(0,2) dynamics from four dimensions

David Kutasov and Jennifer Lin
EFI and Department of Physics, University of Chicago, 5640S. Ellis Avenue, Chicago, Illinois 60637, USA
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We study (0,2) supersymmetric two-dimensional theories obtained by compactifying four-dimensional $N = 1$ supersymmetric theories on a two-torus, with a magnetic field for a global $U(1)$ symmetry, and present evidence that Seiberg duality in four dimensions leads to an identification of different models of this type.

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I. INTRODUCTION

Supersymmetric quantum field theories (SQFTs) with four supercharges in various dimensions ($d \leq 4$) have been extensively studied in the past few decades. Many of their properties have been elucidated using symmetries, anomalies, holomorphy, strong-weak coupling duality and related ideas (see e.g. [1,2] for reviews of theories in $d = 4$), as well as insights from string theory, where many such theories can be realized as low energy theories on branes [3–5]. An interesting aspect of this program was the discovery of connections between the dynamics of such theories in different dimensions (see e.g. [6] for a recent discussion of the $3d$–$4d$ connection).

It is natural to ask how much of this progress can be extended to theories with two supercharges in $d \leq 3$. In three dimensions, such theories have the minimal amount of supersymmetry, $N = 1$, for which many of the techniques mentioned above are inapplicable. In two dimensions, one can consider either (1,1) SQFTs, which suffer from the same problem,[1] or models with chiral, say (0,2), supersymmetry (SUSY) which seem more promising from this point of view, due to their inherent chirality and the fact that they have the minimal amount of supersymmetry for which holomorphy, anomalies, etc., place strong constraints on the dynamics. They are also interesting for applications to (heterotic world sheet) string theory.

To explore the $2d$–$4d$ connection for theories with (0,2) supersymmetry, we need a way to associate a (0,2) model with a given four-dimensional $N = 1$ SQFT. One way to do this in a large class of examples is the following. Given a four-dimensional $N = 1$ SQFT with a global (non-R) symmetry, one can couple the current supermultiplet to an external vector superfield, which consists of a vector field A_μ, a gaugino λ_α, and an auxiliary field D. These fields are nondynamical, but can still take nonzero expectation values. In a spacetime of the form $\mathbb{R}^{1,1} \times \mathrm{T}^2$, labeled by the coordinates $(x^0, x^3) \in \mathbb{R}^{1,1}$, $(x^1, x^2) \in \mathrm{T}^2$, it is natural to turn on an expectation value for the magnetic field through the torus,

$$F_{12} = B. \tag{1.1}$$

This breaks supersymmetry completely, as can be seen by examining the variation of the external gaugino field: $\delta\lambda = F_{\mu\nu}\sigma^{\mu\nu}\epsilon$, with $\mu, \nu = 0, 1, 2, 3$. For a finite magnetic field, $\delta\lambda$ is nonzero for all ϵ. To preserve some of the supersymmetry we also turn on a nonzero D field [7], which modifies the gaugino variation to

$$\delta\lambda = (F_{\mu\nu}\sigma^{\mu\nu} + iD)\epsilon. \tag{1.2}$$

Plugging (1.1) into (1.2), one finds

$$\delta\lambda = i\begin{pmatrix} D - B & 0 \\ 0 & D + B \end{pmatrix}\begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix}, \tag{1.3}$$

where ϵ_- (ϵ_+) generates right (left) moving supersymmetry on $\mathbb{R}^{1,1}$. For generic B and D, supersymmetry is broken as before, but for $D = \pm B$, some of it remains unbroken. For $D = B$, the right-hand side (r.h.s.) of (1.3) vanishes for all ϵ_- (and $\epsilon_+ = 0$); $D = -B$ is the same with $\epsilon_+ \leftrightarrow \epsilon_-$. In other words, for $D = B$ the theory preserves (0,2) SUSY while $D = -B$ gives a theory with (2,0) SUSY. Without loss of generality we can focus on the (0,2) case.

The above construction can be alternatively interpreted in terms of the three-dimensional $N = 2$ SQFT obtained by compactifying the original four-dimensional $N = 1$ supersymmetric theory on a circle. The three-dimensional theory has the $U(1)$ global symmetry of the underlying four-dimensional model. To get a (0,2) SQFT in $1 + 1$ dimensions, we compactify this theory on one more circle and turn on a real mass term associated with the $U(1)$ symmetry that depends on its position along the circle. This point of view will be useful below.

The procedure that associates a two-dimensional (0,2) model with a four-dimensional $N = 1$ SQFT with a global $U(1)$ symmetry is clearly not unique in theories with large global symmetry groups, since then there are many inequivalent ways to choose the $U(1)$ current that figures in

[1]Indeed, one way to construct (1,1) SQFTs in two dimensions is to consider the low energy limit of three-dimensional $N = 1$ SQFTs on a circle.

the construction. Also, it may be that the resulting (0,2) theory breaks supersymmetry when quantum effects are taken into account. The purpose of this paper is to study this class of theories and to address these and other issues. We will see that for some choices of the $U(1)$ symmetry, supersymmetry is broken in the quantum theory, while for others it is not. Even in cases where SUSY is unbroken, quantum effects play an important role in the dynamics.

We will also discuss the dependence of the properties of the two-dimensional (0,2) theory on those of the underlying four-dimensional one. For example, if the higher-dimensional theory exhibits Seiberg duality [8], one can ask whether the two-dimensional theories obtained by compactifying the electric and magnetic models inherit it. On the one hand, one can argue that they should, since no phase transitions are expected as a function of the parameters that the models depend on. On the other hand, one might worry that the duality might be spoiled by the presence of scalar fields in the adjoint representation (coming from dimensional reduction of the four-dimensional gauge field), which are sensitive to the rank of the gauge group. We shall argue that the duality survives in two dimensions and explain how it deals with the adjoint fields. Similarly, one can ask how mirror symmetry in three dimensions [9] is realized after the reduction. We leave a discussion of this issue to another publication.

Although the construction described above is general, we shall mostly focus on the case of $N = 1$ supersymmetric QCD with gauge group $U(N_c)$ and matter in the (anti) fundamental representation. This class of theories can be realized in string theory as low energy theories of systems of D-branes and $NS5$-branes [3–5], and we shall find this description useful for our purposes.

The plan of the paper is as follows. In Sec. II we briefly review the basic structure of (0,2) supersymmetric gauge theories in two dimensions. In Sec. III we describe the effects of background magnetic and D fields on the spectrum of free charged four-dimensional superfields compactified on a torus. In Sec. IV we include interactions and discuss the effect of background fields on a four-dimensional $N = 1$ supersymmetric gauge theory, focusing on $N = 1$ SQCD with gauge group $U(N_c)$ and fundamental matter. In Sec. V we embed this theory in string theory as a low energy theory on branes and explain the effects of the background fields on it. This embedding is known to be useful for studying the classical and quantum low energy dynamics of various gauge theories in different dimensions [5], and this turns out to be the case here as well.

In Sec. VI we describe our construction from the point of view of the three-dimensional $N = 2$ supersymmetric gauge theory obtained by compactifying four-dimensional $N = 1$ SQCD on a circle. In the three-dimensional description, the magnetic field for the background $U(1)$ becomes a real mass term for the fundamentals that depends linearly on one of the spatial directions. This description is useful for analyzing the dynamics, particularly in the brane realization, since the background fields correspond to geometric deformations in the extra dimensions.

In Sec. VI we discuss some properties of the two-dimensional quantum theories obtained from our construction. We show that the classical Coulomb moduli space is lifted in the quantum theory and is replaced by a discrete set of vacua. We describe the low energy theory in each of these vacua and argue that four-dimensional theories related by Seiberg duality give rise to the same low energy (0,2) theories in two dimensions. Section VIII contains a brief discussion of our results. Some technical details appear in the Appendix.

II. (0,2) SUPERSYMMETRY IN TWO DIMENSIONS

In this section we review some aspects of (0,2) supersymmetric field theories in $1 + 1$ dimensions which will play a role in our discussion below. Our main goal is to establish the notation, which follows that of [10].

A. General structure

In addition to the bosonic coordinates (x^0, x^3), (0,2) superspace has two fermionic coordinates $(\theta^+, \bar{\theta}^+)$. The right-moving supercharges act on superspace as follows:

$$Q_+ = \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+(\partial_0 + \partial_3),$$
$$\bar{Q}_+ = -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+(\partial_0 + \partial_3). \tag{2.1}$$

They satisfy $Q_+^2 = \bar{Q}_+^2 = 0$, $\{Q_+, \bar{Q}_+\} = -2i(\partial_0 + \partial_3)$. The supercharges anticommute with the superspace covariant derivatives

$$D_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\partial_0 + \partial_3),$$
$$\bar{D}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+(\partial_0 + \partial_3), \tag{2.2}$$

which satisfy $D_+^2 = \bar{D}_+^2 = 0$, $\{D_+, \bar{D}_+\} = 2i(\partial_0 + \partial_3)$.

To construct (0,2) supersymmetric gauge theory, we extend the superspace derivatives $D_+, \bar{D}_+, \partial_0, \partial_3$ to gauge covariant superderivatives $\mathcal{D}_+, \bar{\mathcal{D}}_+, \mathcal{D}_0, \mathcal{D}_3$. These can be written in a Wess-Zumino–type gauge as

$$\mathcal{D}_0 + \mathcal{D}_3 = \partial_0 + \partial_3 + i(A_0 + A_3),$$
$$\mathcal{D}_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_3),$$
$$\bar{\mathcal{D}}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+(\mathcal{D}_0 + \mathcal{D}_3),$$
$$\mathcal{D}_0 - \mathcal{D}_3 = \partial_0 - \partial_3 + iV, \tag{2.3}$$

where

$$V = A_0 - A_3 - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D \tag{2.4}$$

is a vector superfield which transforms in the adjoint representation of the gauge group G. λ_- is the left-moving gaugino, while D is a nonpropagating auxiliary field. These fields and the gauge field can be combined into the (0,2) field strength

$$\begin{aligned}\Upsilon &= [\bar{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_3] \\ &= -2(\lambda_- - i\theta^+(D - iF_{03}) - i\theta^+\bar{\theta}^+(D_0 + D_3)\lambda_-),\end{aligned} \tag{2.5}$$

which transforms in the adjoint representation of the gauge group and satisfies the chirality constraint $\bar{\mathcal{D}}_+\Upsilon = 0$.

We will focus on two types of matter superfields in a representation r of the gauge group G. One is the bosonic chiral superfield Φ with component expansion

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\bar{\theta}^+(D_0 + D_3)\phi, \tag{2.6}$$

obeying $\bar{\mathcal{D}}_+\Phi = 0$. Here ϕ is a complex scalar field, and ψ_+ is a complex right-moving fermion. The second is the Fermi superfield

$$\Lambda = \psi_- - \sqrt{2}\theta^+\mathcal{F} - i\theta^+\bar{\theta}^+(D_0 + D_3)\psi_- - \sqrt{2}\bar{\theta}^+ E, \tag{2.7}$$

whose only propagating degree of freedom is a complex left-moving fermion ψ_-. Λ obeys the superspace constraint

$$\bar{\mathcal{D}}_+\Lambda = \sqrt{2}E, \qquad \bar{\mathcal{D}}_+ E = 0. \tag{2.8}$$

In (2.5)–(2.7), D_0, D_3 are the usual gauge-covariant derivatives $D_\mu = \partial_\mu + iA^a_\mu T^a$, or equivalently the superderivatives (2.3) evaluated at $\theta^+ = \bar{\theta}^+ = 0$. T^a are the generators of G in the representation r, and E is a chiral superfield, usually taken to be a function of the basic chiral superfields in the theory [10].

The natural supersymmetric actions for the above superfields are

$$\begin{aligned}S_\Upsilon &= \frac{1}{8g^2}\,\mathrm{Tr}\int d^2x d^2\theta \bar{\Upsilon}\Upsilon, \\ S_\Phi &= -\frac{i}{2}\int d^2x d^2\theta \bar{\Phi}(\mathcal{D}_0 - \mathcal{D}_3)\Phi, \\ S_\Lambda &= -\frac{1}{2}\int d^2x d^2\theta \bar{\Lambda}\Lambda,\end{aligned} \tag{2.9}$$

with component expansions

$$\begin{aligned}S_\Upsilon &= \frac{1}{g^2}\,\mathrm{Tr}\int d^2x\left\{\frac{1}{2}F_{03}^2 + i\bar{\lambda}_-(D_0 + D_3)\lambda_- + \frac{1}{2}D^2\right\}, \\ S_\Phi &= \int d^2x\{-|D_\mu\phi|^2 + i\bar{\psi}_+(D_0 - D_3)\psi_+ - i\sqrt{2}\bar{\phi}T^a\lambda^a_-\psi_+ \\ &\quad + i\sqrt{2}\phi T^a\bar{\psi}_+\bar{\lambda}^a_- + \bar{\phi}T^a\phi D^a\}, \\ S_\Lambda &= \int d^2x\left\{i\bar{\psi}_-(D_0 + D_3)\psi_- + |\mathcal{F}|^2 - |E|^2\right. \\ &\quad \left. - \left(\bar{\psi}_-\frac{\partial E}{\partial\phi_i}\psi_{+i} + \bar{\psi}_{+i}\frac{\partial\bar{E}}{\partial\bar{\phi}_i}\psi_-\right)\right\}.\end{aligned} \tag{2.10}$$

To include a nontrivial kinetic term for Φ_i, one can replace the field $\bar{\Phi}_i$ in the action S_{Φ_i} by a more general Kahler form $K_i(\Phi, \bar{\Phi}) = \partial K/\partial\Phi_i$.

Other terms in the action that are often of interest have the general form $\int d^2x d\theta^+(\cdots)|_{\bar{\theta}^+=0} + \text{c.c.}$ where $(\cdots)$ is an anticommuting superfield annihilated by $\bar{\mathcal{D}}_+$. One example is the Fayet-Iliopoulos (FI) term for a $U(1)$ symmetry,

$$\begin{aligned}S_{\mathrm{FI}} &= \frac{t}{4}\int d^2x d\theta^+\Upsilon|_{\bar{\theta}^+=0} + \text{c.c.} \\ &= \frac{it}{2}\int d^2x(D - iF_{01}) + \text{c.c.},\end{aligned} \tag{2.11}$$

where

$$t = ir + \frac{\theta}{2\pi} \tag{2.12}$$

is the complexified FI parameter. Another is the (0,2) superpotential

$$\begin{aligned}S_{\mathcal{W}} &= -\frac{1}{\sqrt{2}}\int d^2x d\theta^+\Lambda_a J^a|_{\bar{\theta}^+=0} + \text{c.c.} \\ &= -\int d^2x\left\{\mathcal{F}_a J^a(\phi_i) + \psi_{-a}\psi_{+i}\frac{\partial J^a}{\partial\phi_i}\right\} + \text{c.c.},\end{aligned} \tag{2.13}$$

where Λ_a are Fermi superfields, and J^a are holomorphic functions of the (bosonic) chiral superfields. Because of (2.8), chirality of the superpotential (2.13) requires that

$$E \cdot J = 0. \tag{2.14}$$

B. Reduction of (2,2) superfields under (0,2) SUSY

Before turning on the background B and D fields, the theories we shall study have (2,2) supersymmetry in two dimensions. The background fields split the (2,2) multiplets into (0,2) ones. Thus, it is useful to recall how (2,2) superfields decompose under (0,2) supersymmetry.

The (2,2) SQFT is conveniently described in a superspace with coordinates $(x^\mu, \theta^\pm, \bar{\theta}^\pm)$, an obvious extension of the (0,2) superspace described above to include the left-moving supercoordinates. The right-moving supercovariant derivatives (2.3) are supplemented by left-moving ones,

$$\mathcal{D}_- = \frac{\partial}{\partial\theta^-} - i\bar{\theta}^-(\mathcal{D}_0 - \mathcal{D}_3),$$
$$\bar{\mathcal{D}}_- = -\frac{\partial}{\partial\bar{\theta}^-} + i\theta^-(\mathcal{D}_0 - \mathcal{D}_3). \tag{2.15}$$

All the (0,2) superfields described above fit naturally into two types of (2,2) superfields. One is the chiral superfield $\Phi^{(2,2)}$, which satisfies the chirality constraints $\bar{\mathcal{D}}_+\Phi^{(2,2)} = \bar{\mathcal{D}}_-\Phi^{(2,2)} = 0$ and has the free field action

$$S_\Phi^{(2,2)} = -\frac{1}{4}\int d^2x d^4\theta \bar{\Phi}_i^{(2,2)}\Phi_i^{(2,2)}. \tag{2.16}$$

The other is the twisted chiral superfield $\Sigma^{(2,2)} = \frac{1}{2\sqrt{2}}\{\bar{\mathcal{D}}_+, \mathcal{D}_-\}$, which describes the gauge field strength and obeys $\bar{\mathcal{D}}_+\Sigma^{(2,2)} = \mathcal{D}_-\Sigma^{(2,2)} = 0$. It has the action

$$S_g^{(2,2)} = -\frac{1}{4g^2}\int d^2x d^4\theta \bar{\Sigma}^{(2,2)}\Sigma^{(2,2)}. \tag{2.17}$$

A (2,2) chiral superfield splits into a (0,2) chiral superfield and a Fermi superfield, as can be seen from the θ^- expansion

$$\Phi^{(2,2)} = \Phi^{(0,2)} + \sqrt{2}\theta^-\Lambda^{(0,2)} - i\theta^-\bar{\theta}^-(\mathcal{D}_0 - \mathcal{D}_3)\Phi^{(0,2)}. \tag{2.18}$$

The (2,2) field strength splits into an adjoint chiral superfield and a (0,2) field strength superfield,

$$\Sigma^{(2,2)} = \Sigma^{(0,2)} + \frac{i}{\sqrt{2}}\bar{\theta}^-\Upsilon^{(0,2)} - i\theta^-\bar{\theta}^-(\mathcal{D}_0 - \mathcal{D}_3)\Sigma^{(0,2)}. \tag{2.19}$$

To reproduce the action of a (2,2) chiral superfield that transforms nontrivially under a gauge symmetry, one must include a nonzero E (2.7) for the Fermi superfield in (2.18),

$$E = i\sqrt{2}\Sigma^a T^a \Phi. \tag{2.20}$$

III. FREE FIELDS IN A MAGNETIC FIELD

In this section we review the spectrum of charged four-dimensional free fields of spin 0 and $1/2$ in the presence of a constant external magnetic field. We start with the noncompact case, and then discuss compactification on a two-torus.

Consider a free massless (complex) scalar ϕ of charge e under a $U(1)$ gauge field A_μ. To study its dynamics in the background magnetic field (1.1), we turn on a background gauge field $A_2 = Bx_1$. The Klein-Gordon equation for ϕ then takes the form

$$(-\partial_0^2 + \partial_3^2 + \partial_1^2 + \tilde{D}_2^2)\phi = 0, \tag{3.1}$$

where $\tilde{D}_2 = \partial_2 + ieBx_1$. The $(1+1)$-dimensional spectrum is obtained by writing

$$\phi(x^0, x^1, x^2, x^3) = \varphi(x^0, x^3)\chi(x^1, x^2). \tag{3.2}$$

If we take χ to be an eigenfunction of

$$H = -(\partial_1^2 + \tilde{D}_2^2) = p_1^2 + (p_2 + eBx_1)^2, \tag{3.3}$$

$H\chi = m^2\chi$, Eq. (3.1) gives rise to a two-dimensional scalar field φ with mass m. The Hamiltonian (3.3) is just that of the Landau problem of a particle in a magnetic field, whose spectrum is given by

$$m_n^2 = (2n+1)|eB|. \tag{3.4}$$

Thus, turning on B leads to a discrete spectrum of $(1+1)$-dimensional massive particles, as expected.

As mentioned in the Introduction, to preserve supersymmetry we must turn on in addition to the magnetic field also the D component of the corresponding vector multiplet. The latter contributes to the scalar Lagrangian the term $eD|\phi|^2$, which shifts (3.5) to

$$m_n^2 = (2n+1)|eB| - eD. \tag{3.5}$$

For the case $B = D$, for which the background fields preserve (0,2) supersymmetry, we see that fields with $eB > 0$ give rise to massless $(1+1)$-dimensional scalars, while those with $eB < 0$ lead to a massive spectrum. Without loss of generality we can restrict to the case $B > 0$ (otherwise, we can take A_μ, D, e to minus themselves), so that fields with positive $U(1)$ charge are the ones that give massless fields in two dimensions. As is familiar from the Landau problem, the spectrum (3.5) is in general degenerate. We will discuss this degeneracy below when we turn to the compact case.

The above discussion can be repeated for spin $1/2$ fields. A four-dimensional charged Weyl fermion in a magnetic field satisfies the wave equation

$$i\begin{pmatrix} -\partial_0 - \partial_3 & -\partial_1 + i\tilde{D}_2 \\ -\partial_1 - i\tilde{D}_2 & -\partial_0 + \partial_3 \end{pmatrix}\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = 0. \tag{3.6}$$

The top component of the spinor (ψ_-) is a left-moving fermion in the two dimensions (x^0, x^3); the bottom component, ψ_+, is right moving. Squaring (3.6) yields decoupled equations for $\psi_\pm$,

$$(-\partial_0^2 + \partial_3^2 + \partial_1^2 + \tilde{D}_2^2 \mp i[\partial_1, \tilde{D}_2])\psi_\pm = 0. \tag{3.7}$$

Using the fact that $[\partial_1, \tilde{D}_2] = [\partial_1, \partial_2 + ieBx_1] = ieB$, Eq. (3.7) is essentially identical to (3.3). So the right- and left-moving fermions have the spectrum

$$m_+^2 = (2n+1)|eB| - eB, \qquad m_-^2 : (2n+1)|eB| + eB. \tag{3.8}$$

Comparing to (3.5) we see that for $B = D$ the right-moving fermions align with the scalars, while the left-moving fermions do not, in agreement with the expectation from (0,2) supersymmetry.

To summarize, a four-dimensional free massless chiral superfield Φ with $U(1)$ charge e reduces in a constant $B = D > 0$ background to a massless (0,2) chiral superfield (2.6) for $e > 0$, and to a massless Fermi superfield (2.7) for $e < 0$. In both cases one also finds an infinite tower of massive two-dimensional chiral and Fermi superfields with the spectrum (3.5) and (3.8), respectively.

So far we have discussed the effects on the spectrum of charged fields of turning on a magnetic field in noncompact four-dimensional spacetime. Since we are interested in turning on a magnetic field on T^2, we need to take the coordinates (x^1, x^2) to be periodic, $x^i \sim x^i + 2\pi R_i$. The background gauge field $A_2 = Bx_1$ is not periodic on the torus; rather, it satisfies

$$A_2(x^1 + 2\pi R_1) = A_2(x^1) + 2\pi R_1 B = A_2(x^1) + \partial_2 \Gamma, \tag{3.9}$$

with $\Gamma(x^2) = 2\pi R_1 B x^2$ a gauge transformation parameter.

A field ϕ of charge e transforms under the gauge transformation generated by $\Gamma(x^2)$ as $\phi \to \exp(ie\Gamma)\phi$. Requiring that $\exp(ie\Gamma)$ is well defined on the torus leads to the Dirac quantization condition for the magnetic field

$$eB\mathcal{A} \in 2\pi \mathrm{Z}, \tag{3.10}$$

where $\mathcal{A} = 2\pi R_1 \times 2\pi R_2$ is the area of the torus.

The eigenvalue problem for wave functions on the magnetized torus is now more complicated, due to the periodicity conditions. One finds (see e.g. [11]) that the spectrum is still given by (3.5) and (3.8), and the degeneracy of states at a given level is

$$n_e = \frac{|e|B\mathcal{A}}{2\pi}. \tag{3.11}$$

Since the charges are proportional to the degeneracies with a universal proportionality constant, one can normalize them such that a field of charge e has degeneracy $|e|$; we shall use this normalization in our discussion below. Thus, a field of charge $e > 0$ gives e massless (0,2) chiral superfields, while one of charge $e < 0$ gives $|e|$ massless Fermi superfields.

IV. FOUR-DIMENSIONAL $N = 1$ SQFT ON A MAGNETIZED TORUS

So far we discussed the effect of a magnetic field on free superfields in four dimensions; in this section we generalize to the interacting case. We take as the starting point an $N = 1$ supersymmetric gauge theory with gauge group G and chiral matter fields Φ_i in representations r_i of the gauge group. There can also be a (gauge invariant) superpotential and other interactions, which we shall discuss later.

If we compactify such a theory on a two-torus without turning on a magnetic field, we find at low energies a two-dimensional (2,2) supersymmetric theory, which contains the (2,2) chiral superfields Φ_i and a twisted chiral superfield Σ in the adjoint representation describing the field strength of G. To reduce the supersymmetry to (0,2) we need to identify a suitable $U(1)$ global symmetry. We assign to the superfields Φ_i global charges e_i and demand that the symmetry be nonanomalous (i.e. conserved in the quantum theory). This leads to the constraints

$$\sum_i e_i T(r_i) = 0, \qquad \sum_i e_i^2 \mathrm{Tr} T^a(r_i) = 0, \tag{4.1}$$

where $T^a(r)$ are the generators of G in the representation r and $T(r)$ is defined by $\mathrm{Tr}_r T^a T^b = T(r)\delta^{ab}$, with $a, b = 1, \ldots, \dim G$. The first condition in (4.1) comes from the anomaly of one global and two gauge currents, while the second is the anomaly of two global and one gauge currents. We need to impose it since there is a nonzero source for the global $U(1)$. Note that we do not need to impose the vanishing of the anomaly of three global currents since $F\tilde{F} = 0$ for it.

In general there may be many solutions to (4.1), which correspond to different $U(1)$ subgroups of the global symmetry group of the model. We will comment on the dependence on the choice of $U(1)$ below. Note also that solutions to (4.1) include $U(1)$ factors of the gauge group. For those, one can show that the B and D fields that play a role in our construction are equivalent to a B field and FI term for the dynamical $U(1)$. We will not discuss these cases in detail here and will choose the $U(1)$ to be orthogonal to the gauge group.

Four-dimensional theories of the sort discussed above typically develop strong coupling in the UV or IR, when studied on $\mathbb{R}^{3,1}$. If $T(\mathrm{adj}) > \sum_i T(r_i)/3$, the four-dimensional theory is asymptotically free in the UV, and in general develops strong coupling in the IR below the dynamically generated scale Λ. One can then ask whether/how we can apply the results of the previous section to study the effects of the B and D fields on such a theory. As we saw, turning on the external fields leads to the appearance of a new energy scale associated with the Landau levels (3.5). The other relevant scale is the

Kaluza-Klein scale associated with the size of the two-torus. If these scales are much larger than Λ, we can use the results of the previous section to analyze the theory, since at the energies at which the B field and compactification modify the dynamics, the four-dimensional theory is still weakly coupled.

If the hierarchy of scales is the other way around, naively we cannot use the results of the previous section and have to search for a description of the four-dimensional theory that is weakly coupled at the scales associated with the B field and compactification. As usual, we may hope that since the two regimes are connected by a continuous deformation (the size of the torus), if there are no phase transitions as we vary the parameters, we nevertheless should find the right result by following the procedure of the previous section. We shall see examples of this later in the paper.

Clearly, any nontrivial solution to (4.1) must involve some positive and negative e_i. As mentioned above, each field with $e_i > 0$ gives e_i (0,2) massless chiral superfields ϕ_i, while fields with $e_i < 0$ give $|e_i|$ Fermi superfields. Fields with $e_i = 0$, which include the gauge multiplet and any chiral superfields that are not charged under the $U(1)$, are not influenced by the background fields and can be treated by standard Kaluza-Klein methods. In particular, the gauge multiplet gives rise to the (0,2) multiplets Υ, Σ as in (2.19), while uncharged chiral superfields give chiral and Fermi superfields (2.18) in the appropriate representations of G.

To find the two-dimensional Lagrangian for these fields, we start with the four-dimensional Lagrangian

$$\mathcal{L} = \mathcal{L}_V + \sum_i \mathcal{L}_i,$$

$$\mathcal{L}_V = -\frac{1}{4g^2}(F^a_{\mu\nu}F^{a\mu\nu} + 4i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda^a - 2D^aD^a),$$

$$\mathcal{L}_i = -D^\mu\bar{\phi}_i D_\mu\phi_i - i\bar{\psi}_i\bar{\sigma}^\mu D_\mu\psi_i + \bar{\mathcal{F}}_i\mathcal{F}_i + i\sqrt{2}(\bar{\phi}_i T^a\psi_i\lambda^a - \bar{\lambda}^a\bar{\psi}_i T^a\phi_i) + \bar{\phi}_i T^a\phi_i D^a, \tag{4.2}$$

where ϕ_i and ψ_i are components of the chiral superfields Φ_i and λ^a are the gauginos of G. We need to couple (4.2) to the background fields described above and reduce it on the torus, using the wave functions of the various fields. Although we are interested in low energy dynamics, some of the contributions of massive modes to (4.2) need to be kept, since integrating them out may give terms in the Lagrangian of the massless modes that are relevant in the infrared.

The Lagrangian of the vector superfield, $\mathcal{L}_V$, gives rise to the standard (2,2) Lagrangian (2.17), or equivalently the Lagrangian for the (0,2) gauge superfield Υ and chiral superfield in the adjoint representation,[2] Σ, given in the first and second lines of (2.9). The bottom component of Σ is a scalar field

$$\sigma = \frac{A_1 + iA_2}{\sqrt{2}} \tag{4.3}$$

that comes from components of the gauge field along the two-torus.

Turning to the chiral superfields Φ_i, we need to discuss separately the cases of fields with positive, negative and zero $U(1)$ charge. For fields with $e_i = 0$, the compactification preserves (2,2) SUSY and the Lagrangian is the usual one, reviewed in Sec. II. Fields with $e_i < 0$ give rise to Fermi multiplets Λ_i, and their Lagrangians are given by the last lines of (2.9) and (2.10). The holomorphic functions E_i (2.8) vanish in the massless sector, but receive nonzero contributions (2.20) from massive modes.

The reduction of fields with $e_i > 0$ is more subtle. These fields give (0,2) chiral superfields, which we shall also denote by Φ_i, dropping the superscript (0,2) in (2.18). Before turning on the magnetic field, the Lagrangian of the scalars ϕ_i that are the bottom components of these superfields contains a potential proportional to $|\sigma|^2|\phi_i|^2$, which in (0,2) language comes from the $|E_i|^2$ terms in (2.10), with E_i given by (2.20). However, after turning on the magnetic field the E_i must vanish in the light sector, for the same reason as in the $e_i < 0$ case: the Fermi superfields associated with Φ_i are lifted, and there are no Fermi superfields with the right quantum numbers in the light sector to give rise to a $|\sigma|^2|\phi_i|^2$ potential.

To see how this happens, consider a massless four-dimensional complex scalar field ϕ charged under a dynamical $U(1)$ gauge field A_μ and under a global $U(1)$ for which we turn on (equal) background B and D fields. The kinetic term for ϕ,

$$\mathcal{L}_\phi = -D_\mu\phi D^\mu\bar{\phi} + eD|\phi|^2 + \cdots, \tag{4.4}$$

contains couplings to the dynamical and background $U(1)$ gauge fields A_μ, $\tilde{A}_\mu$, $D_\mu\phi = (\partial_\mu + iA_\mu + ie\tilde{A}_\mu)\phi$. The terms with $\mu = 1, 2$, in particular, contain the coupling to the background B field and the zero mode of the dynamical gauge field, σ (4.3). Plugging the background gauge potential $\tilde{A}_2 = Bx_1$ into (4.4) and using (4.3), we see that the role of a nonzero σ is to shift the location of the zero mode of ϕ in the x^1 plane. It has the same effect on the wave functions as a Wilson line for $\tilde{A}_\mu$, discussed e.g. in [11].

For non-Abelian gauge groups we do not expect the $|\sigma|^2\phi_i^2$ terms in the potential to completely disappear. The (0,2) D-term potential includes terms of the form $\bar{\phi}_i[\bar{\sigma}, \sigma]\phi_i$. They can be obtained by reducing the four-dimensional action to two dimensions, taking into account the massive modes.

To summarize, starting with a four-dimensional $N = 1$ SUSY gauge theory with gauge group G, and chiral matter superfields Φ_i in representations r_i of the gauge group, and compactifying it on a two-torus with equal background magnetic and D fields for a global $U(1)$ symmetry under

[2] $\Sigma^{(0,2)}$ in (2.19).

which Φ_i have charges $e_i \in \mathbb{Z}$, leads at low energies to a two-dimensional (0,2)-supersymmetric gauge theory containing the following:

(1) A G gauge superfield Υ.
(2) An adjoint chiral superfield Σ.
(3) e_i (0,2) chiral superfields Φ_i in the representation r_i, for fields with $e_i > 0$.
(4) $|e_i|$ Fermi superfields Λ_i in the representation r_i, for $e_i < 0$.
(5) A chiral superfield Φ_i and Fermi superfield Λ_i in the representation r_i, for $e_i = 0$.

In addition to the gauge interactions described in Sec. II, the fields above couple to the massive modes. These couplings are important for understanding some aspects of the dynamics.

The global symmetries of the resulting theory can be described from either the four-dimensional or two-dimensional point of view. Consider, for example, a $U(1)$ symmetry that assigns charges q_i to the chiral superfields Φ_i. In four dimensions, we have to impose the conditions[3]

$$\sum_i q_i T(r_i) = 0,$$
$$\sum_i e_i q_i \mathrm{Tr} T^a(r_i) = 0, \tag{4.5}$$

coming from anomalies of one global and two gauge, and one global, one background and one gauge currents. The first of these is an inherently four-dimensional constraint,[4] while the second has a natural two-dimensional interpretation—it is the two-dimensional gauge anomaly of one global and one gauge current in the gauge theory with matter content (1)–(5).

An example of the above construction that will play a central role in our discussion below is supersymmetric QCD with gauge group $G = U(N_c)$ and N_f flavors of chiral superfields in the fundamental representation of the gauge group, Q^i, $\tilde{Q}_i$, $i = 1, 2, \ldots, N_f$. This theory has been extensively studied in the past and has been found to exhibit a rich dynamical structure; see e.g. [1,2] for reviews. For $0 < N_f < N_c$ it exhibits runaway behavior (no supersymmetric vacuum), while for $N_f \geq N_c$ it has nontrivial infrared behavior and exhibits interesting dynamical phenomena such as quantum deformations of the classical moduli space, confinement, and Seiberg duality.

The global (non-R) symmetry group of this theory is

$$SU(N_f) \times SU(N_f). \tag{4.6}$$

The two factors act by special unitary transformations on the Q's and $\tilde{Q}$'s. For our construction we need to pick a $U(1)$ subgroup of (4.6), which can be done by assigning integer charges e_i to Q^i and $\tilde{e}_i$ to $\tilde{Q}_i$. The anomaly freedom constraints (4.1) imply that

$$\sum_i e_i + \sum_i \tilde{e}_i = 0,$$
$$\sum_i e_i^2 - \sum_i \tilde{e}_i^2 = 0. \tag{4.7}$$

The simplest solution to (4.7) is $e_i = e$, $\tilde{e}_i = -e$. However, this corresponds to picking the $U(1)$ to be the baryon number, which is gauged in our model. If we want the global $U(1)$ used for our construction to be orthogonal to the gauge group, we need to further require

$$\sum_i e_i = \sum_i \tilde{e}_i = 0. \tag{4.8}$$

An example of a solution to (4.8), which exists for all even N_f, is to take $N_f/2$ of the e_i to be equal to $+1$ and the rest equal to -1, and similarly for $\tilde{e}_i$. This breaks the symmetry (4.6) to

$$SU(N_f/2)^4 \times U(1). \tag{4.9}$$

The N_f fundamentals Q^i give rise to $N_f/2$ chiral superfields and $N_f/2$ Fermi superfields in the fundamental, and similarly for $\tilde{Q}$. Much of our discussion below will focus on this example.

More generally, we can take N_+ of e_i to be positive and $N_- = N_f - N_+$ to be negative, and similarly for $\tilde{e}$ [while imposing (4.7) and (4.8)]. The resulting theory has $\sum_{i\in N_+} e_i$ chiral multiplets and $\sum_{i\in N_-} e_i$ Fermi multiplets in the fundamental, and $\sum_{j\in \tilde{N}_+} \tilde{e}_j$ chiral and $\sum_{j\in \tilde{N}_-} \tilde{e}_j$ Fermi multiplets in the antifundamental. We will not study this more general case in detail, but will comment on it later.

The $N = 1$ SQCD in four dimensions exhibits Seiberg duality [8], which is the conjecture that its infrared limit is equivalent to that of a different theory, which has gauge group $U(N_f - N_c)$, N_f chiral multiplets q_i in the fundamental, $\tilde{q}^i$ in the antifundamental, and a meson M^i_j which is a singlet of the gauge group, and is the dual of the gauge invariant composite field $Q^i \tilde{Q}_j$ in the electric theory. The magnetic theory further includes a superpotential coupling the magnetic meson M to the magnetic quarks q, $\tilde{q}$,

$$\mathcal{W} = M^i_j q_i \tilde{q}^j. \tag{4.10}$$

Part of the statement of the duality is the identification of the global symmetry group of the electric theory, Eq. (4.6), with the corresponding symmetry in the magnetic theory. Hence, given a choice of charges e_i, $\tilde{e}_i$ in the electric theory, we can write down the corresponding charges in the magnetic one. The magnetic quarks q_i have charge $-e_i$, $\tilde{q}^i$ have charge $-\tilde{e}_i$, and M^i_j have charge $e_i + \tilde{e}_j$. The spectrum

[3]Note that (4.1) is a special case of this, with $q_i = e_i$.
[4]Therefore, it can be lifted in the two-dimensional infrared theory, if all the terms in the Lagrangian that violate symmetries that do not satisfy this constraint are irrelevant in the IR.

of the low energy (0,2) theory can then be read off the general analysis above. The superpotential (4.10) gives rise at low energies to a (0,2) superpotential, which will be discussed below.

V. BRANE CONSTRUCTION

The $N = 1$ SQCD has a natural embedding in string theory as the low energy effective theory on a system of D-branes and $NS5$-branes.[5] This description was found in the past to be useful for analyzing both supersymmetric and nonsupersymmetric aspects of the dynamics of the theory, and it is natural to ask whether that is the case here as well. In this section we shall describe the gauge theory construction of the previous section in terms of branes.

The brane system whose infrared limit corresponds to the $N = 1$ SQCD is shown in Fig. 1. It contains three types of BPS branes in type IIA string theory: Neveu-Schwarz (NS) five-branes, Dirichlet four-branes and six-branes. All branes are extended in the $3 + 1$ dimensions x^μ, $\mu = 0, 1, 2, 3$. The N_c $D4$-branes are further stretched in the x^6 direction between two differently oriented $NS5$-branes, which we shall refer to as NS and NS'-branes. The former are stretched in (x^4, x^5); the latter in (x^8, x^9). The N_f $D6$-branes are extended in the directions (x^7, x^8, x^9) and are placed between the five-branes.

The configuration of Fig. 1 preserves four supercharges. Thus, the theory that lives at the intersection of the different branes is $N = 1$ supersymmetric in the $3 + 1$ dimensions x^μ; this theory is $N = 1$ SQCD [4]. The $U(N_c)$ vector superfield and corresponding supersymmetric Yang-Mills (SYM) Lagrangian [the first line of (4.2)] comes from taking the low energy limit of $4 - 4$ strings. For infinite $D4$-branes this gives five-dimensional SYM with 16 supercharges; however, here one of the world volume directions (x^6) is a line segment, so the low energy theory is $3 + 1$ dimensional. The boundary conditions at the ends of the line segment, where the four-branes end on five-branes, give mass to all fields other than the $N = 1$ SYM ones.

The N_f fundamental chiral superfields Q^i, $\tilde{Q}_i$ come from 4–6 strings which are localized in x^6. The full global symmetry (4.6) is not manifest in Fig. 1. The diagonal $SU(N_f)$ is visible as the gauge symmetry on the flavor branes[6] (the $D6$-branes). The full symmetry (4.6) is from the point of view of Fig. 1 an accidental symmetry of the low energy theory. One can make it manifest by moving all the $D6$-branes in the x^6 direction to the location of the NS'-brane [12]. Then the $D6$-branes are split into two disconnected components (with $x^7 \geq 0$ and $x^7 \leq 0$, respectively) by the five-brane, and one can perform separate $SU(N_f)$ transformations on the two components.

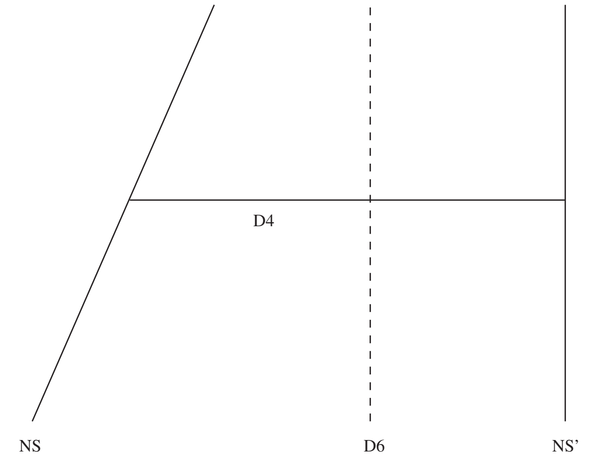

FIG. 1. The brane system that realizes $N = 1$ SQCD in type IIA string theory consists of N_c $D4$-branes stretched in the x^6 direction between two differently oriented $NS5$-branes. N_f $D6$-branes intersect the $D4$-branes at a particular x^6.

To implement our construction, we would like to compactify (x^1, x^2) on a torus, pick a $U(1)$ inside the global symmetry group (4.6), and turn on the magnetic field on the torus and D field for it. As mentioned above, in the brane construction it is slightly simpler to deal with the diagonal $SU(N_f)$; therefore, we shall take the $U(1)$ to be a subgroup of this $SU(N_f)$. It is possible to generalize the discussion to other $U(1)$ symmetries, by placing the $D6$-branes of Fig. 1 at the location of the NS'-brane and using the results of [12].

In terms of the branes, the procedure of turning on the external fields discussed above is described as follows. Each of the N_f $D6$-branes gives rise to one hypermultiplet, Q^i, $\tilde{Q}_i$, which is charged under the $U(1)$ gauge field living on the six-brane. We turn on a magnetic field B_i for this $U(1)$ field and accompany it by a suitable rotation of the $D6$-brane from the x^7 to the x^6 direction. The latter is the brane analog of the D term in the low energy gauge theory, and it preserves SUSY if we tune the rotation angle to correspond to the magnetic field that we turned on. This can be shown directly, but we shall see it in a slightly different language later. The requirement that the B field lies in the diagonal $SU(N_f)$ [i.e. is orthogonal to the gauge group, as in (4.8)] is in this language $\sum_i B_i = 0$. The resulting configuration in the (x^6, x^7) plane is exhibited in Fig. 2. The N_f $D6$-branes are rotated from the x^7 axis by angles θ_i that are determined by the magnetic fields B_i, $\tan\theta_i \propto B_i$. This corresponds in the field theory to giving to the ith flavor charge e_i proportional to B_i for Q^i and $\tilde{e}_i = -e_i$ for $\tilde{Q}_i$.

Since we shall be discussing below the consequences of Seiberg duality in the compactified theory with the background B and D fields, it is useful to review the

[5] We shall not provide a self-contained discussion of this system, instead referring the reader to the review [5].

[6] The gauge symmetry on the six-branes is in fact $U(N_f)$, but $U(1)$ acts on the low energy theory in the same way as the $U(1)$ factor in $U(N_c)$, and so is not an independent symmetry.

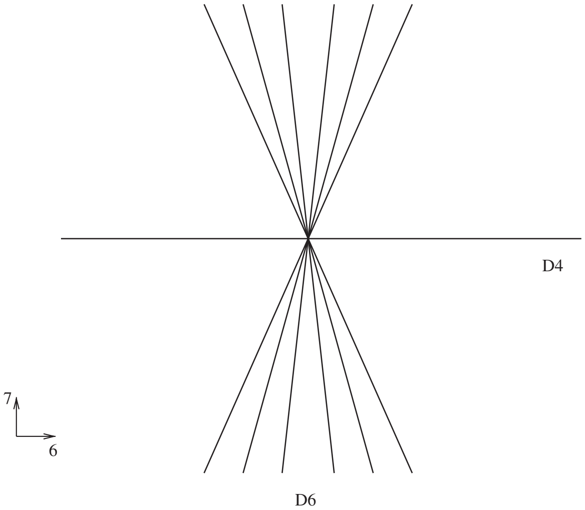

FIG. 2. Turning on the D field gives rise to a configuration in which the $D6$-branes are rotated in the (x^6, x^7) plane.

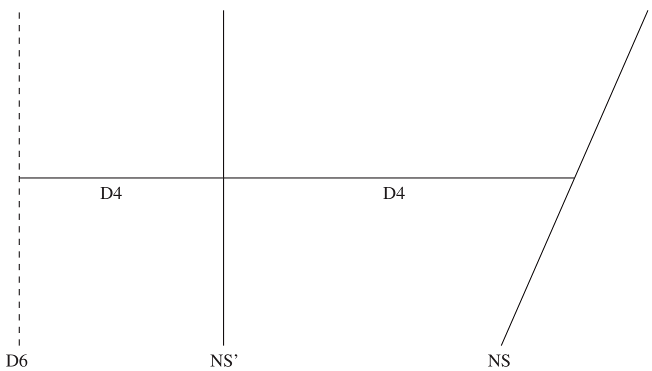

FIG. 3. The brane system that realizes the Seiberg dual of $N = 1$ SQCD includes $N_f - N_c$ color $D4$-branes connecting two $NS5$-branes, and N_f flavor $D4$-branes, each connecting the NS'-brane to one of N_f $D6$-branes.

generalization of the above discussion to the magnetic theory. The brane configuration corresponding to the Seiberg dual theory [4,5] is depicted in Fig. 3.

The $N_f - N_c$ color $D4$-branes connecting the $NS5$-branes give rise to the magnetic gauge group $U(N_f - N_c)$. The N_f flavor $D4$-branes stretched between the NS'-brane and $D6$-branes give the magnetic meson M. The magnetic quarks q, $\tilde{q}$ come from strings stretched between the color and flavor branes, and are thus localized near the NS'-brane.

Turning on the B field on the $D6$-branes and rotating them in the (67) plane by a suitable amount leads to the configuration of Fig. 4. The flavor $D4$-branes are now rotated in the (67) plane by the same angle as the corresponding $D6$-branes. Supersymmetry seems to be superficially violated, but is restored by a nonzero value of the magnetic field on the flavor $D4$-branes.

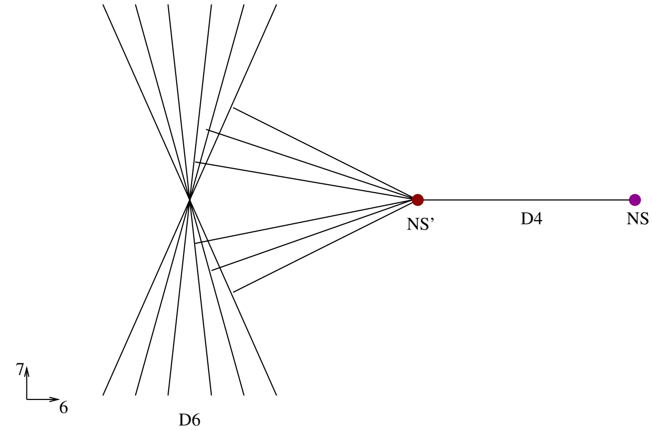

FIG. 4 (color online). The effect of the D field on the magnetic brane system.

In the brane systems described in this section, the D field deformation is described geometrically, but the magnetic field on the torus is not. It would be nice to geometrize the B field as well. To this end we next turn to a description of our system as a compactification of the three-dimensional theory obtained by first compactifying $N = 1$ SQCD on a circle.

VI. THREE-DIMENSIONAL DESCRIPTION

Three-dimensional $N = 2$ SQCD has a brane description which is formally obtained by T dualizing the system of Fig. 1 in, say, the x^2 direction [5]. This does not do anything to the $NS5$-branes, but turns the $D4$-branes into $D3$-branes stretched in (0136), and the $D6$-branes into $D5$-branes stretched in (013789), in type IIB string theory. Since we want to think of the theory as a compactification of four-dimensional $N = 1$ SQCD on a finite circle, we keep the size of the x^2 circle finite.

As before, we want to also compactify x^1 on a circle and turn on the background fields B and D described above. In the four-dimensional description, the B field corresponds to a vector potential $A_2 = Bx^1$ for a global $U(1)$, which is realized in the brane construction as the $U(1)$ gauge field on a $D6$-brane. In three dimensions, A_2 becomes a scalar field in the vector multiplet on a $D5$-brane. Turning on an expectation value for it corresponds to rotating the $D5$-brane in the (x^1, x^2) plane. The quantization of the B field is manifest in the three-dimensional description: it is due to the fact that as the $D5$-brane wraps the x^1 circle once, it has to return to its original position on the torus, and thus must wrap the x^2 circle an integer number of times (see Fig. 5). As is clear from the figure, rotating the $D5$-brane in this way localizes the matter coming from 3–5 strings on the x^1 circle, as is expected from the four-dimensional perspective, where this is due to the magnetic field. The degeneracy (3.11) comes in this language from the fact that a $D5$-brane that wraps e times around the x^2 circle as it goes once

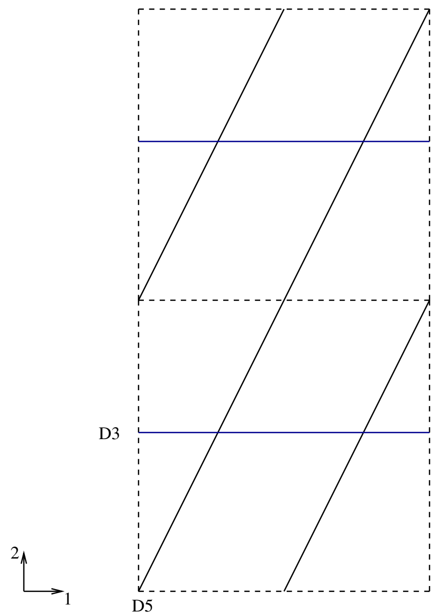

FIG. 5 (color online). Quantization of the magnetic field on the two-torus corresponds in the IIB language to quantization of the angle that the $D5$-brane makes with the x^1 axis. Plotted is a $D5$-brane with $e = 2$ in the double covering space of the torus (parallel dashed lines are identified). As the $D5$-brane wraps the x^1 circle once, it wraps the x^2 circle twice.

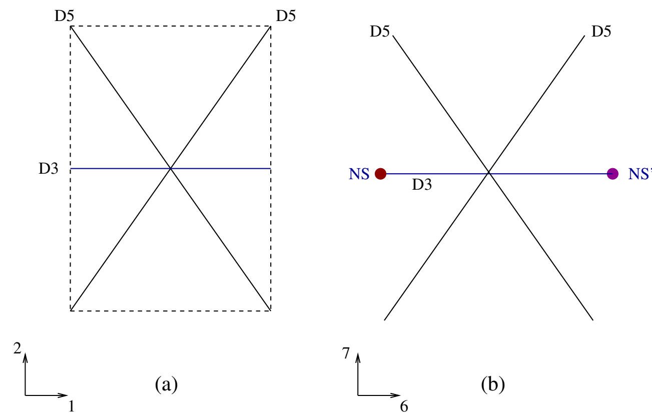

FIG. 6 (color online). Two views of the brane configuration describing compact three-dimensional $N = 2$ SQCD in a background global $U(1)$ under which half of the flavors have charge $+1$ and half -1. (a) The configuration in the (12) plane; the flavor branes wind once around the x^2 circle as they wind once around the x^1 one. (b) The configuration in the (67) plane; turning on the D field for the flavor symmetry corresponds to a rotation of the flavor branes from x^7 to x^6.

around x^1 intersects the $D3$-branes at e points on the x^1 circle.

In the low energy field theory, moving a $D5$-brane in x^2 corresponds to giving equal and opposite real masses to the corresponding chiral superfields Q, $\tilde{Q}$. Rotating it in the (x^1, x^2) plane thus corresponds to a real mass that depends linearly on x^1, which breaks Lorentz symmetry to $SO(1,1)$ and localizes these fields at the minimum of the resulting potential.

Turning on the D field again corresponds to rotating the $D5$-brane in the (x^6, x^7) plane, as in Fig. 2. The fact that the configuration preserves supersymmetry can be seen as in other cases involving rotated branes [13]. Defining $z_1 = x^1 + ix^7$ and $z_2 = x^2 + ix^6$, the $D5$-branes are originally located at $z_2 = 0$ (say), and together with the other branes in the configuration preserve $N = 2$ supersymmetry in the three dimensions (x^0, x^1, x^3). Rotating a $D5$-brane by a general angle θ,

$$z \to \Omega(\theta) z, \tag{6.1}$$

with $\Omega(\theta)$ the standard 2×2 rotation matrix, preserves two of the four supercharges, which form a (0,2) superalgebra in the $1 + 1$ dimensions (x^0, x^3).

We can now combine all of the above elements to describe the models of Sec. IV in terms of branes. Consider e.g. the model in which we give $N_f/2$ of the Q's ($\tilde{Q}$'s) charge $e = +1$ (-1), and to the other $N_f/2$ the opposite charge. The corresponding brane configuration is depicted in Fig. 6. Each intersection of the N_c $D3$-branes with one of the N_f rotated $D5$-branes supports either a (0,2) chiral superfield coming from Q and a Fermi superfield from $\tilde{Q}$ or the other way around, depending on the sign of the rotation angle. The Σ chiral superfield comes from the low lying modes living on the $D3$-branes, as does the gauge superfield Υ. In the next section we shall study the dynamics of these fields.

Note, incidentally, that if we were to remove the constraint (4.8) that the $U(1)$ which we use for the construction is not part of the gauge group, we could consider taking the charges to be $e_i = e$, $\tilde{e}_i = -e$ for all $i = 1, \ldots, N_f$. This corresponds to turning on a magnetic field and D field for the $U(1)$ factor in the gauge group, $U(1)_B$. In the brane picture, this would correspond to rotating all N_f $D5$-branes by the same angle, leading to the brane configuration of Fig. 7(a).

An equivalent brane configuration can be obtained by turning on an FI term for the $U(1)$ factor in the gauge group. Indeed, the latter corresponds to the relative displacement of the two $NS5$-branes in x^7 [5]. Naively, this deformation breaks supersymmetry; however, there are two ways to restore it (classically). If the number of flavors is large enough, one can break the gauge group and go into the Higgs branch, while maintaining $(2+1)$-dimensional Poincaré symmetry. Another option, which is available for all N_f, is to turn on the scalar field ϕ_2 in the $U(1)$ vector multiplet, $\phi_2 \propto x^1$, breaking the $(2+1)$-dimensional Lorentz symmetry down to $1+1$ dimensions. The resulting brane configuration is plotted in Fig. 7(b). Clearly, Figs. 7(a) and 7(b) are related by an overall rotation in the (67) plane and lead to equivalent physics. This is a quick way to see that rotating the flavor branes in the (67) plane

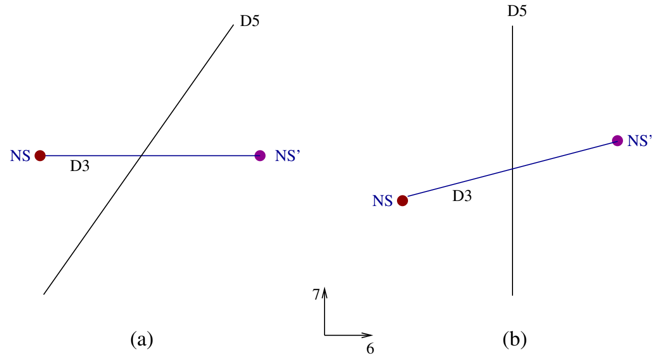

FIG. 7 (color online). Turning on B and D fields for a global symmetry that acts on the dynamical fields in the same way as the $U(1)$ part of the gauge group leads to the brane configuration (a), which is equivalent to the configuration (b) obtained by turning on an FI term and scalar field $\phi_2(x^1)$ for that $U(1)$.

corresponds to turning on a D term for a $U(1)$ symmetry. Note that the above discussion also implies that turning on an FI term for a $U(1)$ gauge symmetry in the higher dimensional theory does not correspond to turning on the FI term (2.11) in the low energy two-dimensional (0,2) theory.

VII. COMMENTS ON THE QUANTUM THEORY

In this section we discuss some aspects of the low energy dynamics of the systems described in the previous sections. We will focus on the choice of charges depicted in Fig. 6, but it is easy to generalize to other cases.

The light spectrum of the resulting model is summarized in Table I, where we indicated the transformation properties of the superfields under the $SU(N_f/2)$ factors in (4.9), as well as under the global $U(1)$ used in our construction, which is denoted by $U(1)_e$ in the table. The gauge transformation properties have been suppressed in the table. The first (last) two fields in the table transform in the (anti) fundamental representation of the $U(N_c)$ gauge group.

In addition to the fields in Table I, there is the (0,2) chiral superfield Σ and the field strength Υ. Both transform in the adjoint of $U(N_c)$ and are singlets under all the symmetries listed in Table I. As explained in Sec. IV, the E functions for the Fermi superfields in Table I vanish in the light sector, and in particular there is no potential of the form $|\sigma|^2|\Phi|^2$ coupling the fundamentals $(\Phi = Q, \tilde{Q})$ and adjoint. This is easy to see from the brane perspective. Before turning on the background B and D fields, this potential could be understood as follows. The imaginary part of σ can be thought of as parametrizing the location of the $D3$-branes in the x^2 direction.[7] For generic positions of the three-branes, strings stretched between them and the five-branes have a

TABLE I. The quantum numbers of the light states of the electric model.

Field	$SU(N_f/2)_1$	$SU(N_f/2)_2$	$SU(N_f/2)_3$	$SU(N_f/2)_4$	$U(1)_e$
Q^1	$N_f/2$	1	1	1	$+1$
Λ^2	1	$N_f/2$	1	1	-1
$\tilde{\Lambda}_1$	1	1	$\overline{N_f/2}$	1	-1
$\tilde{Q}_2$	1	1	1	$\overline{N_f/2}$	$+1$

TABLE II. The quantum numbers of the light states of the magnetic model.

Field	$SU(N_f/2)_1$	$SU(N_f/2)_2$	$SU(N_f/2)_3$	$SU(N_f/2)_4$	$U(1)_e$
λ_1	$\overline{N_f/2}$	1	1	1	-1
q_2	1	$\overline{N_f/2}$	1	1	$+1$
$\tilde{q}^1$	1	1	$N_f/2$	1	$+1$
$\tilde{\lambda}^2$	1	1	1	$N_f/2$	-1
M_1^1, Λ_1^1	$N_f/2$	1	$\overline{N_f/2}$	1	0
M_2^2, Λ_2^2	1	$N_f/2$	1	$\overline{N_f/2}$	0
$M_2^1(\times 2)$	$N_f/2$	1	1	$\overline{N_f/2}$	$+2$
$\Lambda_1^2(\times 2)$	1	$N_f/2$	$\overline{N_f/2}$	1	-2

minimal length proportional to the $D3$–$D5$ separation in the x^2 direction. This gives rise to a mass for the chiral superfields which is captured by the $|\sigma|^2|\Phi|^2$ potential. After turning on the background fields, it is no longer true that changing σ gives a mass to the chiral superfields. As is clear from Fig. 6(a), all it does is change their location in x^1, an effect that we saw in the discussion of the gauge theory Lagrangian (4.4).

One of our goals in this section is to compare the two-dimensional theory obtained from SQCD to the one obtained from its Seiberg dual. To facilitate the comparison, we present in Table II the two-dimensional spectrum and quantum numbers of the charged superfields in the magnetic theory.

Here λ_1 and $\tilde{\lambda}^2$ are Fermi superfields obtained from the four-dimensional chiral superfields q_1 and $\tilde{q}^2$, which have $U(1)_e$ charge -1, using the construction of Sec. III. Similarly, q_2 and $\tilde{q}^1$, which have charge $+1$, give rise to chiral superfields. The four-dimensional singlet meson fields M_i^i, $i = 1, 2$, have $U(1)_e$ charge 0 and thus give upon reduction a Kaluza-Klein (KK) tower of chiral and Fermi superfields, the lowest of which are massless. The off-diagonal meson fields M_2^1 and Λ_1^2 have charges $+2$ and -2, respectively, and give chiral and Fermi superfields with degeneracy two. This degeneracy is due to the fact that the flavor $D3$-branes in the brane configuration dual to that of Fig. 6 intersect twice on the torus. Each intersection supports one copy of the above chiral and Fermi superfields. The magnetic superpotential (4.10) gives rise in two dimensions to an effective (0,2) superpotential of the schematic form

[7]On the Coulomb branch, the $U(N_c)$ gauge symmetry is broken to $U(1)^{N_c}$. The real part of σ can be thought of as the dual of the unbroken gauge field.

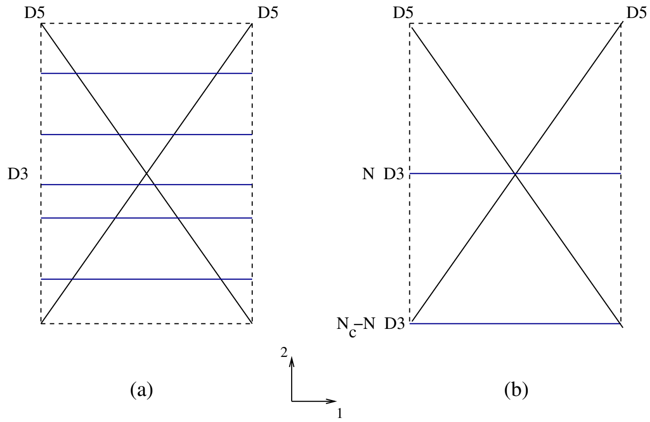

FIG. 8 (color online). (a) At a generic point on the Coulomb branch, the N_c $D3$-branes are separated in x^2. (b) At special points on the Coulomb branch, the Q's and $\tilde{Q}$'s are localized at the same points in x^1.

$$\mathcal{W} = M_1^1\lambda_1\tilde{q}^1 + M_2^2 q_2\tilde{\lambda}^2 + \Lambda_1^2 q_2\tilde{q}^1. \tag{7.1}$$

We would like to understand the low energy dynamics of the electric and magnetic theories described above. Starting with the electric theory, the first issue we would like to discuss is the fate of the Coulomb branch. The D-term potential $\mathrm{Tr}[\sigma,\bar{\sigma}]^2$ can be used as usual to diagonalize the adjoint scalar field σ. The Coulomb branch is labeled by the eigenvalues of this matrix. From the brane perspective it corresponds to displacing the color branes in the x^2 direction, as depicted in Fig. 8(a).

Classically, the potential on the Coulomb branch is flat, but quantum mechanically it is not. At a generic point in the Coulomb branch, the chiral superfields Q^1 and $\tilde{Q}_2$ are localized at different places on the x^1 circle [see Fig. 8(a)]. Locally in x^1 the theory thus looks like two copies of a $U(1)$ gauge theory, one with just Q^1 and Λ^2, the other with $\tilde{\Lambda}_1$, $\tilde{Q}_2$. Each of these theories separately breaks supersymmetry due to a nonzero expectation value of the D term in the quantum theory. Therefore, it is natural to expect that the theory that contains both also has this property. A more detailed discussion of this issue is presented in the Appendix.

The analysis of the Appendix implies that the Coulomb branch of Fig. 8(a) is replaced in the quantum theory by a discrete set of vacua, labeled by an integer N, which in the brane description corresponds to the number of color three-branes that are placed at one of the intersections of the flavor branes. The other $N_c - N$ color branes are placed at the other intersection [see Fig. 8(b)]. Superficially it looks like the integer N runs from 0 to N_c, but we shall next argue that it must satisfy further constraints.

The D-term conditions for the modes localized at one of the intersections in Fig. 8(b), say the one with N color $D3$-branes, are given in the Appendix. They are the same as those for four-dimensional $N = 1$ SQCD with gauge group $U(N)$ and $N_f/2$ flavors Q^1 and $\tilde{Q}_2$. The classical moduli space of that theory has been well studied (see e.g. [1,2]). For $N_f \geq 2N$, the gauge symmetry is generically completely broken, and the moduli space is $N_f N - N^2$ dimensional. For $N_f < 2N$, the gauge symmetry is generically broken to $U(N - \frac{1}{2}N_f)$ by the fundamentals. The classical Higgs moduli space is in this case $N_f^2/4$ dimensional and can be parametrized by the gauge invariant meson fields $Q^1\tilde{Q}_2$. In two dimensions there are also Coulomb moduli associated with the unbroken part of the gauge group, which break it further to the Cartan subalgebra.

Quantum mechanically, we expect supersymmetry to be broken at intersections with $N_f < 2N$. Indeed, in other situations of this sort, such as $N = 1$ SQCD in four dimensions and $N = 2$ SQCD in three dimensions with more colors than flavors, the theory develops a quantum superpotential for some of the classical moduli that pushes them to infinity. We expect the same to happen here, but will not attempt to show that it does.

In the brane picture, the fact that the gauge symmetry is not completely broken along the Higgs moduli space implies that at a generic point in moduli space $N - \frac{1}{2}N_f$ of the $D3$-branes continue to stretch between the $NS5$-branes. Quantum mechanically, such D-branes are typically repelled by other $D3$-branes ending on the five-branes [5]. This generates a potential for the corresponding Coulomb branch moduli that pushes them away from the intersection.

Assuming that the above picture is correct, we conclude that if we want the low energy theory to have a supersymmetric vacuum, N must lie in the range

$$N_c - \frac{N_f}{2} \leq N \leq \frac{N_f}{2}, \tag{7.2}$$

where we also included the constraint that follows from requiring stability of the vacuum at the other intersection. Note that (7.2) implies in particular that $N_f \geq N_c$. This is satisfactory since if N_f and N_c are outside this range, the four-dimensional theory does not have a vacuum even before we compactify it and turn on any background fields.

To summarize, we conclude that quantum vacua of the two-dimensional theory obtained from $N = 1$ SQCD via our construction are labeled by an integer N, which can be thought of as a discrete remnant of the classical Coulomb branch and takes values in the range

$$\max\left(0, N_c - \frac{1}{2}N_f\right) \leq N \leq \min\left(N_c, \frac{1}{2}N_f\right). \tag{7.3}$$

The total number of disconnected branches of moduli space is

$$N_{\mathrm{br}} = \begin{cases} N_f - N_c + 1 & \text{for } N_f \leq 2N_c \\ N_c + 1 & \text{for } N_f \geq 2N_c \end{cases}. \tag{7.4}$$

In each component, the low energy theory consists of theories localized at the two intersections. In general, we expect these theories to be coupled by terms obtained from integrating out massive modes living on the three-branes and exchange of modes living on the five-branes, but these couplings can be suppressed by taking the size to the x^2 circle to infinity. Therefore, below we shall assume that the two theories are decoupled.

The bosonic part of the theory at the intersection corresponding to the N color branes is a σ model on the moduli space of solutions to the $U(N)$ D-term equations. As mentioned above, the complex dimension of this space is $N_f N - N^2$. The right-moving central charge, and the left-moving one that is equal to it due to the absence of a gravitational anomaly in the spectrum of Table I, can be read off by going to large values of Q, $\tilde{Q}$, where the theory becomes weakly coupled. This gives

$$c_R = c_L = 3(N_f N - N^2). \tag{7.5}$$

Near the origin of the Higgs branch, the theory becomes strongly interacting in the IR due to the gauge dynamics. If $N_f \gg N$, one can use vector model techniques to solve it.

We next turn to the magnetic theory, whose brane description is given in Fig. 9. Flavor $D3$-branes are plotted in black in the figure. They split into two groups according to the global $U(1)$ charges (which are ± 1, as before) and intersect at two points on the torus. Color branes, plotted in blue, split into two groups of $\hat{N}$ and $N_f - N_c - \hat{N}$ which are placed at the two intersections, as in the electric theory.

The dynamics is again localized at the two intersections. Consider e.g. the intersection associated with the $\hat{N}$ color

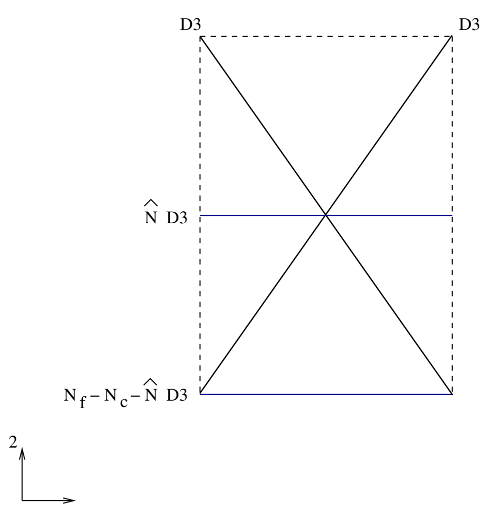

FIG. 9 (color online). The magnetic brane configuration for general N_f, N_c. Vacua are labeled by an integer $\hat{N}$ that runs over the range described in the text. For each $\hat{N}$ the theory splits into two decoupled theories at the intersections.

branes. It is a $U(\hat{N})$ gauge theory with the matter content listed in Table II above and superpotential (7.1). The parameter $\hat{N}$ takes a value in the range $0 \le \hat{N} \le N_f - N_c$, but as in the electric case we expect it to also satisfy the constraints $\hat{N}$, $N_f - N_c - \hat{N} \le N_f/2$. Indeed, if the singlet mesons M^i_j were not coupled to the magnetic theory, the latter would be identical to the electric theory with the replacement $N_c \to N_f - N_c$, $N \to \hat{N}$, and the parameter $\hat{N}$ would satisfy the analog of (7.3),

$$\max\left(0, \frac{1}{2}N_f - N_c\right) \le \hat{N} \le \min\left(N_f - N_c, \frac{1}{2}N_f\right). \tag{7.6}$$

The number of branches of moduli space would again be given by (7.4), which is essentially the statement that this expression is invariant under Seiberg duality, $N_c \to N_f - N_c$. Coupling the meson fields to the magnetic theory is not expected to change the number of branches; hence the vacua of the full magnetic theory are also expected to be labeled by the integer $\hat{N}$ taking value in the range (7.6).

The map between the electric and magnetic vacua can be obtained by comparing the ‘t Hooft anomalies of the two models. In the electric theory, the spectrum of Table I gives rise to the following nonzero anomalies:

$$\begin{aligned} &[SU(N_f/2)_1]^2: && +N, \\ &[SU(N_f/2)_2]^2: && -N, \\ &[SU(N_f/2)_3]^2: && -N, \\ &[SU(N_f/2)_4]^2: && +N. \end{aligned} \tag{7.7}$$

In the magnetic theory we find the same result, with N replaced by $N_f/2 - \hat{N}$. Therefore, we conclude that the map between the electric and magnetic vacua is

$$\hat{N} = \frac{N_f}{2} - N. \tag{7.8}$$

This relation maps the range (7.3)–(7.6).

The equivalence between the electric and magnetic theories can be thought of as a strong-weak coupling duality in the following sense. As explained above, the electric theory, which is a (0,2) sigma model on the Higgs branch of the $U(N)$ gauge theory with $N_f/2$ flavors, becomes weakly coupled in the region of large Q^1, $\tilde{Q}_2$. In the magnetic theory, the field parametrizing the target space of this sigma model is the singlet meson M^1_2. Since it does not appear in the magnetic superpotential (7.1), superficially it appears that this field is free in the infrared. However, one can show that integrating out the massive fields leads to the appearance of interactions of this field with the Fermi superfields that are charged under the gauge group in Table II. Denoting these superfields collectively by λ, the leading couplings take the schematic form

$$\mathcal{L} \sim \int d^2\theta |M_2^1|^2 |\lambda|^2. \tag{7.9}$$

These interactions make the magnetic theory strongly coupled in the large M_2^1 region, where the electric theory is weakly coupled. This is reminiscent of what happens in four dimensions [8], where going along the electric Higgs branch makes the electric (magnetic) theory more weakly (strongly) coupled.

Another sense in which the relation between the electric and magnetic theories is a strong-weak coupling duality is the following. As mentioned above, for $N_f/2 \gg N$, the electric theory becomes a gauge theory with many more flavors than colors, which can be treated using large N vector model techniques. In this sense, the electric gauge dynamics becomes weakly coupled in this limit. The relation (7.8) implies that in this limit the magnetic gauge dynamics is strongly coupled, since the rank of the magnetic gauge group, $\hat{N}$, is comparable to the number of flavors, $N_f/2$. Conversely, for $N_f/2 \gg \hat{N}$, the magnetic gauge dynamics is weakly coupled, since the number of flavors is much larger than the rank of the magnetic gauge group, while the electric theory is strongly coupled since the electric rank and number of flavors are comparable. This is similar to what happens in the three-dimensional analogs of Seiberg duality [14,15].

Earlier in this section we wrote down the central charge of the electric theory localized at an intersection with N color branes, Eq. (7.5). This was done by studying the low energy theory at large values of the scalar fields that parametrize the Higgs branch, where the theory simplifies. In the magnetic theory, the analogous calculation is more involved, due to the presence of the singlet mesons and superpotential (7.1). However, we can attempt to calculate the central charge by using its relation to the anomaly of the $U(1)_R$ symmetry which belongs to the $N = 2$ superconformal multiplet.

In the electric theory this $U(1)_R$ assigns charge zero to Q and $\tilde{Q}$, since they parametrize the target space of the low energy σ model (as in [16]). The fermions in these multiplets thus have R charge -1. The Fermi superfields have R charge zero, being left moving. The adjoint chiral superfield Σ is massive, hence its R charge is one. Finally, Υ has R charge one, as is clear from its mode expansion (2.5). Altogether we find the anomaly

$$k = 2 \times \frac{1}{2} N_f N - N^2, \tag{7.10}$$

with the first contribution coming from the fermions in Q, $\tilde{Q}$, and the second from Υ. Multiplying by three we get the central charge of the theory, Eq. (7.5).

In the magnetic theory, the adjoint fields Υ and Σ have R charge one, as before. The magnetic meson M_2^1 has R charge zero, since it is related to the chiral superfield $Q^1\tilde{Q}_2$ in the electric theory. We will assume that the fields M_1^1 and Λ_1^1 are massive in the quantum theory. Indeed, a coupling of the E-type (2.8), $\bar{\mathcal{D}}_+\Lambda_1^1 = mM_1^1$ is consistent with all the symmetries of the problem, and we expect it to be generated by the quantum dynamics, since the electric field dual to M_1^1, $Q^1\tilde{Q}_1$, is massive. The singlet meson Λ_1^1 in the magnetic theory is dual to $Q^1\tilde{\Lambda}_1$ in the electric one, which has R charge zero. Thus, we assign R charge zero to Λ_1^1 and one to M_1^1. The magnetic chiral superfields $q_2, \tilde{q}^1$ are assigned charge R_q, while the Fermi superfields λ_1 and $\tilde{\lambda}^2$ are assigned charge R_λ.

The magnetic superpotential (7.1) then implies that $R_q + R_\lambda = 0$, and $R(\Lambda_1^2) = 1 - 2R_q$. Thus, all R charges are determined by R_q. To find it we evaluate the $U(1)_R^2$ anomaly in the magnetic theory, which gives

$$k = N_f\hat{N}(R_q - 1)^2 - N_f\hat{N}R_q^2 + \frac{N_f^2}{4} - \frac{N_f^2}{4}(1 - 2R_q)^2 - \hat{N}^2, \tag{7.11}$$

and demand that it be equal to (7.10). This gives[8]

$$R_q = -R_\lambda = \frac{2N}{N_f} - \frac{1}{2} = \frac{N - \hat{N}}{N_f}. \tag{7.12}$$

The electric theory has a nonvanishing $U(1)_e U(1)_R$ anomaly

$$U(1)_e U(1)_R : \ -NN_f. \tag{7.13}$$

One can check that the magnetic theory with the charge assignments above gives the same anomaly.

It is natural to ask whether the result (7.12) can be understood directly in the magnetic theory, rather than by demanding agreement with the electric one. We do not have a full answer to this question, but can offer the following observations. First, note that the $U(1)_R$ anomaly (7.10) can be written in terms of magnetic variables as

$$k = \left(\frac{N_f}{2}\right)^2 - \hat{N}^2. \tag{7.14}$$

If the $\hat{N}^2$ term was absent, it would be natural to interpret it as due to the $(N_f/2)^2$ chiral superfields M_2^1, which do not appear in the superpotential (7.1), and thus, at least naively, seem to be free. In fact, as mentioned above, these fields have interactions with the Fermi superfields charged under the magnetic gauge group, of the schematic form (7.9). These and other interactions presumably reduce the rank of

[8] A second solution, $R_q = 1/2$, can be discarded based on the $U(1)_e U(1)_R$ anomaly.

the part of the M_2^1 matrix that is genuinely free by $\hat{N}$, and give rise to (7.14). We have not studied the detailed way in which this happens, but it seems that at least in the limit $N_f \gg 2\hat{N}$, we can approximately treat the full $\frac{N_f}{2} \times \frac{N_f}{2}$ matrix M_2^1 as consisting of free noncompact chiral superfields, whose R charge is fixed at zero. The R charge R_q can then be evaluated by extremizing k (7.11) subject to this constraint [17,18].[9] This gives $R_q = N/N_f$, which agrees with (7.12), since in the limit $N_f/2 \gg \hat{N}$ one has $N = N_f/2 - \hat{N} \simeq N_f/2$.

For generic $N_f, \hat{N}$, it is less clear what to do. From the duality we expect part of the matrix M_2^1 to give rise to a noncompact moduli space of complex dimension (7.14), which implies that using k extremization is more involved. The derivation of (7.12) in that case remains an open problem.

VIII. DISCUSSION

Our main purpose in this paper was to explore the theories one gets by starting with four-dimensional $N = 1$ supersymmetric gauge theories and reducing them on a two-torus with a nonzero magnetic field and corresponding D term for a global $U(1)$ symmetry. This gives rise at long distances to two-dimensional field theories with (0,2) supersymmetry, and our goal was to explore their dynamics. We focused on the special case of $N = 1$ SQCD and a particular choice of global $U(1)$ symmetry, which is a subgroup of the global symmetry group of the model. We showed that the low energy theory has a nontrivial vacuum structure, and in a given vacuum it gives a (0,2) supersymmetric σ model on the Higgs branch.

We also discussed the fate of Seiberg duality in four dimensions under such a reduction. We presented evidence that it survives the compactification to two dimensions and the reduction of supersymmetry from four to two supercharges. In particular, the vacuum structure of the electric theory appears to be reproduced by the magnetic theory. Our analysis was incomplete in many respects. In particular, parts of the discussion of quantum effects in Sec. VII relied to a large extent on the brane picture, and analogies to what happens in other dimensions. It would be interesting to fill these gaps and understand the dynamics directly in field theory.

Most of our analysis was restricted to the case where the global charges of the fields under the $U(1)$ for which we turn the background fields are $+1$ for half of the flavors and -1 for the other half. One might wonder what happens if we take a more general choice of solution to the constraints (4.7) and (4.8).

A simple generalization is to replace the charges ± 1 in the construction by $\pm n$, with n an integer larger than one. It is particularly easy to see what happens in that case in the type IIB brane construction of Sec. VI. The flavor branes have $2n$ intersections, which split into two groups of n intersections at a given value of x^2. Thus, the low energy theory is a sum of n theories of the sort described above with gauge group $U(N)$ living at one value of x^2, and n theories with gauge group $U(N_c - N)$ living at the other. Thus, taking $n > 1$ does not introduce any genuinely new elements into the discussion.

Other generalizations of our construction involve nontrivial solutions of the equations for the e_i, (4.7), (4.8). As a simple example, consider a $U(1)$ gauge theory with three flavors and charges $e_1 = 2$, $e_2 = e_3 = -1$ (and $\tilde{e}_i = -e_i$). It is easy to see what happens in this model using the brane construction. The three flavors correspond to $D5$-branes, one of which has winding number two (as in Fig. 5), and the other two winding one with the opposite orientation. The color $D3$-brane is again pushed toward the intersection of the $D5$-branes; however, now even at the intersection, the D-term potential leads in the quantum theory to supersymmetry breaking due to an incomplete cancellation between the contributions of Q and $\tilde{Q}$. This is typically what happens for general global charge assignments.

There are many natural generalizations of the discussion in this paper. In particular, one can study more general $N = 1$ gauge theories and the corresponding two-dimensional (0,2) models [20], generalize to models with $N = 2$ SUSY in four dimensions, and embed our construction in a holographic setting (for $N = 4$ SYM this was discussed in [7,18]). Hopefully, such generalizations will help elucidate the relation between the dynamics of the underlying four-dimensional models and the long distance two-dimensional ones, and shed light on both. It would also be interesting to understand the relation between our construction and other ways of getting (0,2) models in two dimensions from higher dimensional theories, such as [21–23].

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APPENDIX: COULOMB BRANCH ANALYSIS

In this appendix we discuss the classical Coulomb branch of the theory discussed in the text and show that it is lifted by quantum corrections.

1. Classical analysis

We begin by studying the classical vacuum structure. To introduce some of the issues in a simpler setting, we

[9] For a discussion of k extremization in the context of gauge/gravity duality, see [19].

start with the case $N_f = N_c = 1$ (depicted in Fig. 5). This case does not satisfy the constraints (4.8) but is a useful warm-up for models with larger N_f and N_c. Before turning on the background fields (in the brane realization when the three-brane and five-brane in Fig. 5 are both stretched in the x^1 direction), the low energy theory is a three-dimensional field theory with $N = 2$ SUSY, $U(1)$ gauge group, and two charged chiral superfields Q, $\tilde{Q}$ with global charges $+1$ and -1, respectively, compactified on a circle. After turning on the background fields, Q and $\tilde{Q}$ give (0,2) chiral and Fermi superfields Q, $\tilde{\Lambda}$, while the vector superfield gives rise to the (0,2) field strength superfield Υ and adjoint chiral superfield Σ.

The classical moduli space consists of a one-dimensional Coulomb branch labeled by σ. This is easy to see in the brane picture, where the Coulomb branch is parametrized by the position of the $D3$-brane in the x^2 direction and the dual of the $U(1)$ gauge field on the three-brane. The two-dimensional theory does not have a Higgs branch; the field Q is set to zero by the D-term potential, which is proportional to $|Q|^4$. We would like to verify this picture in the higher dimensional field theory.

The fields that enter our analysis are the scalar field $\phi = \mathrm{Im}\,\sigma$ parametrizing the position of the $D3$-brane in the x^2 direction, the auxiliary field D in the vector multiplet on the $D3$-brane, and Q, as well as the background fields ϕ_B (which parametrizes the location of the $D5$-brane in x^2) and D_B. In addition, it is necessary to include in the discussion fields Q_n that correspond to wound open strings in the brane picture. The chiral superfield Q is in the brane picture a string stretching from the three-brane to the five-brane. Because x^2 is compact, there are actually infinitely many such fields Q_n, corresponding to strings that stretch between the three-brane and five-brane while winding n times around the x^2 circle.[10] For a given position of the D-branes, at most one of these fields is light; however, relative motions of the branes change which one it is. For example, keeping the three-brane fixed and sending the five-brane around the x^2 circle takes the mass of Q_n to that of $Q_{n\pm1}$, depending on the direction of the motion. In the presence of the nontrivial ϕ_B required by our construction, the position of the five-brane in x^2 changes with x^1, so a similar shift occurs as a function of x^1.

Since we are only interested in the scalar potential, we take all fields to be constant in (x^0, x^3). Denoting the remaining coordinate x^1 by x, the scalar potential is given by

$$U = \int_0^{2\pi R_1} dx \left[\frac{1}{2g^2}(\phi'^2 - D^2) + \sum_{n\in Z}(|Q_n'|^2 - |Q_n|^2(D_B - D) + |Q_n|^2(\phi_B - \phi + 2\pi n R_2 T)^2)\right], \quad \text{(A1)}$$

where $T = 1/2\pi\alpha'$ is the string tension and the prime denotes differentiation in x^1. Equation (A1) includes the kinetic terms for the various fields and the standard couplings (4.2) of charged fields to the auxiliary fields in the vector multiplets. The relative minus sign between the D_B and D couplings is due to the fact that Q corresponds to an oriented 3–5 string, and therefore has opposite charges under the gauge groups on the $D3$- and $D5$-branes.

The mass (last) term is the energy of a string stretching from the three-brane to the five-brane while winding n times around the x^2 circle. Note that we have set the x component of the dynamical gauge field $A_1(x)$ to zero; this can (almost) be done by a $U(1)$ gauge transformation. The only information in A_1 that cannot be gauged away is the Wilson line $\exp(i\int dx A_1)$ or in other words constant A_1, which parametrizes one of the two directions of the compact Coulomb branch of the model (the other being the constant mode of ϕ). We shall set it to zero below.

To arrive at the configuration in Fig. 5, we take the background fields to have the values

$$\phi_B = Bx, \qquad D_B = B. \quad \text{(A2)}$$

Plugging (A2) into (A1) gives

$$U = \int_0^{2\pi R_1} dx \left[\frac{1}{2g^2}(\phi'^2 - D^2) + \sum_n (|Q_n{}' + (\phi_B - \phi + 2\pi n R_2 T)Q_n|^2 - |Q_n|^2(\phi' - D))\right], \quad \text{(A3)}$$

where we used integration by parts and the fact that $\phi_B{}' = D_B$.

We also require that the integrand (A3) be well defined on the circle. While ϕ and D are periodic, ϕ_B satisfies $\phi_B(x + 2\pi R_1) = \phi_B(x) + 2\pi R_1 B$. If B satisfies

$$R_1 B = l R_2 T \quad \text{(A4)}$$

for some integer l, we can absorb this violation of periodicity by demanding that

$$Q_n(x + 2\pi R_1) = Q_{n+l}(x). \quad \text{(A5)}$$

The constraint (A4) on the magnetic field B is the same as the one obtained in the four-dimensional analysis (3.10). The two are related by the standard T-duality relation

[10] These fields owe their existence to the compactness of x^2. From the point of view of the underlying four-dimensional theory they are the momentum modes of the four-dimensional field Q in the x^2 direction.

$R_2^{(IIB)} = \alpha'/R_2^{(IIA)}$. In the following we shall restrict to the case $l = 1$.

The equation of motion for D in (A3) sets it to

$$D = g^2 \sum_n |Q_n|^2. \tag{A6}$$

Plugging this back into (A3) yields the energy function

$$U = \int_0^{2\pi R_1} dx \left[\frac{1}{2g^2} \left(\phi' - g^2 \sum\nolimits_n |Q_n|^2 \right)^2 + \sum_n |Q_n{}' + (\phi_B - \phi + 2\pi n R_2 T) Q_n|^2 \right]. \tag{A7}$$

Note that the first term in (A7) is the square of the supersymmetry variation of the right-moving gaugino λ_+ in the vector multiplet, and the second term is the square of the variation of the right-moving fermion $(\psi_+)_n$ in the chiral multiplet Q_n.

The vanishing of the energy necessary for supersymmetry implies that the two terms in (A7) vanish separately. In particular,

$$\phi' = g^2 \sum_n |Q_n|^2. \tag{A8}$$

However, this relation is inconsistent with the periodicity of ϕ. Indeed, upon integrating (A8) over the x^1 circle, the left-hand side vanishes since ϕ is periodic, while the r.h.s. is positive definite. We conclude that Q_n must vanish for the vacuum to be supersymmetric. On the other hand, configurations with $Q_n = \phi' = 0$ manifestly satisfy $U = 0$ for arbitrary (constant) values of ϕ. The classical moduli space of the model with $G = U(1)$ and one flavor therefore indeed consists of a Coulomb branch parametrized by ϕ.

To get a nontrivial Higgs branch, we add another chiral superfield with opposite $U(1)$ charge, so that there are two flavors $(Q^i, \tilde{Q}_i)$, $i = 1, 2$. This is the simplest model that satisfies (4.8). After turning on the background fields (A2), we arrive at the system of Fig. 6. The brane picture suggests that there is again a Coulomb branch, parametrized by the scalar field σ, whose imaginary part is the position of the $D3$-brane in x^2. At generic points on the Coulomb branch, the chiral superfields $(Q^i, \tilde{Q}_i)$ give rise to two (0,2) chiral superfields $Q = Q^1$ and $\tilde{Q} = \tilde{Q}_2$ (as well as two Fermi superfields, Λ^2 and $\tilde{\Lambda}_1$), which are localized at different points on the x^1 circle.

To analyze this case, we again start from the three-dimensional Lagrangian and turn on the background fields. The analogs of (A6) and (A7) are

$$D = g^2 \sum_n (|Q_n|^2 - |\tilde{Q}_n|^2),$$

$$U = \int_0^{2\pi R_1} dx \left\{ \frac{1}{2g^2} \left[\phi' - g^2 \sum\nolimits_n (|Q_n|^2 - |\tilde{Q}_n|^2) \right]^2 + \sum_n [|Q_n{}' + (\phi_B - \phi + 2\pi n R_2 T) Q_n|^2 + |\tilde{Q}_n{}' + (\phi_B + \phi + 2\pi n R_2 T) \tilde{Q}_n|^2] \right\}. \tag{A9}$$

The three terms in the scalar potential correspond to the supersymmetry variations of the right-moving fermion inside the vector, Q_n and $\tilde{Q}_n$ multiplets, respectively, and all three must vanish for (0,2) supersymmetry to be preserved. It is clear that $U = 0$ when $\phi' = Q_n = \tilde{Q}_n = 0$ for arbitrary values of constant ϕ, corresponding to the classical Coulomb branch. One can also satisfy the conditions for unbroken supersymmetry by taking $Q_n(x) = \tilde{Q}_n(x)$ and $D = \phi' = 0$; this is the Higgs branch mentioned above.

Note that for constant ϕ the condition for the vanishing of the last two terms in U can be thought of as follows. Using the matching conditions (A5) (with $l = 1$), we can construct out of the Q_n a single function $Q(x)$ on the covering space of the x^1 circle, and similarly for $\tilde{Q}$. The vanishing of the last two terms in U is the requirement that $Q(x)$ and $\tilde{Q}(x)$ are ground states of the harmonic oscillator, localized at $x = \phi/B$ and $-\phi/B$, respectively. The first term in U (A9) then requires $\phi = 0$; i.e. it fixes the Coulomb modulus.

In the classical theory there actually seem to be solutions with $U = 0$ that have nontrivial $D = \phi'$ (as well as Q, $\tilde{Q} \neq 0$). We shall not discuss them here since, as we shall see below, they are lifted by quantum effects.

The above picture can be generalized to higher N_f, N_c. The analog of (A9) for the general case is

$$D^a = g^2 \sum_n (\bar{Q}_n^i T^a Q_n^i - \overline{\tilde{Q}_\rho^n} T^a \tilde{Q}_\rho^n) - g^2 [\bar{\sigma}, \sigma]^a,$$

$$U = \int_0^{2\pi R_1} dx \left\{ \frac{1}{2g^2} \left[(\phi^a)' - g^2 \sum\nolimits_n (\bar{Q}_n^i T^a Q_n^i - \overline{\tilde{Q}_\rho^n} T^a \tilde{Q}_\rho^n) \right]^2 + \sum_n [|(Q_n^i)' + (\phi_B - \phi_a T^a + 2\pi n R_2 T) Q_n^i|^2 + |(\tilde{Q}_\rho^n)' + (\phi_B + \phi_a T^a + 2\pi n R_2 T) \tilde{Q}_\rho^n|^2] \right\}. \tag{A10}$$

A new element in this case is the fact that the scalar fields ϕ^a belong to the adjoint representation of $U(N_c)$. We can think of them as forming a Hermitian $N_c \times N_c$ matrix, which can be diagonalized by a $U(N_c)$ gauge transformation. The eigenvalues of ϕ parametrize the Coulomb branch of the model and correspond in the brane desription to the positions of the N_c $D3$-branes in the x^2 direction [as in

Fig. 8(a)]. It is easy to generalize the $U(1)$ analysis to this case.

2. Quantum analysis

We now turn to a discussion of quantum effects. Since we are interested in the Coulomb moduli, we set $Q_n = \tilde{Q}_n = 0$ and ask whether the moduli space labeled by σ survives in the quantum theory. To see the basic physics it is enough to consider the case $N_c = 1$. The generalization to larger N_c is straightforward.

The scalar potential U for this case can be written in the form

$$U = \int_0^{2\pi R_1} dx \left[\frac{1}{2g^2}(\phi'^2 - D^2) + \sum_n (|Q_n' + (\phi_B - \phi + 2\pi n R_2 T)Q_n|^2 + |\tilde{Q}_n{}' + (\phi_B + \phi + 2\pi n R_2 T)\tilde{Q}_n|^2 - (|Q_n|^2 - |\tilde{Q}_n|^2)(\phi' - D))\right], \tag{A11}$$

where we suppressed the flavor indices of Q and $\tilde{Q}$. Although we are interested in the fate of classical vacua with $Q = \tilde{Q} = 0$, in the quantum theory we need to include the effects of fluctuations of these fields on the Coulomb modulus ϕ. At large N_f this can be done by performing the Gaussian integral over Q, $\tilde{Q}$ and treating ϕ and D semi-classically, and we shall do that below. In general ϕ and D should be path integrated over as well, but we do not expect this to change the conclusions.

The fields Q abd $\tilde{Q}$ have an expansion in Landau levels,

$$Q_n(x^0, x^3, x) = \sum_{a=0}^{\infty} Q_a(x^0, x^3)\varphi_n^a(x),$$
$$\tilde{Q}_n(x^0, x^3, x) = \sum_a \tilde{Q}_a(x^0, x^3)\tilde{\varphi}_n^a(x), \tag{A12}$$

where Q_a and $\tilde{Q}_a$ are two-dimensional fields with masses that scale as $M_a^2 = a|eB|$ (3.5). At one loop one has (see e.g. [21] for a related discussion)

$$\langle \bar{Q}_0 Q_0 \rangle = \langle \overline{\tilde{Q}_0} \tilde{Q}_0 \rangle = \frac{N_f}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \mu^2} \sim \frac{N_f}{2} \ln \frac{\Lambda}{\mu}, \tag{A13}$$

where μ is an infrared scale; for the massive fields $Q_{a\neq 0}$ and $\tilde{Q}_{a\neq 0}$, μ is replaced by m_a. Since the infrared scale μ is in general well below m_a, the contributions of the massive fields to the two point function are suppressed relative to the massless ones. Therefore, we need only consider the fluctuations of Q_0 and $\tilde{Q}_0$ in (A11). As discussed above, the wave functions φ_n^0 and $\tilde{\varphi}_n^0$ are segments of the wave function of the ground state of the harmonic oscillator, localized at $x = \phi_0/B$ and $-\phi_0/B$, respectively.

It is convenient to define $\phi_{nz} = \phi - \phi_0$, where ϕ_0 is the average value of ϕ on the x^1 circle, and $\phi' = \phi'_{nz}$. After integrating out Q_0, $\tilde{Q}_0$ (A13), the effective potential for ϕ_{nz} and D reads

$$U_{\rm eff} = \int_0^{2\pi R_1} dx \left[\frac{1}{2g^2}(\phi'^2_{nz} - D^2) + \frac{N_f}{2}\left(\ln\frac{\Lambda}{\mu}\right)|\phi_{nz}|^2 (F_Q(x) + F_{\tilde{Q}}(x)) - \frac{N_f}{2}\left(\ln\frac{\Lambda}{\mu}\right)(F_Q(x) - F_{\tilde{Q}}(x))(\phi'_{nz} - D)\right]. \tag{A14}$$

Here $F_Q(x) = \sum_n |\varphi_n^0(x)|^2$ and $F_{\tilde{Q}}(x) = \sum_n |\tilde{\varphi}_n^0(x)|^2$. The two differ by a shift $F_{\tilde{Q}}(x) = F_Q(x + x_0)$ with x_0 proportional to the Coulomb modulus ϕ_0.

The equations of motion for ϕ'_{nz} and D that follow from (A14) are

$$\frac{1}{g^2}\phi''_{nz} - \frac{N_f}{2}\left(\ln\frac{\Lambda}{\mu}\right)(F'_Q - F'_{\tilde{Q}}) = N_f\left(\ln\frac{\Lambda}{\mu}\right)\phi_{nz}(F_Q + F_{\tilde{Q}}),$$
$$\frac{1}{g^2}D - \frac{N_f}{2}\left(\ln\frac{\Lambda}{\mu}\right)(F_Q - F_{\tilde{Q}}) = 0. \tag{A15}$$

As mentioned previously, $\phi' - D$ is the variation of the right-moving gaugino in the vector multiplet which must vanish in order for (0,2) supersymmetry to be preserved. From the above equations of motion, one can show that this is equivalent to requiring $F_Q = F_{\tilde{Q}}$. But F_Q, $F_{\tilde{Q}}$ are identical up to a shift in x. This condition is satisfied iff the shift is zero, i.e. Q and $\tilde{Q}$ are localized at the same position in x.

To summarize, we find that for $N_c = 1$ the Coulomb moduli space labeled by ϕ is replaced in the quantum theory by two isolated vacua. In the brane picture, they correspond to having a straight $D3$-brane pass through one of the two intersections of the $D5$-branes.

In the non-Abelian case, the same mechanism leads to the collapse of the classical moduli space of Fig. 8(a) to the isolated vacua of Fig. 8(b). The dynamics at a given intersection involves additional phenomena that place further constraints on the integer N, as described in the text.

The above analysis involved in an important way the massive KK modes of the various three-dimensional fields.

It is natural to ask how the quantum effects that fix the Coulomb moduli manifest themselves in the two-dimensional low energy theory of the light modes. We have not understood this in detail, but D-term couplings of the adjoint chiral superfield Σ with the charged Fermi superfields, which can be shown to have the schematic form $\int d^2\theta |\Sigma|^2 |\Lambda|^2$ (for small Σ) seem to play an important role in this problem.

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