\mathcal{D}_a+\frac{1}{2}R\mathcal{D}^2-2\mathrm{i}(1-w)\mathcal{C}^a\mathcal{D}_a+\frac{1}{2}(\mathcal{D}^\alpha R)\mathcal{D}_\alpha$$
$$+2\mathrm{i}(1-w)(\bar{\mathcal{D}}^\alpha\mathcal{S})\mathcal{D}_\alpha+w(2-w)(\mathcal{C}^a\mathcal{C}_a+4\mathcal{S}^2)$$
$$-w\mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathcal{S}+\frac{w}{8}(\bar{\mathcal{D}}^2\bar{R}-\mathcal{D}^2R)\Big\}\chi. \qquad (2.10)$$

This relation turns out to be useful for the component reduction of locally supersymmetric sigma models to be discussed later on.

[9] The rationale for choosing the $\mathrm{U}(1)_R$ charge of χ to be negative is Eq. (2.15).

B. Super-Weyl invariance

The algebra of covariant derivatives (2.7) does not change under a super-Weyl transformation[10] of the covariant derivatives [10,11]

$$\mathcal{D}'_\alpha=\mathrm{e}^{\frac{1}{2}\sigma}(\mathcal{D}_\alpha+(\mathcal{D}^\gamma\sigma)\mathcal{M}_{\gamma\alpha}-(\mathcal{D}_\alpha\sigma)\mathcal{J}), \qquad (2.11a)$$

$$\bar{\mathcal{D}}'_\alpha=\mathrm{e}^{\frac{1}{2}\sigma}(\bar{\mathcal{D}}_\alpha+(\bar{\mathcal{D}}^\gamma\sigma)\mathcal{M}_{\gamma\alpha}+(\bar{\mathcal{D}}_\alpha\sigma)\mathcal{J}), \qquad (2.11b)$$

$$\mathcal{D}'_a=\mathrm{e}^\sigma\Big(\mathcal{D}_a-\frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\mathcal{D}_{(\gamma}\sigma)\bar{\mathcal{D}}_{\delta)}-\frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\bar{\mathcal{D}}_{(\gamma}\sigma)\mathcal{D}_{\delta)}$$
$$+\varepsilon_{abc}(\mathcal{D}^b\sigma)\mathcal{M}^c+\frac{\mathrm{i}}{2}(\mathcal{D}_\gamma\sigma)(\bar{\mathcal{D}}^\gamma\sigma)\mathcal{M}_a$$
$$-\frac{\mathrm{i}}{8}(\gamma_a)^{\gamma\delta}([\mathcal{D}_\gamma,\bar{\mathcal{D}}_\delta]\sigma)\mathcal{J}-\frac{3\mathrm{i}}{4}(\gamma_a)^{\gamma\delta}(\mathcal{D}_\gamma\sigma)(\bar{\mathcal{D}}_\delta\sigma)\mathcal{J}\Big) \qquad (2.11c)$$

accompanied by the following transformation of the torsion tensors:

$$\mathcal{S}'=\mathrm{e}^\sigma\left(\mathcal{S}-\frac{\mathrm{i}}{4}\mathcal{D}_\gamma\bar{\mathcal{D}}^\gamma\sigma\right), \qquad (2.11d)$$

$$\mathcal{C}'_a=\mathrm{e}^\sigma\left(\mathcal{C}_a+\frac{1}{8}(\gamma_a)^{\gamma\delta}[\mathcal{D}_\gamma,\bar{\mathcal{D}}_\delta]\sigma+\frac{1}{4}(\gamma_a)^{\gamma\delta}(\mathcal{D}_\gamma\sigma)\bar{\mathcal{D}}_\delta\sigma\right), \qquad (2.11e)$$

$$R'=\mathrm{e}^\sigma\left(R+\frac{1}{4}\bar{\mathcal{D}}^2\sigma-\frac{1}{4}(\bar{\mathcal{D}}_\gamma\sigma)\bar{\mathcal{D}}^\gamma\sigma\right). \qquad (2.11f)$$

The gauge group of conformal supergravity is defined to be generated by the $\mathcal{K}$ transformation (2.4) and the super-Weyl transformations. The super-Weyl invariance is the reason why the U(1) superspace geometry describes the Weyl multiplet.

Using the above super-Weyl transformation laws, it is an instructive exercise to demonstrate that the real symmetric spinor superfield [15]

[10] The super-Weyl transformation (2.11) is uniquely fixed if one (i) postulates that the components of the inverse vielbein E_A transform as $E'_\alpha=\mathrm{e}^{\frac{1}{2}\sigma}E_\alpha$ and $E'_a=\mathrm{e}^\sigma E_a+$ spinor terms; and (ii) requires that the transformed covariant derivatives preserve the constraints [14] leading to the algebra (2.7).

$$\mathcal{W}_{\alpha\beta} := \frac{\mathrm{i}}{2}[\mathcal{D}^\gamma, \bar{\mathcal{D}}_\gamma]\mathcal{C}_{\alpha\beta} - [\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]\mathcal{S} - 4\mathcal{S}\mathcal{C}_{\alpha\beta} \tag{2.12}$$

transforms homogeneously,

$$\mathcal{W}'_{\alpha\beta} = \mathrm{e}^{2\sigma}\mathcal{W}_{\alpha\beta}. \tag{2.13}$$

This superfield is the $\mathcal{N}=2$ supersymmetric generalization of the Cotton tensor. Using the Bianchi identities, one can obtain an equivalent expression for this super Cotton tensor:

$$\begin{aligned}\mathcal{W}_a &= -\frac{1}{2}(\gamma_a)^{\alpha\beta}\mathcal{W}_{\alpha\beta}\\ &= -\frac{1}{2}(\gamma_a)^{\alpha\beta}[\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]\mathcal{S} + 2\varepsilon_{abc}\mathcal{D}^b\mathcal{C}^c + 4\mathcal{S}\mathcal{C}_a. \end{aligned}\tag{2.14}$$

An application of this relation will be given in Appendix D. A curved superspace is conformally flat if and only if $\mathcal{W}_{\alpha\beta} = 0$; see [57] for the proof.

For our subsequent consideration, it is important to recall one of the results obtained in [10]. Let χ be a covariantly chiral scalar, $\bar{\mathcal{D}}_\alpha\chi = 0$, which is primary under the super-Weyl group, $\delta_\sigma\chi = w\sigma\chi$. Then its super-Weyl weight w and its $\mathrm{U}(1)_R$ charge are equal and opposite [10],

$$\bar{\mathcal{D}}_\alpha\chi = 0, \qquad \mathcal{J}\chi = -w\chi, \qquad \chi' = \mathrm{e}^{w\sigma}\chi. \tag{2.15}$$

Unlike χ itself, its chiral d'Alembertian $\Box_c\chi$, Eq. (2.10), is not a primary superfield under the super-Weyl group.

In what follows, we often consider the infinitesimal super-Weyl transformation and denote the corresponding variation by δ_σ.

III. SUPERSYMMETRIC AND SUPER-WEYL INVARIANT ACTION

There are two (closely related) locally supersymmetric and super-Weyl invariant actions in $\mathcal{N}=2$ supergravity [10]. Given a real scalar Lagrangian $\mathcal{L} = \bar{\mathcal{L}}$ with the super-Weyl transformation law

$$\delta_\sigma\mathcal{L} = \sigma\mathcal{L}, \tag{3.1}$$

the action

$$S = \int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E\mathcal{L}, \qquad E^{-1} = \mathrm{Ber}(E_A{}^M), \tag{3.2}$$

is invariant under the supergravity gauge group. It is also super-Weyl invariant due to the transformation law

$$\delta_\sigma E = -\sigma E. \tag{3.3}$$

Given a covariantly chiral scalar Lagrangian $\mathcal{L}_c$ of super-Weyl weight two,

$$\bar{\mathcal{D}}_\alpha\mathcal{L}_c = 0, \qquad \mathcal{J}\mathcal{L}_c = -2\mathcal{L}_c, \qquad \delta_\sigma\mathcal{L}_c = 2\sigma\mathcal{L}_c, \tag{3.4}$$

the following chiral action

$$S_c = \int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E\frac{\mathcal{L}_c}{R} = \int \mathrm{d}^3x\mathrm{d}^2\theta\mathcal{E}\mathcal{L}_c \tag{3.5}$$

is locally supersymmetric and super-Weyl invariant. The first representation in (3.5), which is only valid when $R \neq 0$, is analogous to that derived by Zumino [29] in 4D $\mathcal{N}=1$ supergravity. The second representation in (3.5) involves integration over the chiral subspace of the full superspace, with $\mathcal{E}$ the chiral density possessing the properties

$$\mathcal{J}\mathcal{E} = 2\mathcal{E}, \qquad \delta_\sigma\mathcal{E} = -2\sigma\mathcal{E}. \tag{3.6}$$

The explicit expression for $\mathcal{E}$ in terms of the supergravity prepotentials is given in [15]. Complex conjugating (3.5) gives the action $\bar{S}_c$ generated by the antichiral Lagrangian $\bar{\mathcal{L}}_c$.

The two actions, (3.2) and (3.5), are related to each other as follows:

$$\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E\mathcal{L} = \int \mathrm{d}^3x\mathrm{d}^2\theta\mathcal{E}\mathcal{L}_c, \qquad \mathcal{L}_c := -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{L}. \tag{3.7}$$

This relation shows that the chiral action, or its conjugate antichiral action, is more fundamental than (3.2).

The chiral action can be reduced to component fields by making use of the prepotential formulation for $\mathcal{N}=2$ conformal supergravity [15] and following the component reduction procedure developed in [3] for $\mathcal{N}=1$ supergravity in four dimensions. Being conceptually straightforward, however, this procedure is technically rather tedious and time consuming. A simpler way to reduce S_c to components consists in making use of the superform approach to the construction of supersymmetric invariants [25–28]. In conjunction with the requirement of super-Weyl invariance, the latter approach turns out to be extremely powerful. As a matter of taste, here we prefer to deal with $\bar{S}_c$, because it turns out that the corresponding closed three-form involves no one-forms $\bar{E}_\alpha$.

The super-Weyl transformation laws of the components of the superspace vielbein

$$E^A := \mathrm{d}z^M E_M{}^A = (E^a, E^\alpha, \bar{E}_\alpha), \tag{3.8}$$

are

$$\delta_\sigma E^a = -\sigma E^a, \tag{3.9a}$$

$$\delta_\sigma E^\alpha = -\frac{1}{2}\sigma E^\alpha + \frac{\mathrm{i}}{2}E^b(\gamma_b)^{\alpha\gamma}\bar{\mathcal{D}}_\gamma\sigma,$$
$$\delta_\sigma \bar{E}_\alpha = -\frac{1}{2}\sigma \bar{E}_\alpha + \frac{\mathrm{i}}{2}E^b(\gamma_b)_{\alpha\gamma}\mathcal{D}^\gamma\sigma. \qquad (3.9b)$$

We are looking for a dimensionless three-form, $\Xi(\bar{\mathcal{L}}_c) = \frac{1}{6}E^C\wedge E^B\wedge E^A\Xi_{ABC}$, such that (i) its components Ξ_{ABC} are linear functions of $\bar{\mathcal{L}}_c$ and covariant derivatives thereof; and (ii) $\Xi(\bar{\mathcal{L}}_c)$ is super-Weyl invariant, $\delta_\sigma\Xi(\bar{\mathcal{L}}_c) = 0$. Modulo an overall numerical factor, such a form is uniquely determined to be

$$\Xi(\bar{\mathcal{L}}_c) = \frac{1}{2}E^\gamma\wedge E^\beta\wedge E^a\Xi_{a\beta\gamma} + \frac{1}{2}E^\gamma\wedge E^b\wedge E^a\Xi_{ab\gamma} + \frac{1}{6}E^c\wedge E^b\wedge E^a\Xi_{abc}, \qquad (3.10)$$

where

$$\Xi_{a\beta\gamma} = 4(\gamma_a)_{\beta\gamma}\bar{\mathcal{L}}_c, \qquad (3.11a)$$

$$\Xi_{ab\gamma} = -\mathrm{i}\varepsilon_{abd}(\gamma^d)_{\gamma\delta}\bar{\mathcal{D}}^\delta\bar{\mathcal{L}}_c, \qquad (3.11b)$$

$$\Xi_{abc} = \frac{1}{4}\varepsilon_{abc}(\bar{\mathcal{D}}^2 - 16R)\bar{\mathcal{L}}_c. \qquad (3.11c)$$

It is easy to check that this three-form is closed,

$$\mathrm{d}\Xi(\bar{\mathcal{L}}_c) = 0, \qquad (3.12)$$

and therefore $\Xi(\bar{\mathcal{L}}_c)$ generates a locally supersymmetric action.

The locally supersymmetric and super-Weyl invariant action associated with $\Xi(\bar{\mathcal{L}}_c)$ is

$$\bar{S}_c = -\int \mathrm{d}^3x\, e\left[\frac{1}{4}\bar{\mathcal{D}}^2 - 4R - \frac{\mathrm{i}}{2}(\gamma^a)_{\gamma\rho}\psi_a{}^\gamma\bar{\mathcal{D}}^\rho + \frac{1}{2}\varepsilon^{abc}(\gamma_a)_{\beta\gamma}\psi_b{}^\beta\psi_c{}^\gamma\right]\bar{\mathcal{L}}_c|, \qquad (3.13)$$

with $e := \det(e_m{}^a)$. Here we have used definitions introduced in the next section.

IV. COMPONENT REDUCTION

In this section we develop a simple universal setup to carry out the component reduction of the general $\mathcal{N}=2$ supergravity-matter systems presented in [10,11]. Our consideration below is very similar to that given in standard textbooks on four-dimensional $\mathcal{N}=1$ supergravity [2,3].

Given a superfield $U(z)$ we define its bar projection $U|$ to be the θ, $\bar\theta$-independent term in the expansion of $U(x,\theta,\bar\theta)$ in powers of θ's and $\bar\theta$'s,

$$U| := U(x,\theta,\bar\theta)|_{\theta=\bar\theta=0}. \qquad (4.1)$$

Thus $U|$ is a field on the spacetime $\mathcal{M}^3$ which is the bosonic body of the curved superspace $\mathcal{M}^{3|4}$.

In a similar way we define the bar projection of the covariant derivatives:

$$\mathcal{D}_A| := E_A{}^M|\partial_M + \frac{1}{2}\Omega_A{}^{bc}|\mathcal{M}_{bc} + \mathrm{i}\Phi_A|\mathcal{J}. \qquad (4.2)$$

More generally, given a differential operator $\hat{\mathcal{O}} := \mathcal{D}_{A_1}\ldots\mathcal{D}_{A_n}$, we define its bar projection, $\hat{\mathcal{O}}|$, by the rule $(\hat{\mathcal{O}}|U)| := (\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_n}U)|$, for any tensor superfield U.

Of special importance is the bar projection of a vector covariant derivative,[11]

$$\mathcal{D}_a| = \mathbf{D}_a - \frac{1}{2}\psi_a{}^\gamma\mathcal{D}_\gamma| - \frac{1}{2}\bar\psi_{a\gamma}\bar{\mathcal{D}}^\gamma|, \qquad (4.3)$$

where

$$\mathbf{D}_a = e_a + \frac{1}{2}\omega_a{}^{bc}\mathcal{M}_{bc} + \mathrm{i}b_a\mathcal{J}, \qquad e_a := e_a{}^m\partial_m \qquad (4.4)$$

is a spacetime covariant derivative with Lorentz and $\mathrm{U}(1)_R$ connections. For some calculations, it will be useful to work with a spacetime covariant derivative without $\mathrm{U}(1)_R$ connection, $\mathfrak{D}_a$, defined by

$$\mathfrak{D}_a = \mathbf{D}_a - \mathrm{i}b_a\mathcal{J}. \qquad (4.5)$$

A. The Wess-Zumino and normal gauges

The freedom to perform general coordinate and local Lorentz transformations can be used to choose a Wess-Zumino (WZ) gauge of the form

$$\mathcal{D}_\alpha| = \delta_\alpha{}^\mu\frac{\partial}{\partial\theta^\mu}, \qquad \bar{\mathcal{D}}^\alpha| = \delta^\alpha{}_\mu\frac{\partial}{\partial\bar\theta_\mu}. \qquad (4.6)$$

In this gauge, it is easy to see that

$$E_a{}^m| = e_a{}^m, \qquad E_a{}^\mu| = -\frac{1}{2}\psi_a{}^\gamma\delta_\gamma{}^\mu, \qquad \bar{E}_{a\mu}| = -\frac{1}{2}\bar\psi_{a\gamma}\delta^\gamma{}_\mu, \qquad (4.7a)$$

$$\Omega_a{}^{bc}| = \omega_a{}^{bc}, \qquad \Phi_a| = b_a. \qquad (4.7b)$$

The gauge condition (4.6) will be used in what follows.

In the WZ gauge, we still have a tail of component fields which originates at higher orders in the θ, $\bar\theta$ expansion of $E_A{}^M$, $\Omega_A{}^{bc}$ and Φ_A and which are pure gauge (that is, they may be completely gauged away). A way to get rid of such

[11]The definition of the gravitino agrees with that used in the 4D case in [1].

a tail of redundant fields is to impose a normal gauge around the bosonic body $\mathcal{M}^3$ of the curved superspace $\mathcal{M}^{3|4}$; see [58] for more details. This gauge is defined by the conditions

$$\Theta^M E_M{}^A(x,\Theta) = \Theta^M \delta_M{}^A, \tag{4.8a}$$

$$\Theta^M \Omega_M{}^{cd}(x,\Theta) = 0, \tag{4.8b}$$

$$\Theta^M \Phi_M(x,\Theta) = 0, \tag{4.8c}$$

where we have introduced

$$\Theta^M \equiv (\Theta^m, \Theta^\mu, \bar{\Theta}_\mu) := (0, \theta^\mu, \bar{\theta}_\mu). \tag{4.9}$$

In (4.8) the connections with world indices are defined in the standard way: $\Omega_M{}^{cd} = E_M{}^A \Omega_A{}^{cd}$ and $\Phi_M = E_M{}^A \Phi_A$. It can be proved [58] that the normal gauge conditions (4.8) allow one to reconstruct the vielbein $E_M{}^A(x,\Theta)$ and the connections $\Omega_M{}^{cd}(x,\Theta)$ and $\Phi_M(x,\Theta)$ as Taylor series in Θ, in which all the coefficients [except the leading Θ-independent terms given by the relations (4.6) and (4.7)] are tensor functions of the torsion, the curvature and their covariant derivatives evaluated at $\Theta = 0$.

In principle, there is no need to introduce the normal gauge which eliminates the tail of superfluous fields. Such fields (once properly defined) are pure gauge and do not show up in the gauge-invariant action. This is similar to the concept of double-bar projection; see e.g. [59].

B. The component field strengths

The spacetime covariant derivatives $\mathbf{D}_a$ defined by (4.3) obey commutation relations of the form

$$[\mathbf{D}_a, \mathbf{D}_b] = \mathcal{T}_{ab}{}^c \mathbf{D}_c + \frac{1}{2}\mathcal{R}_{ab}{}^{cd}\mathcal{M}_{cd} + \mathrm{i}\mathcal{F}_{ab}\mathcal{J}, \tag{4.10}$$

where $\mathcal{T}_{ab}{}^c$ is the torsion, $\mathcal{R}_{ab}{}^{cd}$ the Lorentz curvature, and $\mathcal{F}_{ab}$ the $\mathrm{U}(1)_R$ field strength. These tensors can be related to the superspace geometric objects by bar projecting the (anti-)commutation relations (2.6). A short calculation gives the torsion

$$\mathcal{T}_{ab}{}^c = -\frac{\mathrm{i}}{2}(\bar{\psi}_a\gamma^c\psi_b - \bar{\psi}_b\gamma^c\psi_a). \tag{4.11}$$

The Lorentz connection is

$$\omega_{abc} = \omega_{abc}(e) + \frac{1}{2}[\mathcal{T}_{abc} - \mathcal{T}_{bca} + \mathcal{T}_{cab}], \tag{4.12}$$

where $\omega_{abc}(e)$ denotes the torsion-free connection,

$$\omega_{abc}(e) = -\frac{1}{2}[\mathcal{C}_{abc} - \mathcal{C}_{bca} + \mathcal{C}_{cab}],$$
$$\mathcal{C}_{ab}{}^c := (e_a e_b{}^m - e_b e_a{}^m)e_m{}^c. \tag{4.13}$$

For the gravitino field strength defined by

$$\boldsymbol{\psi}_{ab}{}^\gamma := \mathbf{D}_a\psi_b{}^\gamma - \mathbf{D}_b\psi_a{}^\gamma - \mathcal{T}_{ab}{}^c\psi_c{}^\gamma \tag{4.14}$$

we read off

$$\boldsymbol{\psi}_{ab}{}^\gamma = \Big(\mathrm{i}\varepsilon_{abc}(\gamma^c)^{\alpha\beta}\bar{\boldsymbol{C}}_{\alpha\beta}{}^\gamma - \frac{4\mathrm{i}}{3}\varepsilon_{abc}(\gamma^c)^{\gamma\delta}\bar{\mathcal{D}}_\delta\mathcal{S} - \frac{2}{3}\varepsilon_{abc}(\gamma^c)^{\gamma\delta}\mathcal{D}_\delta R + 2\mathrm{i}\varepsilon_{cd[a}(\gamma^c)^{\gamma\delta}\psi_{b]\delta}\mathcal{C}^d + 2(\gamma_{[a})^{\gamma\delta}\psi_{b]\delta}\mathcal{S} + 2\mathrm{i}(\gamma_{[a})^{\gamma\delta}\bar{\psi}_{b]\delta}R\Big)\Big|. \tag{4.15}$$

This tells us how the gravitino field strength is embedded in the superspace curvature and torsion. A longer calculation is to derive an explicit expression for the Lorentz curvature

$$\mathcal{R}_{ab}{}^{cd} = 2e_{[a}\omega_{b]}{}^{cd} + 2\omega_{[a}{}^{cf}\omega_{b]f}{}^d - \mathcal{C}_{ab}{}^f\omega_f{}^{cd}. \tag{4.16}$$

The result is

$$\begin{aligned}\mathcal{R}_{ab}{}^{cd} = \Big\{ &-\frac{\mathrm{i}}{4}\varepsilon_{abe}(\gamma^e)^{\alpha\beta}\varepsilon^{cdf}(\gamma_f)^{\tau\delta}(\mathcal{D}_{(\tau}\bar{\boldsymbol{C}}_{\delta\alpha\beta)} + \bar{\mathcal{D}}_{(\tau}\boldsymbol{C}_{\delta\alpha\beta)}) + \delta^c_{[a}\delta^d_{b]}\Big[\frac{1}{3}(\mathcal{D}^2 R + \bar{\mathcal{D}}^2\bar{R}) + \frac{4\mathrm{i}}{3}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathcal{S} - 8\bar{R}R - 8\mathcal{S}^2\Big] \\ &+ 4\varepsilon_{abe}\varepsilon^{cdf}\mathcal{C}^e\mathcal{C}_f + \psi_{[a}{}^\beta\Big[(\gamma_{b]})_\beta{}^\gamma\boldsymbol{C}_{\gamma\delta\rho}\varepsilon^{cde}(\gamma_e)^{\delta\rho} + \frac{1}{3}\varepsilon_{b]}{}^{cd}(2\mathcal{D}_\beta\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_\beta\bar{R}) - \frac{4}{3}\delta^{[c}_{b]}(\gamma^{d]})_\beta{}^\gamma(2\mathcal{D}_\gamma\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_\gamma\bar{R})\Big] \\ &+ \bar{\psi}_{[a\beta}\Big[(\gamma_{b]})^{\beta\gamma}\bar{\boldsymbol{C}}_{\gamma\delta\rho}\varepsilon^{cde}(\gamma_e)^{\delta\rho} + \frac{1}{3}\varepsilon_{b]}{}^{cd}(2\bar{\mathcal{D}}^\beta\mathcal{S} - \mathrm{i}\mathcal{D}^\beta R) - \frac{4}{3}\delta^{[c}_{b]}(\gamma^{d]})^{\beta\gamma}(2\bar{\mathcal{D}}_\gamma\mathcal{S} - \mathrm{i}\mathcal{D}_\gamma R)\Big] \\ &+ \varepsilon^{cde}(\gamma_e)_{\gamma\delta}\psi_{[a}{}^\gamma\psi_{b]}{}^\delta\bar{R} - \varepsilon^{cde}(\gamma_e)^{\gamma\delta}\bar{\psi}_{[a\gamma}\bar{\psi}_{b]\delta}R + 2\mathrm{i}\varepsilon^{cde}(\gamma_e)^{\gamma\delta}\psi_{[a\gamma}\bar{\psi}_{b]\delta}\mathcal{S} + 2\psi_{[a}{}^\gamma\bar{\psi}_{b]\gamma}\varepsilon^{cde}\mathcal{C}_e\Big\}\Big|. \end{aligned} \tag{4.17}$$

Finally, for the $\mathrm{U}(1)_R$ field strength

$$\mathcal{F}_{ab} = \mathfrak{D}_a b_b - \mathfrak{D}_b b_a - \mathcal{T}_{ab}{}^c b_c \tag{4.18}$$

we obtain

$$\mathcal{F}_{ab} = \varepsilon_{abc}\left\{\frac{1}{2}(\gamma^c)^{\alpha\beta}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta]\mathcal{S} - \varepsilon^{cef}\mathfrak{D}_e\mathcal{C}_f - 4\mathcal{S}\mathcal{C}^c + \mathrm{i}\varepsilon^{cef}(\gamma_e)^{\gamma\rho}\psi_{f\gamma}\mathcal{D}_\rho\mathcal{S} + \mathrm{i}\varepsilon^{cef}(\gamma_e)^{\beta\gamma}\bar{\psi}_{f\beta}\bar{\mathcal{D}}_\gamma\mathcal{S}\right\}\Big| + \mathrm{i}(\gamma_c)^{\gamma\delta}\psi_{[a_\gamma}\bar{\psi}_{b]_\delta}\mathcal{C}^c| - 2\psi_{[a}{}^\gamma\bar{\psi}_{b]_\gamma}\mathcal{S}|. \tag{4.19}$$

It turns out that the expressions for $\boldsymbol{\psi}_{ab}{}^\gamma$, $\mathcal{R}_{ab}{}^{cd}$, and $\mathcal{F}_{ab}$ drastically simplify if we also partially gauge fix the super-Weyl invariance to choose the so-called Weyl multiplet gauge that will be introduced in Sec. IV D.

C. Residual gauge transformations

In the WZ gauge, there remains a subset of gauge transformations which preserve the conditions (4.6). To work out the structure of this residual gauge freedom, we start from the transformation laws of the inverse vielbein $E_A{}^M$ and of the connections $\Omega_A{}^{cd}$ and Φ_A under the gauge group of conformal supergravity.

Under the $\mathcal{K}$ transformation (2.4), the gauge fields vary as follows:

$$\delta_\mathcal{K} E_A{}^M = \xi^C T_{CA}{}^B E_B{}^M - (\mathcal{D}_A\xi^B)E_B{}^M + \frac{1}{2}K^{cd}(\mathcal{M}_{cd})_A{}^B E_B{}^M + \mathrm{i}\tau(\mathcal{J})_A{}^B E_B{}^M, \tag{4.20a}$$

$$\delta_\mathcal{K}\Omega_A{}^{cd} = \xi^C T_{CA}{}^B\Omega_B{}^{cd} + \xi^B R_{BA}{}^{cd} - (\mathcal{D}_A\xi^B)\Omega_B{}^{cd} + K^{cd}(\mathcal{M}_{cd})_A{}^B\Omega_B{}^{cd} - (\mathcal{D}_A K^{cd}) + \mathrm{i}\tau(\mathcal{J})_A{}^B\Omega_B{}^{cd}, \tag{4.20b}$$

$$\delta_\mathcal{K}\Phi_A = \xi^C T_{CA}{}^B\Phi_B + \xi^B R_{BA} - (\mathcal{D}_A\xi^B)\Phi_B + \frac{1}{2}K^{cd}(\mathcal{M}_{cd})_A{}^B\Phi_B + \mathrm{i}\tau(\mathcal{J})_A{}^B\Phi_B - \mathcal{D}_A\tau. \tag{4.20c}$$

Here we have introduced the Lorentz and $\mathrm{U}(1)_R$ generators $(\mathcal{M}^{cd})_A{}^B$ and $(\mathcal{J})_A{}^B$, respectively, defined by

$$[\mathcal{M}^{cd}, \mathcal{D}_A] = (\mathcal{M}^{cd})_A{}^B\mathcal{D}_B, \qquad [\mathcal{J}, \mathcal{D}_A] = (\mathcal{J})_A{}^B\mathcal{D}_B.$$

The super-Weyl transformation (2.11) acts on the gauge fields as follows:

$$\delta_\sigma E_a{}^M = \sigma E_a{}^M - \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\mathcal{D}_{(\gamma}\sigma)\bar{E}_{\delta)}{}^M - \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\bar{\mathcal{D}}_{(\gamma}\sigma)E_{\delta)}{}^M, \tag{4.21a}$$

$$\delta_\sigma E_\alpha{}^M = \frac{1}{2}\sigma E_\alpha{}^M, \tag{4.21b}$$

$$\delta_\sigma\Omega_a{}^{bc} = \sigma\Omega_a{}^{bc} - \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\mathcal{D}_{(\gamma}\sigma)\bar{\Omega}_{\delta)}{}^{bc} - \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\bar{\mathcal{D}}_{(\gamma}\sigma)\Omega_{\delta)}{}^{bc} + 2(\mathcal{D}^{[b}\sigma)\delta_a^{c]}, \tag{4.21c}$$

$$\delta_\sigma\Omega_\alpha{}^{bc} = \frac{1}{2}\sigma\Omega_\alpha{}^{bc} + (\mathcal{D}^\gamma\sigma)(\gamma_a)_{\gamma\alpha}\varepsilon^{abc}, \tag{4.21d}$$

$$\delta_\sigma\Phi_a = \sigma\Phi_a - \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\mathcal{D}_{(\gamma}\sigma)\bar{\Phi}_{\delta)} - \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\bar{\mathcal{D}}_{(\gamma}\sigma)\Phi_{\delta)} - \frac{1}{8}(\gamma_a)^{\gamma\delta}[\mathcal{D}_\gamma, \bar{\mathcal{D}}_\delta]\sigma, \tag{4.21e}$$

$$\delta_\sigma\Phi_\alpha = \frac{1}{2}\sigma\Phi_\alpha + \mathrm{i}\mathcal{D}_\alpha\sigma. \tag{4.21f}$$

Requiring the WZ gauge to be preserved, $(\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_\alpha| = 0$, gives

$$\mathcal{D}_\alpha\xi^b| = \xi^C T_{C\alpha}{}^b|, \tag{4.22a}$$

$$\mathcal{D}_\alpha\xi^\beta| = \left(\xi^C T_{C\alpha}{}^\beta + \frac{1}{2}K_\alpha{}^\beta + \mathrm{i}\tau\delta_\alpha^\beta + \frac{1}{2}\sigma\delta_\alpha^\beta\right)\Big|, \tag{4.22b}$$

$$\mathcal{D}_\alpha\bar{\xi}_{\underline{\beta}}| = \xi^C T_{C\alpha\underline{\beta}}|, \tag{4.22c}$$

$$\mathcal{D}_\alpha K^{cd}| = (\xi^B R_{B\alpha}{}^{cd} + (\gamma_a)_{\alpha\gamma}\varepsilon^{acd}\mathcal{D}^\gamma\sigma)|, \tag{4.22d}$$

$$\mathcal{D}_\alpha\tau| = (\xi^B R_{B\alpha} + \mathrm{i}\mathcal{D}_\alpha\sigma)|. \tag{4.22e}$$

We see that the residual gauge transformations are constrained. More specifically, only the parameters

$$v^a := \xi^a|, \quad \epsilon^\alpha := \xi^\alpha|, \quad w_{ab} := K_{ab}|, \quad \boldsymbol{\tau} := \tau| \tag{4.23a}$$

are completely unrestricted in the WZ gauge. Here the bosonic parameters correspond to general coordinate (v^a), local Lorentz (w_{ab}) and local R-symmetry ($\boldsymbol{\tau}$) transformations; the fermionic parameter ϵ^α generates a local Q-supersymmetry transformation. However, the parameters $\mathcal{D}_\alpha\xi^A|$, $\mathcal{D}_\alpha K^{cd}|$ and $\mathcal{D}_\alpha\tau|$ are fully determined in terms of those listed in (4.23a) and the following ones:

$$\boldsymbol{\sigma} := \sigma|, \qquad \eta_\alpha := \mathcal{D}_\alpha\sigma|. \tag{4.23b}$$

Here the parameter $\boldsymbol{\sigma}$ and η_α generate the Weyl and local S-supersymmetry transformations respectively. It should be pointed out that there is no parameter generating a local special conformal transformation. As compared with the 3D $\mathcal{N}=2$ superconformal tensor calculus in superspace [57], our formulation corresponds to a gauge in which the dilatation gauge field is switched off by making use of the local special conformal transformations.

The relations (4.22) comprise all the conditions on the residual gauge transformations, which are implied by the WZ gauge. If in addition we also choose the normal gauge (4.8), then all higher-order terms in the Θ expansion of the gauge parameters will be determined in terms of those listed in (4.23).

In what follows, we will be interested in local Q-supersymmetry transformations of the gauge fields $e_m{}^a$, $\psi_m{}^\gamma = e_m{}^a\psi_a{}^\gamma$ and $b_m = e_m{}^a b_a$. Yet we introduce a more general transformation

$$\delta \coloneqq \delta_Q + \delta_S + \delta_W + \delta_R \tag{4.24}$$

which includes the local Q-supersymmetry (ϵ_α) and S-supersymmetry (η_α) transformations, as well as the Weyl ($\boldsymbol{\sigma}$) and local R-symmetry ($\boldsymbol{\tau}$) transformations. There is a simple reason for considering this combination of four transformations. As will be shown in the next two sections, in any off-shell formulation for Poincaré or AdS supergravity, the Q-supersymmetry transformation has to be accompanied by a special S-supersymmetry transformation with parameter $\eta_\alpha(\epsilon)$ and, in some case, by a special $\mathrm{U}(1)_R$ transformation with parameter $\boldsymbol{\tau}(\epsilon)$. Typically, it will hold that $\boldsymbol{\sigma}(\epsilon)=0$. However, since δ_η is part of the super-Weyl transformation, it makes sense to include δ_W into (4.24).

Making use of the relations (4.20), (4.21), (4.22), and (4.23), we read off the transformation laws of the gauge fields under (4.24):

$$\delta e_m{}^a = \mathrm{i}(\epsilon\gamma_a\bar\psi_m + \bar\epsilon\gamma_a\psi_m) - \boldsymbol{\sigma} e_m{}^a, \tag{4.25a}$$

$$\delta\psi_m{}^\alpha = 2\mathbf{D}_m\epsilon^\alpha + 2e_m{}^a(\epsilon^\beta T_{a\beta}{}^\alpha| + \bar\epsilon_{\underline{\beta}} T_a{}^{\underline{\beta}\alpha}|) + \mathrm{i}(\gamma_m)^{\alpha\beta}\bar\eta_\beta - \mathrm{i}\boldsymbol{\tau}\psi_m{}^\alpha + \frac{1}{2}\boldsymbol{\sigma}\psi_m{}^\alpha, \tag{4.25b}$$

$$\delta b_m = \left\{-\frac{1}{2}e_m{}^a\epsilon^\beta\left[\mathrm{i}(\gamma_a)^{\gamma\delta}\boldsymbol{C}_{\beta\gamma\delta}| + \frac{1}{3}(\gamma_a)_\beta{}^\gamma(8\mathrm{i}\mathcal{D}_\gamma\mathcal{S}| - \bar{\mathcal{D}}_\gamma\bar R|)\right] + \epsilon^\beta\bar\psi_{m\delta}(\mathrm{i}\mathcal{C}_\beta{}^\delta| + 2\delta_\beta^\delta\mathcal{S}|) + \frac{\mathrm{i}}{2}\psi_m{}^\delta\eta_\delta + \text{c.c.}\right\} - \mathbf{D}_m\boldsymbol{\tau} - \frac{1}{8}(\gamma_m)^{\gamma\delta}[\mathcal{D}_\gamma,\bar{\mathcal{D}}_\delta]\sigma|. \tag{4.25c}$$

TABLE I. WZ-gauge choices and the parameters used to achieve them.

Gauge choice	σ component
$\mathcal{S}\vert=0$	$[\mathcal{D}^\alpha,\bar{\mathcal{D}}_\alpha]\sigma\vert$
$\mathcal{C}_{\alpha\beta}\vert=0$	$[\mathcal{D}_{(\alpha},\bar{\mathcal{D}}_{\beta)}]\sigma\vert$
$R\vert=\bar R\vert=0$	$\bar{\mathcal{D}}^2\sigma\vert, \mathcal{D}^2\sigma\vert$
$\mathcal{D}_\alpha R\vert=\bar{\mathcal{D}}_\alpha\bar R\vert=0$	$\mathcal{D}_\alpha\bar{\mathcal{D}}^2\sigma\vert, \bar{\mathcal{D}}_\alpha\mathcal{D}^2\sigma\vert$
$\mathcal{D}^2R\vert+\bar{\mathcal{D}}^2\bar R\vert=0$	$\{\mathcal{D}^2,\bar{\mathcal{D}}^2\}\sigma\vert$

The superspace torsion and curvature transform as tensors under the $\mathcal{K}$-gauge group, Eqs. (2.4) and (2.5). Their super-Weyl transformations follow from the transformation rules of the dimension-1 torsion superfields given in the previous section, Eqs. (2.11d)–(2.11f). This allows one to compute the variations of the component field strengths under the supersymmetry transformation (4.24).

D. The Weyl multiplet gauge

The super-Weyl invariance given by Eq. (2.11) preserves the WZ gauge, so we can eliminate a number of component fields. We choose the gauge conditions

$$\mathcal{S}|=0,\qquad \mathcal{C}_{\alpha\beta}|=0,\qquad R|=\bar R|=0,\qquad \mathcal{D}^2R|+\bar{\mathcal{D}}^2\bar R|=0, \tag{4.26}$$

which constitute the Weyl multiplet gauge. In Table 1, we identify those components of the super-Weyl parameter σ which have to be used in order to impose the Weyl multiplet gauge.

In the gauge (4.26), the super-Weyl gauge freedom is not fixed completely. We stay with unbroken Weyl and local S-supersymmetry transformations corresponding to the parameters $\boldsymbol{\sigma}$ and η_α, $\bar\eta_\alpha$ respectively. The only independent component fields are the vielbein $e_m{}^a$, the two gravitini $\psi_m{}^\alpha$ and $\bar\psi_m{}^\alpha$, and the $\mathrm{U}(1)_R$ gauge field b_m. These fields and the associated local symmetries correspond to those describing the $\mathcal{N}=2$ Weyl multiplet [4].

In the Weyl multiplet gauge, the explicit expressions for the gravitino field strength and the curvature tensors simplify drastically. The gravitino field strength becomes

$$\boldsymbol{\psi}_{ab}{}^\gamma = \mathrm{i}\varepsilon_{abc}(\gamma^c)^{\alpha\beta}\bar{\boldsymbol{C}}_{\alpha\beta}{}^\gamma\Big| - \frac{4\mathrm{i}}{3}\varepsilon_{abc}(\gamma^c)^{\gamma\delta}\bar{\mathcal{D}}_\delta\mathcal{S}\Big|. \tag{4.27}$$

The Lorentz curvature takes the form:

$$\mathcal{R}_{ab}{}^{cd} = \left\{ -\frac{\mathrm{i}}{4}\varepsilon_{abe}(\gamma^e)^{\alpha\beta}\varepsilon^{cdf}(\gamma_f)^{\tau\delta}(\mathcal{D}_{(\tau}\bar{\boldsymbol{C}}_{\delta\alpha\beta)} + \bar{\mathcal{D}}_{(\tau}\boldsymbol{C}_{\delta\alpha\beta)}) + \frac{4\mathrm{i}}{3}\delta^c_{[a}\delta^d_{b]}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathcal{S} \right. \\ + \psi_{[a}{}^\beta\left[(\gamma_{b]})_\beta{}^\gamma\boldsymbol{C}_{\gamma\delta\rho}\varepsilon^{cde}(\gamma_e)^{\delta\rho} + \frac{2}{3}\varepsilon_{b]}{}^{cd}\mathcal{D}_\beta\mathcal{S} - \frac{8}{3}\delta^{[c}_{b]}(\gamma^{d]})_\beta{}^\gamma\mathcal{D}_\gamma\mathcal{S}\right] \\ \left. + \bar{\psi}_{[a\beta}\left[(\gamma_{b]})^{\beta\gamma}\bar{\boldsymbol{C}}_{\gamma\delta\rho}\varepsilon^{cde}(\gamma_e)^{\delta\rho} + \frac{2}{3}\varepsilon_{b]}{}^{cd}\bar{\mathcal{D}}^\beta\mathcal{S} - \frac{8}{3}\delta^{[c}_{b]}(\gamma^{d]})^{\beta\gamma}\bar{\mathcal{D}}_\gamma\mathcal{S}\right]\right\}\Bigg|. \tag{4.28}$$

From here we read off the scalar curvature

$$\mathcal{R}(e,\psi) = 4\mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathcal{S}| + \left\{\psi_a{}^\beta\left((\gamma^a)^{\gamma\delta}C_{\beta\gamma\delta}\Big| + \frac{8}{3}(\gamma^a)_\beta{}^\gamma\mathcal{D}_\gamma\mathcal{S}\Big|\right) + \text{c.c.}\right\}. \tag{4.29}$$

An equivalent form for this result is

$$\mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathcal{S}| = \frac{1}{4}(\mathcal{R}(e,\psi) + \mathrm{i}\bar{\psi}_a\gamma_b\boldsymbol{\psi}^{ab} + \mathrm{i}\psi_a\gamma_b\bar{\boldsymbol{\psi}}^{ab}). \tag{4.30}$$

The $\mathrm{U}(1)_R$ field strength becomes

$$\mathcal{F}_{ab} = \varepsilon_{abc}\left\{\frac{1}{2}(\gamma^c)^{\alpha\beta}[\mathcal{D}_\alpha,\bar{\mathcal{D}}_\beta]\mathcal{S} + \mathrm{i}\varepsilon^{cde}(\gamma_d)^{\beta\gamma}[\psi_{e\beta}\mathcal{D}_\gamma\mathcal{S} + \bar{\psi}_{e\beta}\bar{\mathcal{D}}_\gamma\mathcal{S}]\right\}\Bigg|. \tag{4.31}$$

An equivalent form for this result is

$$[\mathcal{D}_{(\alpha},\bar{\mathcal{D}}_{\beta)}]\mathcal{S}| = (\gamma^a)_{\alpha\beta}\left\{\mathcal{F}_a + \frac{1}{4}\psi^b\bar{\boldsymbol{\psi}}_{ab} - \frac{1}{4}\bar{\psi}^b\boldsymbol{\psi}_{ab} + \frac{1}{4}\varepsilon^{abc}(\psi_b\gamma^d\bar{\boldsymbol{\psi}}_{cd} - \bar{\psi}_b\gamma^d\boldsymbol{\psi}_{cd})\right\}, \tag{4.32}$$

where $\mathcal{F}_a := \frac{1}{2}\varepsilon_{abc}\mathcal{F}^{bc}$.

We need to determine those residual gauge transformations which leave invariant the Weyl multiplet gauge. Imposing the conditions $\delta\mathcal{C}_{\alpha\beta}| = \delta\mathcal{S}| = \delta R| = 0$, with the transformation δ defined by (4.24), we obtain

$$[\mathcal{D}_{(\alpha},\bar{\mathcal{D}}_{\beta)}]\sigma| = -\varepsilon_{cab}(\gamma^c)_{\alpha\beta}(\epsilon\bar{\boldsymbol{\psi}}^{ab} - \bar{\epsilon}\boldsymbol{\psi}^{ab}), \tag{4.33a}$$

$$\mathrm{i}\mathcal{D}^\gamma\bar{\mathcal{D}}_\gamma\sigma| = \frac{\mathrm{i}}{2}\varepsilon_{cab}(\epsilon\gamma^c\bar{\boldsymbol{\psi}}^{ab} + \bar{\epsilon}\gamma^c\boldsymbol{\psi}^{ab}), \tag{4.33b}$$

$$\mathcal{D}^2\sigma| = \bar{\mathcal{D}}^2\sigma| = 0. \tag{4.33c}$$

Using these results in (4.25a)–(4.25c), together with the fact that the bar projections of all the dimension-1 curvature superfields vanish, we derive the transformations of the gauge fields in the Weyl multiplet gauge:

$$\delta e_m{}^a = \mathrm{i}(\epsilon\gamma^a\bar{\psi}_m + \bar{\epsilon}\gamma^a\psi_m) - \boldsymbol{\sigma}e_m{}^a, \tag{4.34a}$$

$$\delta\psi_m{}^\alpha = 2\mathbf{D}_m\epsilon^\alpha + \mathrm{i}(\tilde{\gamma}_m\bar{\eta})^\alpha - \mathrm{i}\boldsymbol{\tau}\psi_m{}^\alpha - \frac{1}{2}\boldsymbol{\sigma}\psi_m{}^\alpha, \tag{4.34b}$$

$$\delta b_m = -\frac{1}{2}e_m{}^a\left[\epsilon\gamma^b\bar{\boldsymbol{\psi}}_{ab} + \frac{1}{2}\varepsilon_{abc}\epsilon\bar{\boldsymbol{\psi}}^{bc} - \mathrm{i}\psi_a\eta + \text{c.c.}\right] \\ - \mathbf{D}_m\boldsymbol{\tau}, \tag{4.34c}$$

with the γ matrices with world indices defined by $\gamma_m := e_m{}^a\gamma_a$ and similarly for $\tilde{\gamma}_m$.

The above description of the Weyl multiplet agrees with that given in [4].

E. Alternative gauge fixings

There exist different schemes for component reduction that correspond to alternative choices of fixing the supergravity gauge freedom. Here we mention two possible options that are most useful in the context of type I or type II supergravity formulations.

The super-Weyl and local $\mathrm{U}(1)_R$ gauge freedom can be used to impose the gauge condition [10]

$$\mathcal{S} = 0, \qquad \Phi_\alpha = 0, \qquad \Phi_a = \mathcal{C}_a. \tag{4.35}$$

Since the resulting $\mathrm{U}(1)_R$ connection is a tensor superfield, we may equally well work with covariant derivatives ∇_A without $\mathrm{U}(1)_R$ connection, which are defined by

$$\nabla_\alpha := \mathcal{D}_\alpha, \qquad \nabla_a := \mathcal{D}_a - \mathrm{i}\mathcal{C}_a\mathcal{J}. \tag{4.36}$$

The gauge condition (4.35) does not fix completely the super-Weyl and local $\mathrm{U}(1)_R$ gauge freedom. The residual transformation is generated by a covariantly chiral scalar parameter λ, $\bar\nabla_\alpha\lambda = 0$, and has the form [11]

$$\nabla'_\alpha = \mathrm{e}^{\frac{1}{2}(3\bar\lambda-\lambda)}(\nabla_\alpha + (\nabla^\gamma\lambda)\mathcal{M}_{\gamma\alpha}), \tag{4.37a}$$

$$\nabla'_a = \mathrm{e}^{\lambda+\bar\lambda}\left(\nabla_a - \frac{\mathrm{i}}{2}(\gamma_a)^{\alpha\beta}(\nabla_\alpha\lambda)\bar\nabla_\beta - \frac{\mathrm{i}}{2}(\gamma_a)^{\alpha\beta}(\bar\nabla_\alpha\bar\lambda)\nabla_\beta + \varepsilon_{abc}(\nabla^b(\lambda+\bar\lambda))\mathcal{M}^c - \frac{\mathrm{i}}{2}(\nabla^\gamma\lambda)(\bar\nabla_\gamma\bar\lambda)\mathcal{M}_a\right). \tag{4.37b}$$

The dimension-1 torsion superfields transform as

$$\mathcal{C}'_a = \mathrm{e}^{\lambda+\bar\lambda}\left(\mathcal{C}_a - \frac{\mathrm{i}}{2}\nabla_a(\lambda-\bar\lambda) + \frac{1}{4}(\gamma_a)^{\alpha\beta}(\nabla_\alpha\lambda)\bar\nabla_\beta\bar\lambda\right), \tag{4.38a}$$

$$R' = -\frac{1}{4}\mathrm{e}^{3\lambda}(\bar\nabla^2 - 4R)\mathrm{e}^{-\bar\lambda}. \tag{4.38b}$$

This formulation is very similar to the old minimal 4D $\mathcal{N}=1$ supergravity, see e.g. [3] for a review. It is best suited when dealing with type I minimal supergravity-matter systems.

The super-Weyl freedom can be used to impose the gauge condition [10]

$$R = 0, \tag{4.39}$$

with the local $\mathrm{U}(1)_R$ group being unbroken. This superspace geometry is most suitable for the type II minimal supergravity. The gauge condition (4.39) does not completely fix the super-Weyl group. The residual super-Weyl transformation is generated by a real superfield σ constrained by $\mathcal{D}^2\mathrm{e}^{-\sigma} = \bar{\mathcal{D}}^2\mathrm{e}^{-\sigma} = 0$.

Each of the two restricted superspace geometries considered, (4.35) and (4.39), is suitable for describing the Weyl multiplet of conformal supergravity. In each case, we can define a Wess-Zumino gauge and a Weyl multiplet gauge.

Some alternative gauge conditions will be used in Sec. VIII.

V. TYPE I MINIMAL SUPERGRAVITY

This off-shell supergravity theory and its matter couplings are analogous to the old minimal formulation for 4D $\mathcal{N}=1$ supergravity; see [1–3] for reviews. Its specific feature is that its conformal compensators are a covariantly chiral superfield Φ of super-Weyl weight $w=1/2$,

$$\bar{\mathcal{D}}_\alpha\Phi = 0, \qquad \mathcal{J}\Phi = -\frac{1}{2}\Phi, \qquad \delta_\sigma\Phi = \frac{1}{2}\sigma\Phi, \tag{5.1}$$

and its conjugate $\bar\Phi$.

A. Pure supergravity

As a warm-up exercise, we first analyze the action for pure type I supergravity with a cosmological term. It is obtained from (5.15) by switching off the matter sector, that is by setting $K=0$ and $W=\mu=\mathrm{const}$,

$$S_{\rm SG} = -4\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta E\bar\Phi\Phi + \mu\int \mathrm{d}^3x\mathrm{d}^2\theta\mathcal{E}\Phi^4 + \bar\mu\int \mathrm{d}^3x\mathrm{d}^2\bar\theta\,\bar{\mathcal{E}}\,\bar\Phi^4. \tag{5.2}$$

The second and third terms in the action generate a supersymmetric cosmological term, with the parameter $|\mu|^2$ being proportional to the cosmological constant. The dynamics of this theory was analyzed in superspace in [11]. Here we reduce the action (5.2) to components.

In the Weyl multiplet gauge, the super-Weyl gauge freedom is not fixed completely. We can use the residual Weyl and local $\mathrm{U}(1)_R$ symmetries to impose the gauge condition

$$\Phi| = 1. \tag{5.3a}$$

In addition, the local S-supersymmetry invariance can be used to make the gauge choice

$$\mathcal{D}_\alpha\Phi| = 0. \tag{5.3b}$$

The only surviving component field of Φ may be defined as

$$M := \mathcal{D}^2\Phi|. \tag{5.4}$$

To perform the component reduction of the kinetic term in (5.2), the first step is to associate with it, by applying the relation (3.7), the equivalent antichiral Lagrangian $\bar{\mathcal{L}}_{\rm c} = (\mathcal{D}^2 - 4\bar R)(\bar\Phi\Phi)$. After that we can use (3.13) to reduce the action to components. The antichiral Lagrangian corresponding to the $\bar\mu$ term in (5.2) is $\bar{\mathcal{L}}_{\rm c} = \bar\mu\bar\Phi^4$. Finally, the component version of the μ term in (5.2) is the complex conjugate of the $\bar\mu$ term.

Direct calculations lead to the supergravity Lagrangian

$$L_{\rm SG} = \frac{1}{2}\mathcal{R}(e,\psi) + \frac{\mathrm{i}}{4}\varepsilon^{abc}(\bar\psi_{ab}\psi_c + \bar\psi_a\psi_{bc}) - \frac{1}{4}\bar MM + b^ab_a - \bar\mu\left(\bar M - \frac{1}{2}\varepsilon^{abc}\psi_a\gamma_b\psi_c\right) - \mu\left(M + \frac{1}{2}\varepsilon^{abc}\bar\psi_a\gamma_b\bar\psi_c\right), \tag{5.5}$$

where the gravitino field strength is defined as

$$\psi_{ab} := \mathfrak{D}_a\psi_b - \mathfrak{D}_b\psi_a - \mathcal{T}_{ab}{}^c\psi_c, \tag{5.6}$$

which differs from (4.14). We recall that the covariant derivative $\mathfrak{D}_a$, Eq. (4.5), has no $\mathrm{U}(1)_R$ connection. It is natural to use $\mathfrak{D}_a$ since the local $\mathrm{U}(1)_R$ symmetry has been fixed. The type I supergravity multiplet consists of the following fields: the dreibein $e_m{}^a$, the gravitini $\psi_m{}^\alpha$ and $\bar\psi_{m\alpha}$, and the auxiliary fields M, $\bar M$ and b_m.

Upon elimination of the auxiliary fields, the Lagrangian becomes

$$L_{\mathrm{SG}} = \frac{1}{2}\mathcal{R}(e,\psi) + \frac{\mathrm{i}}{4}\varepsilon^{abc}(\bar\psi_{ab}\psi_c + \bar\psi_a\psi_{bc}) + 4\bar\mu\mu + \frac{\bar\mu}{2}\varepsilon^{abc}\psi_a\gamma_b\psi_c - \frac{\mu}{2}\varepsilon^{abc}\bar\psi_a\gamma_b\bar\psi_c. \tag{5.7}$$

This Lagrangian describes (1,1) anti-de Sitter supergravity for $\mu \neq 0$ [5].

B. Supersymmetry transformations

The gauge conditions (5.3a) and (5.3b) completely fix the Weyl, local $\mathrm{U}(1)_R$ and S-supersymmetry invariances. However, performing just a single Q-supersymmetry transformation, with ϵ_α and $\bar\epsilon_\alpha$ the only nonzero parameters in (4.34), does not preserve these gauge conditions. To restore the gauge defined by (5.3a) and (5.3b), the Q-supersymmetry transformation has to be accompanied by a compensating S-supersymmetry transformation. Indeed, applying the transformation (4.24) to $\Phi|$ gives

$$\delta\Phi| = \epsilon^\beta\mathcal{D}_\beta\Phi| + \frac{1}{2}(\boldsymbol{\sigma} - \mathrm{i}\boldsymbol{\tau})\Phi| = \frac{1}{2}(\boldsymbol{\sigma} - \mathrm{i}\boldsymbol{\tau}), \tag{5.8}$$

where we have used Eqs. (5.3a) and (5.3b). Setting $\delta\Phi| = 0$ gives

$$\boldsymbol{\sigma} = \boldsymbol{\tau} = 0. \tag{5.9}$$

On the other hand, the transformation of $\mathcal{D}_\alpha\Phi|$ is

$$\begin{aligned}\delta\mathcal{D}_\alpha\Phi| &= \epsilon^\beta\mathcal{D}_\beta\mathcal{D}_\alpha\Phi| + \bar\epsilon_\beta\bar{\mathcal{D}}^\beta\mathcal{D}_\alpha\Phi| - \eta_\alpha\left(\mathcal{J}\Phi\Big| - \frac{1}{2}\Phi\Big|\right)\\ &= -\frac{1}{2}\epsilon_\alpha M + (\gamma^a\bar\epsilon)_\alpha b_a + \eta_\alpha,\end{aligned} \tag{5.10}$$

where here we have used (5.3a) and (5.3b). Setting $\delta\mathcal{D}_\alpha\Phi| = 0$ gives

$$\eta_\alpha(\epsilon) = \frac{1}{2}\epsilon_\alpha M - (\gamma^a\bar\epsilon)_\alpha b_a. \tag{5.11}$$

Using these results in (4.34), we obtain the supersymmetry transformation laws of the gauge fields:

$$\delta_\epsilon e_m{}^a = \mathrm{i}(\epsilon\gamma^a\bar\psi_m + \bar\epsilon\gamma^a\psi_m), \tag{5.12a}$$

$$\delta_\epsilon\psi_m{}^\alpha = 2\mathfrak{D}_m\epsilon^\alpha - \mathrm{i}b_m\epsilon^\alpha + \mathrm{i}e_m{}^a\varepsilon_{abc}b^b(\tilde\gamma^c\epsilon)^\alpha + \frac{\mathrm{i}}{2}\bar M(\tilde\gamma_m\bar\epsilon)^\alpha, \tag{5.12b}$$

$$\begin{aligned}\delta_\epsilon b_m = -\frac{1}{2}e_m{}^a\Big\{&\epsilon\gamma^b\bar\psi_{ab} + \frac{1}{2}\varepsilon_{abc}\epsilon\bar\psi^{bc} + \mathrm{i}\varepsilon_{abc}b^b\epsilon\bar\psi^c\\ &+ \mathrm{i}(b_a\epsilon\gamma^b\bar\psi_b - 2b_b\epsilon\gamma^b\bar\psi_a) - \frac{\mathrm{i}}{2}M\epsilon\psi_a\Big\} + \mathrm{c.c.}\end{aligned} \tag{5.12c}$$

The supergravity multiplet also includes the auxiliary scalar $M = \mathcal{D}^2\Phi|$. Due to (5.9) and since $\mathcal{D}^2\sigma| = 0$, Eq. (4.33c), the supersymmetry transformation of M is

$$\delta_\epsilon M = \epsilon^\beta\mathcal{D}_\beta\mathcal{D}^2\Phi| + \bar\epsilon_\beta\bar{\mathcal{D}}^\beta\mathcal{D}^2\Phi| = \bar\epsilon_\beta[\bar{\mathcal{D}}^\beta,\mathcal{D}^2]\Phi|. \tag{5.13}$$

Making use of the algebra of covariant derivatives gives

$$\delta_\epsilon M = -\varepsilon_{cab}\bar\epsilon\tilde\gamma^c\bar\psi^{ab} - \mathrm{i}M\bar\epsilon\tilde\gamma^a\psi_a - 2\mathrm{i}b_a\bar\epsilon\bar\psi^a. \tag{5.14}$$

C. Matter-coupled supergravity

We consider a general locally supersymmetric nonlinear σ-model

$$S = -4\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta E\bar\Phi \mathrm{e}^{-K/4}\Phi + \int \mathrm{d}^3x\mathrm{d}^2\theta\mathcal{E}\Phi^4 W + \int \mathrm{d}^3x\mathrm{d}^2\bar\theta\,\bar{\mathcal{E}}\,\bar\Phi^4\bar W. \tag{5.15}$$

Here the Kähler potential, $K = K(\varphi^I,\bar\varphi^{\bar J})$, is a real function of the covariantly chiral superfields φ^I and their conjugates $\bar\varphi^{\bar I}$, $\bar{\mathcal{D}}_\alpha\varphi^I = 0$. The superpotential, $W = W(\varphi^I)$, is a holomorphic function of φ^I alone. The matter superfields φ^I and $\bar\varphi^{\bar J}$ are chosen to be inert under the super-Weyl and local $\mathrm{U}(1)_R$ transformations. This guarantees the super-Weyl invariance of the action. In Appendix B, we describe a different parametrization of the nonlinear σ-model (5.15) in which the dynamical variables Φ and φ^I are replaced by covariantly chiral superfields $\phi^i = (\phi^0,\phi^I)$ of super-Weyl weight $w = 1/2$ that parametrize a Kähler cone.

The action (5.15) is also invariant under a target-space Kähler transformation

$$K(\varphi,\bar\varphi) \to K(\varphi,\bar\varphi) + \Lambda(\varphi) + \bar\Lambda(\bar\varphi), \tag{5.16a}$$

$$W(\varphi) \to \mathrm{e}^{-\Lambda(\varphi)}W(\varphi), \tag{5.16b}$$

provided the compensator changes as

$$\Phi \to \mathrm{e}^{\Lambda(\varphi)/4}\Phi, \tag{5.16c}$$

with $\Lambda(\varphi^I)$ an arbitrary holomorphic function.

We first compute the component form of (5.15) in the special case $W=0$,

$$S_{\rm kinetic} = -4\int d^3x d^2\theta d^2\bar\theta E\bar\Phi e^{-K(\varphi,\bar\varphi)/4}\Phi. \tag{5.17}$$

Associated with $S_{\rm kinetic}$ is the antichiral Lagrangian

$$\bar{\mathcal{L}}_c = (\mathcal{D}^2 - 4\bar R)(\bar\Phi e^{-K/4}\Phi), \tag{5.18}$$

which has to be used for computing the component action using the general formula (3.13).

Our consideration in this and the next sections is similar to that in 4D $\mathcal{N}=1$ supergravity [60,61]. To reduce the action to components, we impose the following Weyl and local S-supersymmetry gauge conditions:

$$(\bar\Phi e^{-K/4}\Phi)| = 1, \tag{5.19a}$$

$$\mathcal{D}_\alpha(\bar\Phi e^{-K/4}\Phi)| = 0. \tag{5.19b}$$

Both gauge conditions are manifestly Kähler invariant. It turns out that the condition (5.19a) leads to the correct Einstein-Hilbert gravitational Lagrangian at the component level. On the other hand, the condition (5.19b) guarantees that no cross terms $\mathcal{D}^\alpha\mathcal{S}|\bar{\mathcal{D}}_\alpha K|$ are generated at the component level. See Appendix B for more details. Finally we fix the local $U(1)_R$ symmetry by imposing the gauge condition

$$\Phi| = \bar\Phi| = e^{K/8}. \tag{5.19c}$$

The auxiliary scalar fields contained in Φ and $\bar\Phi$ may be defined in a manifestly Kähler-invariant way as

$$\mathbb{M} := \mathcal{D}^2(\bar\Phi e^{-\frac14 K}\Phi)|, \qquad \bar{\mathbb{M}} := \bar{\mathcal{D}}^2(\bar\Phi e^{-\frac14 K}\Phi)|. \tag{5.20}$$

To make the gauge condition (5.19c) Kähler invariant, the Kähler transformation generated by a parameter Λ has to be accompanied by a special $U(1)_R$ transformation with parameter $\tau = \frac{i}{4}(\bar\Lambda - \Lambda)$ such that the component vector field b_a, which belongs to the Weyl multiplet and is defined by Eq. (4.4), transforms as

$$b_a \to b_a + \frac{i}{4}\mathfrak{D}_a(\Lambda - \bar\Lambda). \tag{5.21}$$

We define the component fields of φ^I as follows:

$$X^I := \varphi^I|, \tag{5.22a}$$

$$\lambda^I_\alpha := \mathcal{D}_\alpha\varphi^I|, \tag{5.22b}$$

$$F^I := -\frac14[\mathcal{D}^2\varphi^I + \Gamma^I_{JK}(\mathcal{D}^\alpha\varphi^J)\mathcal{D}_\alpha\varphi^K]|. \tag{5.22c}$$

Under a holomorphic reparametrization, $X^I \to f^I(X)$, of the target Kähler space, the fields λ^I_α and F^I transform as holomorphic vector fields. Direct calculations lead to the following component Lagrangian:

$$\begin{aligned} L_{\rm kinetic} &= \frac12\mathcal{R}(e,\psi) + \frac{i}{4}\varepsilon^{abc}(\tilde{\bar\psi}_{ab}\psi_c + \bar\psi_a\tilde\psi_{bc}) - \frac14\bar{\mathbb{M}}\mathbb{M} + \mathbb{B}^a\mathbb{B}_a \\ &\quad + g_{I\bar J}\Big[F^I\bar F^{\bar J} - (\mathfrak{D}^a X^I)\mathfrak{D}_a\bar X^{\bar J} - \frac{i}{4}\lambda^I\gamma^a\overset{\leftrightarrow}{\tilde{\mathfrak{D}}}_a\bar\lambda^{\bar J} + \frac18\lambda^I\bar\lambda^{\bar J}(\varepsilon^{abc}\bar\psi_a\gamma_b\psi_c - \bar\psi^a\psi_a) \\ &\quad - \frac18\lambda^I\gamma^a\bar\lambda^{\bar J}(\bar\psi^b\gamma_a\psi_b + \varepsilon_{abc}\bar\psi^b\psi^c) - \frac12\psi^a\gamma_b\tilde\gamma_a\lambda^I\mathfrak{D}^b\bar X^{\bar J} - \frac12\bar\lambda^{\bar J}\tilde\gamma_a\gamma_b\bar\psi^a\mathfrak{D}^b X^I\Big] \\ &\quad + \frac{1}{16}R_{I\bar K J\bar L}\lambda^I\lambda^J\bar\lambda^{\bar K}\bar\lambda^{\bar L} - \frac{1}{64}(g_{I\bar J}\lambda^I\bar\lambda^{\bar J})^2, \end{aligned} \tag{5.23}$$

where the auxiliary vector field $\mathbb{B}_a$ is defined by the rule

$$\mathbb{B}_a := b_a - \frac18 g_{I\bar J}\lambda^I\gamma_a\bar\lambda^{\bar J} - \frac{i}{4}(K_I\mathfrak{D}_a X^I - K_{\bar I}\mathfrak{D}_a\bar X^{\bar I}), \tag{5.24}$$

and is invariant under the Kähler transformations, in accordance with (5.21). The gravitino field strength in (5.23) differs from that introduced earlier in (5.6):

$$\tilde\psi_{ab} = \tilde{\mathfrak{D}}_a\psi_b - \tilde{\mathfrak{D}}_b\psi_a - \mathcal{T}_{ab}{}^c\psi_c, \tag{5.25}$$

where the Kähler-covariant derivative $\tilde{\mathfrak{D}}_a$ is defined (similarly to the 4 D case; see e.g. [1]) as follows:

$$\tilde{\mathfrak{D}}_a\psi_b := \mathfrak{D}_a\psi_b + \frac14(K_J\mathfrak{D}_a X^J - K_{\bar J}\mathfrak{D}_a\bar X^{\bar J})\psi_b, \tag{5.26a}$$

$$\tilde{\mathfrak{D}}_a\lambda^I := \mathfrak{D}_a\lambda^I - \frac14(K_J\mathfrak{D}_a X^J - K_{\bar J}\mathfrak{D}_a\bar X^{\bar J})\lambda^I + \lambda^J\Gamma^I_{JK}\mathfrak{D}_a X^K. \tag{5.26b}$$

In (5.23), as usual $g_{I\bar J}$ denotes the Kähler metric, $g_{I\bar J} = K_{I\bar J}$, and $R_{I\bar K J\bar L}$ the Riemann curvature,

$$R_{I\bar{K}J\bar{L}} = K_{IJ\bar{K}\bar{L}} - g_{M\bar{N}}\Gamma^M_{IJ}\Gamma^{\bar{N}}_{\bar{K}\bar{L}}, \tag{5.27}$$

with $\Gamma^I_{JK} = g^{I\bar{L}}K_{JK\bar{L}}$ the Christoffel symbol.

We now turn to the third term in (5.15). The corresponding antichiral Lagrangian is $\bar{\mathcal{L}}_c = \bar{\Phi}^4\bar{W}(\bar{\varphi})$. To reduce this functional to components, we again make use of the general rule (3.13) in conjunction with the relations (5.19) and (5.22) which define the component fields of Φ and φ^I. The second term in (5.15) is just the complex conjugate of the third term.

The component Lagrangian corresponding to the second and third terms in (5.15) is

$$L_{\text{potential}} = -e^{K/2}\bigg[\left(\bar{\mathbb{M}} - \frac{1}{2}\varepsilon^{abc}\psi_a\gamma_b\psi_c\right)\bar{W} + \left(\mathbb{M} + \frac{1}{2}\varepsilon^{abc}\bar{\psi}_a\gamma_b\bar{\psi}_c\right)W - \left(\bar{F}^{\bar{I}} + \frac{i}{2}\psi_a\gamma^a\bar{\lambda}^{\bar{I}}\right)\nabla_{\bar{I}}\bar{W} - \left(F^I + \frac{i}{2}\bar{\psi}_a\gamma^a\lambda^I\right)\nabla_I W + \frac{1}{4}\bar{\lambda}^{\bar{I}}\bar{\lambda}^{\bar{J}}\nabla_{\bar{I}}\nabla_{\bar{J}}\bar{W} + \frac{1}{4}\lambda^I\lambda^J\nabla_I\nabla_J W\bigg]. \tag{5.28}$$

Here we have introduced the Kähler-covariant derivatives

$$\nabla_I W := W_I + K_I W, \tag{5.29a}$$

$$\nabla_I\nabla_J W := W_{IJ} + 2K_{(I}\nabla_{J)}W - \Gamma^L_{IJ}\nabla_L W + K_{IJ}W + K_I K_J W. \tag{5.29b}$$

The component Lagrangian corresponding to the supergravity-matter system (5.15) is $L = L_{\text{kinetic}} + L_{\text{potential}}$. Putting together (5.23) and (5.28) gives

$$\begin{aligned} L ={}& \frac{1}{2}\mathcal{R}(e,\psi) + \frac{i}{4}\varepsilon^{abc}(\tilde{\bar{\psi}}_{ab}\psi_c + \bar{\psi}_a\tilde{\psi}_{bc}) - \frac{1}{4}\bar{\mathbb{M}}\mathbb{M} + \mathbb{B}^a\mathbb{B}_a + g_{I\bar{J}}\bigg[F^I\bar{F}^{\bar{J}} - (\mathfrak{D}^a X^I)\mathfrak{D}_a\bar{X}^{\bar{J}} - \frac{i}{4}\lambda^I\gamma^a\overset{\leftrightarrow}{\tilde{\mathfrak{D}}}_a\bar{\lambda}^{\bar{J}} \\ &+ \frac{1}{8}\lambda^I\bar{\lambda}^{\bar{J}}(\varepsilon^{abc}\bar{\psi}_a\gamma_b\psi_c - \bar{\psi}^a\psi_a) - \frac{1}{8}\lambda^I\gamma^a\bar{\lambda}^{\bar{J}}(\bar{\psi}^b\gamma_a\psi_b + \varepsilon_{abc}\bar{\psi}^b\psi^c) - \frac{1}{2}\psi^a\gamma_b\tilde{\gamma}_a\lambda^I\mathfrak{D}^b\bar{X}^{\bar{J}} - \frac{1}{2}\bar{\lambda}^{\bar{J}}\tilde{\gamma}_a\gamma_b\bar{\psi}^a\mathfrak{D}^b X^I\bigg] \\ &+ \frac{1}{16}R_{I\bar{K}J\bar{L}}\lambda^I\lambda^J\bar{\lambda}^{\bar{K}}\bar{\lambda}^{\bar{L}} - \frac{1}{64}(g_{I\bar{J}}\lambda^I\bar{\lambda}^{\bar{J}})^2 - e^{K/2}\bigg[\left(\bar{\mathbb{M}} - \frac{1}{2}\varepsilon^{abc}\psi_a\gamma_b\psi_c\right)\bar{W} + \left(\mathbb{M} + \frac{1}{2}\varepsilon^{abc}\bar{\psi}_a\gamma_b\bar{\psi}_c\right)W \\ &- \left(\bar{F}^{\bar{I}} + \frac{i}{2}\psi_a\gamma^a\bar{\lambda}^{\bar{I}}\right)\nabla_{\bar{I}}\bar{W} - \left(F^I + \frac{i}{2}\bar{\psi}_a\gamma^a\lambda^I\right)\nabla_I W + \frac{1}{4}\bar{\lambda}^{\bar{I}}\bar{\lambda}^{\bar{J}}\nabla_{\bar{I}}\nabla_{\bar{J}}\bar{W} + \frac{1}{4}\lambda^I\lambda^J\nabla_I\nabla_J W\bigg]. \end{aligned} \tag{5.30}$$

Upon elimination of the auxiliary fields, the Lagrangian turns into

$$\begin{aligned} L ={}& \frac{1}{2}\mathcal{R}(e,\psi) + \frac{i}{4}\varepsilon^{abc}(\tilde{\bar{\psi}}_{ab}\psi_c + \bar{\psi}_a\tilde{\psi}_{bc}) + \frac{1}{16}R_{I\bar{K}J\bar{L}}\lambda^I\lambda^J\bar{\lambda}^{\bar{K}}\bar{\lambda}^{\bar{L}} - \frac{1}{64}(g_{I\bar{J}}\lambda^I\bar{\lambda}^{\bar{J}})^2 + g_{I\bar{J}}\bigg[-(\mathfrak{D}^a X^I)\mathfrak{D}_a\bar{X}^{\bar{J}} - \frac{i}{4}\lambda^I\gamma^a\overset{\leftrightarrow}{\tilde{\mathfrak{D}}}_a\bar{\lambda}^{\bar{J}} \\ &+ \frac{1}{8}\lambda^I\bar{\lambda}^{\bar{J}}(\varepsilon^{abc}\bar{\psi}_a\gamma_b\psi_c - \bar{\psi}^a\psi_a) - \frac{1}{8}\lambda^I\gamma^a\bar{\lambda}^{\bar{J}}(\bar{\psi}^b\gamma_a\psi_b + \varepsilon_{abc}\bar{\psi}^b\psi^c) - \frac{1}{2}\psi^a\gamma_b\tilde{\gamma}_a\lambda^I\mathfrak{D}^b\bar{X}^{\bar{J}} - \frac{1}{2}\bar{\lambda}^{\bar{J}}\tilde{\gamma}_a\gamma_b\bar{\psi}^a\mathfrak{D}^b X^I\bigg] \\ &+ e^{K/2}\bigg[\frac{1}{2}\varepsilon^{abc}(\psi_a\gamma_b\psi_c\bar{W} - \bar{\psi}_a\gamma_b\bar{\psi}_c W) + \frac{i}{2}\psi_a\gamma^a\bar{\lambda}^{\bar{I}}\nabla_{\bar{I}}\bar{W} + \frac{i}{2}\bar{\psi}_a\gamma^a\lambda^I\nabla_I W - \frac{1}{4}\bar{\lambda}^{\bar{I}}\bar{\lambda}^{\bar{J}}\nabla_{\bar{I}}\nabla_{\bar{J}}\bar{W} - \frac{1}{4}\lambda^I\lambda^J\nabla_I\nabla_J W\bigg] \\ &- e^K(g^{I\bar{J}}\nabla_I W\bar{\nabla}_{\bar{J}}\bar{W} - 4W\bar{W}). \end{aligned} \tag{5.31}$$

The potential generated, $P_{3D} = e^K(g^{I\bar{J}}\nabla_I W\bar{\nabla}_{\bar{J}}\bar{W} - 4W\bar{W})$, differs slightly from the famous four-dimensional result $P_{4D} = e^K(g^{I\bar{J}}\nabla_I W\bar{\nabla}_{\bar{J}}\bar{W} - 3W\bar{W})$, see e.g. [1].

D. Supersymmetry transformations in Einstein frame

In matter coupled supergravity, the gauge conditions (5.19) depend on the matter fields. As a consequence, the supersymmetry transformation laws of the supergravity fields will differ from those given in Sec. V B. To preserve the gauge condition $\Phi| = e^{K/8}$, we have to choose

$$\boldsymbol{\sigma}(\epsilon) = 0, \qquad \boldsymbol{\tau}(\epsilon) = -\frac{i}{4}(K_I\epsilon\lambda^I - K_{\bar{I}}\bar{\epsilon}\bar{\lambda}^{\bar{I}}). \tag{5.32}$$

To preserve the gauge condition $\mathcal{D}_\alpha(\bar{\Phi}e^{-K/4}\Phi)| = 0$, we have to apply the compensating S-supersymmetry transformation with parameter

$$\eta_\alpha(\epsilon) = \frac{1}{2}\epsilon_\alpha\mathbb{M} + \bar\epsilon^\beta\Big[-b_{\alpha\beta} + \frac{\mathrm{i}}{4}(K_I\mathfrak{D}_{\alpha\beta}X^I - K_{\bar I}\mathfrak{D}_{\alpha\beta}\bar X^{\bar I}) + \frac{1}{8}g_{I\bar J}(\varepsilon_{\alpha\beta}\lambda^I\bar\lambda^{\bar J} + 2\lambda^I_{(\alpha}\bar\lambda^{\bar J}_{\beta)})\Big]. \tag{5.33}$$

Making use of the parameters $\boldsymbol{\tau}(\epsilon)$ and $\eta_\alpha(\epsilon)$ in (4.34), one may derive the supersymmetry transformations of the supergravity fields $e_m{}^a$ and $\psi_m{}^\gamma$ and b_m. These expressions are not illuminating, and here we do not give them. We only comment upon the derivation of the supersymmetry transformation of $\mathbb{M}$. Its transformation follows from the fact that $\mathbb{M}$ is defined to be the lowest component of the scalar superfield $\mathcal{D}^2(\bar\Phi e^{-\frac{1}{4}K}\Phi)$. Making use of (5.32) and (5.33), after some algebra we get

$$\delta_\epsilon\mathbb{M} = -\varepsilon_{cab}\bar\epsilon\tilde\gamma^c\bar\psi^{ab} - \mathrm{i}\bar\epsilon\tilde\gamma^a\psi_a\mathbb{M} - 2\mathrm{i}b_a\bar\epsilon\bar\psi^a + g_{I\bar J}F^I\bar\epsilon\bar\lambda^{\bar J} - \mathrm{i}g_{I\bar J}\bar\epsilon\tilde\gamma^a\lambda^I\mathfrak{D}_a\bar X^{\bar J} - \frac{1}{2}\bar\epsilon\tilde\gamma^a\gamma^b\bar\psi_a(K_{\bar I}\mathfrak{D}_b\bar X^{\bar I} - K_I\mathfrak{D}_bX^I) + 2\mathrm{i}\boldsymbol{\tau}(\epsilon)\mathbb{M}. \tag{5.34}$$

In conclusion, we give the transformation rules of the component fields of φ^I:

$$\delta_\epsilon X^I = \epsilon\lambda^I, \tag{5.35a}$$

$$\delta_\epsilon\lambda^I_\alpha = 2\epsilon_\alpha\left(F^I + \frac{1}{4}\Gamma^I_{JK}\lambda^J\lambda^K\right) + 2\mathrm{i}(\gamma^a\bar\epsilon)_\alpha\left(\mathfrak{D}_aX^I - \frac{1}{2}\psi_a\lambda^I\right) + \mathrm{i}\boldsymbol{\tau}(\epsilon)\lambda^I_\alpha, \tag{5.35b}$$

$$\delta_\epsilon F^I = -\epsilon\lambda^J\Gamma^I_{JK}F^K + \frac{1}{2}\lambda^I\eta(\epsilon) + 2\mathrm{i}\boldsymbol{\tau}(\epsilon)F^I + \mathrm{i}\bar\epsilon\gamma^a(\mathfrak{D}_a + \mathrm{i}b_a)\lambda^I - \frac{1}{4}g^{I\bar L}R_{J\bar LK\bar P}\bar\epsilon\bar\lambda^{\bar P}\lambda^J\lambda^K + \mathrm{i}\bar\epsilon\gamma^a\lambda^J\Gamma^I_{JK}\mathfrak{D}_aX^K - \mathrm{i}\bar\epsilon\gamma^a\psi_aF^I - \bar\epsilon\gamma^a\tilde\gamma^b\bar\psi_a\left(\mathfrak{D}_bX^I - \frac{1}{2}\psi_b\lambda^I\right). \tag{5.35c}$$

These can be derived by using the definition of the components of φ^I (5.22).

VI. TYPE II MINIMAL SUPERGRAVITY

This supergravity theory is a 3D analogue of the new minimal formulation for 4D $\mathcal{N}=1$ supergravity [21] (see [2,3] for reviews). Its conformal compensator is a real covariantly linear scalar $\mathbb{G}$,

$$(\mathcal{D}^2 - 4\bar R)\mathbb{G} = (\bar{\mathcal{D}}^2 - 4R)\mathbb{G} = 0, \tag{6.1}$$

chosen to be nowhere vanishing, $\mathbb{G}\neq 0$. The super-Weyl transformation of $\mathbb{G}$ is uniquely fixed by the constraint (6.1) to be

$$\delta_\sigma\mathbb{G} = \sigma\mathbb{G}. \tag{6.2}$$

A. Real linear scalar

A general solution of the off-shell constraint (6.1) is

$$\mathbb{G} = \mathrm{i}\bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha G = \mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha G, \tag{6.3}$$

where the real *unconstrained* scalar G is defined modulo gauge transformations of the form

$$\delta G = \Lambda + \bar\Lambda, \qquad \mathcal{J}\Lambda = 0, \qquad \bar{\mathcal{D}}_\alpha\Lambda = 0. \tag{6.4}$$

This gauge freedom allows us to interpret G as the gauge prepotential for an Abelian massless vector multiplet, and $\mathbb{G}$ as the gauge invariant field strength.[12] The prepotential can be chosen to be inert under the super-Weyl transformations,[13]

$$\delta_\sigma G = 0. \tag{6.5}$$

Then the field strength $\mathbb{G}$, defined by Eq. (6.3), transforms according to (6.2).

Making use of the constraint (6.1), we deduce the important identity

$$\mathcal{D}^{\alpha\beta}\mathbb{G}_{\alpha\beta} = 8\{(\mathcal{D}^\alpha\mathcal{S})\bar{\mathcal{D}}_\alpha - (\bar{\mathcal{D}}^\alpha\mathcal{S})\mathcal{D}_\alpha\}\mathbb{G} + 4\mathrm{i}\{(\mathcal{D}^\alpha R)\mathcal{D}_\alpha + (\bar{\mathcal{D}}^\alpha\bar R)\bar{\mathcal{D}}_\alpha\}\mathbb{G}, \tag{6.6}$$

where we have denoted

$$\mathbb{G}_{\alpha\beta} = (\gamma^a)_{\alpha\beta}\mathbb{G}_a \coloneqq [\mathcal{D}_{(\alpha},\bar{\mathcal{D}}_{\beta)}]\mathbb{G} + 4\mathcal{C}_{\alpha\beta}\mathbb{G}. \tag{6.7}$$

In the Weyl multiplet gauge (4.26), it follows from (6.6) that

$$\mathbb{G}^a| = \mathcal{H}^a - \varepsilon^{abc}\bar\psi_b\psi_c\mathbb{G}| - \mathrm{i}\varepsilon^{abc}(\psi_b\gamma_c\mathcal{D}\mathbb{G}| + \bar\psi_b\gamma_c\bar{\mathcal{D}}\mathbb{G}|), \tag{6.8}$$

where $\mathcal{H}^a$ denotes the Hodge-dual of the field strength of a U(1) gauge field a_a,

$$\mathcal{H}^a = \frac{1}{2}\varepsilon^{abc}\mathcal{H}_{bc}, \qquad \mathcal{H}_{ab} = \mathfrak{D}_aa_b - \mathfrak{D}_ba_a - \mathcal{T}_{ab}{}^ca_c. \tag{6.9}$$

The other independent component fields of $\mathbb{G}$ may be chosen as follows:

[12] In four dimensions, the real linear superfield is naturally interpreted as the gauge invariant field strength of a massless tensor multiplet [62].

[13] The transformation law (6.5) is consistent with the requirement that the gauge parameter Λ in (6.4) be super-Weyl inert.

$$\mathbb{G}|, \qquad \mathcal{D}_\alpha\mathbb{G}|, \qquad \bar{\mathcal{D}}_\alpha\mathbb{G}|, \qquad \mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathbb{G}|. \tag{6.10}$$

B. Poincaré supergravity

The off-shell action for type II supergravity without a cosmological term [10,11] is

$$S_{\text{Poincaré}} = 4\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E(\mathbb{G}\ln\mathbb{G} - 4G\mathcal{S}). \tag{6.11}$$

The action can be written in a different but equivalent form:

$$S_{\text{Poincaré}} = 4\mathrm{i}\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}EG\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\ln\frac{\mathbb{G}}{\bar{\Phi}\Phi}, \tag{6.12}$$

where Φ is a nowhere vanishing covariantly chiral superfield of the type Eq. (5.1). One may see that the variables Φ and $\bar{\Phi}$ are purely gauge degrees of freedom.

The theory (6.11) was shown in [11] to be classically equivalent to type I supergravity without a cosmological term, the latter being defined by Eq. (5.2) with $\mu = 0$. The above action can equivalently be described by the antichiral Lagrangian

$$\bar{\mathcal{L}}_c = -(\mathcal{D}^2 - 4\bar{R})(\mathbb{G}\ln\mathbb{G} - 4G\mathcal{S}), \tag{6.13}$$

which has to be used to carry out the component reduction of (6.11) by applying the general rule (3.13).

Component reduction is often greatly simplified if suitable gauge conditions are imposed. Making use of the Weyl and local S-supersymmetry transformations allow us to choose the gauge conditions

$$\mathbb{G}| = 1, \tag{6.14a}$$

$$\mathcal{D}_\alpha\mathbb{G}| = 0. \tag{6.14b}$$

The compensator also contains a real scalar component field that can be defined as

$$Z := \mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathbb{G}|. \tag{6.15}$$

It is also useful to choose a WZ gauge for the U(1) gauge symmetry (6.4). A standard choice is

$$G| = 0, \tag{6.16a}$$

$$\mathcal{D}_\alpha G| = 0, \tag{6.16b}$$

$$\mathcal{D}^2 G| = 0. \tag{6.16c}$$

It then follows from (6.14) and (6.16) that

$$\mathcal{D}^2\bar{\mathcal{D}}_\alpha G| = 0, \tag{6.17a}$$

$$\bar{\mathcal{D}}^\alpha\mathcal{D}^2 G| = \frac{\mathrm{i}}{2}(\bar{\psi}^b\gamma_a\tilde{\gamma}_b)^\alpha a^a + (\bar{\psi}^a\tilde{\gamma}_a)^\alpha, \tag{6.17b}$$

$$-\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}^2 G| = \frac{\mathrm{i}}{2}\mathbf{D}_a a^a + \frac{1}{4}\bar{\psi}^b\gamma_a\psi_b a^a + \frac{\mathrm{i}}{2}\bar{\psi}^a\psi_a + \frac{1}{2}Z. \tag{6.17c}$$

The only independent component fields of G are

$$[\mathcal{D}_{(\alpha},\bar{\mathcal{D}}_{\beta)}]G| = \frac{1}{2}a_{\alpha\beta}, \tag{6.18a}$$

$$(\bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha)^2 G| = -Z. \tag{6.18b}$$

By construction, the scalar Z is invariant under the gauge transformations (6.4).

The component supergravity Lagrangian is

$$L_{\text{Poincaré}} = \frac{1}{2}\mathcal{R}(e,\psi) + \frac{\mathrm{i}}{4}\varepsilon^{abc}(\bar{\boldsymbol{\psi}}_{ab}\psi_c + \bar{\psi}_a\boldsymbol{\psi}_{bc}) + a_a\mathcal{F}^a - \frac{1}{4}\tilde{\mathcal{H}}_a\tilde{\mathcal{H}}^a - \frac{1}{4}Z^2, \tag{6.19}$$

where we have introduced the combination

$$\tilde{\mathcal{H}}^a := \mathcal{H}^a - \varepsilon^{abc}\bar{\psi}_b\psi_c. \tag{6.20}$$

The gravitino field strength is defined as in (4.14), with $\mathbf{D}_a = \mathbf{D}_a(e,\psi,b)$ the covariant derivative containing the $\mathrm{U}(1)_R$ connection b_a. In this formulation, the supergravity multiplet consists of the following fields: the dreibein $e_m{}^a$, the gravitini $\psi_m{}^\alpha$ and $\bar{\psi}_{m\alpha}$, the two gauge fields a_m and b_m, and the auxiliary scalar Z.

It is not difficult to demonstrate that the vector fields a_a and b_a have no propagating degrees of freedom for the dynamical system (6.19). To see this, let us work out the equation of motion for the $\mathrm{U}(1)_R$ gauge field b_a. In the supergravity Lagrangian (6.19), this field appears both in the Rarita-Schwinger and Chern-Simons terms. We note that

$$\int \mathrm{d}^3x e a_a\mathcal{F}^a = \int \mathrm{d}^3x e b_a\mathcal{H}^a, \tag{6.21}$$

modulo a total derivative. Another relevant observation is that the Rarita-Schwinger Lagrangian depends on b_a only via the linear term $-\varepsilon^{abc}b_a\bar{\psi}_b\psi_c$. As a result, the equation of motion for b_a is

$$\tilde{\mathcal{H}}^a = 0. \tag{6.22}$$

This equation tells us that a_a has no independent degrees of freedom on the mass shell. Now, varying (6.19) with respect to a_a and making use of (6.22) gives

$$\mathcal{F}^a = 0, \tag{6.23}$$

and therefore the $\mathrm{U}(1)_R$ connection b_a is flat and may completely be gauged away.

The off-shell Lagrangian (6.19) does not coincide with that proposed in [14] to describe (2,0) Poincaré supergravity (in our terminology, type II supergravity without a cosmological term),; see Eq. (4.1) in [14]. In particular, the Lagrangian given in [14] contains no $\mathcal{H}_a\mathcal{H}^a$ term. The two Lagrangians are actually equivalent modulo a total derivative and a redefinition of the b_a field.[14] Indeed, making use of (6.21) and defining

$$b_a \to b'_a = b_a - \frac{1}{4}\tilde{\mathcal{H}}_a, \tag{6.24}$$

the Lagrangian (6.19) takes the form

$$L_{\text{Poincaré}} = \frac{1}{2}\mathcal{R}(e,\psi) + \frac{\mathrm{i}}{4}\varepsilon^{abc}(\bar{\boldsymbol{\psi}}'_{ab}\psi_c + \bar{\psi}_a\boldsymbol{\psi}'_{bc}) + b'_a\mathcal{H}^a - \frac{1}{4}Z^2, \tag{6.25}$$

where the gravitino field strength $\boldsymbol{\psi}'_{ab}$ is defined as (4.14) but with the $\mathrm{U}(1)_R$ connection b_a replaced by b'_a. The Lagrangian (6.25) is equivalent to the one given in [14].

C. (2,0) anti-de Sitter supergravity

The main difference between type II supergravity and the new minimal formulation for $\mathcal{N}=1$ supergravity in four dimensions is that the action (6.11) can be deformed by adding a gauge-invariant cosmological term

$$S_{\text{cosm}} = -4\xi\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}EG\mathbb{G}. \tag{6.26}$$

To evaluate its component form, we have to make use of the supersymmetric action principle (3.13) with

$$\bar{\mathcal{L}}_c = \xi(\mathcal{D}^2 - 4\bar{R})(G\mathbb{G}) = \xi\mathbb{G}\mathcal{D}^2G + 2\xi(\mathcal{D}^\alpha\mathbb{G})\mathcal{D}_\alpha G. \tag{6.27}$$

A short calculation that makes use of (6.17c) leads to

$$L_{\text{cosm}} = \xi\left(Z + \frac{1}{4}a_a\mathcal{H}^a - \frac{\mathrm{i}}{2}\varepsilon^{abc}\bar{\psi}_a\gamma_b\psi_c\right). \tag{6.28}$$

The superfield action for (2,0) AdS supergravity is

$$S_{\text{AdS}} = 4\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E(\mathbb{G}\ln\mathbb{G} - 4G\mathcal{S} - \xi G\mathbb{G}). \tag{6.29}$$

[14] G. T.-M. is grateful to Daniel Butter for pointing out the same situation in the new minimal formulation for 4D $\mathcal{N}=1$ supergravity (see, e.g., [63,64] for the relevant discussions).

The component Lagrangian for off-shell (2,0) AdS supergravity is

$$L_{\text{AdS}} = \frac{1}{2}\mathcal{R}(e,\psi) + \frac{\mathrm{i}}{4}\varepsilon^{abc}(\bar{\boldsymbol{\psi}}_{ab}\psi_c + \bar{\psi}_a\boldsymbol{\psi}_{bc}) + a_a\mathcal{F}^a - \frac{1}{4}\tilde{\mathcal{H}}_a\tilde{\mathcal{H}}^a - \frac{1}{4}Z^2 + \xi\left(Z + \frac{1}{4}a_a\mathcal{H}^a - \frac{\mathrm{i}}{2}\varepsilon^{abc}\bar{\psi}_a\gamma_b\psi_c\right). \tag{6.30}$$

In this theory, the equation of motion for the $\mathrm{U}(1)_R$ gauge field b_a is still given by (6.22). As concerns the equation of motion for a_a, it becomes

$$\mathcal{F}^a + \frac{1}{2}\xi\mathcal{H}^a = 0. \tag{6.31}$$

We see that the local $\mathrm{U}(1)_R$ gauge freedom can be completely fixed by imposing the condition $a_a = -\frac{2}{\xi}b_a$.

Dynamics described by the off-shell theory (6.30) is equivalent to that generated by

$$\tilde{L}_{\text{AdS}} = \frac{1}{2}\mathcal{R}(e,\psi) + \frac{\mathrm{i}}{4}\varepsilon^{abc}(\bar{\boldsymbol{\psi}}_{ab}\psi_c + \bar{\psi}_a\boldsymbol{\psi}_{bc} - 2\xi\bar{\psi}_a\gamma_b\psi_c) + \xi^2 - \frac{1}{\xi}b_a\mathcal{F}^a. \tag{6.32}$$

One can recognize (6.32) to be the standard on-shell Lagrangian for (2,0) AdS supergravity [5] (see also [7]). The third term in the parentheses in (6.32) may be absorbed into the gravitino field strength by introducing a modified covariant derivative

$$\hat{\mathbf{D}}_a\psi_b{}^\beta = \mathbf{D}_a\psi_b{}^\beta - \frac{1}{2}\xi(\gamma_a)_\gamma{}^\beta\psi_b{}^\gamma. \tag{6.33}$$

D. Supersymmetry transformations

The gauge conditions (6.14) completely fix the Weyl and local S-supersymmetry freedom. To preserve the condition $\mathbb{G}| = 1$, no residual Weyl invariance remains, $\boldsymbol{\sigma} = 0$. However, each Q-supersymmetry transformation has to be accompanied by a compensating S-supersymmetry transformation in order to preserve the condition $\mathcal{D}_\alpha\mathbb{G}| = 0$. Indeed, the field $\mathcal{D}_\alpha\mathbb{G}|$ transforms as

$$(\delta_Q + \delta_S)\mathcal{D}_\alpha\mathbb{G}| = \epsilon^\beta\mathcal{D}_\beta\mathcal{D}_\alpha\mathbb{G}| + \bar{\epsilon}_\beta\bar{\mathcal{D}}^\beta\mathcal{D}_\alpha\mathbb{G}| + \eta_\alpha\mathbb{G}| = \frac{1}{2}\tilde{\mathcal{H}}_a(\gamma^a\bar{\epsilon})_\alpha - \frac{\mathrm{i}}{2}Z\bar{\epsilon}_\alpha + \eta_\alpha, \tag{6.34}$$

where here we have used the identities (6.6)–(6.8), (6.14), and (6.15). We have to require $(\delta_Q + \delta_S)\mathcal{D}_\alpha\mathbb{G}| = 0$, and therefore

$$\eta_\alpha(\epsilon) = -\frac{1}{2}\tilde{\mathcal{H}}_a(\gamma^a\bar{\epsilon})_\alpha + \frac{\mathrm{i}}{2}Z\bar{\epsilon}_\alpha. \tag{6.35}$$

Choosing $\boldsymbol{\sigma} = \boldsymbol{\tau} = 0$ and $\eta_\alpha = \eta_\alpha(\epsilon)$ in (4.34), we obtain the supersymmetry transformations of the gauge fields $e_m{}^a$, $\psi_m{}^\gamma$ and b_m:

$$\delta_\epsilon e_m{}^a = \mathrm{i}(\epsilon\gamma^a\bar{\psi}_m + \bar{\epsilon}\gamma^a\psi_m), \tag{6.36a}$$

$$\delta_\epsilon\psi_m{}^\alpha = 2\mathbf{D}_m\epsilon^\alpha + \frac{\mathrm{i}}{2}\tilde{\mathcal{H}}_m\epsilon^\alpha - \frac{\mathrm{i}}{2}e_m{}^a\varepsilon_{abc}\tilde{\mathcal{H}}^c(\tilde{\gamma}^b\epsilon)^\alpha + \frac{1}{2}Z(\tilde{\gamma}_m\epsilon)^\alpha, \tag{6.36b}$$

$$\delta_\epsilon b_m = -\frac{1}{4}e_m{}^a\{\varepsilon_{abc}\epsilon\bar{\boldsymbol{\psi}}^{bc} + 2\epsilon\gamma^b\bar{\boldsymbol{\psi}}_{ab} - \mathrm{i}\epsilon\gamma^b\bar{\psi}_a\tilde{\mathcal{H}}_b - \epsilon\bar{\psi}_a Z\} + \text{c.c.} \tag{6.36c}$$

The supergravity multiplet also includes the fields a_m and Z. The supersymmetry transformation of a_m follows from its definition $a_m = e_m{}^a a_a$, with a_a originating as a component field of G, Eq. (6.18a). Note that, in order to preserve the WZ gauge (6.16), in computing the supersymmetry transformations of a_m it is necessary to include a compensating ϵ-dependent U(1) gauge transformation (6.4) with parameter $\Lambda(\epsilon)$ such that

$$\Lambda(\epsilon)| = 0, \tag{6.37a}$$

$$\mathcal{D}_\alpha\Lambda(\epsilon)| = -\frac{1}{4}(\bar{\epsilon}\gamma^b)_\alpha a_b + \frac{\mathrm{i}}{2}\bar{\epsilon}_\alpha, \tag{6.37b}$$

$$\mathcal{D}^2\Lambda(\epsilon)| = \frac{\mathrm{i}}{2}\bar{\epsilon}\gamma^a\tilde{\gamma}^b\bar{\psi}_a a_b - \bar{\epsilon}\gamma^a\bar{\psi}_a. \tag{6.37c}$$

We then obtain

$$\begin{aligned}\delta_\epsilon a_m &= (\delta_\epsilon e_m{}^a)a_a - e_m{}^a(\gamma_a)^{\alpha\beta} \\ &\quad\times(\epsilon^\gamma\mathcal{D}_\gamma[\mathcal{D}_\alpha,\bar{\mathcal{D}}_\beta]G| + 2\mathrm{i}\mathcal{D}_{\alpha\beta}\Lambda(\epsilon)| + \text{c.c.}) \\ &= \mathrm{i}\epsilon\gamma^a\bar{\psi}_m a_a + (\gamma_m)^{\alpha\beta}\epsilon^\gamma\mathcal{D}_\gamma \\ &\quad\times\{\mathcal{D}_\alpha,\bar{\mathcal{D}}_\beta\}G| + 4\mathrm{i}\psi_m{}^\alpha\mathcal{D}_\alpha\Lambda(\epsilon)| + \text{c.c.}\end{aligned} \tag{6.38}$$

Evaluating this variation gives

$$\delta_\epsilon a_m = -2(\epsilon\bar{\psi}_m + \bar{\epsilon}\psi_m). \tag{6.39}$$

The scalar field Z originates as a component field of $\mathbb{G}$, Eq. (6.15), and therefore its supersymmetry transformation is

$$\delta_\epsilon Z = \frac{\mathrm{i}}{2}\epsilon_\alpha\mathcal{D}^2\bar{\mathcal{D}}^\alpha\mathbb{G}| + \frac{\mathrm{i}}{2}\bar{\epsilon}_\alpha\bar{\mathcal{D}}^2\mathcal{D}^\alpha\mathbb{G}| + \mathrm{i}(\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\sigma)\mathbb{G}|. \tag{6.40}$$

Making use of (4.33b), we then derive

$$\begin{aligned}\delta_\epsilon Z &= -\frac{\mathrm{i}}{2}\epsilon\gamma^a\bar{\psi}_a Z - \frac{1}{2}\varepsilon^{abc}\epsilon\gamma_a\bar{\psi}_b\tilde{\mathcal{H}}_c + \frac{1}{2}\epsilon\bar{\psi}_a\tilde{\mathcal{H}}^a \\ &\quad + \frac{\mathrm{i}}{2}\varepsilon^{abc}\epsilon\gamma_a\bar{\boldsymbol{\psi}}_{bc} + \text{c.c.}\end{aligned} \tag{6.41}$$

For completeness, let us also work out the supersymmetry transformation of the field strength $\tilde{\mathcal{H}}_a$. Making use of the definition of $\tilde{\mathcal{H}}_a$ gives

$$\begin{aligned}\delta_\epsilon\tilde{\mathcal{H}}^a &= -\frac{1}{2}(\gamma^a)^{\alpha\beta}\{\epsilon^\gamma\mathcal{D}_\gamma[\mathcal{D}_\alpha,\bar{\mathcal{D}}_\beta]\mathbb{G}| + \bar{\epsilon}_\gamma\bar{\mathcal{D}}^\gamma[\mathcal{D}_\alpha,\bar{\mathcal{D}}_\beta]\mathbb{G}| \\ &\quad + ([\mathcal{D}_\alpha,\bar{\mathcal{D}}_\beta]\sigma)\mathbb{G}|\}.\end{aligned} \tag{6.42}$$

With the aid of (4.33a) we obtain

$$\begin{aligned}\delta_\epsilon\tilde{\mathcal{H}}^a &= -\frac{\mathrm{i}}{2}\varepsilon^{abc}\epsilon\bar{\psi}_b\tilde{\mathcal{H}}_c + \mathrm{i}\epsilon\gamma^{[a}\bar{\psi}_b\tilde{\mathcal{H}}^{b]} \\ &\quad + \frac{1}{2}\varepsilon^{abc}\epsilon\gamma_b\bar{\psi}_c Z - \varepsilon^{abc}\epsilon\bar{\boldsymbol{\psi}}_{bc} + \text{c.c.}\end{aligned} \tag{6.43}$$

E. Matter-coupled supergravity

The action for a locally supersymmetric σ-model coupled to type II supergravity is

$$S_{\text{matter}} = \int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E\mathbb{G}K(\varphi,\bar{\varphi}). \tag{6.44}$$

Here the Kähler potential $K(\varphi,\bar{\varphi})$ and the matter superfields are the same as in Sec. V. In particular, the covariantly chiral superfields φ^I are super-Weyl and $\mathrm{U}(1)_R$ neutral, $\delta_\sigma\varphi^I = \mathcal{J}\varphi^I = 0$. The action is invariant under the Kähler transformations (5.16a) due to the identity

$$\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E\mathbb{G}\Lambda(\varphi) = 0. \tag{6.45}$$

In order to carry out the component reduction of S_{matter}, we associate with (6.44) the antichiral Lagrangian

$$\bar{\mathcal{L}}_c = -\frac{1}{4}(\mathcal{D}^2 - 4\bar{R})(\mathbb{G}K) = -\frac{1}{4}\mathbb{G}\mathcal{D}^2K - \frac{1}{2}(\mathcal{D}^\alpha\mathbb{G})\mathcal{D}_\alpha K. \tag{6.46}$$

The component fields of φ^I are defined as in (5.22). Unlike the type I supergravity case, now we do not have to modify the gauge conditions on the compensator in the presence of matter. Direct calculations lead to the following component Lagrangian:

$$L_{\rm matter} = g_{I\bar J}\Big[F^I\bar F^{\bar J} - (\mathfrak{D}_a X^I)\mathfrak{D}^a\bar X^{\bar J} - \frac{i}{4}\lambda^I\gamma^a\overset{\leftrightarrow}{\tilde{\mathbf{D}}}_a\bar\lambda^{\bar J} - \frac{1}{2}\bar\psi^a\bar\lambda^{\bar J}\mathfrak{D}_a X^I + \frac{1}{2}\psi_a\lambda^I\mathfrak{D}^a\bar X^{\bar J}$$
$$- \frac{1}{2}\varepsilon^{abc}(\psi_a\gamma_b\lambda^I\mathfrak{D}_c\bar X^{\bar J} - \bar\psi_a\gamma_b\bar\lambda^{\bar J}\mathfrak{D}_c X^I) + \frac{1}{8}\psi_a\gamma^b\bar\psi^a\lambda^I\gamma_b\bar\lambda^{\bar J} - \frac{1}{8}\psi_a\bar\psi^a\lambda^I\bar\lambda^{\bar J}$$
$$+ \frac{1}{8}\varepsilon^{abc}(\psi_a\bar\psi_b\lambda^I\gamma_c\bar\lambda^{\bar J} + \psi_a\gamma_b\bar\psi_c\lambda^I\bar\lambda^{\bar J}) + \frac{1}{8}\lambda^I\gamma_a\bar\lambda^{\bar J}\tilde{\mathcal{H}}^a - \frac{i}{8}Z\lambda^I\bar\lambda^{\bar J}\Big]$$
$$+ \frac{i}{4}\mathcal{H}^a[K_{\bar I}\mathfrak{D}_a\bar X^{\bar I} - K_I\mathfrak{D}_a X^I] + \frac{1}{16}R_{I\bar K J\bar L}\lambda^I\lambda^J\bar\lambda^{\bar K}\bar\lambda^{\bar L}. \qquad (6.47)$$

Here we have introduced the Kähler-covariant derivative

$$\tilde{\mathbf{D}}_a\lambda^I \coloneqq \mathbf{D}_a\lambda^I + \lambda^J\Gamma^I_{JK}\mathfrak{D}_a X^K$$
$$= \mathfrak{D}_a\lambda^I + i b_a\lambda^I + \lambda^J\Gamma^I_{JK}\mathfrak{D}_a X^K. \qquad (6.48)$$

The σ-model action generated by the Lagrangian (6.47) proves to be invariant under the Kähler transformations. The first term in the fourth line of (6.47) is the only one which varies under the Kähler transformations. The corresponding contribution to the action is indeed Kähler invariant due to the identity $\int d^3x e\mathcal{H}^a\mathfrak{D}_a\Lambda = 0$.

As may be seen from (6.47), the gauge fields b_a and a_a couple to conserved currents of completely different types. The $U(1)_R$ gauge field couples to the $U(1)_R$ Noether current

$$\mathcal{J}^a_{\rm Noether} = \varepsilon^{abc}\bar\psi_b\psi_c + \frac{1}{2}g_{I\bar J}\lambda^I\gamma^a\bar\lambda^{\bar J}. \qquad (6.49)$$

As regards the gauge field a_a, it couples to the topological current

$$\mathcal{J}^a_{\rm top} = \frac{1}{2}\varepsilon^{abc}(\mathfrak{D}_b\mathfrak{R}_c - \mathfrak{D}_c\mathfrak{R}_b - \mathcal{T}_{bc}{}^d\mathfrak{R}_d),$$
$$\mathfrak{R}_a \coloneqq i(K_{\bar I}\mathfrak{D}_a\bar X^{\bar I} - K_I\mathfrak{D}_a X^I), \qquad (6.50)$$

which is identically conserved. These properties were pointed out in [14].

Now, we consider a complete supergravity-matter system described by the action [11]

$$S = 4\int d^3x d^2\theta d^2\bar\theta E\left(\mathbb{G}\left\{\ln\mathbb{G} + \frac{1}{4}K(\varphi,\bar\varphi)\right\} - 4G\mathcal{S}\right). \qquad (6.51)$$

It describes Poincaré supergravity coupled to the locally supersymmetric σ-model. As shown in [11], this theory is dual to the type I supergravity-matter system (5.17). To compute the corresponding component Lagrangian, we combine $L_{\rm matter}$ given by (6.47) with the type II supergravity Lagrangian without cosmological term, Eq. (6.19). The result is

$$L = \frac{1}{2}\mathcal{R}(e,\psi) + \frac{i}{4}\varepsilon^{abc}(\bar{\boldsymbol\psi}_{ab}\psi_c + \bar\psi_a\boldsymbol{\psi}_{bc}) + a_a\mathcal{F}^a - \frac{1}{4}\tilde{\mathcal{H}}_a\tilde{\mathcal{H}}^a - \frac{1}{4}\mathbb{Z}^2$$
$$+ g_{I\bar J}\Big[F^I\bar F^{\bar J} - (\mathfrak{D}_a X^I)\mathfrak{D}^a\bar X^{\bar J} - \frac{i}{4}\lambda^I\gamma^a\overset{\leftrightarrow}{\tilde{\mathbf{D}}}_a\bar\lambda^{\bar J} - \frac{1}{2}\bar\psi^a\bar\lambda^{\bar J}\mathfrak{D}_a X^I + \frac{1}{2}\psi_a\lambda^I\mathfrak{D}^a\bar X^{\bar J}$$
$$- \frac{1}{2}\varepsilon^{abc}(\psi_a\gamma_b\lambda^I\mathfrak{D}_c\bar X^{\bar J} - \bar\psi_a\gamma_b\bar\lambda^{\bar J}\mathfrak{D}_c X^I) + \frac{1}{8}\psi_a\gamma^b\bar\psi^a\lambda^I\gamma_b\bar\lambda^{\bar J} - \frac{1}{8}\psi_a\bar\psi^a\lambda^I\bar\lambda^{\bar J}$$
$$+ \frac{1}{8}\varepsilon^{abc}(\psi_a\bar\psi_b\lambda^I\gamma_c\bar\lambda^{\bar J} + \psi_a\gamma_b\bar\psi_c\lambda^I\bar\lambda^{\bar J}) + \frac{1}{8}g_{I\bar J}\lambda^I\gamma_a\bar\lambda^{\bar J}\tilde{\mathcal{H}}^a\Big]$$
$$+ \frac{i}{4}\mathcal{H}^a[K_{\bar I}\mathfrak{D}_a\bar X^{\bar I} - K_I\mathfrak{D}_a X^I] + \frac{1}{16}R_{I\bar K J\bar L}\lambda^I\lambda^J\bar\lambda^{\bar K}\bar\lambda^{\bar L} - \frac{1}{64}(g_{I\bar J}\lambda^I\bar\lambda^{\bar J})^2, \qquad (6.52)$$

where we have defined

$$\mathbb{Z} \coloneqq Z + \frac{i}{4}g_{I\bar J}\lambda^I\bar\lambda^{\bar J}. \qquad (6.53)$$

Let us show that the dynamical system (6.52) is equivalent to the type I supergravity-matter system (5.31) with $W=0$. Integrating out Z gives

$$\mathbb{Z} = 0. \qquad (6.54)$$

The equation of motion for the gauge field b_a is

$$\mathbb{H}^a \coloneqq \tilde{\mathcal{H}}^a - \frac{1}{2}g_{I\bar J}\lambda^I\gamma^a\bar\lambda^{\bar J}$$
$$= \mathcal{H}^a - \varepsilon^{abc}\bar\psi_b\psi_c - \frac{1}{2}g_{I\bar J}\lambda^I\gamma^a\bar\lambda^{\bar J} = 0. \qquad (6.55)$$

Let us consider the equation of motion for the gauge field a_a. It can be represented in the form

$$\mathfrak{D}_a \mathbb{B}_b - \mathfrak{D}_b \mathbb{B}_a - \mathcal{T}_{ab}{}^c \mathbb{B}_c = 0, \tag{6.56}$$

where $\mathbb{B}_a$ is defined in (5.24). This equation tells us that the local $\mathrm{U}(1)_R$ gauge freedom can be completely fixed by choosing the condition

$$b_a = \frac{1}{8} g_{I\bar{J}} \lambda^I \gamma_a \bar{\lambda}^{\bar{J}} + \frac{\mathrm{i}}{4}(K_I \mathfrak{D}_a X^I - K_{\bar{I}} \mathfrak{D}_a \bar{X}^{\bar{I}}). \tag{6.57}$$

Making use of the equations (6.54), (6.55), and (6.57) reduces the supergravity-matter system (6.52) to that described by the Lagrangian (5.31) with $W = 0$.

To preserve the gauge condition (6.57), any Kähler transformation generated by a parameter Λ has to be accompanied by a special $\mathrm{U}(1)_R$ transformation with parameter $\tau = \frac{\mathrm{i}}{4}(\bar{\Lambda} - \Lambda)$; see also Eq. (5.21).

Finally, we generalize the supergravity-matter system (6.51) to include a cosmological term. The manifestly supersymmetric action is

$$S = 4 \int \mathrm{d}^3x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} E \left(\mathbb{G} \left\{ \ln \mathbb{G} + \frac{1}{4} K(\varphi, \bar{\varphi}) \right\} - 4G\mathcal{S} - \xi G \mathbb{G} \right). \tag{6.58}$$

The corresponding component Lagrangian is obtained from the supergravity-matter Lagrangian (6.52) by adding the cosmological term (6.28). The result is

$$\begin{aligned} L = {}& \frac{1}{2}\mathcal{R}(e,\psi) + \frac{\mathrm{i}}{4}\varepsilon^{abc}(\bar{\boldsymbol{\psi}}_{ab}\psi_c + \bar{\psi}_a \boldsymbol{\psi}_{bc}) + a_a \mathcal{F}^a - \frac{1}{4}\tilde{\mathcal{H}}_a \tilde{\mathcal{H}}^a - \frac{1}{4}(\mathbb{Z} - 2\xi)^2 + \frac{1}{4}\xi a_a \mathcal{H}^a - \frac{\mathrm{i}}{2}\xi \varepsilon^{abc} \bar{\psi}_a \gamma_b \psi_c + \xi^2 \\ & + g_{I\bar{J}} \bigg[F^I \bar{F}^{\bar{J}} - (\mathfrak{D}_a X^I)\mathfrak{D}^a \bar{X}^{\bar{J}} - \frac{\mathrm{i}}{4}\lambda^I \gamma^a \overset{\leftrightarrow}{\tilde{\mathbf{D}}}_a \bar{\lambda}^{\bar{J}} - \frac{1}{2}\bar{\psi}^a \bar{\lambda}^{\bar{J}} \mathfrak{D}_a X^I + \frac{1}{2}\psi_a \lambda^I \mathfrak{D}^a \bar{X}^{\bar{J}} - \frac{1}{2}\varepsilon^{abc}(\psi_a \gamma_b \lambda^I \mathfrak{D}_c \bar{X}^{\bar{J}} - \bar{\psi}_a \gamma_b \bar{\lambda}^{\bar{J}} \mathfrak{D}_c X^I) \\ & + \frac{1}{8}\psi_a \gamma^b \bar{\psi}^a \lambda^I \gamma_b \bar{\lambda}^{\bar{J}} - \frac{1}{8}\psi_a \bar{\psi}^a \lambda^I \bar{\lambda}^{\bar{J}} + \frac{1}{8}\varepsilon^{abc}(\psi_a \bar{\psi}_b \lambda^I \gamma_c \bar{\lambda}^{\bar{J}} + \psi_a \gamma_b \bar{\psi}_c \lambda^I \bar{\lambda}^{\bar{J}}) + \frac{1}{8} g_{I\bar{J}} \lambda^I \gamma_a \bar{\lambda}^{\bar{J}} \tilde{\mathcal{H}}^a - \frac{\mathrm{i}}{4}\xi \lambda^I \bar{\lambda}^{\bar{J}} \bigg] \\ & + \frac{\mathrm{i}}{4}\mathcal{H}^a [K_{\bar{I}} \mathfrak{D}_a \bar{X}^{\bar{I}} - K_I \mathfrak{D}_a X^I] + \frac{1}{16} R_{I\bar{K}J\bar{L}} \lambda^I \lambda^J \bar{\lambda}^{\bar{K}} \bar{\lambda}^{\bar{L}} - \frac{1}{64}(g_{I\bar{J}} \lambda^I \bar{\lambda}^{\bar{J}})^2. \end{aligned} \tag{6.59}$$

We conclude this section by giving the supersymmetry transformations of the component field of φ^I:

$$\delta_\epsilon X^I = \epsilon \lambda^I, \tag{6.60a}$$

$$\delta_\epsilon \lambda^I_\alpha = 2\epsilon_\alpha \left(F^I + \frac{1}{4}\Gamma^I_{JK} \lambda^J \lambda^K \right) + 2\mathrm{i}(\gamma^a \bar{\epsilon})_\alpha \left(\mathbf{D}_a X^I - \frac{1}{2}\psi_a \lambda^I \right), \tag{6.60b}$$

$$\begin{aligned} \delta_\epsilon F^I = {}& -\epsilon \lambda^J \Gamma^I_{JK} F^K + \frac{1}{2}\lambda^I \eta(\epsilon) + \mathrm{i}\bar{\epsilon}\gamma^a \mathbf{D}_a \lambda^I - \frac{1}{4} g^{I\bar{L}} R_{J\bar{L}K\bar{P}} \bar{\epsilon} \bar{\lambda}^{\bar{P}} \lambda^J \lambda^K + \mathrm{i}\bar{\epsilon}\gamma^a \lambda^J \Gamma^I_{JK} \mathbf{D}_a X^K - \frac{\mathrm{i}}{2}\bar{\epsilon}\gamma^a \psi_a F^I \\ & - \bar{\epsilon}\gamma^a \tilde{\gamma}^b \bar{\psi}_a \left(\mathbf{D}_b X^I - \frac{1}{2}\psi_b \lambda^I \right). \end{aligned} \tag{6.60c}$$

It is a useful exercise for the reader to derive these transformation laws.

F. R-invariant sigma models

Type II minimal supergravity admits more general matter couplings [11] than those we have so far studied. In particular, it can be coupled to R-invariant σ-models, similarly to the new minimal $\mathcal{N} = 1$ supergravity in four dimensions (see, e.g., [65] for more details). Here we briefly discuss such theories.

We consider a system of covariantly chiral scalars ϕ^I of super-Weyl weights r_I,

$$\bar{\mathcal{D}}_\alpha \phi^I = 0, \qquad \mathcal{J}\phi^I = -r_I \phi^I, \qquad \delta_\sigma \phi^I = r_I \sigma \phi^I. \tag{6.61}$$

We introduce a supergravity-matter system of the form

$$\begin{aligned} S = {}& 4 \int \mathrm{d}^3x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} E \\ & \times \left(\mathbb{G} \left\{ \ln \mathbb{G} + \frac{1}{4}\boldsymbol{K}(\phi^I/\mathbb{G}^{r_I}, \bar{\phi}^{\bar{J}}/\mathbb{G}^{r_J}) \right\} - 4G\mathcal{S} \right) \\ & + \left\{ \int \mathrm{d}^3x \mathrm{d}^2\theta \mathcal{E} \boldsymbol{W}(\phi^I) + \text{c.c.} \right\}. \end{aligned} \tag{6.62}$$

This action is super-Weyl invariant if the superpotential $\boldsymbol{W}(\phi^I)$ obeys the homogeneity equation

$$\sum_I r_I \phi^I \boldsymbol{W}_I = 2\boldsymbol{W}. \tag{6.63}$$

The action is invariant under the local $\mathrm{U}(1)_R$ transformations if the Kähler potential $\boldsymbol{K}(\phi^I, \bar{\phi}^{\bar{J}})$ obeys the equation

$$\sum_I r_I \phi^I \boldsymbol{K}_I = \sum_{\bar{I}} r_I \bar{\phi}^{\bar{I}} \boldsymbol{K}_{\bar{I}}. \tag{6.64}$$

In a flat superspace limit, the theory (6.62) reduces to a general R-invariant nonlinear σ-model.

The action (6.62) may be reduced to components using the formalism developed above. In general, however, the Weyl and S-supersymmetry gauge conditions (6.14) have to be replaced with matter-dependent ones [similar to the gauge conditions (5.19) in type I supergravity] if we want the gravitational action to be given in Einstein frame. We will not give such an analysis here.

VII. TOPOLOGICALLY MASSIVE SUPERGRAVITY

Consider $\mathcal{N}=2$ conformal supergravity (CSG) coupled to matter supermultiplets. The supergravity-matter action is

$$S = \frac{1}{g} S_{\mathrm{CSG}} + S_{\mathrm{matter}}, \tag{7.1}$$

where S_{CSG} denotes the conformal supergravity action [4,66] and S_{matter} the matter action [10,11]. Both terms in (7.1) must be super-Weyl invariant. As regards S_{CSG}, the formulation given in [4] is purely component, and the concept of super-Weyl transformations is not defined within this approach. However, the super-Weyl invariance of S_{CSG} is manifest in the superspace formulation given recently in [66]; see Appendix D for a review. Requiring the super-Weyl invariance of S_{matter} is equivalent to the fact that this action will describe an $\mathcal{N}=2$ superconformal field theory in a flat superspace limit.

The equation of motion for conformal supergravity is

$$-\frac{4}{g}\mathcal{W}_{\alpha\beta} + \mathcal{J}_{\alpha\beta} = 0, \tag{7.2}$$

where $\mathcal{W}_{\alpha\beta}$ is the $\mathcal{N}=2$ super Cotton tensor, Eq. (2.12), and $\mathcal{J}_{\alpha\beta}$ is the matter supercurrent. This equation is obtained by varying S with respect to the real vector prepotential $H^{\alpha\beta} = H^{\beta\alpha}$ of conformal supergravity [15],

$$\mathcal{W}_{\alpha\beta} \propto \frac{\boldsymbol{\delta}}{\boldsymbol{\delta} H^{\alpha\beta}} S_{\mathrm{CSG}}, \qquad \mathcal{J}_{\alpha\beta} \propto \frac{\boldsymbol{\delta}}{\boldsymbol{\delta} H^{\alpha\beta}} S_{\mathrm{matter}}, \tag{7.3}$$

with $\boldsymbol{\delta}/\boldsymbol{\delta} H^{\alpha\beta}$ a covariantized variational derivative with respect to $H^{\alpha\beta}$. Equation (7.2) and the matter equations of motion determine the dynamics of the supergravity-matter system.

A. Properties of the supercurrent

The fundamental properties of the super Cotton tensor are (i) its super-Weyl transformation law (2.13); and (ii) the transversality condition [57]

$$\mathcal{D}^\beta \mathcal{W}_{\alpha\beta} = \bar{\mathcal{D}}^\beta \mathcal{W}_{\alpha\beta} = 0. \tag{7.4}$$

The matter supercurrent must have analogous properties. Specifically, it is characterized by the super-Weyl transformation law

$$\mathcal{J}'_{\alpha\beta} = \mathrm{e}^{2\sigma} \mathcal{J}_{\alpha\beta} \tag{7.5}$$

and obeys the conservation equation

$$\mathcal{D}^\beta \mathcal{J}_{\alpha\beta} = \bar{\mathcal{D}}^\beta \mathcal{J}_{\alpha\beta} = 0. \tag{7.6}$$

These must hold when the matter fields are subject to their equations of motion. Of course, the relations (7.5) and (7.6) may be viewed as the consistency conditions for the equation of motion (7.2). However, there is an independent way to justify (7.5) and (7.6) that follows from the definition of $\mathcal{J}_{\alpha\beta}$ as the covariantized variational derivative with respect to $H^{\alpha\beta}$. Here we only sketch the proof. For a more complete derivation, it is necessary to develop a background-quantum formalism for 3D $\mathcal{N}=2$ supergravity similar to that given by Grisaru and Siegel for $\mathcal{N}=1$ supergravity in four dimensions [67,68] (see [3] for a pedagogical review).

As demonstrated in [15], in complete analogy with the 4D case [69], the gravitational superfield originates via $\exp(-2\mathrm{i}H)$, where

$$H = \bar{H} = H^m \partial_m + H^\mu D_\mu + \bar{H}_\mu \bar{D}^\mu \tag{7.7}$$

and D_μ and $\bar{D}^\mu$ are the spinor covariant derivatives of Minkowski superspace. By construction, the superfields $H^M = (H^m, H^\mu, \bar{H}_\mu)$ are super-Weyl invariant. The supergravity gauge group can be used to gauge away H^μ and its conjugate, leaving us with the only unconstrained prepotential H^m. This prepotential possesses a highly nonlinear gauge transformation

$$\delta_L H_{\alpha\beta} = \bar{D}_{(\alpha} L_{\beta)} - D_{(\alpha} \bar{L}_{\beta)} + O(H), \tag{7.8}$$

where the gauge parameter L_α is an unconstrained complex spinor. Due to the nonlinear nature of this transformation, the gravitational superfield is not a tensor object, and special care is required in order to represent the variation of the action induced by a variation $H^m \to H^m + \delta H^m$ in a covariant way. This is what the background-quantum splitting in supergravity [67,68] is about.

It turns out that giving the gravitational superfield a finite displacement is equivalent to a deformation of the covariant derivatives that can be represented, in a chiral representation, as follows:

$$\bar{\mathcal{D}}^\alpha \to \bar{\mathcal{F}}\bar{\mathcal{D}}^\alpha + \cdots, \tag{7.9a}$$

$$\mathcal{D}_\alpha \to \mathrm{e}^{-2\mathrm{i}\mathbf{H}}(\mathcal{N}_\alpha{}^\beta \mathcal{F}\mathcal{D}_\beta + \cdots)\mathrm{e}^{2\mathrm{i}\mathbf{H}}, \quad \det(\mathcal{N}_\alpha{}^\beta) = 1, \tag{7.9b}$$

where

$$\mathbf{H} = -\frac{1}{2}\mathbf{H}^{\alpha\beta}\mathcal{D}_{\alpha\beta} - \frac{\mathrm{i}}{6}(\mathcal{D}_\beta \mathbf{H}^{\alpha\beta})\bar{\mathcal{D}}_\alpha - \frac{\mathrm{i}}{6}(\bar{\mathcal{D}}_\beta \mathbf{H}^{\alpha\beta})\mathcal{D}_\alpha + \cdots \tag{7.10}$$

The ellipses in these expressions denote all terms with Lorentz and $U(1)_R$ generators. The deformed covariant derivatives must obey the same constraints as the original ones $\mathcal{D}_A$. This can be shown to imply that the complex scalar $\mathcal{F}$ and the unimodular 2×2 matrix $\mathcal{N}$ are determined in terms of $\mathbf{H}^a$. The vector superfield $\mathbf{H}^a$ describes the finite deformation of the gravitational superfield. A crucial property of the first-order operator $\mathbf{H}$ is that it is super-Weyl invariant when acting on any super-Weyl inert real scalar $U = \bar{U}$,

$$\delta_\sigma \mathbf{H}\cdot U = 0, \tag{7.11}$$

provided $\mathbf{H}_{\alpha\beta}$ transforms as

$$\delta_\sigma \mathbf{H}_{\alpha\beta} = -\sigma \mathbf{H}_{\alpha\beta}. \tag{7.12}$$

The superfield $\mathbf{H}_{\alpha\beta}$ proves to be defined modulo gauge transformations of the form

$$\delta_L \mathbf{H}_{\alpha\beta} = \bar{\mathcal{D}}_{(\alpha}L_{\beta)} - \mathcal{D}_{(\alpha}\bar{L}_{\beta)} + O(\mathbf{H}), \tag{7.13}$$

which are compatible with the super-Weyl transformation (7.12) provided the gauge parameter is endowed with the properties

$$\mathcal{J}L_\alpha = L_\alpha, \qquad \delta_\sigma L_\alpha = -\frac{3}{2}\sigma L_\alpha. \tag{7.14}$$

Giving the gravitational superfield an infinitesimal displacement, $\mathbf{H}_a = \delta\mathbf{H}_a$, the matter action changes as

$$\delta S_{\mathrm{matter}} = \int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E\delta\mathbf{H}^a\mathcal{J}_a \equiv \int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E\delta\mathbf{H}^a\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}H^a}S_{\mathrm{matter}}. \tag{7.15}$$

This functional must be super-Weyl invariant. Due to Eqs. (3.3) and (7.12), and since the matter equations of motion hold, we conclude that the super-Weyl transformation of the supercurrent is given by Eq. (7.5). Since S_{matter} is invariant under the supergravity gauge transformations, choosing $\delta\mathbf{H}_{\alpha\beta} = \bar{\mathcal{D}}_{(\alpha}L_{\beta)} - \mathcal{D}_{(\alpha}\bar{L}_{\beta)}$ in (7.15) should give $\delta S_{\mathrm{matter}} = 0$ if the matter equations of motion hold. Since L_α is completely arbitrary, this is possible if and only if the conservation equation (7.6) holds.

B. Topologically massive minimal supergravity: Type I

Let us choose S_{matter} to be the superconformal sigma model (B2). The corresponding supercurrent proves to be

$$\mathcal{J}_{\alpha\beta} = N_{i\bar{j}}\mathcal{D}_{(\alpha}\phi^i\bar{\mathcal{D}}_{\beta)}\bar{\phi}^{\bar{j}} - \frac{1}{4}[\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]N - \mathcal{C}_{\alpha\beta}N. \tag{7.16}$$

The matter equations of motion are

$$-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4\bar{R})N_i + P_i = 0. \tag{7.17}$$

The relative coefficients in (7.16) are uniquely fixed if one demands the transversality condition (7.6) to hold on the mass shell, Eq. (7.17). Alternatively, it may be shown that the relative coefficients in (7.16) are uniquely fixed if one requires the super-Weyl transformation law (7.5). In the flat superspace limit, the supercurrent (7.16) reduces to the one given in [11].

We now turn to considering topologically massive type I supergravity. It is described by the action

$$S_{\mathrm{TMSG}} = \frac{1}{g}S_{\mathrm{CSG}} - S_{\mathrm{SG}}, \tag{7.18}$$

where S_{SG} is the action for type I supergravity with a cosmological term, Eq. (5.2). In topologically massive gravity [70] and its supersymmetric extensions [32,33], the Einstein term appears with the "wrong" sign. In the context of the σ-model action (B2), the matter sector in (7.18) corresponds to the choice $N = 4\bar{\Phi}\Phi$ and $P = -\mu\Phi^4$. The equation of motion for Φ is

$$\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\bar{\Phi} + \mu\Phi^3 = 0. \tag{7.19}$$

The equation of motion for the gravitational superfield (7.2) becomes

$$-\frac{4}{g}\left\{\frac{\mathrm{i}}{2}[\mathcal{D}^\gamma, \bar{\mathcal{D}}_\gamma]\mathcal{C}_{\alpha\beta} - [\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]\mathcal{S} - 4\mathcal{S}\mathcal{C}_{\alpha\beta}\right\} + 4\mathcal{D}_{(\alpha}\Phi\bar{\mathcal{D}}_{\beta)}\bar{\Phi} - [\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}](\bar{\Phi}\Phi) - 4\mathcal{C}_{\alpha\beta}\bar{\Phi}\Phi = 0. \tag{7.20}$$

As shown in [10], the freedom to perform the super-Weyl and local $\mathrm{U}(1)_R$ transformations can be used to impose the gauge[15]

[15]Upon gauge-fixing Φ to become constant, there still remain rigid scale and $\mathrm{U}(1)_R$ transformations that allow us to make f in (7.21) have any given value. The choice $f = 1$ leads to a canonically normalized Einstein-Hilbert term at the component level.

$$\Phi = \sqrt{f} = \text{const}, \tag{7.21}$$

which implies the conditions (4.35). Then, the matter equation of motion (7.19) turns into

$$R = \mu = \text{const}. \tag{7.22}$$

Using the identity $\mathcal{D}^\beta \mathcal{C}_{\alpha\beta} = -\frac{1}{2}\bar{\mathcal{D}}_\alpha \bar{R} - 2\mathrm{i}\mathcal{D}_\alpha \mathcal{S}$, which follows from (2.8c), we also obtain

$$\mathcal{D}^\beta \mathcal{C}_{\alpha\beta} = \bar{\mathcal{D}}^\beta \mathcal{C}_{\alpha\beta} = 0. \tag{7.23}$$

Now, the conformal supergravity equation (7.20) drastically simplifies

$$\frac{\mathrm{i}}{2}[\mathcal{D}^\gamma, \bar{\mathcal{D}}_\gamma]\mathcal{C}_{\alpha\beta} + gf\mathcal{C}_{\alpha\beta} = 0. \tag{7.24}$$

Equations (7.23) and (7.24) have a solution $\mathcal{C}_a = 0$, which corresponds to (i) a flat superspace for $\mu = 0$, or (ii) (1,1) anti-de Sitter superspace if $\mu \neq 0$. In the case $\mu = 0$, we can linearize Eq. (7.24) around Minkowski superspace. Its obvious implication is $(\Box - m^2)\mathcal{C}_a = 0$, where $m = \frac{1}{2}fg$.

Combining the Lagrangians (5.5) and (D12), we obtain the component Lagrangian for topologically massive type I supergravity

$$L_{\text{TMSG}} = \frac{1}{4g}\varepsilon^{abc}[\mathcal{R}_{bcfg}\omega_a{}^{fg} + \frac{2}{3}\omega_{af}{}^g\omega_{bg}{}^h\omega_{ch}{}^f - 4\mathcal{F}_{ab}b_c + \mathrm{i}\bar{\boldsymbol{\psi}}_{bc}\gamma_d\tilde{\gamma}_a\varepsilon^{def}\boldsymbol{\psi}_{ef}] - \frac{1}{2}\mathcal{R}(e,\psi) - \frac{\mathrm{i}}{4}\varepsilon^{abc}(\bar{\psi}_{ab}\psi_c + \bar{\psi}_a\psi_{bc})$$
$$+ \frac{1}{4}\bar{M}M - b^a b_a + \bar{\mu}\left(\bar{M} - \frac{1}{2}\varepsilon^{abc}\psi_a\gamma_b\psi_c\right) + \mu\left(M + \frac{1}{2}\varepsilon^{abc}\bar{\psi}_a\gamma_b\bar{\psi}_c\right). \tag{7.25}$$

The Lagrangian is computed in the Weyl, local $\mathrm{U}(1)_R$ and S-supersymmetry gauge (5.3). However, it is possible to avoid the use of (5.3). To achieve this the component form of S_{SG} has to be computed using the results of Appendix B.

C. Topologically massive minimal supergravity: Type II

Topologically massive type II supergravity is described by the action

$$S_{\text{TMSG}} = \frac{1}{g}S_{\text{CSG}} - S_{\text{AdS}}, \tag{7.26}$$

where S_{AdS} is the action for (2,0) AdS supergravity, Eq. (6.29). We can think of the theory with action (6.29) as a model for the vector multiplet coupled to background supergravity. Then, the equation of motion for G is

$$\mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha \ln \mathbb{G} - 4\mathcal{S} - 2\xi\mathbb{G} = 0. \tag{7.27}$$

The supercurrent corresponding to the action $S_{\text{matter}} = -S_{\text{AdS}}$ is

$$\mathcal{J}_{\alpha\beta} = \frac{4}{\mathbb{G}}\mathcal{D}_{(\alpha}\mathbb{G}\bar{\mathcal{D}}_{\beta)}\mathbb{G} - [\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]\mathbb{G} - 4\mathcal{C}_{\alpha\beta}\mathbb{G}. \tag{7.28}$$

It is an instructive exercise to show that $\mathcal{J}_{\alpha\beta}$ possesses the super-Weyl transformation law (7.5) and obeys the conservation equation (7.6) provided (7.27) holds. In the flat superspace limit, the supercurrent (7.28) reduces to the one given in [11].

Instead of (7.20), now the equation of motion for the gravitational superfield is

$$-\frac{4}{g}\left\{\frac{\mathrm{i}}{2}[\mathcal{D}^\gamma, \bar{\mathcal{D}}_\gamma]\mathcal{C}_{\alpha\beta} - [\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]\mathcal{S} - 4\mathcal{S}\mathcal{C}_{\alpha\beta}\right\}$$
$$+ \frac{4}{\mathbb{G}}\mathcal{D}_{(\alpha}\mathbb{G}\bar{\mathcal{D}}_{\beta)}\mathbb{G} - [\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]\mathbb{G} - 4\mathcal{C}_{\alpha\beta}\mathbb{G} = 0. \tag{7.29}$$

As shown in [10], the freedom to perform the super-Weyl transformations can be used to impose the gauge

$$\mathbb{G} = f = \text{const}, \tag{7.30}$$

which implies the constraint (4.39). Then the equation of motion (7.27) tells us that

$$\mathcal{S} = -\frac{\xi}{2} = \text{const}. \tag{7.31}$$

These properties lead to the constraint (7.23). As a result, the conformal supergravity equation (7.29) turns into

$$\frac{\mathrm{i}}{2}[\mathcal{D}^\gamma, \bar{\mathcal{D}}_\gamma]\mathcal{C}_{\alpha\beta} + (gf + 2\xi)\mathcal{C}_{\alpha\beta} = 0. \tag{7.32}$$

Equations (7.23) and (7.32) have a solution $\mathcal{C}_a = 0$, which corresponds either to a flat superspace for $\xi = 0$ or (2,0) anti-de Sitter superspace if $\xi \neq 0$.

Combining the Lagrangians (6.30) and (D12), we obtain the component Lagrangian for topologically massive type II supergravity

$$L_{\rm TMSG} = \frac{1}{4g}\varepsilon^{abc}\left[\mathcal{R}_{bcfg}\omega_a{}^{fg} + \frac{2}{3}\omega_{af}{}^g\omega_{bg}{}^h\omega_{ch}{}^f - 4\mathcal{F}_{ab}b_c + {\rm i}\bar{\boldsymbol{\psi}}_{bc}\gamma_d\tilde{\gamma}_a\varepsilon^{def}\boldsymbol{\psi}_{ef}\right]$$
$$-\frac{1}{2}\mathcal{R}(e,\psi) - \frac{\rm i}{4}\varepsilon^{abc}(\bar{\boldsymbol{\psi}}_{ab}\psi_c + \bar{\psi}_a\boldsymbol{\psi}_{bc}) - a_a\mathcal{F}^a + \frac{1}{4}\tilde{\mathcal{H}}_a\tilde{\mathcal{H}}^a + \frac{1}{4}Z^2 - \xi\left(Z + \frac{1}{4}a_a\mathcal{H}^a - \frac{\rm i}{2}\varepsilon^{abc}\bar{\psi}_a\gamma_b\psi_c\right). \quad (7.33)$$

The Lagrangian is computed in the Weyl and local S-supersymmetry gauge 6.14. However, one can avoid the use of 6.14. To achieve this the component form of $S_{\rm AdS}$ has to be computed using the results of Appendix C.

D. Topologically massive nonminimal supergravity

Topologically massive nonminimal supergravity is described by the action

$$S_{\rm TMSG} = \frac{1}{g}S_{\rm CSG} - S_{\rm AdS}, \quad (7.34)$$

where $S_{\rm AdS}$ denotes the action for nonminimal (1,1) AdS supergravity [11]

$$S_{\rm AdS} = -2\int {\rm d}^3x{\rm d}^2\theta{\rm d}^2\bar{\theta}E(\bar{\Gamma}\Gamma)^{-1/2}. \quad (7.35)$$

The dynamical variable Γ is a deformed complex linear scalar Γ obeying the constraint (1.2). If we think of (7.35) as the action describing the dynamics of matter superfields Γ and $\bar{\Gamma}$ in a background curved superspace, then this theory is dual to the type I minimal model (5.2); see [11] for more details. As a result, topologically massive nonminimal supergravity is dual to that constructed in Sec. VII B. To relate the two theories, it suffices to note that when Γ and $\bar{\Gamma}$ are subject to their equations of motion, we can represent

$$\Gamma = \Phi^{-3}\bar{\Phi}, \quad (7.36)$$

where Φ is a chiral scalar of super-Weyl weight 1/2 under the equation of motion (7.19).

VIII. SYMMETRIES OF CURVED SUPERSPACE

In this section we derive the conditions for a curved superspace to possess (conformal) isometries. After that we concentrate on a discussion of curved backgrounds admitting conformal and rigid supersymmetries.

A. Conformal isometries

Consider some background superspace $\mathcal{M}^{3|4}$ such that its geometry is of the type described in Sec. II A. In order to formulate rigid superconformal or rigid supersymmetric field theories on $\mathcal{M}^{3|4}$, it is necessary to determine all (conformal) isometries of this superspace. This can be done similarly to the case of 4D $\mathcal{N}=1$ supergravity described in detail in [3] and reviewed in [51]. In this subsection we study the infinitesimal conformal isometries of $\mathcal{M}^{3|4}$.

Let $\xi = \xi^A E_A$ be a real supervector field on $\mathcal{M}^{3|4}$, $\xi^A \equiv (\xi^a, \xi^\alpha, \bar{\xi}_\alpha)$. It is called conformal Killing if one can associate with ξ a supergravity gauge transformation[16] (2.4) and an infinitesimal super-Weyl transformation (2.11) such that their combined action does not change the covariant derivatives,

$$(\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_A = 0. \quad (8.1)$$

Since the vector covariant derivative $\mathcal{D}_a$ is given in terms of an anticommutator of two spinor ones, it suffices to analyze the implications of (8.1) for the case $A=\alpha$. A short calculation gives

$$(\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_\alpha = \left\{\frac{1}{2}(\sigma + 2{\rm i}\tau)\varepsilon_{\alpha\beta} + \mathcal{D}_\alpha\xi_\beta + {\rm i}\xi_{(\alpha}{}^\gamma\mathcal{C}_{\beta)\gamma} - \xi_{\alpha\beta}\mathcal{S} - \frac{1}{2}K_{\alpha\beta}\right\}\mathcal{D}^\beta - \{\mathcal{D}_\alpha\bar{\xi}_\beta + {\rm i}\xi_{\alpha\beta}\bar{R}\}\bar{\mathcal{D}}^\beta + \left\{\frac{1}{2}\mathcal{D}_\alpha\xi_{\beta\gamma} - 2{\rm i}\varepsilon_{\alpha(\beta}\bar{\xi}_{\gamma)}\right\}\mathcal{D}^{\beta\gamma}$$
$$- \left[\varepsilon_{\alpha(\beta}(\mathcal{D}_{\gamma)}\sigma) + \frac{1}{2}\mathcal{D}_\alpha K_{\beta\gamma} - 4\varepsilon_{\alpha(\beta}\xi_{\gamma)}\bar{R} - 4{\rm i}\varepsilon_{\alpha(\beta}\bar{\xi}_{\gamma)}\mathcal{S} - 2\bar{\xi}_\alpha\mathcal{C}_{\beta\gamma} + \xi_\alpha{}^\delta\boldsymbol{C}_{\beta\gamma\delta}\right.$$
$$\left. - \frac{2}{3}\varepsilon_{\alpha(\beta}\xi_{\gamma)\delta}(2\mathcal{D}^\delta\mathcal{S} + {\rm i}\bar{\mathcal{D}}^\delta\bar{R}) + \frac{1}{6}(2\mathcal{D}_\alpha\mathcal{S} + {\rm i}\bar{\mathcal{D}}_\alpha\bar{R})\xi_{\beta\gamma}\right]\mathcal{M}^{\beta\gamma}$$
$$- \left[\mathcal{D}_\alpha(\sigma + {\rm i}\tau) - 2\bar{\xi}^\beta\mathcal{C}_{\alpha\beta} - 4{\rm i}\bar{\xi}_\alpha\mathcal{S} + \frac{1}{2}\xi^{\beta\gamma}\boldsymbol{C}_{\alpha\beta\gamma} - \frac{1}{6}\xi_{\alpha\gamma}(8\mathcal{D}^\gamma\mathcal{S} + {\rm i}\bar{\mathcal{D}}^\gamma\bar{R})\right]\mathcal{J}. \quad (8.2)$$

[16]Strictly speaking, the parameters of gauge transformations are usually restricted to have compact support in spacetime; see e.g. [71]. The (conformal) Killing vector and spinor fields do not have this property.

The right-hand side of (8.2) is a linear combination of the five linearly independent operators $\mathcal{D}^\beta$, $\bar{\mathcal{D}}^\beta$, $\mathcal{D}^{\beta\gamma}$, $\mathcal{M}^{\beta\gamma}$ and $\mathcal{J}$. Therefore, demanding $(\delta_\mathcal{K}+\delta_\sigma)\mathcal{D}_\alpha=0$ gives five different equations. Let us first consider the equations associated with the operators $\mathcal{D}^\beta$ and $\mathcal{D}^{\beta\gamma}$ in the right-hand side of (8.2),

$$\mathcal{D}_\alpha\xi_\beta=-\frac{1}{2}\varepsilon_{\alpha\beta}(\sigma+2\mathrm{i}\tau)-\mathrm{i}\xi_{(\alpha}{}^\gamma\mathcal{C}_{\beta)\gamma}+\xi_{\alpha\beta}\mathcal{S}+\frac{1}{2}K_{\alpha\beta}, \qquad (8.3\mathrm{a})$$

$$\mathcal{D}_\alpha\xi_{\beta\gamma}=4\mathrm{i}\varepsilon_{\alpha(\beta}\bar{\xi}_{\gamma)}, \qquad (8.3\mathrm{b})$$

as well as their complex conjugate equations. These relations imply, in particular, that the parameters $\xi^\alpha, \bar{\xi}_\alpha, K_{\alpha\beta}, \sigma$ and τ are uniquely expressed in terms of ξ^a and its covariant derivatives as follows:

$$\xi^\alpha=-\frac{\mathrm{i}}{6}\bar{\mathcal{D}}_\beta\xi^{\beta\alpha}, \qquad \bar{\xi}_\alpha=-\frac{\mathrm{i}}{6}\mathcal{D}^\beta\xi_{\beta\alpha}, \qquad (8.4\mathrm{a})$$

$$\sigma=\frac{1}{2}(\mathcal{D}_\alpha\xi^\alpha+\bar{\mathcal{D}}^\alpha\bar{\xi}_\alpha), \qquad (8.4\mathrm{b})$$

$$\tau=-\frac{\mathrm{i}}{4}(\mathcal{D}_\alpha\xi^\alpha-\bar{\mathcal{D}}^\alpha\bar{\xi}_\alpha), \qquad (8.4\mathrm{c})$$

$$K_{\alpha\beta}=\mathcal{D}_{(\alpha}\xi_{\beta)}-\bar{\mathcal{D}}_{(\alpha}\bar{\xi}_{\beta)}-2\xi_{\alpha\beta}\mathcal{S}. \qquad (8.4\mathrm{d})$$

This is why we may also use the notation $\mathcal{K}=\mathcal{K}[\xi]$ and $\sigma=\sigma[\xi]$. In accordance with (8.3b), the remaining vector parameter ξ^a satisfies the equation[17]

$$\mathcal{D}_{(\alpha}\xi_{\beta\gamma)}=0 \qquad (8.5)$$

and its complex conjugate. Immediate corollaries of (8.5) are

$$(\mathcal{D}^2+4\bar{R})\xi_a=(\bar{\mathcal{D}}^2+4R)\xi_a=0, \qquad (8.6\mathrm{a})$$

$$\mathcal{D}_a\xi_b=\eta_{ab}\sigma-\varepsilon_{abc}K^c. \qquad (8.6\mathrm{b})$$

The latter relation implies the conformal Killing equation

$$\mathcal{D}_a\xi_b+\mathcal{D}_b\xi_a=\frac{2}{3}\eta_{ab}\mathcal{D}^c\xi_c. \qquad (8.7)$$

If Eq. (8.5) holds and the conditions (8.4a)–(8.4d) are adopted, it can be shown that the conditions (8.1) are satisfied identically. Therefore, (8.5) is the fundamental equation containing all the information about the conformal Killing supervector fields. As a consequence, we can give an alternative definition of the conformal Killing supervector field. It is a real supervector field

$$\xi=\xi^A E_A, \quad \xi^A\equiv(\xi^a,\xi^\alpha,\bar{\xi}_\alpha)=\left(\xi^a,-\frac{\mathrm{i}}{6}\bar{\mathcal{D}}_\beta\xi^{\beta\alpha},-\frac{\mathrm{i}}{6}\mathcal{D}^\beta\xi_{\beta\alpha}\right) \qquad (8.8)$$

which obeys the master equation (8.5).

If ξ_1 and ξ_2 are two conformal Killing supervector fields, their Lie bracket $[\xi_1,\xi_2]$ is a conformal Killing supervector field. It is obvious that, for any real c numbers r_1 and r_2, the linear combination $r_1\xi_1+r_2\xi_2$ is a conformal Killing supervector field. Thus the set of all conformal Killing supervector fields is a super Lie algebra. The conformal Killing supervector fields generate the symmetries of a superconformal field theory on $\mathcal{M}^{3|4}$.

Making use of (8.2), the condition $(\delta_\mathcal{K}+\delta_\sigma)\mathcal{D}_\alpha=0$ leads to several additional relations which can be represented in the form

$$\mathcal{D}_\alpha\bar{\xi}_\beta=-\mathrm{i}\xi_{\alpha\beta}\bar{R}, \qquad (8.9\mathrm{a})$$

$$\begin{aligned}\mathcal{D}_\alpha K_{\beta\gamma}=&4\mathcal{C}_{(\alpha\beta}\bar{\xi}_{\gamma)}-2\boldsymbol{C}_{\delta(\alpha\beta}\xi_{\gamma)}{}^\delta-\frac{1}{3}(\mathrm{i}\bar{\mathcal{D}}_{(\alpha}\bar{R}+2\mathcal{D}_{(\alpha}\mathcal{S})\xi_{\beta\gamma)}\\&+\varepsilon_{\alpha(\beta}\Big[-2\mathcal{D}_{\gamma)}\sigma+8\bar{R}\xi_{\gamma)}+8\mathrm{i}\mathcal{S}\bar{\xi}_{\gamma)}+\frac{8}{3}\mathcal{C}_{\gamma)\delta}\bar{\xi}^\delta\\&-\frac{4}{3}\boldsymbol{C}_{\gamma)\delta\rho}\xi^{\delta\rho}+\frac{10}{9}\xi_{\gamma)\delta}(\mathrm{i}\bar{\mathcal{D}}^\delta\bar{R}+2\mathcal{D}^\delta\mathcal{S})\Big], \qquad (8.9\mathrm{b})\end{aligned}$$

$$\begin{aligned}\mathcal{D}_\alpha\tau=&\mathrm{i}\mathcal{D}_\alpha\sigma+4\mathcal{S}\bar{\xi}_\alpha-2\mathrm{i}\mathcal{C}_{\alpha\delta}\bar{\xi}^\delta+\frac{\mathrm{i}}{2}\boldsymbol{C}_{\alpha\delta\rho}\xi^{\delta\rho}\\&+\frac{1}{6}(\bar{\mathcal{D}}^\beta\bar{R}-8\mathrm{i}\mathcal{D}^\beta\mathcal{S})\xi_{\alpha\beta}. \qquad (8.9\mathrm{c})\end{aligned}$$

Actually these relations have nontrivial implications. Equations (8.3) and (8.9) tell us that the spinor covariant derivatives of the parameters $\Upsilon:=(\xi^B,K^{\beta\gamma},\tau)$ can be represented as linear combinations of Υ, σ, $\mathcal{D}_\alpha\sigma$ and $\bar{\mathcal{D}}_\alpha\sigma$. It turns out that the vector covariant derivative of Υ can be represented as a linear combination of Υ, σ and $\mathcal{D}_A\sigma$. In order to prove this assertion, the key observation is that, because of (8.1), the torsion tensor $T_{AB}{}^C$, the Lorentz and $\mathrm{U}(1)_R$ curvature tensors $R_{AB}{}^{cd}$ and R_{AB}, all defined by Eq. (2.6), as well as their covariant derivatives are invariant under the transformation $\delta=\delta_\mathcal{K}+\delta_\sigma$ generated by the conformal Killing supervector field. In particular, the dimension-1 torsion tensors $\mathcal{S}$, R and $\mathcal{C}_a$ are invariant, and therefore

$$-\frac{\mathrm{i}}{4}\mathcal{D}^\beta\bar{\mathcal{D}}_\beta\sigma=(\xi^B\mathcal{D}_B+\sigma)\mathcal{S}, \qquad (8.10\mathrm{a})$$

$$-\frac{1}{4}\bar{\mathcal{D}}^2\sigma=(\xi^B\mathcal{D}_B+\sigma)R-2\mathrm{i}\tau R, \qquad (8.10\mathrm{b})$$

$$-\frac{1}{8}(\gamma_a)^{\beta\gamma}[\mathcal{D}_\beta,\bar{\mathcal{D}}_\gamma]\sigma=(\xi^B\mathcal{D}_B+\sigma)\mathcal{C}_a+K_a{}^b\mathcal{C}_b. \qquad (8.10\mathrm{c})$$

[17] Equation (8.5) is analogous to the conformal Killing equation, $\mathfrak{D}_{(\alpha\beta}V_{\gamma\delta)}=0$, on a (pseudo)Riemannian three-dimensional manifold.

To complete the proof, it only remains to make use of Eq. (2.7b).

It is an instructive exercise to derive the following identity

$$\mathcal{D}_\alpha \mathcal{D}_{\beta\gamma}\sigma = \frac{2}{3}\varepsilon_{\alpha(\beta}\Big\{2\mathrm{i}\mathcal{C}_{\gamma)\delta}\mathcal{D}^\delta\sigma + 4\mathcal{S}\mathcal{D}_{\gamma)}\sigma + 3\mathrm{i}\bar{R}\bar{\mathcal{D}}_{\gamma)}\sigma - \frac{\mathrm{i}}{4}\bar{\mathcal{D}}_{\gamma)}(\mathcal{D}^2\sigma) - \frac{\mathrm{i}}{2}\mathcal{D}_{\gamma)}(\mathcal{D}^\delta\bar{\mathcal{D}}_\delta\sigma)\Big\} - \frac{\mathrm{i}}{2}\mathcal{D}_{(\alpha}([\mathcal{D}_\beta, \bar{\mathcal{D}}_{\gamma)}]\sigma) \tag{8.11}$$

and its complex conjugate. In conjunction with Eqs. (8.10), they tell us that $\mathcal{D}_A\mathcal{D}_B\sigma$ can be represented as a linear combination of Υ, σ and $\mathcal{D}_C\sigma$. We have already established that $\mathcal{D}_A\Upsilon$ is a linear combination of Υ, σ and $\mathcal{D}_C\sigma$. These properties mean that the super Lie algebra of the conformal Killing vector fields on $\mathcal{M}^{3|4}$ is finite dimensional. The number of its even and odd generators cannot exceed those in the $\mathcal{N}=2$ superconformal algebra $\mathfrak{osp}(2|4)$.

To study supersymmetry transformations at the component level, it is useful to spell out one of the implications of (8.1) with $A=a$. Specifically, we consider the equation $(\delta_\mathcal{K}+\delta_\sigma)\mathcal{D}_a = 0$ and read off its part proportional to a linear combination of the spinor covariant derivatives $\mathcal{D}_\beta$. The result is

$$0 = \mathcal{D}_a\xi_\alpha + \frac{\mathrm{i}}{2}(\gamma_a)_\alpha{}^\beta\bar{\mathcal{D}}_\beta\sigma - \mathrm{i}\varepsilon_{abc}(\gamma^b)_\alpha{}^\beta\mathcal{C}^c\xi_\beta - (\gamma_a)_\alpha{}^\beta(\xi_\beta\mathcal{S} + \bar{\xi}_\beta R) - \frac{1}{2}\varepsilon_{abc}\xi^b(\gamma^c)^{\beta\gamma}\left(\mathrm{i}\bar{\mathcal{C}}_{\alpha\beta\gamma} - \frac{4\mathrm{i}}{3}\varepsilon_{\alpha(\beta}\bar{\mathcal{D}}_{\gamma)}\mathcal{S} - \frac{2}{3}\varepsilon_{\alpha(\beta}\mathcal{D}_{\gamma)}R\right). \tag{8.12}$$

B. Conformally related superspaces

Consider a curved superspace $\hat{\mathcal{M}}^{3|4}$ that is conformally related to $\mathcal{M}^{3|4}$. This means that the covariant derivatives $\mathcal{D}_A$ and $\hat{\mathcal{D}}_A$, which correspond to $\mathcal{M}^{3|4}$ and $\hat{\mathcal{M}}^{3|4}$ respectively, are related to each other in accordance with (2.11),

$$\hat{\mathcal{D}}_\alpha = \mathrm{e}^{\frac{1}{2}\omega}(\mathcal{D}_\alpha + (\mathcal{D}^\gamma\omega)\mathcal{M}_{\gamma\alpha} - (\mathcal{D}_\alpha\omega)\mathcal{J}), \tag{8.13a}$$

$$\hat{\mathcal{D}}_a = \mathrm{e}^{\omega}\bigg(\mathcal{D}_a - \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\mathcal{D}_{(\gamma}\omega)\bar{\mathcal{D}}_{\delta)} - \frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}(\bar{\mathcal{D}}_{(\gamma}\omega)\mathcal{D}_{\delta)} + \frac{\mathrm{i}}{2}(\mathcal{D}_\gamma\omega)(\bar{\mathcal{D}}^\gamma\omega)\mathcal{M}_a + \varepsilon_{abc}(\mathcal{D}^b\omega)\mathcal{M}^c - \frac{\mathrm{i}}{8}(\gamma_a)^{\gamma\delta}([\mathcal{D}_\gamma, \bar{\mathcal{D}}_\delta]\omega)\mathcal{J} - \frac{3\mathrm{i}}{4}(\gamma_a)^{\gamma\delta}(\mathcal{D}_\gamma\omega)(\bar{\mathcal{D}}_\delta\omega)\mathcal{J}\bigg), \tag{8.13b}$$

for some super-Weyl parameter ω. The two superspaces $\mathcal{M}^{3|4}$ and $\hat{\mathcal{M}}^{3|4}$ prove to have the same conformal Killing supervector fields. Given such a supervector field ξ, it can be represented in two different forms

$$\xi = \xi^A E_A = \hat{\xi}^A\hat{E}_A, \tag{8.14}$$

where $\hat{E}_A$ is the inverse vielbein associated with the covariant derivatives $\hat{\mathcal{D}}_A$. The parameters ξ^A and $\hat{\xi}^A$ are related to each others as follows:

$$\hat{\xi}^a = \mathrm{e}^{-\omega}\xi^a, \qquad \hat{\xi}^\alpha = \mathrm{e}^{-\frac{1}{2}\omega}\left(\xi^\alpha + \frac{\mathrm{i}}{2}\xi^{\beta\alpha}\bar{\mathcal{D}}_\beta\omega\right). \tag{8.15}$$

One may prove that the following identities hold:

$$\sigma[\hat{\xi}] = \sigma[\xi] - \xi\omega, \tag{8.16a}$$

$$\tau[\hat{\xi}] = \tau[\xi] - \mathrm{i}\xi^\alpha\mathcal{D}_\alpha\omega + \mathrm{i}\bar{\xi}_\alpha\bar{\mathcal{D}}^\alpha\omega + \frac{1}{8}\xi^{\alpha\beta}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta]\omega - \frac{1}{4}\xi^{\alpha\beta}(\mathcal{D}_\alpha\omega)\bar{\mathcal{D}}_\beta\omega, \tag{8.16b}$$

$$K_{\alpha\beta}[\hat{\xi}] = K_{\alpha\beta}[\xi] - 2\xi_{(\alpha}\mathcal{D}_{\beta)}\omega + 2\bar{\xi}_{(\alpha}\bar{\mathcal{D}}_{\beta)}\omega + \varepsilon^{abc}(\gamma_c)_{\alpha\beta}\xi_a\mathcal{D}_b\omega + \frac{\mathrm{i}}{2}\xi_{\alpha\beta}(\mathcal{D}^\gamma\omega)\bar{\mathcal{D}}_\gamma\omega. \tag{8.16c}$$

These identities imply the following important relation:

$$\mathcal{K}[\hat{\xi}] := \hat{\xi}^A\hat{\mathcal{D}}_A + \frac{1}{2}K^{cd}[\hat{\xi}]\mathcal{M}_{cd} + \mathrm{i}\tau[\hat{\xi}]\mathcal{J} = \mathcal{K}[\xi]. \tag{8.17}$$

C. Isometries

In order to describe $\mathcal{N}=2$ Poincaré or anti-de Sitter supergravity theories, the Weyl multiplet has to be coupled to a certain conformal compensator Ξ and its conjugate. In general, the latter is a scalar superfield of super-Weyl weight $w \neq 0$ and $\mathrm{U}(1)_R$ charge q,

$$\delta_\sigma\Xi = w\sigma\Xi, \qquad \mathcal{J}\Xi = q\Xi, \tag{8.18}$$

chosen to be nowhere vanishing, $\Xi \neq 0$. It is assumed that $q=0$ if and only if Ξ is real, which is the case for type II supergravity. Different off-shell supergravity theories correspond to different superfield types of Ξ.

Once Ξ and its conjugate have been fixed, the off-shell supergravity multiplet is completely described in terms of the following data: (i) the U(1) superspace geometry described earlier; (ii) the conformal compensator and its conjugate. Given a supergravity background, its isometries should preserve both of these inputs. This leads us to the concept of Killing supervector fields.

A conformal Killing supervector field $\xi = \xi^A E_A$ on $\mathcal{M}^{3|4}$ is said to be Killing if the following conditions hold:

$$\left[\xi^B\mathcal{D}_B+\frac{1}{2}K^{bc}[\xi]\mathcal{M}_{bc}+\mathrm{i}\tau[\xi]\mathcal{J},\mathcal{D}_A\right]+\delta_{\sigma[\xi]}\mathcal{D}_A=0, \tag{8.19a}$$

$$(\xi^B\mathcal{D}_B+\mathrm{i}q\tau[\xi]+w\sigma[\xi])\Xi=0, \tag{8.19b}$$

with the parameters $K^{bc}[\xi]$, $\tau[\xi]$ and $\sigma[\xi]$ defined as in (8.4). The set of all Killing supervector fields on $\mathcal{M}^{3|4}$ is a super Lie algebra. The Killing supervector fields generate the symmetries of rigid supersymmetric field theories on $\mathcal{M}^{3|4}$.

The Killing equation (8.19) are super-Weyl invariant. Specifically, if $(\mathcal{D}_A,\Xi)$ and $(\hat{\mathcal{D}}_A,\hat{\Xi})$ are conformally related supergravity backgrounds,

$$\hat{\mathcal{D}}_\alpha=\mathrm{e}^{\frac{1}{2}\omega}(\mathcal{D}_\alpha+(\mathcal{D}^\gamma\omega)\mathcal{M}_{\gamma\alpha}-(\mathcal{D}_\alpha\omega)\mathcal{J}),\qquad \hat{\Xi}=\mathrm{e}^{w\sigma}\Xi, \tag{8.20}$$

then Eq. (8.19) imply that $\xi=\xi^BE_B=\hat{\xi}^B\hat{E}_B$ is also a Killing supervector field with respect to $(\hat{\mathcal{D}}_A,\hat{\Xi})$. In particular, it holds that

$$(\hat{\xi}^B\hat{\mathcal{D}}_B+\mathrm{i}q\tau[\hat{\xi}]+w\sigma[\hat{\xi}])\hat{\Xi}=0. \tag{8.21}$$

The super-Weyl and local $\mathrm{U}(1)_R$ symmetries allow us to choose a useful gauge

$$\Xi=1 \tag{8.22}$$

which characterizes the off-shell supergravity formulation chosen. If $q\neq 0$, there remains no residual super-Weyl and local $\mathrm{U}(1)_R$ freedom in this gauge. Otherwise, the local $\mathrm{U}(1)_R$ symmetry remains unbroken while the super-Weyl freedom is completely fixed.

In the gauge (8.22), the Killing equation (8.19b) becomes

$$\mathrm{i}q(\xi^B\Phi_B+\tau[\xi])+w\sigma[\xi]=0. \tag{8.23}$$

Hence, the isometry transformations are generated by those conformal Killing supervector fields which respect the conditions

$$\sigma[\xi]=0, \tag{8.24a}$$

$$\tau[\xi]=-\xi^B\Phi_B,\qquad q\neq 0. \tag{8.24b}$$

These properties provide the main rationale for choosing the gauge condition (8.22) which is for any off-shell supergravity formulation; the isometry transformations are characterized by the condition $\sigma[\xi]=0$.

Since for $q\neq 0$ the local $\mathrm{U}(1)_R$ symmetry is completely fixed in the gauge (8.22), it is reasonable to switch to new covariant derivatives without $\mathrm{U}(1)_R$ connection which are defined by $\mathcal{D}_A\to\nabla_A:=\mathcal{D}_A-\mathrm{i}\Phi_A\mathcal{J}$.[18] The original $\mathrm{U}(1)_R$ connection Φ_A turns into a tensor superfield.

D. Charged conformal Killing spinors

We wish to look for those curved superspace backgrounds which admit at least one conformal supersymmetry. By definition, such a superspace possesses a conformal Killing supervector field ξ^A with the property

$$\xi^a|=0,\qquad \epsilon^\alpha:=\xi^\alpha|\neq 0. \tag{8.25}$$

All other bosonic parameters will also be assumed to vanish, $\sigma|=\tau|=K_{\alpha\beta}|=0$. Our analysis will be restricted to U(1) superspace backgrounds without covariant fermionic fields, that is

$$\mathcal{D}_\alpha\mathcal{S}|=0,\qquad \mathcal{D}_\alpha R|=0,\qquad \mathcal{D}_\alpha\mathcal{C}_{\beta\gamma}|=0. \tag{8.26}$$

These conditions mean that the gravitini can completely be gauged away such that the projection (4.3) becomes

$$\mathcal{D}_a|=\mathbf{D}_a\Longleftrightarrow\psi_m{}^\alpha=0. \tag{8.27}$$

In what follows, we always assume that the gravitini have been gauged away.

The above definitions provide a superspace realization for what is usually called a "supersymmetric spacetime." For instance, according to [14], it is a supergravity background "for which all fermions *and their supersymmetry variations* vanish for some nonzero supersymmetry parameter."

We introduce scalar and vector fields associated with the superfield torsion:

$$s:=\mathcal{S}|,\qquad r:=R|,\qquad c_a:=\mathcal{C}_a|. \tag{8.28}$$

We also recall that the S-supersymmetry parameter is $\eta_\alpha:=\mathcal{D}_\alpha\sigma|$. Bar projecting Eq. (8.12) gives

$$0=\mathbf{D}_a\epsilon^\alpha+\frac{\mathrm{i}}{2}(\tilde{\gamma}_a\bar{\eta})^\alpha+\mathrm{i}\varepsilon_{abc}c^b(\tilde{\gamma}^c\epsilon)^\alpha-s(\tilde{\gamma}_a\epsilon)^\alpha-\mathrm{i}r(\tilde{\gamma}_a\bar{\epsilon})^\alpha. \tag{8.29}$$

This is equivalent to the following two equations:

$$0=(\mathbf{D}_{(\alpha\beta}-\mathrm{i}c_{(\alpha\beta})\epsilon_{\gamma)}, \tag{8.30a}$$

$$\bar{\eta}_\alpha=-\frac{2\mathrm{i}}{3}((\gamma^a\mathbf{D}_a\epsilon)_\alpha+2\mathrm{i}(\gamma^a\epsilon)_\alpha c_a+3s\epsilon_\alpha+3\mathrm{i}r\bar{\epsilon}_\alpha). \tag{8.30b}$$

Equation (8.30a) tells us that the supersymmetry parameter is a *charged conformal Killing spinor*, since (8.30a) can be rewritten in the form

[18]This is similar to the 4D procedure of degauging introduced by Howe [18] and reviewed in [2].

$$\tilde{\mathbf{D}}_{(\alpha\beta}\epsilon_{\gamma)} = 0, \qquad \tilde{\mathbf{D}}_{\alpha\beta}\epsilon_\gamma := \mathfrak{D}_{\alpha\beta}\epsilon_\gamma - \mathrm{i}(b_{\alpha\beta} + c_{\alpha\beta})\epsilon_\gamma. \quad (8.31)$$

Let us choose ϵ_α to be a bosonic (commuting) spinor. Then it follows from (8.31) that the real vector field $V_a := (\gamma_a)^{\alpha\beta}\bar{\epsilon}_\alpha\epsilon_\beta$ has the following properties: (i) V_a is a conformal Killing vector field, $\mathfrak{D}_{(\alpha\beta}V_{\gamma\delta)} = 0$; and (ii) V_a is null or timelike, since $V^aV_a = (\bar{\epsilon}^\alpha\epsilon_\alpha)^2 \le 0$. This vector field is null if and only if $\bar{\epsilon}_\alpha \propto \epsilon_\alpha$. As a result, we have reproduced two of the main results of [35].

By construction, the conditions (8.26) are supersymmetric, that is

$$(\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_\alpha\mathcal{S} = 0, \qquad (\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_\alpha R = 0,$$
$$(\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_\alpha\mathcal{C}_{\beta\gamma} = 0. \quad (8.32)$$

Evaluating the bar projection of these variations gives, respectively,

$$\mathcal{D}^2\bar{\mathcal{D}}_\alpha\sigma| = \bar{\epsilon}^\beta(8\mathfrak{D}_{\alpha\beta}s - 4\mathrm{i}[\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]\mathcal{S}| - 4\mathrm{i}\varepsilon_{\alpha\beta}\mathcal{D}^\gamma\bar{\mathcal{D}}_\gamma\mathcal{S}|) + 16\mathrm{i}\epsilon_\alpha\bar{r}s - 8\mathrm{i}\eta_\alpha s + 6\bar{\eta}_\alpha\bar{r}, \quad (8.33a)$$

$$\mathcal{D}^2\bar{\mathcal{D}}_\alpha\sigma| = \epsilon^\beta(8\mathrm{i}(\mathfrak{D}_{\alpha\beta} + 2\mathrm{i}b_{\alpha\beta} - 2\mathrm{i}c_{\alpha\beta})\bar{r} - 32\mathrm{i}\varepsilon_{\alpha\beta}s\bar{r}) + 2\bar{\epsilon}_\alpha\bar{\mathcal{D}}^2\bar{R}| - 4\mathrm{i}\mathbf{D}_{\alpha\beta}\eta^\beta + 4\mathrm{i}\eta_\alpha s - 6\bar{\eta}_\alpha\bar{r}, \quad (8.33b)$$

$$0 = \epsilon_{(\alpha}(\mathfrak{D}_{\beta\gamma)} + 2\mathrm{i}b_{\beta\gamma)} + 4\mathrm{i}c_{\beta\gamma)})\bar{r} + \bar{\epsilon}^\delta\bigg\{\mathfrak{D}_{(\alpha\beta}c_{\gamma\delta)} - \frac{1}{2}(\mathcal{D}_{(\alpha}\bar{\boldsymbol{C}}_{\beta\gamma\delta)} + \bar{\mathcal{D}}_{(\alpha}\boldsymbol{C}_{\beta\gamma\delta)})| + \varepsilon_{\delta(\alpha}\bigg[[\mathcal{D}_\beta, \bar{\mathcal{D}}_{\gamma)}]\mathcal{S}| - \mathrm{i}\mathfrak{D}_{\beta\gamma)}s + \frac{3}{2}\varepsilon^{cab}(\gamma_c)_{\beta\gamma)}\mathfrak{D}_a c_b + 6c_{\beta\gamma)}s\bigg]\bigg\} - \frac{1}{2}\mathbf{D}_{(\alpha\beta}\eta_{\gamma)} - \frac{3\mathrm{i}}{2}c_{(\alpha\beta}\eta_{\gamma)}. \quad (8.33c)$$

E. Supersymmetric backgrounds

In order to describe a *rigid supersymmetry transformation*, the structure equations given in the previous subsection have to be supplemented by the additional condition

$$\sigma[\xi] = 0 \Longrightarrow \eta_\alpha = 0, \quad (8.34)$$

in accordance with (8.24a). Here we do not specify any particular compensator. However, we assume that some compensator has been chosen and the gauge condition (8.22) has been imposed.

Because of (8.34), Eq. (8.29) turns into

$$\mathbf{D}_a\epsilon^\alpha = -\mathrm{i}\varepsilon_{abc}c^b(\tilde{\gamma}^c\epsilon)^\alpha + s(\tilde{\gamma}_a\epsilon)^\alpha + \mathrm{i}r(\tilde{\gamma}_a\bar{\epsilon})^\alpha. \quad (8.35)$$

In the spinor notation, this equation reads

$$\mathbf{D}_{\alpha\beta}\epsilon_\gamma = \mathrm{i}c_{(\alpha\beta}\epsilon_{\gamma)} - 2\mathrm{i}\varepsilon_{\gamma(\alpha}\rho_{\beta)},$$
$$\rho_\alpha := \frac{2}{3}c_a(\gamma^a\epsilon)_\alpha - \mathrm{i}s\epsilon_\alpha + r\bar{\epsilon}_\alpha. \quad (8.36)$$

This relation shows that, in a neighborhood of any given point x_0, the supersymmetry parameter $\epsilon_\gamma(x)$ is determined by its value at x_0. As a result, any nonzero solution of Eq. (8.35) or, equivalently, (8.36), is nowhere vanishing if the spacetime $\mathcal{M}^3$ is a connected manifold.[19]

F. Supersymmetric backgrounds with four supercharges

The existence of rigid supersymmetries imposes nontrivial restrictions on the background fields. For simplicity, here we work out these restrictions in the case of four supercharges. Since $\sigma = 0$, one may deduce from (8.33) the following conditions:

$$\mathcal{D}^2R| = \mathcal{D}^\gamma\bar{\mathcal{D}}_\gamma\mathcal{S}| = [\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}]\mathcal{S}| = (\mathcal{D}_{(\alpha}\bar{\boldsymbol{C}}_{\beta\gamma\delta)} + \bar{\mathcal{D}}_{(\alpha}\boldsymbol{C}_{\beta\gamma\delta)})| = 0. \quad (8.37)$$

It is an instructive exercise to demonstrate that these conditions constrain the background fields s, r and c_a as follows:

$$\mathfrak{D}_a s = 0, \quad (8.38a)$$

$$\mathfrak{D}_a r = 2\mathrm{i}b_a r, \quad (8.38b)$$

$$\mathfrak{D}_a c_b = 2\varepsilon_{abc}c^c s, \quad (8.38c)$$

$$rs = 0, \quad (8.38d)$$

$$rc_a = 0. \quad (8.38e)$$

It follows from (8.38c) that c_a is a Killing vector field,

$$\mathfrak{D}_a c_b + \mathfrak{D}_b c_a = 0. \quad (8.39)$$

The $\mathrm{U}(1)_R$ field strength proves to vanish,

$$\mathcal{F}_{ab} = 0. \quad (8.40)$$

The Einstein tensor (A12) is uniquely fixed to be

$$\mathcal{G}_{ab} = 4[c_a c_b + \eta_{ab}(s^2 + \bar{r}r)]. \quad (8.41)$$

[19]This can be proved as follows. Let us assume that $\epsilon_\gamma(x)$ vanishes at some point $x_0 \in \mathcal{M}^3$. We can expand $\epsilon_\gamma(x)$ in a covariant Taylor series centered at x_0 (see, e.g., [58]) in an open neighborhood U of x_0. Then, due to (8.36), $\epsilon_\gamma(x)$ is equal to zero on U. It is also clear that $\epsilon_\gamma(x)$ vanishes on the closure $\bar{U}$ of U. Now we can introduce the subset $\mathcal{W} \in \mathcal{M}^3$ consisting of all zeros of $\epsilon_\gamma(x)$. It follows that this subset is open and closed, and therefore it coincides with $\mathcal{M}^3$ since the latter is connected.

We recall that in three dimensions the Riemann tensor is determined in terms of the Einstein tensor according to Eq. (A12). For the Cotton tensor (A14) we obtain

$$\mathcal{W}_{ab} = -24s\left[c_a c_b - \frac{1}{3}\eta_{ab}c^d c_d\right]. \tag{8.42}$$

The spacetime is conformally flat if $sc_a = 0$.

So far we have not specified any compensator. We now turn to considering the known off-shell supergravity formulations [11].

G. Type I minimal backgrounds with four supercharges

In type I supergravity, the conformal compensators are a covariantly chiral superfield Φ of super-Weyl weight $w = 1/2$ and its complex conjugate $\bar{\Phi}$. We recall that the properties of Φ are given by Eq. (5.1). The freedom to perform the super-Weyl and local $\mathrm{U}(1)_R$ transformations can be used to impose the gauge

$$\Phi = 1. \tag{8.43}$$

Such a gauge fixing is accompanied by the consistency conditions [10]

$$\begin{aligned} 0 &= \mathcal{D}_\alpha\Phi = -\frac{\mathrm{i}}{2}\Phi_\alpha, \\ 0 &= \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\}\Phi = -\Phi_{\alpha\beta} + \mathcal{C}_{\alpha\beta} - 2\mathrm{i}\varepsilon_{\alpha\beta}\mathcal{S}, \end{aligned} \tag{8.44}$$

and therefore

$$\mathcal{S} = 0, \qquad \Phi_\alpha = 0, \qquad \Phi_{\alpha\beta} = \mathcal{C}_{\alpha\beta}. \tag{8.45}$$

Since the local $\mathrm{U}(1)_R$ invariance is completely fixed in this gauge, it is more convenient to make use of covariant derivatives without $\mathrm{U}(1)_R$ connection,

$$\nabla_A := \mathcal{D}_A - \mathrm{i}\Phi_A\mathcal{J}, \tag{8.46}$$

which satisfy the anticommutation relations

$$\{\nabla_\alpha, \nabla_\beta\} = -4\bar{R}\mathcal{M}_{\alpha\beta}, \qquad \{\bar{\nabla}_\alpha, \bar{\nabla}_\beta\} = 4R\mathcal{M}_{\alpha\beta}, \tag{8.47a}$$

$$\{\nabla_\alpha, \bar{\nabla}_\beta\} = -2\mathrm{i}\nabla_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta}. \tag{8.47b}$$

The Killing spinor equation (8.35) becomes

$$\mathfrak{D}_a\epsilon^\alpha = \mathrm{i}c_a\epsilon^\alpha - \mathrm{i}\varepsilon_{abc}c^b(\tilde{\gamma}^c\epsilon)^\alpha + \mathrm{i}r(\tilde{\gamma}_a\bar{\epsilon})^\alpha. \tag{8.48}$$

The supersymmetric backgrounds with four supercharges are characterized by the properties

$$rc_a = 0, \tag{8.49a}$$

$$\mathfrak{D}_a r = 0, \tag{8.49b}$$

$$\mathfrak{D}_a c_b = 0. \tag{8.49c}$$

The Einstein tensor is

$$\mathcal{G}_{ab} = 4[c_a c_b + \eta_{ab}\bar{r}r]. \tag{8.50}$$

Such a spacetime is necessarily conformally flat,

$$\mathcal{W}_{ab} = 0. \tag{8.51}$$

The solution with $c_a = 0$ corresponds to the (1,1) AdS superspace [11].

The Killing spinor equation (8.48) is equivalent to the condition that the gravitino variation (5.12b) vanishes,

$$\begin{aligned} -\frac{1}{2}\delta_\epsilon\psi_m{}^\alpha &= -\mathfrak{D}_m\epsilon^\alpha + \frac{\mathrm{i}}{2}b_m\epsilon^\alpha - \frac{\mathrm{i}}{2}e_m{}^a\varepsilon_{abc}b^b(\tilde{\gamma}^c\epsilon)^\alpha \\ &\quad - \frac{\mathrm{i}}{4}\bar{M}(\tilde{\gamma}_m\bar{\epsilon})^\alpha = 0, \end{aligned} \tag{8.52}$$

provided we replace

$$c_a \to \frac{1}{2}b_a, \qquad r \to -\frac{1}{4}\bar{M}. \tag{8.53}$$

H. Type II minimal backgrounds with four supercharges

Type II minimal supergravity is obtained by coupling the Weyl multiplet to a real linear compensator $\mathbb{G}$ with the super-Weyl transformation law given by Eq. (6.2). The super-Weyl invariance allows us to choose the gauge

$$\mathbb{G} = 1. \tag{8.54}$$

Because the compensator is real, its $\mathrm{U}(1)_R$ charge (8.18) is equal to zero, and thus the local $\mathrm{U}(1)_R$ group remains unbroken in the gauge chosen. The consistency condition for (8.54) is

$$R = \bar{R} = 0. \tag{8.55}$$

Then, the anticommutators of spinor covariant derivatives become

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 0, \tag{8.56a}$$

$$\begin{aligned} \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} &= -2\mathrm{i}\mathcal{D}_{\alpha\beta} - 2\mathcal{C}_{\alpha\beta}\mathcal{J} - 4\mathrm{i}\varepsilon_{\alpha\beta}\mathcal{S}\mathcal{J} \\ &\quad + 4\mathrm{i}\mathcal{S}\mathcal{M}_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta}. \end{aligned} \tag{8.56b}$$

The Killing spinor equation for type II minimal supergravity is

$$\mathbf{D}_a\epsilon^\alpha = -\mathrm{i}\varepsilon_{abc}c^b(\tilde{\gamma}^c\epsilon)^\alpha + s(\tilde{\gamma}_a\epsilon)^\alpha. \tag{8.57}$$

All supersymmetric backgrounds with four supercharges are characterized by the conditions

$$\mathfrak{D}_a s = 0, \tag{8.58a}$$

$$\mathfrak{D}_a c_b = 2\varepsilon_{abc} c^c s. \tag{8.58b}$$

The Einstein tensor is

$$\mathcal{G}_{ab} = 4[c_a c_b + \eta_{ab} s^2], \tag{8.59}$$

and the Cotton tensor is given by Eq. (8.42). The solution with $c_a = 0$ corresponds to the (2,0) AdS superspace [11]. In the case $c_a \neq 0$, the traceless Ricci tensor is

$$\mathcal{R}_{ab} - \frac{1}{3}\eta_{ab}\mathcal{R} = 4\left[c_a c_b - \frac{1}{3}\eta_{ab} c^2\right]. \tag{8.60}$$

From this we conclude (see, e.g., Table 1 in [72]) that spacetime is of type N (for c_a null), type D_s (for c_a spacelike) or D_t (for c_a timelike) in the Petrov-Segre classification. For D_t and D_s it is shown in [72] that spacetime is necessarily biaxially squashed AdS_3.

The Killing spinor equation (8.57) is equivalent to the condition that, in the gauge $\psi_m{}^\alpha = 0$, the gravitino variation (6.36b) vanishes,

$$-\frac{1}{2}\delta_\epsilon \psi_m{}^\alpha = -\mathbf{D}_m \epsilon^\alpha - \frac{\mathrm{i}}{4}\mathcal{H}_m \epsilon^\alpha + \frac{\mathrm{i}}{4} e_m{}^a \varepsilon_{abc}\mathcal{H}^c (\tilde{\gamma}^b \epsilon)^\alpha - \frac{1}{4} Z(\tilde{\gamma}_m \epsilon)^\alpha = 0, \tag{8.61}$$

provided we make the replacements

$$b_a \to b_a - \frac{1}{4}\mathcal{H}_a, \quad c_a \to -\frac{1}{2}\mathcal{H}_a, \quad s \to -\frac{1}{4} Z. \tag{8.62}$$

I. Nonminimal backgrounds with four supercharges

Nonminimal supergravity in three dimensions was studied in [10,11]. It is obtained by coupling the Weyl multiplet to a complex linear compensator Σ and its conjugate. Here Σ obeys the constraint

$$(\bar{\mathcal{D}}^2 - 4R)\Sigma = 0 \tag{8.63}$$

and is subject to no reality condition. By definition, the compensator Σ is chosen to be nowhere vanishing and transforms as a primary field of weight $w \neq 0, 1$ under the super-Weyl group. Then, the $\mathrm{U}(1)_R$ charge of Σ is uniquely determined [10],

$$\delta_\sigma \Sigma = w\sigma\Sigma \Longrightarrow \mathcal{J}\Sigma = (1 - w)\Sigma. \tag{8.64}$$

The super-Weyl and local $\mathrm{U}(1)_R$ symmetries can be used to impose the gauge condition

$$\Sigma = 1. \tag{8.65}$$

In this gauge, some restrictions on the geometry occur [10]. To describe them, it is useful to split the covariant derivatives as

$$\mathcal{D}_\alpha = \nabla_\alpha + \mathrm{i}T_\alpha \mathcal{J}, \qquad \bar{\mathcal{D}}_\alpha = \bar{\nabla}_\alpha + \mathrm{i}\bar{T}_\alpha \mathcal{J}, \tag{8.66}$$

where the original $\mathrm{U}(1)_R$ connection Φ_α has been renamed as T_α. In the gauge (8.65), the constraint $(\bar{\mathcal{D}}^2 - 4R)\Sigma = 0$ turns into

$$R = \frac{1 - w}{4}(\mathrm{i}\bar{\nabla}_\alpha \bar{T}^\alpha + w\bar{T}_\alpha \bar{T}^\alpha). \tag{8.67}$$

We see that R becomes a descendant of $\bar{T}_\alpha$. Equation (8.67) is not the only consistency condition implied by the gauge fixing (8.65). Evaluating explicitly $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\}\Sigma$ and $\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\}\Sigma$ and then setting $\Sigma = 1$ gives

$$\nabla_{(\alpha} T_{\beta)} = 0, \quad \mathcal{S} = \frac{1}{8}(\bar{\nabla}^\alpha T_\alpha - \nabla^\alpha \bar{T}_\alpha + 2\mathrm{i}T^\alpha \bar{T}_\alpha), \tag{8.68a}$$

$$\Phi_{\alpha\beta} = \mathcal{C}_{\alpha\beta} + \frac{\mathrm{i}}{2}\nabla_{(\alpha}\bar{T}_{\beta)} + \frac{\mathrm{i}}{2}\bar{\nabla}_{(\alpha}T_{\beta)} + T_{(\alpha}\bar{T}_{\beta)}. \tag{8.68b}$$

If we define a new vector covariant derivative ∇_a by $\mathcal{D}_a = \nabla_a + \mathrm{i}\Phi_a \mathcal{J}$, then the algebra of the covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}_\alpha)$ proves to be

$$\{\nabla_\alpha, \nabla_\beta\} = -2\mathrm{i}T_{(\alpha}\nabla_{\beta)} - \mathrm{i}(w - 1)(\nabla^\gamma T_\gamma + \mathrm{i}wT^\gamma T_\gamma)\mathcal{M}_{\alpha\beta}, \tag{8.69a}$$

$$\{\nabla_\alpha, \bar{\nabla}_\beta\} = -2\mathrm{i}\nabla_{\alpha\beta} - \mathrm{i}\bar{T}_\beta \nabla_\alpha + \mathrm{i}T_\alpha \bar{\nabla}_\beta - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta} + \frac{\mathrm{i}}{2}(\bar{\nabla}^\gamma T_\gamma - \nabla^\gamma \bar{T}_\gamma + 2\mathrm{i}T^\gamma \bar{T}_\gamma)\mathcal{M}_{\alpha\beta}. \tag{8.69b}$$

The Killing spinor equation in this case is

$$\mathfrak{D}_a \epsilon^\alpha = \mathrm{i}\Phi_a| \epsilon^\alpha - \mathrm{i}\varepsilon_{abc} c^b (\tilde{\gamma}^c \epsilon)^\alpha + s(\tilde{\gamma}_a \epsilon)^\alpha + \mathrm{i}r(\tilde{\gamma}_a \bar{\epsilon})^\alpha. \tag{8.70}$$

It should be kept in mind that R, $\mathcal{S}$ and Φ_a are now composite superfields constructed in terms of T_α, $\bar{T}_\alpha$ and their covariant derivatives, in accordance with Eqs. (8.67), (8.68a) and (8.68b) respectively. Supersymmetric backgrounds with four supercharges are very constrained in the nonminimal case. Indeed, the requirement that $T_\alpha| = 0$ be invariant under the isometry transformations leads to the condition

$$0 = \epsilon^\gamma \nabla_\gamma T_\alpha| - \bar{\epsilon}^\gamma \bar{\nabla}_\gamma T_\alpha|, \tag{8.71}$$

which implies $\nabla_\alpha T_\beta = \bar{\nabla}_\alpha T_\beta = 0$. Due to (8.67)–(8.68b), we deduce that

$$r=0, \qquad s=0, \qquad \Phi_a|=c_a, \tag{8.72}$$

and then c_b is covariantly constant,

$$\mathfrak{D}_a c_b = 0. \tag{8.73}$$

The Einstein tensor becomes

$$\mathcal{G}_{ab} = 4c_a c_b. \tag{8.74}$$

Such a spacetime is necessarily conformally flat, $\mathcal{W}_{ab}=0$.

Nonminimal supergravity is the only off-shell supergravity formulation which does not allow for anti-de Sitter backgrounds. However, there exists an alternative nonminimal formulation in the case $w=-1$ [11], inspired by the 4D construction in [24], which admits an anti-de Sitter solution.

J. Nonminimal AdS backgrounds with four supercharges

In the case $w=-1$, the complex linear constraint (8.63) admits a nontrivial deformation. We introduce a new conformal compensator Γ that has the transformation properties

$$\delta_\sigma\Gamma = -\sigma\Gamma, \qquad \mathcal{J}\Gamma = 2\Gamma \tag{8.75}$$

and obeys the *improved* linear constraint[20]

$$-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Gamma = \mu = \text{const}, \tag{8.76}$$

with the complex parameter $\mu \neq 0$ inducing a cosmological constant. This constraint is super-Weyl invariant.

The super-Weyl and local $\mathrm{U}(1)_R$ symmetries allow us to impose the gauge condition

$$\Gamma = 1. \tag{8.77}$$

As in the previous subsection, this gauge condition implies some restrictions on the geometry. Indeed, the constraint $(\bar{\mathcal{D}}^2-4R)\Gamma=\mu$ turns into

$$R = \mu + \frac{\mathrm{i}}{2}(\bar{\nabla}_\alpha\bar{T}^\alpha + \mathrm{i}\bar{T}_\alpha\bar{T}^\alpha). \tag{8.78}$$

We see that R becomes a descendant of $\bar{T}_\alpha$. Next, evaluating the expressions $\{\mathcal{D}_\alpha,\mathcal{D}_\beta\}\Gamma$ and $\{\mathcal{D}_\alpha,\bar{\mathcal{D}}_\beta\}\Gamma$ and then setting $\Gamma=1$, we again obtain the relations (8.68a) and (8.68b). As in the previous subsection, we can introduce covariant derivatives without $\mathrm{U}(1)_R$ connection, $\nabla_A=(\nabla_a,\nabla_\alpha,\bar{\nabla}_\alpha)$. Their algebra proves to be

[20] In the case $w=-1$, there exists a more general deformation, $(\bar{\mathcal{D}}^2-4R)\Gamma=-4W(\varphi)$, where $W(\varphi^I)$ is a matter superpotential depending on super-Weyl inert chiral superfields φ^I. This super-Weyl invariant constraint reduces to (8.76) for $W=\mu$.

$$\{\nabla_\alpha,\nabla_\beta\} = -2\mathrm{i}T_{(\alpha}\nabla_{\beta)} - 4\bar{\mu}\mathcal{M}_{\alpha\beta} + 2\mathrm{i}(\nabla^\gamma T_\gamma - \mathrm{i}T^\gamma T_\gamma)\mathcal{M}_{\alpha\beta}, \tag{8.79a}$$

$$\{\nabla_\alpha,\bar{\nabla}_\beta\} = -2\mathrm{i}\nabla_{\alpha\beta} - \mathrm{i}\bar{T}_\beta\nabla_\alpha + \mathrm{i}T_\alpha\bar{\nabla}_\beta - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta} + \frac{\mathrm{i}}{2}(\bar{\nabla}^\gamma T_\gamma - \nabla^\gamma\bar{T}_\gamma + 2\mathrm{i}T^\gamma\bar{T}_\gamma)\mathcal{M}_{\alpha\beta}. \tag{8.79b}$$

The Killing spinor equation coincides with (8.70). Unlike the nonminimal formulation studied in the previous subsection, the scalar R is now given by Eq. (8.78). This modified expression for R leads to different backgrounds with four supercharges. Due to the presence of the parameter μ in (8.78), demanding the existence of four supersymmetries gives

$$\mathcal{S}|=0, \qquad R|=\mu, \qquad \Phi_a|=\mathcal{C}_a|=c_a. \tag{8.80}$$

Moreover, one also finds the condition $\mathcal{C}_c|R|=0$. Since $R|=\mu\neq 0$, we conclude that

$$c_a = 0. \tag{8.81}$$

The Einstein tensor is

$$\mathcal{G}_{ab} = 4\eta_{ab}\bar{\mu}\mu. \tag{8.82}$$

This background corresponds to the (1,1) anti-de Sitter superspace [11].

After this work was completed, there appeared a new paper [50] which has some overlap with our results in Secs. VII B and VIII G.

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APPENDIX A: NOTATION, CONVENTIONS AND SOME TECHNICAL DETAILS

Our 3D notation and conventions follow those used in [10]. In particular, the vector indices are denoted by lower case Latin letters from the beginning of the alphabet, for instance $a, b = 0, 1, 2$. The Minkowski metric is $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$, and the Levi-Civita tensor ε_{abc} is normalized by $\varepsilon_{012} = -1$, and hence $\varepsilon^{012} = 1$. The spinor indices are denoted by small Greek letters from the beginning of the alphabet, for instance $\alpha, \beta = 1, 2$.

To deal with spinors, we introduce a basis of *real* symmetric 2×2 matrices

$$\gamma_a = (\gamma_a)_{\alpha\beta} = (\gamma_a)_{\beta\alpha} = (\mathbb{1}, \sigma_1, \sigma_3), \tag{A1a}$$

and also define

$$\tilde{\gamma}_a = (\gamma_a)^{\alpha\beta} = (\gamma_a)^{\beta\alpha} := \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(\gamma_a)_{\gamma\delta}, \tag{A1b}$$

with σ_1 and σ_3 two of the three Pauli matrices. The spinor indices are raised and lowered using the $\mathrm{SL}(2, \mathbb{R})$ invariant tensors

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A2}$$

as follows:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta. \tag{A3}$$

The 3D Dirac γ matrices are

$$\hat{\gamma} = (\gamma_a)_\alpha{}^\beta := \varepsilon^{\beta\gamma}(\gamma_a)_{\alpha\gamma}, \qquad \hat{\gamma}_a\hat{\gamma}_b = \eta_{ab}\mathbb{1} + \varepsilon_{abc}\hat{\gamma}^c. \tag{A4}$$

In this representation of the γ matrices, the Majorana spinors are real.

In $\mathcal{N} = 2$ supersymmetry, we usually deal with complex spinors. *Only* in the case of complex spinors, we use throughout this paper the following types of index contraction:

$$\psi\chi := \psi^\alpha\chi_\alpha, \qquad \psi\bar{\chi} := \psi^\alpha\bar{\chi}_\alpha,$$
$$\bar{\psi}\chi := \bar{\psi}^\alpha\chi_\alpha, \qquad \bar{\psi}\bar{\chi} := \bar{\psi}_\alpha\bar{\chi}^\alpha; \tag{A5a}$$

$$(\gamma_a\psi)_\alpha := (\gamma_a)_{\alpha\beta}\psi^\beta = (\psi\gamma_a)_\alpha,$$
$$(\tilde{\gamma}_a\psi)^\alpha := (\gamma_a)^{\alpha\beta}\psi_\beta = (\psi\tilde{\gamma}_a)^\alpha; \tag{A5b}$$

$$\psi\gamma_a\chi := \psi^\alpha(\gamma_a)_{\alpha\beta}\chi^\beta, \qquad \psi\tilde{\gamma}_a\chi := \psi_\alpha(\gamma_a)^{\alpha\beta}\chi_\beta. \tag{A5c}$$

In particular, contractions of two spinor covariant derivatives are defined as

$$\mathcal{D}^2 := \mathcal{D}^\alpha\mathcal{D}_\alpha, \qquad \bar{\mathcal{D}}^2 = \bar{\mathcal{D}}_\alpha\bar{\mathcal{D}}^\alpha. \tag{A6}$$

Any three-vector F_a can equivalently be realized as a symmetric spinor $F_{\alpha\beta} = F_{\beta\alpha}$. The relationship between F_a and $F_{\alpha\beta}$ is as follows:

$$F_{\alpha\beta} := (\gamma^a)_{\alpha\beta}F_a = F_{\beta\alpha}, \qquad F_a = -\frac{1}{2}(\gamma_a)^{\alpha\beta}F_{\alpha\beta}. \tag{A7}$$

We can also describe the one-form F_a in terms of its Hodge-dual two-form $F_{ab} = -F_{ba}$,

$$F_{ab} := -\varepsilon_{abc}F^c, \qquad F_a = \frac{1}{2}\varepsilon_{abc}F^{bc}. \tag{A8}$$

Then, the symmetric spinor $F_{\alpha\beta} = F_{\beta\alpha}$, which is associated with F_a, can equivalently be defined in terms of F_{ab}:

$$F_{\alpha\beta} := (\gamma^a)_{\alpha\beta}F_a = \frac{1}{2}(\gamma^a)_{\alpha\beta}\varepsilon_{abc}F^{bc}. \tag{A9}$$

These three algebraic objects, F_a, F_{ab} and $F_{\alpha\beta}$, are in one-to-one correspondence to each other. Their inner products are related as follows:

$$-F^aG_a = \frac{1}{2}F^{ab}G_{ab} = \frac{1}{2}F^{\alpha\beta}G_{\alpha\beta}. \tag{A10}$$

An equivalent form of the commutation relations (2.7c) and (2.7d) is

$$[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma] = -\mathrm{i}\varepsilon_{\gamma(\alpha}\mathcal{C}_{\beta)\delta}\mathcal{D}^\delta + \mathrm{i}\mathcal{C}_{\gamma(\alpha}\mathcal{D}_{\beta)} - 2\varepsilon_{\gamma(\alpha}\mathcal{S}\mathcal{D}_{\beta)} - 2\mathrm{i}\varepsilon_{\gamma(\alpha}\bar{R}\bar{\mathcal{D}}_{\beta)} + 2\varepsilon_{\gamma(\alpha}\boldsymbol{C}_{\beta)\delta\rho}\mathcal{M}^{\delta\rho} - \frac{4}{3}(2\mathcal{D}_{(\alpha}\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_{(\alpha}\bar{R})\mathcal{M}_{\beta)\gamma}$$
$$+ \frac{1}{3}(2\mathcal{D}_\gamma\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_\gamma\bar{R})\mathcal{M}_{\alpha\beta} + \left(\boldsymbol{C}_{\alpha\beta\gamma} + \frac{1}{3}\varepsilon_{\gamma(\alpha}(8\mathcal{D}_{\beta)}\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_{\beta)}\bar{R})\right)\mathcal{J}, \tag{A11a}$$

$$[\mathcal{D}_{\alpha\beta}, \bar{\mathcal{D}}_\gamma] = \mathrm{i}\varepsilon_{\gamma(\alpha}\mathcal{C}_{\beta)\delta}\bar{\mathcal{D}}^\delta - \mathrm{i}\mathcal{C}_{\gamma(\alpha}\bar{\mathcal{D}}_{\beta)} - 2\varepsilon_{\gamma(\alpha}\mathcal{S}\bar{\mathcal{D}}_{\beta)} + 2\mathrm{i}\varepsilon_{\gamma(\alpha}R\mathcal{D}_{\beta)} + 2\varepsilon_{\gamma(\alpha}\bar{\boldsymbol{C}}_{\beta)\delta\rho}\mathcal{M}^{\delta\rho} - \frac{4}{3}(2\bar{\mathcal{D}}_{(\alpha}\mathcal{S} - \mathrm{i}\mathcal{D}_{(\alpha}R)\mathcal{M}_{\beta)\gamma}$$
$$+ \frac{1}{3}(2\bar{\mathcal{D}}_\gamma\mathcal{S} - \mathrm{i}\mathcal{D}_\gamma R)\mathcal{M}_{\alpha\beta} - \left(\bar{\boldsymbol{C}}_{\alpha\beta\gamma} + \frac{1}{3}\varepsilon_{\gamma(\alpha}(8\bar{\mathcal{D}}_{\beta)}\mathcal{S} - \mathrm{i}\mathcal{D}_{\beta)}R)\right)\mathcal{J}. \tag{A11b}$$

These relations are very useful for actual calculations.

In three dimensions, the Weyl tensor is identically zero, and the Riemann tensor $\mathcal{R}_{abcd}$ is related to the Einstein tensor by the simple rule

$$\frac{1}{4}\varepsilon^{acd}\varepsilon^{bef}\mathcal{R}_{cdef}=\mathcal{G}^{ab}:=\mathcal{R}^{ab}-\frac{1}{2}\eta^{ab}\mathcal{R}, \\ \mathcal{R}_{abcd}=\varepsilon_{abe}\varepsilon_{cdf}\mathcal{G}^{ef}. \tag{A12}$$

As a consequence, the Riemann tensor is expressed in terms of the Ricci tensor $\mathcal{R}_{ab}:=\mathcal{R}^{c}{}_{acb}$ and the scalar curvature $\mathcal{R}:=\eta^{ab}\mathcal{R}_{ab}$ as follows:

$$\mathcal{R}_{abcd}=\eta_{ac}\mathcal{R}_{bd}-\eta_{ad}\mathcal{R}_{bc}+\eta_{bd}\mathcal{R}_{ac}-\eta_{bc}\mathcal{R}_{ad} \\ -\frac{1}{2}(\eta_{ac}\eta_{bd}-\eta_{ad}\eta_{bc})\mathcal{R}. \tag{A13}$$

The Cotton tensor is defined as follows

$$\mathcal{W}_{ab}:=\frac{1}{2}\varepsilon_{acd}\mathcal{W}^{cd}{}_{b}=\mathcal{W}_{ba}, \\ \mathcal{W}_{abc}=2\mathfrak{D}_{[a}\mathcal{R}_{b]c}+\frac{1}{2}\eta_{c[a}\mathfrak{D}_{b]}\mathcal{R}. \tag{A14}$$

A spacetime is conformally flat if and only if $\mathcal{W}_{ab}=0$ [73] (see [57] for a modern proof).

APPENDIX B: SUPERCONFORMAL SIGMA MODEL

In this appendix we consider an alternative parametrization of the supergravity-matter system (5.15) and reduce it to components without gauge fixing the Weyl, local $\mathrm{U}(1)_R$ and S-supersymmetry transformations.

In the new parametrization, the matter sector of the theory is described in terms of several covariantly chiral superfields $\phi^i=(\phi^0,\phi^I)$ of super-Weyl weight $w=1/2$,

$$\bar{\mathcal{D}}_\alpha\phi^i=0, \qquad \mathcal{J}\phi^i=-\frac{1}{2}\phi^i, \qquad \delta_\sigma\phi^i=\frac{1}{2}\sigma\phi^i. \tag{B1}$$

The action is defined to be

$$S=\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E\,N(\phi^i,\bar{\phi}^{\bar{j}}) \\ +\left\{\int \mathrm{d}^3x\mathrm{d}^2\theta\,\mathcal{E}\;P(\phi^i)+\text{c.c.}\right\} \\ \equiv S_{\text{kinetic}}+S_{\text{potential}} \tag{B2}$$

and may naturally be interpreted as a locally supersymmetric σ-model. For the action to be super-Weyl and $\mathrm{U}(1)_R$ invariant, the Kähler potential N and the superpotential P should obey the homogeneity conditions

$$\sum_i\phi^iN_i=\sum_{\bar{i}}\bar{\phi}^{\bar{i}}N_{\bar{i}}=N, \tag{B3a}$$

$$\sum_i\phi^iP_i=4P. \tag{B3b}$$

Equation (B3a) means that the σ-model target space is a Kähler cone [74].

Before reducing the action to components, we introduce several standard σ-model definitions. As usual, multiple derivatives of the Kähler potential are denoted as

$$N_{i_1\ldots i_p\bar{j}_1\ldots\bar{j}_q}:=\frac{\partial^{(p+q)}}{\partial\phi^{i_1}\ldots\partial\phi^{i_p}\partial\bar{\phi}^{\bar{j}_1}\ldots\partial\bar{\phi}^{\bar{j}_q}}N. \tag{B4}$$

The Kähler metric[21] $N_{i\bar{j}}=N_{\bar{j}i}$ is assumed to be nonsingular, with its inverse being denoted $N^{\bar{i}j}=N^{j\bar{i}}$,

$$N_{i\bar{k}}N^{\bar{k}j}=\delta_i^j, \qquad N^{\bar{i}k}N_{k\bar{j}}=\delta_{\bar{j}}^{\bar{i}}. \tag{B5}$$

The Christoffel symbols γ^k_{ij} are

$$\gamma^k_{ij}:=N_{ij\bar{l}}N^{\bar{l}k}, \qquad \gamma^{\bar{k}}_{\bar{i}\bar{j}}:=N_{l\bar{i}\bar{j}}N^{l\bar{k}}, \tag{B6}$$

and the Riemann curvature $\mathfrak{R}_{i\bar{k}j\bar{l}}$ is

$$\mathfrak{R}_{i\bar{k}j\bar{l}}=\mathfrak{R}_{i\bar{k}j}{}^{p}N_{p\bar{l}}=(\partial_{\bar{k}}\gamma^p_{ij})N_{p\bar{l}}. \tag{B7}$$

We define the component fields of ϕ^i as follows:

$$\rho^i_\alpha:=\mathcal{D}_\alpha\phi^i|, \tag{B8a}$$

$$\mathcal{F}^i:=-\frac{1}{4}[\mathcal{D}^2\phi^i+\gamma^i_{jk}(\mathcal{D}^\alpha\phi^j)\mathcal{D}_\alpha\phi^k]|. \tag{B8b}$$

The physical scalar $\phi^i|$ will be denoted by the same symbol as the chiral superfield ϕ^i itself.

To reduce the kinetic term in (B2) to components, we associate with it the antichiral Lagrangian

$$\bar{\mathcal{L}}_c=-\frac{1}{4}(\mathcal{D}^2-4\bar{R})N \tag{B9}$$

and make use of the action principle (3.13). The resulting component Lagrangian is

[21] We do not assume the Kähler metric to be positive definite.

$$L_{\rm kinetic} = -\frac{1}{8}\left[\mathcal{R} + \frac{\rm i}{2}\varepsilon^{abc}(\psi_a\bar{\boldsymbol{\psi}}_{bc} + \bar{\psi}_a\boldsymbol{\psi}_{bc})\right]N + N_{i\bar{j}}\left[\mathcal{F}^i\bar{\mathcal{F}}^{\bar{j}} - (\mathbf{D}^a\phi^i)\mathbf{D}_a\bar{\phi}^{\bar{j}} - \frac{\rm i}{4}\bar{\rho}^{\bar{j}}\gamma^a\tilde{\mathbf{D}}_a\rho^i - \frac{\rm i}{4}\rho^i\gamma^a\tilde{\mathbf{D}}_a\bar{\rho}^{\bar{j}} + \frac{\rm i}{2}\psi_a\rho^i\mathbf{D}^a\bar{\phi}^{\bar{j}}\right.$$
$$-\frac{\rm i}{2}\bar{\psi}_a\bar{\rho}^{\bar{j}}\mathbf{D}^a\phi^i + \frac{\rm i}{2}\varepsilon^{abc}(\bar{\psi}_a\gamma_b\bar{\rho}^{\bar{j}}\mathbf{D}_c\phi^i - \psi_a\gamma_b\rho^i\mathbf{D}_c\bar{\phi}^{\bar{j}}) - \frac{1}{8}\psi^a\bar{\psi}_a\rho^i\bar{\rho}^{\bar{j}} + \frac{1}{8}\psi^a\gamma_b\bar{\psi}_a\rho^i\gamma^b\bar{\rho}^{\bar{j}}$$
$$\left.+\frac{1}{8}\varepsilon^{abc}(\psi_a\gamma_b\bar{\psi}_c\rho^i\bar{\rho}^{\bar{j}} + \psi_a\bar{\psi}_b\rho^i\gamma_c\bar{\rho}^{\bar{j}})\right]$$
$$+\frac{1}{8}\varepsilon^{abc}[\bar{\boldsymbol{\psi}}_{ab}\gamma_c\bar{\rho}^{\bar{i}}N_{\bar{i}} - \boldsymbol{\psi}_{ab}\gamma_c\rho^iN_i] + \frac{\rm i}{4}\varepsilon^{abc}\psi_a\bar{\psi}_b[N_i\mathbf{D}_c\phi^i - N_{\bar{i}}\mathbf{D}_c\bar{\phi}^{\bar{i}}] + \frac{1}{16}\mathfrak{R}_{i\bar{k}j\bar{l}}\rho^i\rho^j\bar{\rho}^{\bar{k}}\bar{\rho}^{\bar{l}}, \qquad \text{(B10)}$$

where we have introduced the target-space covariant derivative

$$\tilde{\mathbf{D}}_a\rho^i_\alpha := \mathbf{D}_a\rho^i_\alpha + \gamma^i_{jk}\rho^j_\alpha\mathbf{D}_a\phi^k. \qquad \text{(B11)}$$

A short calculation of the component Lagrangian corresponding to $S_{\rm potential}$ gives

$$L_{\rm potential} = \mathcal{F}^iP_i - \frac{1}{4}(P_{jk} - \gamma^i_{jk}P_i)\rho^j\rho^k + \frac{\rm i}{2}\bar{\psi}_a\gamma^a\rho^jP_j$$
$$-\frac{1}{2}\varepsilon^{abc}\bar{\psi}_a\gamma_b\bar{\psi}_cP + \text{c.c.} \qquad \text{(B12)}$$

Both Lagrangians (B10) and (B12) are quite compact.

Now, we relate the theory under consideration to the σ-model (5.15). We assume that the chiral scalar ϕ^0 from the set $\phi^i = (\phi^0, \phi^I)$ is nowhere vanishing, $\phi^0 \neq 0$, and therefore it may be chosen to play the role of conformal compensator. We introduce a new parametrization of the dynamical chiral superfields defined by

$$\phi^0 = \Phi, \quad \phi^I = \Phi\varphi^I. \qquad \text{(B13)}$$

Here the chiral scalars φ^I are neutral under the super-Weyl and $\mathrm{U}(1)_R$ transformations. Since Φ is nowhere vanishing, $N(\phi,\bar{\phi})$ and $P(\phi)$ may be represented in the form

$$N(\phi,\bar{\phi}) = -4\bar{\Phi}{\rm e}^{-\frac{1}{4}K(\varphi,\bar{\varphi})}\Phi, \qquad P(\phi) = \Phi^4W(\varphi). \qquad \text{(B14)}$$

We assume that $K(\varphi,\bar{\varphi})$ is the Kähler potential of a Kähler manifold with positive definite metric $g_{I\bar{J}} := K_{I\bar{J}}$.

Let us express the geometric objects in terms of the new coordinates introduced. A short calculation gives

$$N_{i\bar{j}} = {\rm e}^{-\frac{1}{4}K}\begin{pmatrix} -4 & \bar{\Phi}K_{\bar{J}} \\ \Phi K_I & \Phi\bar{\Phi}\hat{K}_{I\bar{J}} \end{pmatrix}, \qquad \text{(B15a)}$$

where we have denoted

$$K_I := \frac{\partial K}{\partial\varphi^I}, \qquad \hat{K}_{I\bar{J}} := g_{I\bar{J}} - \frac{1}{4}K_IK_{\bar{J}}. \qquad \text{(B15b)}$$

It follows from (B15a) that the conditions $\det(N_{i\bar{j}}) \neq 0$ and $\det(g_{I\bar{J}}) \neq 0$ are equivalent. For the inverse metric we obtain

$$N^{\bar{i}j} = {\rm e}^{\frac{1}{4}K}\begin{pmatrix} -\frac{1}{4}\left(1 - \frac{1}{4}K^LK_L\right) & \frac{1}{4\Phi}K^J \\ \frac{1}{4\bar{\Phi}}K^{\bar{I}} & \frac{1}{\Phi\bar{\Phi}}K^{\bar{I}J} \end{pmatrix}, \qquad \text{(B16a)}$$

where we have denoted

$$K^I := g^{I\bar{J}}K_{\bar{J}}, \qquad K^{\bar{I}} := g^{\bar{I}J}K_J. \qquad \text{(B16b)}$$

For the Christoffel symbols γ^i_{kl} we read off

$$\gamma^0_{kl} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4}\Phi\left(\Gamma^I_{KL}K_I - K_{KL} - \frac{1}{4}K_KK_L\right) \end{pmatrix},$$
$$\gamma^I_{kl} = \begin{pmatrix} 0 & \frac{1}{\Phi}\delta^I_L \\ \frac{1}{\Phi}\delta^I_K & \Gamma^I_{KL} - \frac{1}{2}K_{(K}\delta^I_{L)} \end{pmatrix}, \qquad \text{(B17)}$$

where Γ^I_{KL} is the Christoffel symbol for the Kähler metric $g_{I\bar{J}}$. Since $\partial_{\bar{0}}\gamma^k_{ij} = 0$, the Riemann tensor is characterized by the properties

$$\mathfrak{R}_{0\bar{k}j\bar{l}} = \mathfrak{R}_{i\bar{0}j\bar{l}} = \mathfrak{R}_{i\bar{k}0\bar{l}} = \mathfrak{R}_{i\bar{k}j\bar{0}} = 0. \qquad \text{(B18)}$$

Thus the only nonzero components of the Riemann tensor are

$$\mathfrak{R}_{I\bar{K}J\bar{L}} = \Phi\bar{\Phi}{\rm e}^{-\frac{1}{4}K}\left(R_{I\bar{K}J\bar{L}} - \frac{1}{4}(K_{I\bar{K}}K_{J\bar{L}} + K_{J\bar{K}}K_{I\bar{L}})\right),$$
$$R_{I\bar{K}J\bar{L}} = g_{P\bar{L}}\partial_{\bar{K}}\Gamma^P_{IJ}. \qquad \text{(B19)}$$

Our next step is to express the auxiliary fields $\mathcal{F}^i$, Eq. (B8b), and the spinor fields $\tilde{\mathbf{D}}_a\rho^i_\alpha$, Eq. (B11), in terms of the component fields of Φ and φ^I. We recall that the component fields of φ^I are defined in (5.22). We do not introduce special names for the component fields of Φ; we simply write them as Φ, $\mathcal{D}_\alpha\Phi$ and $\mathcal{D}^2\Phi$, with the bar projection being always assumed here and in what follows. For the auxiliary fields $\mathcal{F}^i$ we get

$$\mathcal{F}^0 = -\frac{1}{4}\mathcal{D}^2\Phi - \frac{1}{16}\Phi\left(\Gamma^I_{KL}K_I - K_{KL} - \frac{1}{4}K_KK_L\right)\lambda^K\lambda^L, \qquad \text{(B20a)}$$

$$\mathcal{F}^I = F^I - \frac{1}{2\Phi}\lambda^{\alpha I}\mathcal{D}_\alpha\Phi + \frac{1}{8}\lambda^I\lambda^JK_J. \qquad \text{(B20b)}$$

For the spinor fields $\tilde{\mathbf{D}}_a\rho^i_\alpha$ we derive

$$\tilde{\mathbf{D}}_a\mathcal{D}_\alpha\Phi=\tilde{\mathbf{D}}_a\rho^0_\alpha=\mathbf{D}_a\mathcal{D}_\alpha\Phi+\frac{1}{4}\Phi\left(\Gamma^I_{KL}K_I-K_{KL}-\frac{1}{4}K_KK_L\right)\lambda^K_\alpha\mathbf{D}_aX^L, \tag{B21a}$$

$$\tilde{\mathbf{D}}_a\lambda^I_\alpha=\mathbf{D}_a\lambda^I_\alpha+\Gamma^I_{JK}\lambda^J_\alpha\mathbf{D}_aX^K+\frac{2}{\Phi}(\mathcal{D}_\alpha\Phi)\mathbf{D}_aX^I-\frac{1}{2}K_J\lambda^{(I}_\alpha\mathbf{D}_aX^{J)}. \tag{B21b}$$

Using the above results, for the kinetic term (B10) we obtain

$$\begin{aligned}
L_{\text{kinetic}}=&\left[\frac{1}{2}\mathcal{R}+\frac{\mathrm{i}}{4}\varepsilon^{abc}(\psi_a\bar{\boldsymbol{\psi}}_{bc}+\bar{\psi}_a\boldsymbol{\psi}_{bc})\right]\bar{\Phi}\mathrm{e}^{-\frac{1}{4}K}\Phi-4\Big[\mathcal{F}^0\bar{\mathcal{F}}^{\bar{0}}-(\mathbf{D}^a\Phi)\mathbf{D}_a\bar{\Phi}-\frac{\mathrm{i}}{4}((\bar{\mathcal{D}}\bar{\Phi})\gamma^a\tilde{\mathbf{D}}_a\mathcal{D}\Phi+(\mathcal{D}\Phi)\gamma^a\tilde{\mathbf{D}}_a\bar{\mathcal{D}}\bar{\Phi})\\
&+\frac{1}{2}\psi_a(\mathcal{D}\Phi)\mathbf{D}^a\bar{\Phi}+\frac{1}{2}\bar{\psi}_a(\bar{\mathcal{D}}\bar{\Phi})\mathbf{D}^a\Phi+\frac{1}{2}\varepsilon^{abc}(\bar{\psi}_a\gamma_b(\bar{\mathcal{D}}\bar{\Phi})\mathbf{D}_c\Phi-\psi_a\gamma_b(\mathcal{D}\Phi)\mathbf{D}_c\bar{\Phi})\\
&+\frac{1}{8}\varepsilon^{abc}(\psi_a\gamma_b\bar{\psi}_c(\mathcal{D}\Phi)\bar{\mathcal{D}}\bar{\Phi}+\psi_a\bar{\psi}_b(\mathcal{D}\Phi)\gamma_c\bar{\mathcal{D}}\bar{\Phi})-\frac{1}{8}\psi^a\bar{\psi}_a(\mathcal{D}\Phi)\bar{\mathcal{D}}\bar{\Phi}+\frac{1}{8}\psi^a\gamma^c\bar{\psi}_a(\mathcal{D}\Phi)\gamma_b\bar{\mathcal{D}}\bar{\Phi}\Big]\mathrm{e}^{-\frac{1}{4}K}\\
&+\Big[\mathcal{F}^0\bar{\mathcal{F}}^{\bar{I}}-(\mathbf{D}^a\Phi)\mathbf{D}_a\bar{X}^{\bar{I}}-\frac{\mathrm{i}}{4}(\bar{\lambda}^{\bar{I}}\gamma^a\tilde{\mathbf{D}}_a\mathcal{D}\Phi+(\mathcal{D}\Phi)\gamma^a\tilde{\mathbf{D}}_a\bar{\lambda}^{\bar{I}})+\frac{1}{2}\psi_a(\mathcal{D}\Phi)\mathbf{D}^a\bar{X}^{\bar{I}}+\frac{1}{2}\bar{\psi}_a\bar{\lambda}^{\bar{I}}\mathbf{D}^a\Phi\\
&+\frac{1}{2}\varepsilon^{abc}(\bar{\psi}_a\gamma_b\bar{\lambda}^{\bar{I}}\mathbf{D}_c\Phi-\psi_a\gamma_b(\mathcal{D}\Phi)\mathbf{D}_c\bar{\varphi}^{\bar{I}})+\frac{1}{8}\varepsilon^{abc}(\psi_a\gamma_b\bar{\psi}_c\bar{\lambda}^{\bar{I}}\mathcal{D}\Phi-\psi_a\bar{\psi}_b\bar{\lambda}^{\bar{I}}\gamma_c\mathcal{D}\Phi)\\
&-\frac{1}{8}\psi^a\bar{\psi}_a\bar{\lambda}^{\bar{I}}\mathcal{D}\Phi-\frac{1}{8}\psi^a\gamma^b\bar{\psi}_a\bar{\lambda}^{\bar{I}}\gamma_b\mathcal{D}\Phi\Big]\mathrm{e}^{-\frac{1}{4}K}\bar{\Phi}K_{\bar{I}}+\Big[\mathcal{F}^I\bar{\mathcal{F}}^{\bar{0}}-(\mathbf{D}^aX^I)\mathbf{D}_a\bar{\Phi}-\frac{\mathrm{i}}{4}(\lambda^I\gamma^a\tilde{\mathbf{D}}_a\bar{\mathcal{D}}\bar{\Phi}+(\bar{\mathcal{D}}\bar{\Phi})\gamma^a\tilde{\mathbf{D}}_a\lambda^I)\\
&+\frac{1}{2}\psi_a\lambda^I\mathbf{D}^a\bar{\Phi}+\frac{1}{2}\bar{\psi}_a(\bar{\mathcal{D}}\bar{\Phi})\mathbf{D}^aX^I+\frac{1}{2}\varepsilon^{abc}(\bar{\psi}_a\gamma_b(\bar{\mathcal{D}}\bar{\Phi})\mathbf{D}_cX^I-\psi_a\gamma_b\lambda^I\mathbf{D}_c\bar{\Phi})\\
&+\frac{1}{8}\varepsilon^{abc}(\psi_a\gamma_b\bar{\psi}_c\lambda^I\bar{\mathcal{D}}\bar{\Phi}+\psi_a\bar{\psi}_b\lambda^I\gamma_c\bar{\mathcal{D}}\bar{\Phi})-\frac{1}{8}\psi^a\bar{\psi}_a\lambda^I\bar{\mathcal{D}}\bar{\Phi}+\frac{1}{8}\psi^a\gamma^b\bar{\psi}_a\lambda^I\gamma_b\bar{\mathcal{D}}\bar{\Phi}\Big]\mathrm{e}^{-\frac{1}{4}K}\Phi K_I\\
&+\Big[\mathcal{F}^I\bar{\mathcal{F}}^{\bar{J}}-(\mathbf{D}^aX^I)\mathbf{D}_a\bar{X}^{\bar{J}}-\frac{\mathrm{i}}{4}(\bar{\lambda}^{\bar{J}}\gamma^a\tilde{\mathbf{D}}_a\lambda^I+\lambda^I\gamma^a\tilde{\mathbf{D}}_a\bar{\lambda}^{\bar{J}})+\frac{1}{2}\psi_a\lambda^I\mathbf{D}^a\bar{X}^{\bar{J}}+\frac{1}{2}\bar{\psi}_a\bar{\lambda}^{\bar{J}}\mathbf{D}^aX^I\\
&+\frac{1}{2}\varepsilon^{abc}(\bar{\psi}_a\gamma_b\bar{\lambda}^{\bar{J}}\mathbf{D}_cX^I-\psi_a\gamma_b\lambda^I\mathbf{D}_c\bar{X}^{\bar{J}})+\frac{1}{8}\varepsilon^{abc}(\psi_a\gamma_b\bar{\psi}_c\lambda^I\bar{\lambda}^{\bar{J}}+\psi_a\bar{\psi}_b\lambda^I\gamma_c\bar{\lambda}^{\bar{J}})\\
&-\frac{1}{8}\psi^a\bar{\psi}_a\lambda^I\bar{\lambda}^{\bar{J}}+\frac{1}{8}\psi^a\gamma^b\bar{\psi}_a\lambda^I\gamma_b\bar{\lambda}^{\bar{J}}\Big]\mathrm{e}^{-\frac{1}{4}K}\Phi\bar{\Phi}\hat{K}_{I\bar{J}}\\
&-\frac{1}{4}\varepsilon^{abc}\left[\bar{\Phi}\left(\mathrm{i}\psi_a\bar{\psi}_b(4\mathbf{D}_c\Phi-\Phi K_I\mathbf{D}_cX^I)-\frac{1}{2}\boldsymbol{\psi}_{ab}\gamma_c(4\mathcal{D}\Phi-\Phi K_I\lambda^I)\right)+\text{c.c.}\right]\mathrm{e}^{-\frac{1}{4}K}\\
&+\frac{1}{16}\left[R_{I\bar{K}J\bar{L}}-\frac{1}{2}g_{I\bar{K}}g_{J\bar{L}}\right]\lambda^I\lambda^J\bar{\lambda}^{\bar{K}}\bar{\lambda}^{\bar{L}}.
\end{aligned} \tag{B22}$$

The potential term (B12) becomes

$$\begin{aligned}
L_{\text{potential}}=\Phi^4\Big[&F^IW_I-\Phi^{-1}W\mathcal{D}^2\Phi-3\Phi^{-2}W(\mathcal{D}\Phi)\mathcal{D}\Phi-2\Phi^{-1}W_I\lambda^I\mathcal{D}\Phi-\frac{1}{4}\lambda^I\lambda^J(W_{IJ}-\Gamma^K_{IJ}W_K)\\
&+\frac{\mathrm{i}}{2}\bar{\psi}_a\gamma^a(4\Phi^{-1}W\mathcal{D}\Phi+W_I\lambda^I)+\frac{1}{2}W\varepsilon^{abc}\bar{\psi}_a\gamma_b\bar{\psi}_c\Big]+\text{c.c.}
\end{aligned} \tag{B23}$$

The sum of the expressions (B22) and (B23) constitutes the component Lagrangian of the theory (5.15) with no gauge condition on the chiral compensator Φ imposed. Looking at the explicit form of (B22), it is easy to understand why the gauge conditions (5.19) have been chosen. First of all, it is seen from the first line of (B22) the canonically normalized Hilbert-Einstein gravitational Lagrangian corresponds to the Weyl gauge condition (5.19a). Secondly, consider the terms in (B22) which involve the gravitino field strength coupled to the matter fermions. These consist of

$$\bar{\Phi}\boldsymbol{\psi}_{ab}\gamma_c\left(\mathrm{e}^{-\frac{1}{4}K}\mathcal{D}_\alpha\Phi - \frac{1}{4}\Phi\,\mathrm{e}^{-\frac{1}{4}}K_I\lambda^I_\alpha\right) = \boldsymbol{\psi}_{ab}\gamma_c\mathcal{D}_\alpha(\bar{\Phi}\,\mathrm{e}^{-\frac{1}{4}K}\Phi) \tag{B24}$$

and its complex conjugate. To eliminate these cross terms, we have to impose the S-supersymmetry gauge condition (5.19b). Finally, the $\mathrm{U}(1)_R$ gauge condition (5.19c) eliminates an overall phase factor in the superpotential (B23). In the gauge (5.19), the only remaining field in Φ occurs at the θ^2 component. It can be defined in the Kähler invariant way (5.20).

In the gauge (5.19), the following useful relations hold

$$\mathcal{D}_\alpha\Phi = \frac{1}{4}\,\mathrm{e}^{\frac{1}{8}K}\lambda^L_\alpha K_L, \tag{B25a}$$

$$\mathcal{F}^0 = -\frac{1}{4}\,\mathrm{e}^{-\frac{1}{8}K}\mathbb{M} + \frac{1}{4}\,\mathrm{e}^{\frac{1}{8}K}F^I K_I, \tag{B25b}$$

$$\mathcal{F}^I = F^I. \tag{B25c}$$

As usual, the bar projection is assumed here. Using these relations one may obtain the component Lagrangians (5.23) and (5.28) from (B22) and (B23).

APPENDIX C: VECTOR MULTIPLET MODEL

In this appendix we present the component Lagrangian for the model of an Abelian vector multiplet coupled to conformal supergravity. As in Sec. VI A, we denote by G the gauge prepotential of the vector multiplet, and by $\mathbb{G}$ the corresponding guage-invariant field strength. The vector multiplet action is

$$S_{\mathrm{VM}} = -4\int \mathrm{d}^3x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}E(\mathbb{G}\ln\mathbb{G} - 4G\mathcal{S} - \kappa G\mathbb{G}), \tag{C1}$$

with κ a constant parameter.

We define the component fields of the vector multiplet as follows:

$$\mathcal{Y} := \mathbb{G}|, \tag{C2}$$

$$v_\alpha := \mathcal{D}_\alpha\mathbb{G}|, \tag{C3}$$

$$\mathcal{Z} := \mathrm{i}\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathbb{G}|, \tag{C4}$$

$$\begin{aligned}\mathcal{B}^a &:= -\frac{1}{2}(\gamma^a)^{\alpha\beta}[\mathcal{D}_{(\alpha},\bar{\mathcal{D}}_{\beta)}]\mathbb{G}| \\ &= \mathcal{H}^a - \varepsilon^{abc}\mathcal{Y}\bar{\psi}_b\psi_c - \mathrm{i}\varepsilon^{abc}(\psi_b\gamma_c v + \bar{\psi}_b\gamma_c\bar{v}).\end{aligned} \tag{C5}$$

As in Sec. VI, $\mathcal{H}^a$ denotes the Hodge dual of the component field strength,

$$\mathcal{H}^a = \frac{1}{2}\varepsilon^{abc}\mathcal{H}_{bc}, \quad \mathcal{H}_{ab} = \mathfrak{D}_a a_b - \mathfrak{D}_b a_a - \mathcal{T}_{ab}{}^c a_c. \tag{C6}$$

We also choose the WZ gauge (6.16) for the vector multiplet. Then, the other component fields of G are

$$[\mathcal{D}_{(\alpha},\bar{\mathcal{D}}_{\beta)}]G| = \frac{1}{2}a_{\alpha\beta}, \tag{C7}$$

$$\mathcal{D}^2\bar{\mathcal{D}}_\alpha G| = 2\mathrm{i}v_\alpha, \tag{C8}$$

$$\begin{aligned}\bar{\mathcal{D}}^2\mathcal{D}^2G| &= -2\mathrm{i}\mathbf{D}_b a^b - 2(\psi_a\gamma^a v + \bar{\psi}_a\gamma^a\bar{v}) \\ &\quad - 2\mathrm{i}\mathcal{Y}\bar{\psi}_a\psi^a - \bar{\psi}_a\gamma^b\psi^a a_b - 2\mathcal{Z}.\end{aligned} \tag{C9}$$

The component Lagrangian corresponding to the action (C1) is

$$\begin{aligned}L_{\mathrm{VM}} = {}& \frac{1}{4}\mathcal{Y}^{-1}\mathcal{B}^a\mathcal{B}_a - a_a\mathcal{F}^a - \mathcal{Y}^{-1}(\hat{\mathbf{D}}^a\mathcal{Y})\hat{\mathbf{D}}_a\mathcal{Y} - \frac{1}{2}\mathcal{Y}\mathcal{R} + \frac{1}{4}\mathcal{Y}^{-1}\mathcal{Z}^2 - \mathrm{i}\mathcal{Y}^{-1}(\bar{v}\gamma^a\hat{\mathbf{D}}_a v + v\gamma^a\hat{\mathbf{D}}_a\bar{v}) - \frac{1}{2}\mathcal{Y}^{-2}\mathcal{B}_a\bar{v}\gamma^a v \\ & - \frac{\mathrm{i}}{2}\mathcal{Y}^{-2}\mathcal{Z}\bar{v}v - \frac{1}{2}\mathcal{Y}^{-3}v^2\bar{v}^2 + \kappa\left[-\mathcal{Y}\mathcal{Z} - \frac{1}{4}a_a\mathcal{B}^b - \mathrm{i}\bar{v}v + \frac{\mathrm{i}}{2}\varepsilon^{abc}\mathcal{Y}^2\bar{\psi}_a\gamma_b\psi_c - \frac{1}{4}\varepsilon^{abc}\mathcal{Y}a_c\bar{\psi}_a\psi_b\right] \\ & + \left\{-\frac{1}{4}\varepsilon^{abc}(2\bar{v}\gamma_a\bar{\boldsymbol{\psi}}_{bc} + \mathrm{i}\mathcal{Y}\psi_a\bar{\boldsymbol{\psi}}_{bc}) - \frac{1}{2}\mathcal{Y}^{-1}\psi_a\gamma^a\tilde{\gamma}^b v\left(\hat{\mathbf{D}}_b\mathcal{Y} - \frac{\mathrm{i}}{2}\mathcal{B}_b\right) + \frac{1}{4}\mathcal{Y}^{-1}\mathcal{Z}\psi_a\gamma^a v - \frac{\mathrm{i}}{4}\mathcal{Y}^{-2}\psi_a\gamma^a\bar{v}v^2\right. \\ & \left. + \frac{1}{4}\mathcal{Y}^{-1}\varepsilon^{abc}\psi_a\gamma_b\psi_c v^2 + \kappa\left[\mathcal{Y}\psi_a\gamma^a v - \frac{\mathrm{i}}{4}\varepsilon^{abc}a_a\psi_b\gamma_c v\right] + \mathrm{c.c.}\right\}.\end{aligned} \tag{C10}$$

Here we have introduced new covariant derivatives

$$\hat{\mathbf{D}}_a\mathcal{Y} := \mathbf{D}_a\mathcal{Y} - \frac{1}{2}\psi_a v - \frac{1}{2}\bar{\psi}_a\bar{v}, \tag{C11}$$

$$\hat{\mathbf{D}}_a v^\alpha := \mathbf{D}_a v^\alpha - \frac{\mathrm{i}}{2}\bar{\psi}_a{}^\alpha\mathcal{Z} + \frac{\mathrm{i}}{2}(\bar{\psi}_a\tilde{\gamma^b})^\alpha\left(\hat{\mathbf{D}}_b\mathcal{Y} - \frac{\mathrm{i}}{2}\mathcal{B}_b\right). \tag{C12}$$

APPENDIX D: THE ACTION FOR CONFORMAL SUPERGRAVITY

In the family of $\mathcal{N} = 2$ locally supersymmetric theories in three dimensions, conformal supergravity [4] is one of the oldest. Originally it was constructed by gauging the 3D $\mathcal{N} = 2$ superconformal algebra, $\mathfrak{osp}(2|4)$, in ordinary spacetime, as a direct generalization of the formulation for $\mathcal{N} = 1$ conformal supergravity [75] (the latter theory being a natural reformulation of topologically massive $\mathcal{N} = 1$ supergravity [32,33]). The construction in [4] was soon generalized to the case of $\mathcal{N}$-extended conformal supergravity [76]. In accordance with [76], the action for $\mathcal{N}$-extended conformal supergravity is a locally supersymmetric completion of the gravitational and SO($\mathcal{N}$) gauge Chern-Simons terms. This action is on shell for $\mathcal{N} \geq 3$, and therefore its applications are rather limited.[22] As concerns the off-shell $\mathcal{N} = 1$ and $\mathcal{N} = 2$ component actions [4,75], it appears useful to recast them in a superfield form, simply because all $\mathcal{N} = 1$ and $\mathcal{N} = 2$ locally supersymmetric matter systems are naturally formulated in superspace.

As mentioned in the Introduction, Refs. [10,11] described the most general matter couplings to conformal supergravity in the cases $1 \leq \mathcal{N} \leq 4$, including the off-shell formulations for Poincaré and AdS supergravity theories. But no conformal supergravity action was considered in these publications, due to the fact that an alternative action principle is required in order to describe pure conformal supergravity. Building on the earlier incomplete results in [2,12,79], the action for $\mathcal{N} = 1$ conformal supergravity has recently been constructed in terms of the superfield connection as a superspace integral [80]. However, such a construction becomes impossible starting at $\mathcal{N} = 2$.[23] This is because (i) the spinor and vector parts of the superfield connection have positive dimension equal to $1/2$ and 1 respectively; and (ii) the dimension of the full superspace measure is $(\mathcal{N} - 3)$. As a result, it is not possible to construct contributions to the action that are cubic in the superfield connection for $\mathcal{N} \geq 2$.

Nevertheless, it was argued in [80] that off-shell conformal supergravity actions (assuming their existence) may be realized in terms of the curved superspace geometry given in [10,14] [also known as SO($\mathcal{N}$) superspace] provided one makes use of the superform approach for the construction of supersymmetric invariants. Such a realization was explicitly worked out in [80] for the case $\mathcal{N} = 1$, and a general method of constructing conformal supergravity actions for $\mathcal{N} > 1$ was outlined. However, it turns out that SO($\mathcal{N}$) superspace [10,14] is not an optimum setting to carry out this program; see [57] for a detailed discussion. From a technical point of view, the derivation of the conformal supergravity actions greatly simplifies if one makes use of the so-called $\mathcal{N}$-extended conformal superspace of [57], which is a novel formulation for conformal supergravity. The SO($\mathcal{N}$) superspace of [10,14] is obtained from the $\mathcal{N}$-extended conformal superspace by gauge fixing certain local symmetries; see [57] for more details. In conformal superspace, the action for $\mathcal{N} = 2$ conformal supergravity is simply the Chern-Simons term associated with $\mathfrak{osp}(2|4)$ [66]. Below we reformulate this action in SO(2) superspace.

[22]Recently, off-shell conformal supergravity actions have been constructed for the cases $\mathcal{N} = 3, 4, 5$ [66] and $\mathcal{N} = 6$ [77,78]. Upon elimination of the auxiliary fields, these actions reduce to those proposed in [76] only for $\mathcal{N} = 3, 4, 5$. In the case $\mathcal{N} = 6$, however, the on-shell version of the off-shell action in [77,78] contains an additional U(1) gauge Chern-Simons term as compared with [76].

[23]If a prepotential formulation is available, the conformal supergravity action may be written as a superspace integral in terms of the prepotentials. Currently, the prepotential formulations are known only for the cases $\mathcal{N} = 1$ [2] and $\mathcal{N} = 2$ [15].

1. Conformal superspace and SO(2) superspace

Conceptually, the $\mathcal{N} = 2$ conformal superspace of [57] corresponds to a certain gauging of the superconformal algebra $\mathfrak{osp}(2|4)$ in superspace [57]. The corresponding covariant derivatives ∇_A include two types of connections: (i) the Lorentz and $\mathrm{U}(1)_R$ connections [as in SO(2) superspace]; and (ii) those associated with the dilatation ($\mathbb{D}$), special conformal (K_a) and S-supersymmetry $(S_\alpha, \bar{S}^\alpha)$ generators of the $\mathcal{N} = 2$ superconformal algebra. To emphasize this grouping, the covariant derivatives ∇_A can be written in the form[24]

$$\nabla_A = \hat{\mathcal{D}}_A + B_A \mathbb{D} + \mathfrak{F}_A{}^b K_b + \mathfrak{F}_A{}^\beta S_\beta + \tilde{\mathfrak{F}}_{A\beta} \bar{S}^\beta, \quad \text{(D1a)}$$

where we have denoted

$$\hat{\mathcal{D}}_A = E_A{}^M \partial_M - \hat{\Omega}_A{}^a \mathcal{M}_a + \mathrm{i} \hat{\Phi}_A \mathcal{J}. \quad \text{(D1b)}$$

By construction, the operators ∇_A are subject to certain covariant constraints [57] such that the entire algebra of covariant derivatives is expressed in terms of a single primary superfield—the super Cotton tensor $\mathcal{W}_{\alpha\beta}$.

As demonstrated in [57], the conformal superspace is intimately related to the SO(2) superspace via a degauging procedure. The crucial observation here is that the local special conformal and S-supersymmetry gauge freedom can be used to switch off the dilatation connection, $B_A = 0$. In this gauge, there remains no residual special conformal and S-supersymmetry gauge freedom, but the covariant derivatives (D1a) still include the connections associated with the generators K_b, S_β and $\bar{S}^\beta$. These connections are tensor superfields with respect to the remaining local Lorentz and $\mathrm{U}(1)_R$ symmetries. From the constraints obeyed by the conformal covariant derivatives, one may deduce that the operators $\hat{\mathcal{D}}_A$ look like

$$\hat{\mathcal{D}}_a = \mathcal{D}_a + \mathrm{i}\, \mathcal{C}_a \mathcal{J}, \qquad \hat{\mathcal{D}}_\alpha = \mathcal{D}_\alpha, \qquad \hat{\bar{\mathcal{D}}}^\alpha = \bar{\mathcal{D}}^\alpha, \quad \text{(D2)}$$

where $\mathcal{D}_A$ are the covariant derivatives of the SO(2) superspace, as defined in Sec. II A, and $\mathcal{C}_a$ is one of the

[24]The connections in D 1 differ in sign from those used in [57].

corresponding torsion superfields. The connections $\mathfrak{F}$'s are uniquely determined as functionals of the torsion superfields of the SO(2) superspace. In terms of the one-forms $\mathfrak{F}^\alpha := E^B\mathfrak{F}_B{}^\alpha$ and $\tilde{\mathfrak{F}}_\alpha := E^B\tilde{\mathfrak{F}}_{B\alpha}$, one obtains

$$\mathfrak{F}^\alpha = E^b\left[-\frac{1}{2}(\gamma_b)_{\beta\gamma}\boldsymbol{C}^{\alpha\beta\gamma} + \frac{1}{6}(\gamma_b)^{\alpha\beta}(\mathrm{i}\bar{\mathcal{D}}_\beta\bar{R} + \mathcal{D}_\beta\mathcal{S})\right] - E^\alpha\bar{R} + \bar{E}_\beta[\mathcal{C}^{\beta\alpha} + \mathrm{i}\varepsilon^{\beta\alpha}\mathcal{S}], \tag{D3a}$$

$$\tilde{\mathfrak{F}}_\alpha = E^b\left[-\frac{1}{2}(\gamma_b)^{\beta\gamma}\bar{\boldsymbol{C}}_{\alpha\beta\gamma} + \frac{1}{6}(\gamma_b)_{\alpha\beta}(\mathrm{i}\mathcal{D}^\beta R - \bar{\mathcal{D}}^\beta\mathcal{S})\right] - E^\beta[\mathcal{C}_{\beta\alpha} + \mathrm{i}\varepsilon_{\beta\alpha}\mathcal{S}] - \bar{E}^\alpha R. \tag{D3b}$$

2. Curvature two-forms

In SO(2) superspace, there exists a two-parameter freedom to define the vector covariant derivative. Instead of dealing with $\mathcal{D}_a$, one may work equally well with a deformed covariant derivative $\mathfrak{D}_a$ defined by

$$\mathfrak{D}_a = \mathcal{D}_a + \lambda\mathcal{S}M_a + \rho\mathrm{i}\mathcal{C}_a\mathcal{J}, \tag{D4}$$

where λ and ρ are real parameters. A natural question is the following: What is special about the deformation (D2)? Here we answer this question.

Let us introduce the torsion and curvature tensors for the covariant derivatives (D2),

$$[\hat{\mathcal{D}}_A, \hat{\mathcal{D}}_B\} = \hat{T}_{AB}{}^C\hat{\mathcal{D}}_C - \hat{R}_{AB}{}^c\mathcal{M}_c + \mathrm{i}\hat{R}_{AB}\mathcal{J}. \tag{D5}$$

Associated with the Lorentz and $\mathrm{U}(1)_R$ curvature tensors are the following two-forms: $\hat{R}^c = \frac{1}{2}E^B\wedge E^A\hat{R}_{AB}{}^c$ and $\hat{R} = \frac{1}{2}E^B\wedge E^A\hat{R}_{AB}$. The explicit expressions for these two-forms are

$$\begin{aligned}\hat{R}^c = {}& \frac{1}{2}E^\beta\wedge E^\alpha[4\bar{R}(\gamma^c)_{\alpha\beta}] + \bar{E}_\beta\wedge E^\alpha[-4\mathrm{i}\mathcal{S}(\gamma^c)_\alpha{}^\beta - 4\delta_\alpha^\beta\mathcal{C}^c] + \frac{1}{2}\bar{E}_\beta\wedge\bar{E}_\alpha[-4R(\gamma^c)^{\alpha\beta}] \\ &+ E^\beta\wedge E^a\left[(\gamma_a)_\beta{}^\gamma(\gamma^c)^{\delta\rho}\boldsymbol{C}_{\gamma\delta\rho} + \frac{1}{3}(\delta_\beta^\gamma\delta_a^c + 2\varepsilon_{ab}{}^c(\gamma^b)_\beta{}^\gamma)(2\mathcal{D}_\gamma\mathcal{S} + \mathrm{i}\bar{\mathcal{D}}_\gamma\bar{R})\right] \\ &+ \bar{E}_\beta\wedge E^a\left[(\gamma_a)^{\beta\gamma}(\gamma^c)^{\delta\rho}\bar{\boldsymbol{C}}_{\gamma\delta\rho} + \frac{1}{3}(\varepsilon^{\beta\gamma}\delta_a^c + 2\varepsilon_{ab}{}^c(\gamma^b)^{\beta\gamma})(2\bar{\mathcal{D}}_\gamma\mathcal{S} - \mathrm{i}\mathcal{D}_\gamma R)\right] \\ &+ \frac{1}{2}E^b\wedge E^a\varepsilon_{abd}\left[\frac{1}{4}(\gamma^d)^{\alpha\beta}(\gamma^c)^{\tau\delta}(\mathrm{i}\mathcal{D}_{(\tau}\bar{\boldsymbol{C}}_{\delta\alpha\beta)} + \mathrm{i}\bar{\mathcal{D}}_{(\tau}\boldsymbol{C}_{\delta\alpha\beta)}) - 4\mathcal{C}^d\mathcal{C}^c\right. \\ &\left.+ \eta^{cd}\left(\frac{2\mathrm{i}}{3}(\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\mathcal{S}) + \frac{1}{6}(\mathcal{D}^2R + \bar{\mathcal{D}}^2\bar{R}) - 4\mathcal{S}^2 - 4\bar{R}R\right)\right],\end{aligned} \tag{D6a}$$

$$\begin{aligned}\hat{R} = {}& \bar{E}_\beta\wedge E^\alpha[4\mathrm{i}\mathcal{C}_\alpha{}^\beta + 4\delta_\alpha^\beta\mathcal{S}] + E^\beta\wedge E^a\left[\mathrm{i}(\gamma_a)^{\gamma\delta}\boldsymbol{C}_{\beta\gamma\delta} - \frac{1}{3}(\gamma_a)_\beta{}^\gamma(\bar{\mathcal{D}}_\gamma\bar{R} - 2\mathrm{i}\mathcal{D}_\gamma\mathcal{S})\right] \\ &+ \bar{E}_\beta\wedge E^a\left[\mathrm{i}(\gamma_a)_{\gamma\delta}\bar{\boldsymbol{C}}^{\beta\gamma\delta} - \frac{1}{3}(\gamma_a)^\beta{}_\gamma(\mathcal{D}^\gamma R + 2\mathrm{i}\bar{\mathcal{D}}^\gamma\mathcal{S})\right] - \frac{1}{2}E^b\wedge E^a[\varepsilon_{abc}\mathcal{W}^c],\end{aligned} \tag{D6b}$$

with $\mathcal{W}^c$ the super Cotton tensor, Eq. (2.14). For completeness, we also reproduce the expressions for the two-forms $R^c = \frac{1}{2}E^B\wedge E^A R_{AB}{}^c$ and $R = \frac{1}{2}E^B\wedge E^A R_{AB}$, where the curvature tensors are those which appear in (2.6). Direct calculations give

$$\hat{R}^c = R^c, \tag{D7a}$$

$$\begin{aligned}\hat{R} = {}& R + \bar{E}_\beta\wedge E^\alpha[2\mathrm{i}\mathcal{C}_\alpha{}^\beta] + E^\beta\wedge E^a\left[\frac{\mathrm{i}}{2}(\gamma_a)^{\gamma\delta}\boldsymbol{C}_{\beta\gamma\delta}\right. \\ &\left.-\frac{1}{6}(\gamma_a)_\beta{}^\gamma(\bar{\mathcal{D}}_\gamma\bar{R} + 4\mathrm{i}\mathcal{D}_\gamma\mathcal{S})\right] \\ &+ \bar{E}_\beta\wedge E^a\left[\frac{\mathrm{i}}{2}(\gamma_a)_{\gamma\delta}\bar{\boldsymbol{C}}^{\beta\gamma\delta} - \frac{1}{6}(\gamma_a)^\beta{}_\gamma(\mathcal{D}^\gamma R - 4\mathrm{i}\bar{\mathcal{D}}^\gamma\mathcal{S})\right] \\ &- \frac{1}{2}E^b\wedge E^a[\varepsilon_{abc}\varepsilon^{cef}\mathcal{D}_e\mathcal{C}_f].\end{aligned} \tag{D7b}$$

The unique feature of the deformation (D2) is that the top component of the U(1) curvature two-form (D6b) is a primary superfield equivalent to the super-Cotton tensor.[25]

3. Closed three-form

In $\mathcal{N} = 2$ conformal superspace, the Chern-Simons three-form Σ_{CS} is characterized by the following properties [66]: (i) it is closed, $\mathrm{d}\Sigma_{\mathrm{CS}} = 0$; and (ii) under the gauge transformations, it is invariant modulo exact terms. This three-form generates the off-shell action for $\mathcal{N} = 2$ conformal supergravity. In this subsection, we follow the degauging procedure of [57] to obtain an expression for this closed three-form in SO(2) superspace,

[25] S. M. K. and G. T.-M. are grateful to Joseph Novak for this observation.

$\mathfrak{J} := \Sigma_{\rm CS}|_{\text{de-gauged}}$. The calculations are straightforward, and we present only the final result.

The three-form $\mathfrak{J}$ turns out to be

$$\mathfrak{J} = -\hat{R}^a\wedge\hat{\Omega}_a + \frac{1}{6}\hat{\Omega}^c\wedge\hat{\Omega}^b\wedge\hat{\Omega}^a\varepsilon_{abc} - 2\hat{R}\wedge\hat{\Phi} - 8\mathrm{i}E^a\wedge\mathfrak{F}^\alpha\wedge\bar{\mathfrak{F}}_\beta(\gamma_a)_\alpha{}^\beta. \tag{D8}$$

The expression for $\mathfrak{J}$ is naturally written in terms of the deformed covariant derivatives $\hat{\mathcal{D}}_A$. Making use of (D2), it is a simple exercise to rewrite (D8) in terms of the original covariant derivatives $\mathcal{D}_A$.

It is interesting to note that the closed three-form $\mathfrak{J}$ can be written as

$$\mathfrak{J} = \hat{\Sigma}_{\rm CS} - \Sigma_T, \tag{D9a}$$

where we have introduced

$$\Sigma_T = -8\mathrm{i}E^a\wedge\mathfrak{F}^\alpha\wedge\tilde{\mathfrak{F}}_\beta(\gamma_a)_\alpha{}^\beta, \tag{D9b}$$

$$\hat{\Sigma}_{\rm CS} = \hat{R}^a\wedge\hat{\Omega}_a - \frac{1}{6}\hat{\Omega}^c\wedge\hat{\Omega}^b\wedge\hat{\Omega}^a\varepsilon_{abc} + 2\hat{R}\wedge\hat{\Phi}. \tag{D9c}$$

The three-form $\hat{\Sigma}_{\rm CS}$ is a sum of the Lorentz and $\mathrm{U}(1)_R$ Chern-Simons three-forms associated with the covariant derivatives $\hat{\mathcal{D}}_A$. The components of Σ_T are functions of the torsion tensor and its covariant derivatives only; this is why Σ_T was called the torsion induced three-form in [80]. The three-forms $\hat{\Sigma}_{\rm CS}$ and Σ_T satisfy the equations

$$\mathrm{d}\Sigma_T = \mathrm{d}\hat{\Sigma}_{\rm CS} = \hat{R}^a\wedge\hat{R}_a + 2\hat{R}\wedge\hat{R}. \tag{D10}$$

By construction, the closed three-form $\mathfrak{J}$ is invariant under the super-Weyl transformations modulo exact terms. In fact, the relative coefficient between the Lorentz and $\mathrm{U}(1)_R$ Chern-Simons terms in (D10) is fixed by the condition that $\mathfrak{J}$ be super-Weyl invariant modulo exact terms.

The covariant derivatives (D2) and the closed three-form (D9) constitute the unique solution to the $\mathcal{N}=2$ problem posed in [80].

4. Conformal supergravity action

Using the three-form $\mathfrak{J} = \frac{1}{3!}E^C\wedge E^B\wedge E^A\mathfrak{J}_{ABC} = \frac{1}{3!}\mathrm{d}z^P\wedge\mathrm{d}z^N\wedge\mathrm{d}z^M\mathfrak{J}_{MNP}$, we can write down the locally supersymmetric and super-Weyl invariant action ($\varepsilon^{mnp} := \varepsilon^{abc}e_a{}^m e_b{}^n e_c{}^p$)

$$S = \int_{\mathcal{M}^3}\mathfrak{J} = \int \mathrm{d}^3x\, e^*\mathfrak{J}|_{\theta=0}, \quad {}^*\mathfrak{J} = \frac{1}{3!}\varepsilon^{mnp}\mathfrak{J}_{mnp}. \tag{D11}$$

Upon implementing the component and gauge fixing reduction described in Sec. IV, the action becomes

$$S = \frac{1}{4}\int \mathrm{d}^3x\, e\varepsilon^{abc}\Big[\mathcal{R}_{bcfg}\omega_a{}^{fg} + \frac{2}{3}\omega_{af}{}^g\omega_{bg}{}^h\omega_{ch}{}^f - 4\mathcal{F}_{ab}b_c + \mathrm{i}\bar{\psi}_{bc}\gamma_d\tilde{\gamma}_a\varepsilon^{def}\psi_{ef}\Big]. \tag{D12}$$

This is the component action for $\mathcal{N}=2$ conformal supergravity of [4].

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