

Källén-Lehmann representation of noncommutative quantum electrodynamics

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Noncommutative (NC) quantum field theory is the subject of many analyses on formal and general aspects looking for deviations and, therefore, potential noncommutative spacetime effects. Within of this large class, we may now pay some attention to the quantization of NC field theory on lower dimensions and look closely at the issue of dynamical mass generation to the gauge field. This work encompasses the quantization of the two-dimensional massive quantum electrodynamics and three-dimensional topologically massive quantum electrodynamics. We begin by addressing the problem on a general dimensionality making use of the perturbative Seiberg-Witten map to, thus, construct a general action, to only then specify the problem to two and three dimensions. The quantization takes place through the Källén-Lehmann spectral representation and Yang-Feldman-Källén formulation, where we calculate the respective spectral density function to the gauge field. Furthermore, regarding the photon two-point function, we discuss how its infrared behavior is related to the term generated by quantum corrections in two dimensions, and, moreover, in three dimensions, we study the issue of nontrivial θ -dependent corrections to the dynamical mass generation.

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I. INTRODUCTION

The quantization of spacetime, in particular, a quantum fuzzy spacetime, comes as a natural consequence to support the coexistence of quantum field theory and gravity demanding, thus, a radical change in our concepts of geometry [1]. Noncommutative (NC) spacetime characterized by canonical Heisenberg-like (commutation relations) Moyal brackets

$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu} \quad (1)$$

and its associated \star product (Weyl symbols)

$$\phi(x) \star \phi(x) = \phi(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu\right) \phi(x) \quad (2)$$

have received much attention over the course of the last two decades. Moreover, since $\theta^{\mu\nu}$ is a constant antisymmetric object, theories defined on such noncommutative spacetime are considered to violate Lorentz invariance, a subtle but highly important issue that has been richly debated [2,3]. Actually, noncommutative spatial coordinates have a well-known realization in physics: the quantized confined motion of particles in a constant magnetic field, sufficiently strong so that the projection on the lowest Landau level can be justified, is described by noncommutative coordinates.

Within the context of field theory, it was only after the work of Seiberg and Witten [4] that the NC gauge theories have acquired a prominent place in several discussions. Many of the intriguing fundamental phenomena regarding NC gauge theories are results from the nonlocality of their interactions, for instance, UV/IR mixing problem [5], loss of unitarity [6], and violation of Lorentz symmetry [7]. Nevertheless, investigation on these features has pointed out, for instance, that a suitable definition of time-ordering operation restores the unitarity in a NC field theory [8] and many other interesting issues [9].

A successful and extensively studied approach to a perturbative analysis of (*nonlocal*) NC field theory is obtained if one exploits the Seiberg-Witten (SW) map [4,10]. Namely, they have shown that a NC \star -gauge theory should be a gauge equivalent to an ordinary counterpart defined on a commutative spacetime, i.e., the SW transformation maps a \star -gauge invariant NC expression into a gauge invariant ordinary expression. In such approach, one can study systematically in a perturbative way the effects of noncommutativity in a *local* quantum field framework. Furthermore, a different route may be taken in obtaining the aforementioned SW map without referring to string theory. This consists of letting the theory be an enveloping algebra valued one that one, thus, can write the SW map of NC fields for arbitrary non-Abelian gauge groups, such as $SU(N)$ [11]. Therefore, it is clear that the SW map of NC fields presents itself as an interesting tool in the understanding of the physical predictions and also to check the behavior of the NC theory itself, such as renormalizability.

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Based on the above discussion, it is interesting to know how the noncommutativity affects established properties of conventional theories, i.e., the study of NC extensions of well-studied quantum field theories and to look then for NC effects on its deviations, since it is generally found that such extensions behave in interesting and nontrivial ways. Therefore, for this purpose, we will investigate here the NC effects first on the celebrated massive Schwinger model [12,13] [quantum electrodynamics (QED) in two dimensions] and next on the QED in three dimensions, Abelian gauge theory, Maxwell-Chern-Simons, or simply Maxwell electrodynamics coupled to fermions [14–19]. One may say that the main goal of physics is to explain phenomena in nature and perhaps even to explain why physical nature dwells in four dimensions; however, the means that we have come to employ in reaching this goal so far are sufficiently intricate that it has proven useful to wander into lower-dimensional worlds, with the wishful thought that in a simpler setting we can learn useful things about the well-recognized four-dimensional problems. Both classes of field theory have been extensively investigated along the years and have long been recognized as laboratories where important theoretical ideas, such as infrared problems, dynamical mass generation, and confinement, to name few, were tested in a simple setting, especially on condensed matter systems, e.g., the quantum Hall effect. Actually, some NC properties of them were discussed previously in Refs. [20,21] on massless QED₂ and in Refs. [22,23] on QED₃.

To analyze the quantized theory, we will make use of a general framework possessing a solid physical basis which is the spectral representation of Källén-Lehmann [24]. Furthermore, in order to accomplish that, we will make use of the Yang-Feldman-Källén formalism [25] which is based on the Heisenberg operators equations of motion and asymptotic conditions. Also, extensions of these formalisms were proposed and studied in different scenarios: a noncommutative spacetime [26] and also for a Lorentz-violating field theory [27].

In this paper, we revisit QED₂ and QED₃ embedded in a noncommutative spacetime from the point of view of Källén-Lehmann formalism, in particular, their radiative corrections at one loop and the *possible* higher-order term [21] and dynamic mass generation, respectively. We start by discussing the general properties and symmetries of NCQED in a generic dimension ω in Sec. II. Next, we discuss and make use of the Seiberg-Witten map to determine the action. Next, we present the expressions for the Lagrangian functions where we will perform the analysis; both expressions are obtained by taking into account the dimensionality of spacetime. From that, we derive the Euler-Lagrange equations of motion, and from the Yang-Feldman-Källén formalism, we construct the solution for these equations. Moreover, another important feature of Yang-Feldman-Källén formalism is that one works exclusively with the free asymptotic fields so that all quantities are mathematically well defined. In Sec. III,

we review and discuss, in general terms, the Källén-Lehmann representation and obtain the spectral density function for the gauge field. In Sec. IV, we present the calculation and results at two-particle contribution to the spectral density function of the gauge field, both calculated on two and three dimensions. In particular, based on a previous result [21], we discuss in detail the correct expression for the self-interaction contribution of the gauge field spectral function in the two-dimensional case, which originates from the infrared sector. In Sec. V, we summarize the results and present our final remarks and prospects.

II. SEIBERG-WITTEN MAP ON NCQED

Before addressing our problem, *per se*, we start this section by reviewing the quantum electrodynamics of massive fermions in an ω -dimensional noncommutative Minkowski spacetime. As a consequence of the nontriviality of the star product, the ordinary theory acquires a non-Abelian-like structure, namely, the NCQED model action is defined in the following way¹:

$$\mathcal{A} = \int d^\omega x \left[-\frac{1}{4} \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} + \hat{\psi} \star (i\gamma^\mu \hat{D}_\mu \hat{\psi} - m\hat{\psi}) \right], \quad (3)$$

where the field strength and covariant derivative are defined as

$$\begin{aligned} \hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig[\hat{A}_\mu, \hat{A}_\nu]_\star, \\ \hat{D}_\mu \hat{\psi} &= \partial_\mu \hat{\psi} - ig\hat{A}_\mu \star \hat{\psi}. \end{aligned} \quad (4)$$

The action is invariant under the finite noncommutative $U(1)$ \star -gauge transformations:

$$\begin{aligned} \hat{\psi}'(x) &= U(x) \star \hat{\psi}(x), \\ \hat{A}'_\mu(x) &= \frac{i}{g} U(x) \star \hat{D}_\mu U^{-1}(x), \end{aligned} \quad (5)$$

or, rather, in its infinitesimal form,

$$\begin{aligned} \hat{\delta}_\sigma \hat{\psi} &= i\hat{\sigma}(x) \star \hat{\psi}(x), \\ \hat{\delta}_\sigma \hat{A}_\mu &= \frac{1}{g} \partial_\mu \hat{\sigma}(x) - i[\hat{A}_\mu, \hat{\sigma}]_\star, \end{aligned} \quad (6)$$

where $U(x) = (e^{i\sigma(x)})_\star$ is defined by an infinite series of multiple *star* products of scalar function $\hat{\sigma}(x)$.

Given the properties of the Moyal product, the product of two functions (distributions) is integrated, giving the same result as the ordinary product of functions. This implies that in this framework, the free propagator expressions do not change; i.e., they have the same expression as in the commutative (ordinary) theory. Therefore, in the case of

¹According to our notation, “hatted” quantities represent objects in noncommutative spacetime, and “unhatted” ones are objects in ordinary spacetime.

covariant perturbative analysis, the signal of noncommutativity is encompassed on the theory's vertices. However, we are interested in studying effects of noncommutativity in a perturbative way by making use of the SW map [4], which allows one to write a NC \star -gauge theory as a gauge theory in ordinary spacetime.

The SW map of NC fields in a gauge invariant theory can be derived from a gauge equivalence relation as it stands for the gauge field and parameter

$$\begin{aligned}\hat{\delta}_\sigma \hat{A}_\mu(A; \theta) &= \hat{A}_\mu(A + \delta_\sigma A; \theta) - \hat{A}_\mu(A; \theta) \\ &= \delta_\sigma \hat{A}_\mu(A; \theta),\end{aligned}\quad (7)$$

$$\hat{\delta}_\sigma \hat{\sigma}(\sigma, A; \theta) = \delta_\sigma \hat{\sigma}(\sigma, A; \theta),\quad (8)$$

as for the matter field

$$\hat{\delta}_\sigma \hat{\psi}(\psi, A; \theta) = \delta_\sigma \hat{\psi}(\psi, A; \theta),\quad (9)$$

in such a way that the NC fields are functionals of the ordinary fields: A_μ , ψ , and σ are the ordinary gauge field, fermion field, and gauge transformation parameter, respectively. To the lowest nontrivial order in θ , one finds the solution to the SW map,²

$$\hat{A}_\mu = A_\mu - \frac{g}{2} \theta^{\lambda\sigma} A_\lambda (2\partial_\sigma A_\mu - \partial_\mu A_\sigma) + O(\theta^2),\quad (10)$$

$$\hat{\psi} = \psi - \frac{g}{2} \theta^{\lambda\sigma} A_\lambda \partial_\sigma \psi + O(\theta^2),\quad (11)$$

$$\hat{\sigma} = \sigma - \frac{g}{2} \theta^{\lambda\sigma} A_\lambda \partial_\sigma \sigma + O(\theta^2).\quad (12)$$

Consequently, an important feature of this map is that it preserves gauge orbits, and so \star -gauge invariance is, therefore, rendered into an ordinary gauge invariance as contained in Eqs. (7)–(9). Utilizing this map, in terms of the ordinary quantities, we arrive at the following $O(\theta)$ modified form for the NC quantum electrodynamics action [28]:

$$\begin{aligned}\mathcal{A} &= \int d^{\omega}x \left[-\frac{1}{4} \left(1 - \frac{g}{2} (\theta.F) \right) F_{\mu\nu} F^{\mu\nu} \right. \\ &\quad - \frac{g}{2} \theta^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma} F^{\mu\nu} + \left(1 - \frac{g}{4} (\theta.F) \right) \bar{\psi} i\gamma.D\psi \\ &\quad \left. + \frac{g}{2} \theta^{\lambda\sigma} \bar{\psi} i\gamma^\mu F_{\lambda\mu} D_\sigma \psi - m \left(1 - \frac{g}{4} (\theta.F) \right) \bar{\psi} \psi \right].\end{aligned}\quad (13)$$

An interesting modification seen on the gauge sector at leading order in θ is the presence of self-interaction terms.

²Because of the freedom in the solutions, ambiguities are present on different solutions of the maps and are reflected in different coefficients obtained in each of the solutions constructed. For the interested reader, we refer to Ref. [10] for a complete disclosure of the general features of the SW map.

Although the NC gauge sector resembles the Yang-Mills theories, it is the noncommutative structure of spacetime which causes the nonlinearity of the field strength in the gauge connection. Furthermore, for $\omega > 2$, the action (13) provides a suitable framework on the investigation of a Lorentz-violating extension of QED once all the θ -dependent terms violate Lorentz symmetry [2].

It should be emphasized that the only needed ingredient in our development in the Källén-Lehmann representation is the translational invariance,³ which is always satisfied in the noncommutative theory. Nevertheless, the Lorentz symmetry (rotations and boosts) is preserved only if $\theta^{\mu\nu}$ transforms as a tensor [3], taking different constant values in different frames. Moreover, it also follows that the Seiberg-Witten map has an explicit Lorentz-invariant form provided that θ transforms like a Lorentz tensor, in accordance with the previous discussion [3].

A. (1 + 1) and (2 + 1) NCQED

At this point, we have considered a field theory defined on an ω -dimensional spacetime. Looking for simplifications of the θ terms of action (13), we shall consider separately two particular cases: (1 + 1)- and (2 + 1)-dimensional spacetime. Thus, taking into account the dimensionality of spacetime, one arrives at

$$\begin{aligned}\mathcal{L}_{1+1} &= \bar{\psi} (i\gamma.D - m)\psi + \frac{mg}{4} (\theta.F) \bar{\psi} \psi \\ &\quad - \frac{1}{4} \left(1 + \frac{g}{2} (\theta.F) \right) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2,\end{aligned}\quad (14)$$

for a two-dimensional spacetime, whereas, for three dimensions, one gets

$$\begin{aligned}\mathcal{L}_{2+1} &= \bar{\psi} (i\gamma.D - m)\psi + \frac{mg}{4} (\theta.F) \bar{\psi} \psi - \frac{\mu}{4} \epsilon^{\sigma\lambda} A_\sigma F_{\nu\lambda} \\ &\quad - \frac{1}{4} \left(1 + \frac{g}{2} (\theta.F) \right) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2,\end{aligned}\quad (15)$$

where in both Lagrangian functions, a gauge-fixing term on the Lorenz condition was inserted. Moreover, on the three-dimensional Lagrangian (15), $\epsilon^{\mu\nu\lambda}$ is the totally antisymmetric Levi-Civita tensor, μ denotes the coupling of the topological term, and we have also made use of the NC extension of the Chern-Simons action derived in Ref. [23], where it was shown that under the SW map, the NC Chern-Simons theory reduces to its commutative counterpart to all orders of θ .

Once we have developed the models of interest, i.e., obtained the Lagrangian functions for (1 + 1) and (2 + 1) dimensions, Eqs. (14) and (15), respectively, we are ready to proceed with our development. Since we aim to discuss both theories on the framework of Källén-Lehmann

³The Lorentz invariance is only required in a way in deriving the dependence and tensor structure of the spectral function with the momentum p as it will be shown next.

representation [24], we have to calculate next the equations of motion of the fields and subsequently find their solution in terms of the Yang-Feldman-Källén equations for Heisenberg operators [25].

On the fermion sector, we obtain the same equation of motion for both cases,

$$(i\gamma^\mu D_\mu - m)\psi = -\frac{mg}{4}(\theta.F)\psi, \quad (16)$$

whereas for the gauge sector, we obtain from Eq. (14),

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} + \frac{1}{\xi} \partial^\beta \partial_\lambda A^\lambda + g\bar{\psi}\gamma^\beta\psi \\ + \frac{g}{4} \partial_\alpha [\theta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + 2(\theta.F)F^{\alpha\beta} - 2m\theta^{\alpha\beta}\bar{\psi}\psi] = 0, \end{aligned} \quad (17)$$

and from Eq. (15), it yields

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} + \frac{1}{\xi} \partial^\beta \partial_\lambda A^\lambda - \frac{\mu}{2} \epsilon^{\alpha\mu\beta} F_{\alpha\mu} + g\bar{\psi}\gamma^\beta\psi \\ + \frac{g}{4} \partial_\alpha [\theta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + 2(\theta.F)F^{\alpha\beta} - 2m\theta^{\alpha\beta}\bar{\psi}\psi] = 0. \end{aligned} \quad (18)$$

In accordance with the general statement of the Yang-Feldman-Källén formulation, with suitable boundary conditions at $t = \pm\infty$, we find from Eq. (16) that the Heisenberg operators ψ and $\bar{\psi}$ satisfy the equations [29]

$$\begin{aligned} \psi_A(x) &= \psi_A^{\text{in}}(x) - \int d^\omega y S_{AB}^{\text{ret}}(x-y)\eta_B(y), \\ \bar{\psi}_A(x) &= \bar{\psi}_A^{\text{in}}(x) - \int d^\omega y \zeta_B(y) S_{AB}^{\text{adv}}(y-x), \end{aligned} \quad (19)$$

where the currents are given by

$$\eta = g\gamma.A\psi - \frac{mg}{4}(\theta.F)\psi, \quad \zeta = g\bar{\psi}\gamma.A - \frac{mg}{4}(\theta.F)\bar{\psi}. \quad (20)$$

The retarded (advanced) fermionic Green's function is

$$S^{\text{ret(adv)}}(x) = \frac{1}{(2\pi)^\omega} \int d^\omega p \frac{1}{\gamma.p - m \mp i\epsilon p_0} e^{-ipx}. \quad (21)$$

In such formulation the asymptotic *in*- and *out*fields (operator-valued fields) satisfy free-field equations and, thus, can be decomposed into positive and negative frequency parts [29]. Furthermore, the Yang-Feldman-Källén equation for the gauge field solution of Eqs. (17) and (18) takes the form

$$A_\mu(x) = A_\mu^{\text{in}}(x) - \int d^\omega y \Delta_{\mu\sigma}^{\text{ret}}(x-y)j^\sigma(y), \quad (22)$$

where

$$\begin{aligned} j^\beta &= g\bar{\psi}\gamma^\beta\psi + \frac{g}{4} \partial_\alpha [\theta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + 2(\theta.F)F^{\alpha\beta} \\ &\quad - 2m\theta^{\alpha\beta}\bar{\psi}\psi]. \end{aligned} \quad (23)$$

The first term in the current Eq. (23) is the usual $U(1)$ gauge interaction term, the second and third terms are related with the photon self-interaction, whereas the fourth term is a Yukawa interaction type. Aside from the integral dimensionality, the difference between the two- and three-dimensional solutions are the Green's functions. That for $(1+1)$ dimensions reads

$$\Delta_{\mu\sigma}^{\text{ret(adv)}}(x) = \int \frac{d^2 h}{(2\pi)^2} \left[\eta_{\mu\sigma} - \frac{h_\mu h_\sigma}{h^2} \right] \frac{e^{-ihx}}{h^2 \mp i\epsilon h_0}, \quad (24)$$

whereas for $(2+1)$ dimensions, it reads

$$\begin{aligned} \Delta_{\lambda\nu}^{\text{ret(adv)}}(x) &= \int \frac{d^3 h}{(2\pi)^3} \left[\eta_{\lambda\nu} - \frac{h_\lambda h_\nu}{h^2} - \frac{i\mu}{h^2} \epsilon_{\lambda\nu\alpha} h^\alpha \right] \\ &\quad \times \frac{e^{-ihx}}{h^2 - \mu^2 \mp i\epsilon h_0}. \end{aligned} \quad (25)$$

As we have succeeded in deriving the Lagrangian densities for two- and three-dimensional spacetime, and from them found the equations of motion and, subsequently, the Yang-Feldman-Källén solution for the (operator-valued) fields A_μ , ψ , and $\bar{\psi}$, we are now in position to calculate the spectral density functions for the contributions of one and two particles for the gauge field. But before such calculation, we will briefly review the Källén-Lehmann spectral representation.

III. KÄLLÉN-LEHMANN REPRESENTATION: EXACT PROPAGATORS

Once we have obtained the Yang-Feldman-Källén equations on the ordinary spacetime, Eqs. (19) and (22), the next step consists (different from the dispersion relations approach to noncommutative models [26]) of the investigation of the spectral functions following the ordinary Källén-Lehmann representation [24,29]. However, before that calculation, let us write some lines and describe the general prescription of the Källén-Lehmann representation. Let $A_\mu(x)$ be a vector field on the Heisenberg representation.⁴ The vacuum expectation value of the product of two fields at different points can be expressed as

$$\langle \Omega | A_\mu(x) A_\nu(y) | \Omega \rangle = \sum_n \langle \Omega | A_\mu(x) | n \rangle \langle n | A_\nu(y) | \Omega \rangle, \quad (26)$$

where the completeness relation of the physical spectrum has been used $\{|n\rangle\}$, i.e., $p_\mu |n\rangle = p_\mu^{(n)} |n\rangle$, and n represents all quantum numbers specifying a state. Based only on general arguments about invariance and the spectrum of p_μ , we will be able to determine the general expression of the exact photon propagator. Furthermore, we use the

⁴Here we will maintain a discussion of the vector fields embedded in a four-dimensional spacetime, but the derivation for other fields and dimensionality is rather direct.

translational invariance [3] of the theory $[A_\mu(x), p_\alpha] = i\partial_\alpha A_\mu(x)$, i.e., write $A_\mu(x) = e^{ip(x-x_0)} \times A_\mu(x_0) e^{-ip(x-x_0)}$, to obtain the following expression of the Wightman's function:

$$\langle \Omega | A_\mu(x) A_\nu(y) | \Omega \rangle = \int_0^\infty d\chi \rho_{\mu\nu}(\chi) \Delta^{(+)}(x-y; \chi), \quad (27)$$

where $\Delta^{(+)}$ is the positive frequency part of the Pauli-Jordan function. The theory's content, perturbative or nonperturbative, is fully encoded on the spectral density function $\rho_{\mu\nu}$. According to the Lorentz invariance, the dependence of ρ on p is only through p^2 , in a such way that allows one to write⁵

$$\rho_{\mu\nu}(q^2) \tau(q_0) = (2\pi)^3 \sum_n \delta^{(4)}(p^{(n)} - q) \times \langle \Omega | A_\mu(0) | n \rangle \langle n | A_\nu(0) | \Omega \rangle. \quad (28)$$

This quantity is null for $q^2 < 0$, and it is real and non-negative for $q^2 \geq 0$. In possess of the Wightman's function (27), one can construct any propagator of its interest. For instance, it follows that the exact Feynman propagator for the gauge field in the Källén-Lehmann spectral representation (27) is given by

$$iD_{\mu\nu}(k^2) = \int_0^\infty d\chi \rho_{\mu\nu}(\chi) \frac{1}{k^2 - \chi - i\epsilon}. \quad (29)$$

IV. SPECTRAL DENSITY FUNCTION: PERTURBATIVE EXAMPLE

After having obtained the Yang-Feldman-Källén equations of the dynamical fields of theory at two and three dimensions, and subsequently, as a brief review, derived the main points of the spectral density function for the gauge field and its relation with the Feynman propagator as well, we are in position to perform an explicit calculation for the

one- and two-particle contributions. Moreover, we will make no distinction between the usual and noncommutative contribution along our calculation, although it will be rather transparent in our resulting expressions, in such a way that the reader can follow how the noncommutative contribution accounts for the usual one. In order for unitarity to hold in the three-dimensional case, we will assume that only $\theta_{ij} \neq 0$ [6], while the time coordinate commutes with the space coordinates.⁶

A. (1 + 1)-dimensional case

On the Källén-Lehmann formulation, the quantity to be initially evaluated is the spectral density function (28). Moreover, because of its tensor structure and Lorentz invariance, $\rho_{\mu\nu}$ can be written as

$$\rho_{\mu\nu}(k^2) = \left(\eta_{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \rho(k^2), \quad (30)$$

where we have introduced the scalar spectral function ρ . The first example here lies in evaluating the one-particle contribution,

$$\frac{\rho_{\mu\nu}(k^2) \tau(k_0)}{2\pi} = \int d p_1 \delta^{(2)}(p_1 - k) \times \sum_j \langle \Omega | A_\mu(0) | p_1, j \rangle \langle p_1, j | A_\nu(0) | \Omega \rangle, \quad (31)$$

in which the one-particle state, a photon carrying momentum p_1 and polarization j , is constructed such as $|p_1, j\rangle = a_j^\dagger(p_1) | \Omega \rangle$. After some straightforward calculation, it follows that the scalar function ρ has the expression $\rho^{(0)}(k^2) = 2\delta(k^2)$. Now, for the two-particle contribution, we have to be cautious and pay attention to the theory's interaction structure. The general expression for such contribution is given by

$$\begin{aligned} \frac{\rho^{(1)}(k^2) \tau(k_0)}{2\pi} &= \int d p_1 d p_2 \delta^{(2)}(k - p_1 - p_2) \sum_{n,m} \langle \Omega | A_\mu(0) | p_1, n; p_2, m \rangle \langle p_1, n; p_2, m | A^\mu(0) | \Omega \rangle, \\ &= \int d p_1 d p_2 \int d^2 z d^2 w \delta^{(2)}(k - p_1 - p_2) \eta_{\mu\nu} \Delta^{\mu\sigma(\text{ret})}(-z) \Delta^{\nu\rho(\text{ret})}(-w) \\ &\quad \times \sum_{n,m} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, n; p_2, m \rangle \langle p_1, n; p_2, m | j_\rho^{\text{in}}(w) | \Omega \rangle. \end{aligned} \quad (32)$$

From the first to the second equality, we have made use of the Yang-Feldman-Källén equation (22), where (j^{in}) stands for the current (23) written in terms of the *infields* and $\Delta_{\mu\sigma}^{\text{ret}}$ is given by Eq. (24). Now, by analyzing the interaction structure of the current (23), we see that it

contains, besides the usual QED term, a Yukawa kind of term and a photon self-interaction as well, information that leads us to construct the two-particle state taking into account all the intermediate interaction in the

⁵Here, τ stands for the step function.

⁶Therefore, noncommutative quantum field theories in two-dimensional spacetime are not unitary.

following form:

$$\begin{aligned} \sum_{n,m} |p_1, n; p_2, m\rangle &= \sum_{i,j} a_i^\dagger(p_1) a_j^\dagger(p_2) |\Omega\rangle \\ &+ \sum_{r,s} d_r^\dagger(p_1) b_s^\dagger(p_2) |\Omega\rangle. \end{aligned} \quad (33)$$

The first term is characterized by two photons carrying momenta and polarization as (p_1, i) and (p_2, j) , whereas the second term corresponds to a fermion and antifermion pair characterized by their momenta and spin (p_1, r) and (p_2, s) , respectively. Therefore, from the above discussion, we find the following expression for the sum of the matrix elements:

$$\begin{aligned} \sum_{n,m} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, n; p_2, m \rangle \langle p_1, n; p_2, m | j_\rho^{\text{in}}(w) | \Omega \rangle \\ = \sum_{r,s} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, r; p_2, s \rangle \langle p_1, r; p_2, s | j_\rho^{\text{in}}(w) | \Omega \rangle \\ + \sum_{i,j} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, i; p_2, j \rangle \langle p_1, i; p_2, j | j_\rho^{\text{in}}(w) | \Omega \rangle. \end{aligned} \quad (34)$$

From the last result, we clearly see that we are left to calculate four matrix elements: two of them are the usual contribution while the others are related to the

noncommutative contribution. As an example, we shall evaluate the matrix elements related with the fermionic interaction. Making use of the explicit expression for the current, for instance, we have

$$\begin{aligned} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, r; p_2, s \rangle &= g \langle \Omega | \bar{\psi}^{\text{in}}(z) \gamma_\sigma \psi^{\text{in}}(z) | p_1, r; p_2, s \rangle \\ &- \frac{mg}{2} \langle \Omega | \tilde{\partial}_\sigma [\bar{\psi}^{\text{in}} \psi^{\text{in}}](z) | p_1, r; p_2, s \rangle, \end{aligned} \quad (35)$$

where we have introduced the notation $\tilde{a}^\alpha = a_\mu \theta^{\mu\alpha}$. Now, using the free solutions of the (operator-valued) fields [29], one obtains

$$\begin{aligned} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, r; p_2, s \rangle &= -\frac{mg}{(2\pi)} \sqrt{\frac{1}{E_{p_1} E_{p_2}}} e^{-iz(p_1+p_2)} \\ &\times \bar{v}(p_1, r) \left[\gamma_\sigma + \frac{im}{2} (\tilde{p}_1 + \tilde{p}_2)_\sigma \right] \\ &\times u(p_2, s), \end{aligned} \quad (36)$$

with $p_k = (E_{p_k}, \vec{p}_k)$ and $E_{p_k} = \sqrt{\vec{p}_k^2 + m^2}$. The remaining matrix elements can be evaluated in a similar fashion. It follows from the second term of Eq. (34), the matrix elements from the photon self-interaction contribution,

$$\begin{aligned} \rho_{\text{self-int}}^{(1)}(k^2) \tau(k_0) &= \int dp_1 dp_2 \int d^2z d^2w \delta^{(2)}(k - p_1 - p_2) \eta_{\mu\nu} \Delta^{\mu\sigma(\text{ret})}(-z) \Delta^{\nu\rho(\text{ret})}(-w) \\ &\times \sum_{i,j} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, i; p_2, j \rangle \langle p_1, i; p_2, j | j_\rho^{\text{in}}(w) | \Omega \rangle, \\ &= \frac{g^2}{4\pi} \frac{1}{(k^2)^2} \int d^2p \tau(p_0) \delta(p^2) \tau(k_0 - p_0) \delta((k - p)^2) \left[(p \circ p) - (k \circ k) - (k \circ p) + \frac{6}{k^2} (k \times p)^2 \right], \end{aligned} \quad (37)$$

where $(a \circ b) = a_\alpha \theta^{\alpha\lambda} \theta_{\lambda\sigma} b^\sigma$ and $(a \times b) = a_\alpha \theta^{\alpha\lambda} b_\lambda$. Nevertheless, from a straightforward computation of the remaining matrix elements, Eq. (34) is, thus, written as

$$\begin{aligned} \rho^{(1)}(k^2) \tau(k_0) &= \frac{g^2}{4\pi(k^2)^2} \left\{ 8m^2 \left[1 + \frac{(k \circ k)}{16} (k^2 - 4m^2) \right] \int d^2p \tau(p_0) \tau(k_0 - p_0) \delta(p^2 - m^2) \delta((k - p)^2 - m^2) \right. \\ &\left. + \int d^2p \tau(p_0) \delta(p^2) \tau(k_0 - p_0) \delta((k - p)^2) \left[(p \circ p) - (k \circ k) - (k \circ p) + \frac{6}{k^2} (k \times p)^2 \right] \right\}. \end{aligned} \quad (38)$$

Evaluating the momentum integral, we find the result for the propagator

$$iD(k^2) = \frac{2}{k^2} + \frac{g^2 m^2}{\pi k^2} \int_{4m^2}^{\infty} \frac{d\chi}{\chi^2 (k^2 - \chi - i\epsilon)} \frac{[1 + \frac{(k \circ k)}{16} (\chi - 4m^2)]}{\sqrt{1 - \frac{4m^2}{\chi}}} + \frac{g^2 \theta^2}{4\pi k^2} \int_{\lambda^2}^{\infty} \frac{d\chi}{\chi (k^2 - \chi - i\epsilon)}, \quad (39)$$

finally yielding the expression

$$iD(k^2) = \frac{2}{k^2} + \frac{g^2}{8\pi k^4} \left[(4 + k^2 \theta^2 m^2) - \frac{m^2 (16 - k^2 \theta^2 (k^2 - 4m^2))}{\sqrt{k^2 (4m^2 - k^2)}} \csc^{-1} \left[\frac{2m}{\sqrt{k^2}} \right] \right] + \frac{g^2 \theta^2}{4\pi k^4} \ln \left[1 - \frac{k^2}{\lambda^2} \right]. \quad (40)$$

It is the first term inside the brackets of Eq. (40) that, when the limit $m \rightarrow 0$ is taken and the vacuum polarization bubbles are summed, gives rise to the well-known Schwinger mass $m_A^2 = \frac{g^2}{\pi}$ of the photon [12]. In order to obtain a simplified expression of Eq. (40) and to compare with a previous result [21], we have made use of the fact that in a two-dimensional spacetime, the noncommutative matrix $\theta^{\mu\nu}$ can be expressed as $\theta^{\mu\nu} = \theta \epsilon^{\mu\nu}$, where $\epsilon^{\mu\nu}$ is the two-dimensional Levi-Civita tensor. Therefore, with this particular choice, we have obtained that the self-interaction contribution is

$$i\mathcal{D}_{\text{self-int}}^{(1)}(k^2) = \frac{g^2 \theta^2}{4\pi k^4} \ln \left[1 - \frac{k^2}{\lambda^2} \right]. \quad (41)$$

This result is in contrast with the one obtained previously in Ref. [21], where it was discussed, through one-loop diagram evaluation, that this contribution gives rise to a higher-order term (when summed into the complete propagator) dynamically generated by quantum corrections and that it is ultraviolet finite. Moreover, in Ref. [21], the propagator behavior in the infrared sector was not clear, which becomes clearly important and transparent in the framework of dispersion integrals. Furthermore, in order to make the integral infrared finite, we had to introduce a finite λ photon mass in Eq. (39), $\tau(k^2) \rightarrow \tau(k^2 - \lambda^2)$, therefore, showing that this term is, in fact, a purely infrared effect, as it is the mechanism behind the photon mass generation and that an analysis on that plays an important role in the correct interpretation of this term [19]. However, it is clear from the expression (41) that this contribution is not actually a higher-order term in any plausible limit when the smallness of λ is taken into account.

B. (2 + 1)-dimensional case

In the hope of learning useful things about the intriguing well-recognized four-dimensional problems, a lot of

attention has been paid to the analysis of general properties of the simple setting of three-dimensional field theories over the years [30], in particular, QED₃. For instance, Ref. [16] provided an unambiguous answer to the question about whether the dynamically generated photon mass is different from zero [14]. Now we revisit this issue in light of Ref. [16] but for a noncommutative theory looking at whether a noncommutative contribution is present. The calculation of the spectral density function on the QED₃ follows the same lines as presented above for the two-dimensional case. However, before starting the calculation, let us recall important points about the general structure of $\rho_{\mu\nu}$. It follows from the gauge and Lorentz invariance that $\rho_{\mu\nu}$ can be expressed as

$$\rho_{\mu\nu}(k^2) = \left(\eta_{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \rho_S(k^2) + i \epsilon_{\mu\nu\sigma} k^\sigma \rho_A(k^2), \quad (42)$$

where it was added the scalar functions ρ_i to the symmetric and antisymmetric sectors. Furthermore, these functions are determined as follows:

$$\rho_S(k^2) = \frac{1}{2} \rho_\mu^\mu(k^2), \quad (43)$$

$$\rho_A(k^2) = -\frac{i}{2k^2} k_\lambda \epsilon^{\mu\nu\lambda} \rho_{\mu\nu}(k^2). \quad (44)$$

For instance, for the one-particle contribution, one obtains from Eqs. (28) and (43) the following result for the symmetric form factor: $\rho_S^{(0)}(k^2) = \frac{3}{2} \delta(k^2 - \mu^2)$. Now, for the two-particle contribution, we make use again of the same arguments presented above to construct the intermediate state, which led to the expression (34). Therefore, by the same arguments, it follows that the spectral density function is given by

$$\begin{aligned} \frac{\rho^{(1)\mu\nu}(k^2)\tau(k_0)}{(2\pi)^2} &= \int d^2 p_1 d^2 p_2 \int d^3 z d^3 w \delta^{(3)}(k - p_1 - p_2) \Delta^{\mu\sigma(\text{ret})}(-z) \Delta^{\nu\rho(\text{ret})}(-w) \\ &\times \left\{ \sum_{r,s} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, r; p_2, s \rangle \langle p_1, r; p_2, s | j_\rho^{\text{in}}(w) | \Omega \rangle + \sum_{i,j} \langle \Omega | j_\sigma^{\text{in}}(z) | p_1, i; p_2, j \rangle \langle p_1, i; p_2, j | j_\rho^{\text{in}}(w) | \Omega \rangle \right\}. \quad (45) \end{aligned}$$

Where j is given by Eq. (23) and the retarded Green's function by Eq. (25). Next, by a straightforward but rather lengthy calculation on the matrix elements, one can obtain the following expression for the symmetric two-particle contribution (43):

$$\begin{aligned} \rho_S^{(1)}(k^2)\tau(k_0) &= \frac{g^2}{2(2\pi)^2} \frac{1}{(k^2 - \mu^2)} \left\{ \left[\left(1 + \frac{4m^2}{k^2} \right) + \frac{m^2}{4} (k \circ k) \left(1 - \frac{4m^2}{k^2} \right) \right] \right. \\ &\quad \times \int d^3 p \tau(p_0) \tau(k_0 - p_0) \delta(p^2 - m^2) \delta((k - p)^2 - m^2) \\ &\quad \left. - \int d^3 p \tau(p_0) \tau(k_0 - p_0) \delta(p^2) \delta((k - p)^2) [3(k \times p)^2 - k^2(k \circ k)] \right\}, \quad (46) \end{aligned}$$

whereas for the antisymmetric contribution (44), one finds

$$\rho_A^{(1)}(k^2)\tau(k_0) = -\frac{ig^2}{(2\pi)^2} \frac{1}{k^2 k^2 (k^2 - \mu^2)} \int d^3 p \tau(p_0) \tau(k_0 - p_0) \delta(p^2 - m^2) \delta((k - p)^2 - m^2) \times [2im^2 \epsilon_{\lambda\sigma\pi} p^\lambda \tilde{k}^\sigma k^\pi + mk^2(2i - (k \times p))]. \quad (47)$$

Therefore, it follows, by solving the remaining momentum integration, the explicit expressions

$$\rho_S^{(1)}(k^2) = \frac{\alpha}{4} \frac{1}{\sqrt{k^2}(k^2 - \mu^2)} \left[\frac{11k^2}{8} (k \circ k) \tau(k^2) + \left[\left(1 + \frac{4m^2}{k^2}\right) + \frac{m^2}{4} (k \circ k) \left(1 - \frac{4m^2}{k^2}\right) \right] \tau(k^2 - 4m^2) \right], \quad (48)$$

and

$$\rho_A^{(1)}(k^2) = \frac{\alpha}{k^2(k^2 - \mu^2)} \frac{m}{\sqrt{k^2}} \tau(k^2 - 4m^2), \quad (49)$$

where $\alpha = \frac{g^2}{4\pi}$. As said above, it is a well-known feature that in three dimensions the photon field acquires a non-null mass [16]; hence, we are interested in analyzing here whether this mass of photon is changed due to NC effects. By means of that, we consider the proper vacuum polarization insertions,

$$(\mathcal{D}^{-1})_{\mu\nu} = (D^{-1})_{\mu\nu} - i\Pi_{\mu\nu}, \quad (50)$$

where the free propagator in a general gauge parameter ξ is

$$(D^{-1})_{\mu\nu} = ik^2 \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \frac{i\mu}{k^2} \epsilon_{\mu\nu\lambda} k^\lambda + \frac{1}{\xi} \frac{k_\mu k_\nu}{k^2} \right), \quad (51)$$

whereas the vacuum polarization tensor is

$$\Pi_{\mu\nu}(k) = \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi_S(k^2) + i\epsilon_{\mu\nu\sigma} k^\sigma \Pi_A(k^2). \quad (52)$$

Furthermore, the scalar polarization functions Π_S and Π_A are related with the scalar spectral functions ρ_S and ρ_A by means of the relation

$$\Pi_{\mu\nu}(k) = \int_0^\infty d\chi \sigma_{\mu\nu}(\chi) \frac{1}{k^2 - \chi - i\epsilon}, \quad (53)$$

where the spectral function $\sigma_{\mu\nu}$ includes contributions of all loops to $\rho_{\mu\nu}$. In order to compute \mathcal{D} from Eq. (50), one can make use of the following set of orthogonal projection operators,

$$\begin{aligned} P_{\mu\nu}^{(1)} &= \frac{1}{2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + i\epsilon_{\mu\nu\lambda} \frac{k^\lambda}{\sqrt{k^2}} \right), \\ P_{\mu\nu}^{(2)} &= \frac{1}{2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - i\epsilon_{\mu\nu\lambda} \frac{k^\lambda}{\sqrt{k^2}} \right), \\ P_{\mu\nu}^{(3)} &= \frac{k_\mu k_\nu}{k^2}, \end{aligned} \quad (54)$$

to, therefore, obtain

$$\begin{aligned} \mathcal{D}^{-1} &= i \left(k^2 \left(1 + \frac{\mu}{\sqrt{k^2}} \right) - \Pi_S - \sqrt{k^2} \Pi_A \right) P^{(1)} \\ &+ i \left(k^2 \left(1 - \frac{\mu}{\sqrt{k^2}} \right) - \Pi_S + \sqrt{k^2} \Pi_A \right) P^{(2)} \\ &+ \frac{i}{\xi} k^2 P^{(3)}. \end{aligned} \quad (55)$$

Finally, after some algebraic manipulation, one can find

$$i\mathcal{D}_{\lambda\nu} = \frac{1}{k^2 - \Pi(k^2)} \left[\eta_{\lambda\nu} - \frac{k_\lambda k_\nu}{k^2} - \frac{i}{k^2} \epsilon_{\lambda\nu\sigma} k^\sigma \frac{\mu - \Pi_A}{1 - \frac{\Pi_S}{k^2}} \right], \quad (56)$$

for $\xi = 0$, where we have defined

$$\Pi(k^2) = \Pi_S + \frac{(\mu - \Pi_A)^2}{1 - \frac{\Pi_S}{k^2}}. \quad (57)$$

It follows that, in the second-order perturbation theory, we have the following expression for the photon mass

$$\Pi^{(1)}(0) = \Pi_S^{(1)}(0) + \frac{(\mu - \Pi_A^{(1)}(0))^2}{1 - \frac{\Pi_S^{(1)}(0)}{k^2}}. \quad (58)$$

Now, by calculating the functions $\Pi_S^{(1)}$ and $\Pi_A^{(1)}$ through Eqs. (53), (48), and (49) evaluating the integration in χ for the region of interest $k^2 < 4m^2$, and, at last, expanding the expressions to $k^2 \rightarrow 0$, we find

$$\lim_{k^2 \rightarrow 0} \frac{\Pi_S^{(1)}(k^2)}{k^2} = \frac{\alpha}{8\mu^3} \left\{ 2m\mu(4 - m^2(k \circ k)) + [(k \circ k)m^2(4m^2 - \mu^2) - 4(\mu^2 + 4m^2)] \coth^{-1} \left[\frac{2m}{\mu} \right] \right\}, \quad (59)$$

and

$$\lim_{k^2 \rightarrow 0} \Pi_A^{(1)}(k^2) = -\frac{g^2}{4\pi} \frac{2m}{\mu} \coth^{-1} \left[\frac{2m}{\mu} \right]. \quad (60)$$

In order to trace a parallel with the known result for QED [16], we take the limit $\mu \rightarrow 0$ in the above expressions (59) and (60). It follows, therefore, a non-null mass for the photon field

$$\Pi(0) = \frac{\alpha^2}{1 + \frac{\alpha}{12m} [8 + m^2(k \circ k)]} \neq 0. \quad (61)$$

Such expression reproduces the known result from QED₃ for $k_{nc}^2 = (k \circ k) \rightarrow 0$. In QED₃ there is a proof in Ref. [31] where it was shown that all contributions to the mass from other graphs vanish identically; therefore, it is plausible to say for QED₃ that a nonperturbative mass is dynamically generated. However, there is no known proof (or discussion) of the noncommutative QED₃ counterpart, which makes it impossible for us to say anything about whether or not the above result (61) is nonperturbative in nature.

V. CONCLUDING REMARKS

This paper presents a study of the two- and three-dimensional NC quantum electrodynamics in light of the Källén-Lehmann spectral representation. Our main interest here is studying NC extensions of well-known quantum field theories to look for NC effects on its deviations. These two models QED₂ and QED₃ were extensively studied in several areas of theoretical physics and have long been recognized as laboratories of testing new ideas on a simpler setting, especially on condensed matter and statistical systems, for instance, the quantum Hall effect.

We begin by discussing NC quantum electrodynamics defined on an ω -dimensional spacetime. After discussing some of the symmetries present on the action, we make use of the Seiberg-Witten map to determine an action. Interesting features are present on that; for instance, it is due to the noncommutativity of spacetime that there are nonlinear terms on the field strength. Furthermore, for $\omega > 2$, this action also provides a suitable framework for studies on Lorentz-violating extension. Next, restricting our attention to the cases of two and three dimensions, it is possible to obtain two Lagrangian functions from which we derive the respective equations of motion for the gauge and fermionic fields. Furthermore, based on the Yang-Feldman-Källén formalism,

it is possible to find solutions to the Heisenberg operator equations of motion— solutions that are an important ingredient of the Källén-Lehmann representation.

On the Källén-Lehmann representation, we first introduce the spectral density function for the gauge field followed by a discussion on its general properties and present, finally, its relation with the exact Feynman propagators as well. The main aim of the present paper is to evaluate the one- and two-particle contribution to the gauge field spectral function in two and three dimensions. Based on that, for the photon field, after calculating the two-particle contribution (g^2 order), we obtain in two dimensions for the self-interaction contribution an interesting result which differs from a previous one [21], where it states that this same contribution gives rise to a higher-derivative term. The main difference we show here is that this term depends explicitly on the infrared parameter, a photon mass λ , and in any plausible limit taking into account the smallness of λ , this term does not generate a higher-order term as stated in Ref. [21]. This consists of an analysis that was not described earlier and clearly plays an important role in order to obtain a consistent interpretation and correct expression arising from the infrared sector, and it is automatically fulfilled in the dispersion relation calculation. Furthermore, based on the well-known result in three dimensions, where a dynamical mass for the photon is generated from radiative calculation, we derive the relation between the self-energy function with the propagator pole in order to see whether and how the noncommutativity affects the value of the photon mass.

For different reasons, much interest is still present in studying lower-dimensional field theory in the most different quantization process, since they always provide a rich testing ground to study the most diverse ideas of more realistic systems, especially in the well-recognized four-dimensional problems. We still have interest in studying some thermodynamical features of these systems [32], in particular, three-dimensional NCQED. Moreover, nonrelativistic field theories around Lifshitz points [33] with anisotropic scaling between space and time have come to our attention and certainly deserve to be discussed in lower-dimensional theories carefully and in further detail [34]. These issues and others will be further elaborated, investigated, and reported elsewhere.

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APPENDIX A: NOTATION AND IDENTITIES

In this appendix, we fix our notation and review some useful identities for two- and three-dimensional spacetime. First, for two dimensions, our convention for the metric

and the γ -matrices representation are

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\gamma_0 = \sigma_1, \quad \gamma_1 = -i\sigma_2, \quad \gamma_5 = \gamma_0\gamma_1 = \sigma_3,$$

where $\{\sigma_i\}$ are the Pauli matrices. We have

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu}, \quad [\gamma_\mu, \gamma_\nu] = -2\epsilon_{\mu\nu}\gamma_5, \quad (\text{A1})$$

from which it follows $\gamma_\mu\gamma_5 = \epsilon_{\mu\nu}\gamma^\nu$ and $\epsilon_{\mu\nu}\epsilon^{\alpha\beta} = \delta_\mu^\alpha\delta_\nu^\beta - \delta_\mu^\beta\delta_\nu^\alpha$. Moreover, for the three-dimensional case, we have

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and

$$\gamma_0 = \sigma_3, \quad \gamma_1 = i\sigma_1, \quad \gamma_2 = i\sigma_2.$$

We also have the algebra

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu}, \quad [\gamma_\mu, \gamma_\nu] = -2i\epsilon_{\mu\nu\lambda}\gamma^\lambda, \quad (\text{A2})$$

and some relevant properties

$$\gamma_\mu\gamma^\mu = 3, \quad \epsilon_{\alpha\beta\lambda}\epsilon^{\alpha\beta\lambda} = 3!, \quad (\text{A3})$$

$$\text{Tr}(\gamma_\mu) = 0, \quad \text{Tr}(\gamma_\mu\gamma_\nu) = 2\eta_{\mu\nu}, \quad \text{Tr}(\gamma_\mu\gamma_\nu\gamma_\sigma) = -2i\epsilon_{\mu\nu\sigma}. \quad (\text{A4})$$

APPENDIX B: INTEGRAL CALCULATION

In this appendix, we provide a calculation of some relevant integrals. We have here three types of dispersion integrals: a scalar, vector, and tensor type. First, the two-dimensional integrals,

$$I^{(1)}(k) = \int d^2 p \tau(p_0)\tau(k_0 - p_0) \times \delta(p^2 - m^2)\delta((k - p)^2 - m^2), \quad (\text{B1})$$

for a timelike k , we can go to a Lorentz frame where $k = (k_0, 0)$. In this frame follows

$$I^{(1)}(k_0) = \int \frac{dp}{2E_p} \tau(k_0 - E_p)\delta(k_0^2 - 2k_0E_p), \quad (\text{B2})$$

where

$$E_p = \sqrt{\mathbf{p}^2 + m^2} = \frac{k_0}{2},$$

which implies $|\mathbf{p}| = \sqrt{k_0^2/4 - m^2}$. Therefore, returning to an arbitrary Lorentz frame, it follows that

$$I^{(1)}(k) = \frac{1}{4k^2} \frac{1}{\sqrt{1 - \frac{4m^2}{k^2}}} \tau(k^2 - 4m^2)\tau(k_0). \quad (\text{B3})$$

Another relevant two-dimensional integral is

$$I^{(2)}(p) = \int d^2 k \tau(k_0)\tau(p_0 - k_0)\delta(k^2 - m^2)\delta((p - k)^2), \quad (\text{B4})$$

and

$$= \int \frac{dk}{2E_k} \tau(p_0 - E_k)\delta(p_0^2 - 2p_0E_k + m^2),$$

where we had taken again a timelike p in the form $p = (p_0, 0)$, and from the above expression follows $|\mathbf{k}| = \frac{p_0^2 - m^2}{2p_0}$ and for $p_0 > 0$, $p_0^2 - m^2 > 0$. Therefore,

$$= \frac{1}{4|p_0|} \tau(p_0^2 - m^2)\tau(p_0) \int dE_k \frac{\delta(\frac{p_0^2 + m^2}{2p_0} - E_k)}{\sqrt{E_k^2 - m^2}}.$$

At last,

$$I^{(2)}(p) = \frac{1}{2} \frac{1}{\sqrt{(p^2 - m^2)^2}} \tau(p^2 - m^2)\tau(p_0). \quad (\text{B5})$$

Moreover, for three dimensions, the scalar integrals are obtained following the same steps as before. We evaluate, thus, a vector and tensor integral for three dimensions. First,

$$I_\mu^{(3)}(k) = \int d^3 p \tau(p_0)\tau(k_0 - p_0) \times \delta(p^2 - m^2)\delta((k - p)^2 - m^2)p_\mu, \quad (\text{B6})$$

and it follows for $\mu \neq 0$ that $I_\mu^{(3)} = 0$ by symmetry. Thus, we have

$$I_\mu^{(3)}(k) = \frac{\pi k_\mu}{4\sqrt{k^2}} \tau(k^2 - 4m^2)\tau(k_0). \quad (\text{B7})$$

It also follows, without further complication, that

$$I_\mu^{(4)}(p) = \int d^3 k \tau(k_0)\tau(p_0 - k_0) \times \delta(k^2 - m^2)\delta((p - k)^2)k_\mu, \\ = \frac{\pi p_\mu}{4\sqrt{p^2}} \left(1 + \frac{m^2}{p^2}\right) \tau(p^2 - m^2)\tau(p_0). \quad (\text{B8})$$

Now, the tensor integral,

$$\begin{aligned}
 I_{\mu\nu}^{(5)}(p) &= \int d^3q \tau(q_0) \tau(p_0 - q_0) \delta(q^2) \delta((p - q)^2) q_\mu q_\nu \\
 &= \left(-\eta_{\mu\nu} + 3 \frac{p_\mu p_\nu}{p^2} \right) \frac{\pi \sqrt{p^2}}{16} \tau(p^2) \tau(p_0). \quad (\text{B9})
 \end{aligned}$$

As remarked in the beginning of this appendix, we listed and evaluated explicitly here some relevant dispersion integrals. Moreover, it is worth of saying that the remaining integrals are simpler and follow the same lines.

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- [1] R. J. Szabo, *Phys. Rep.* **378**, 207 (2003).
- [2] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane, and T. Okamoto, *Phys. Rev. Lett.* **87**, 141601 (2001); A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda, and R. Wulkenhaar, *Eur. Phys. J. C* **24**, 165 (2002); L. Álvarez-Gaumé and M. A. Vázquez-Mozo, *Nucl. Phys. B* **668**, 293 (2003); M. Chaichian, P. P. Kulishb, K. Nishijima, and A. Tureanu, *Phys. Lett. B* **604**, 98 (2004); M. Chaichian, P. Prešnajder, and A. Tureanu, *Phys. Rev. Lett.* **94**, 151602 (2005); A. Tureanu, *Phys. Lett. B* **638**, 296 (2006).
- [3] R. Banerjee, B. Chakraborty, and K. Kumar, *Phys. Rev. D* **70**, 125004 (2004).
- [4] N. Seiberg and E. Witten, *J. High Energy Phys.* 09 (1999) 032.
- [5] S. Minwalla, M. van Raamsdonk, and N. Seiberg, *J. High Energy Phys.* 02 (2000) 020.
- [6] J. Gomis and T. Mehen, *Nucl. Phys. B* **591**, 265 (2000); D.-W. Chiou and O. J. Ganor, *J. High Energy Phys.* 03 (2004) 050.
- [7] N. Seiberg, L. Susskind, and N. Toumbas, *J. High Energy Phys.* 06 (2000) 044; S. Ghosh, *Mod. Phys. Lett. A* **19**, 2505 (2004).
- [8] D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli, *Phys. Lett. B* **533**, 178 (2002).
- [9] N. Ishibashi, S. Iso, H. Kawai, and Y. Kitazawa, *Nucl. Phys. B* **573**, 573 (2000); I. Y. Arefeva, D. M. Belov, and A. S. Koshelev, *Phys. Lett. B* **476**, 431 (2000); M. Hayakawa, *Phys. Lett. B* **478**, 394 (2000); arXiv:hep-th/9912167; J. L. Hewett, F. J. Petriello, and T. G. Rizzo, *Phys. Rev. D* **64**, 075012 (2001); A. P. Balachandran, P. Padmanabhan, and A. R. de Queiroz, *Phys. Rev. D* **84**, 065020 (2011).
- [10] K. Ülker and B. Yapiskan, *Phys. Rev. D* **77**, 065006 (2008).
- [11] J. Madore, S. Schraml, P. Schupp, and J. Wess, *Eur. Phys. J. C* **16**, 161 (2000); B. Jurco, S. Schraml P. Schupp, and J. Wess, *Eur. Phys. J. C* **17**, 521 (2000); B. Jurco, L. Moller, S. Schraml, P. Schupp, and J. Wess, *Eur. Phys. J. C* **21**, 383 (2001).
- [12] J. Schwinger, *Phys. Rev.* **128**, 2425 (1962).
- [13] J. H. Lowenstein and J. A. Swieca, *Ann. Phys. (N.Y.)* **68**, 172 (1971).
- [14] S. Deser, R. Jackiw, and S. Templeton, *Ann. Phys. (N.Y.)* **140**, 372 (1982); J. Schonfeld, *Nucl. Phys. B* **185**, 157 (1981).
- [15] M. de Roo and K. Stam, *Nucl. Phys. B* **246**, 335 (1984); Y. Hoshino, *Nucl. Phys. A* **790**, 614c (2007).
- [16] G. Scharf, W. F. Wreszinski, B. M. Pimentel, and J. L. Tomazelli, *Ann. Phys. (N.Y.)* **231**, 185 (1994); B. M. Pimentel and J. L. Tomazelli, *Prog. Theor. Phys.* **95**, 1217 (1996).
- [17] S. Deser and R. Jackiw, *Phys. Lett. B* **451**, 73 (1999); V. S. Alves, B. Charneski, M. Gomes, L. Nascimento, and F. Peña, *Phys. Rev. D* **88**, 067703 (2013).
- [18] I. J. R. Aitchison, N. E. Mavromatos, and D. McNeill, *Phys. Lett. B* **402**, 154 (1997); M. Lavelle and Z. Mazumder, *Phys. Lett. B* **596**, 70 (2004); P. M. Lo and E. S. Swanson, *Phys. Lett. B* **697**, 164 (2011).
- [19] A. K. Das, J. Frenkel, and C. Schubert, *Phys. Lett. B* **720**, 414 (2013); O. M. Del Cima, D. H. T. Franco, and O. Piguet, *Phys. Rev. D* **89**, 065001 (2014).
- [20] R. Bamarjee and S. Ghosh, *Phys. Lett. B* **533**, 162 (2002); A. Saha, A. Rahaman, and P. Mukherjee, *Phys. Lett. B* **638**, 292 (2006); *Mod. Phys. Lett. A* **23**, 2947 (2008); M. Ghasemkhani and N. Sadooghi, *Phys. Rev. D* **81**, 045014 (2010); F. Ardalan, M. Ghasemkhani, and N. Sadooghi, *Eur. Phys. J. C* **71**, 1606 (2011); A. C. Lehum, J. R. Nascimento, and A. Yu. Petrov, *Phys. Rev. D* **86**, 025011 (2012).
- [21] K. M. Keita, F. Wu, and M. Zhong, *Phys. Lett. B* **681**, 367 (2009).
- [22] K. Kaminsky, Y. Okawa, and H. Ooguri, *Nucl. Phys. B* **663**, 33 (2003); S. Ghosh, *Phys. Lett. B* **583**, 347 (2004); *Mod. Phys. Lett. A* **20**, 1227 (2005).
- [23] N. Grandi and G. A. Silva, *Phys. Lett. B* **507**, 345 (2001).
- [24] G. Källén, *Helv. Phys. Acta* **25**, 417 (1952); H. Lehmann, *Nuovo Cimento* **11**, 342 (1954).
- [25] C. N. Yang and D. Feldman, *Phys. Rev.* **79**, 972 (1950); G. Källén, *Quantum Electrodynamics* (Springer-Verlag, New York, 1973), 1st ed.
- [26] Y. Liao and K. Sibold, *Phys. Lett. B* **549**, 352 (2002); M. Chaichian, M. N. Mnatsakanova, A. Tureanu, and Y. S. Vernov, *Nucl. Phys. B* **673**, 476 (2003); J. Zahn, *Phys. Rev. D* **82**, 105033 (2010).
- [27] R. Potting, *Phys. Rev. D* **85**, 045033 (2012).
- [28] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda and R. Wulkenhaar, arXiv:hep-th/0102103.
- [29] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965); S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1962).
- [30] R. Jackiw, in *Fields and Particles*, edited by H. Mitter and W. Schweiger (Springer-Verlag, Berlin, 1990), pp. 117–169; Republished in *Nucl. Phys. B, Proc. Suppl.* **18**, 107 (1990).
- [31] S. Coleman and B. Hill, *Phys. Lett.* **159B**, 184 (1985); T. Lee, *Phys. Lett. B* **171**, 247 (1986).

- [32] F. T. Brandt, A. Das, J. Frenkel, S. Pereira, and J. C. Taylor, *Phys. Rev. D* **67**, 105010 (2003); F. T. Brandt, J. Frenkel, and C. Muramoto, *Nucl. Phys.* **B754**, 146 (2006); C. D. Fosco and G. A. Silva, *J. High Energy Phys.* 01 (2008) 011; H.-T. Feng, M. He, W.-M. Sun, and H.-S. Zong, *Phys. Lett. B* **688**, 178 (2010).
- [33] M. Visser, *Phys. Rev. D* **80**, 025011 (2009); B. Chen and Q. G. Huang, *Phys. Lett. B* **683**, 108 (2010); P. Hořava, *Phys. Lett. B* **694**, 172 (2010); J. Alexandre, *Int. J. Mod. Phys. A* **26**, 4523 (2011), and references therein.
- [34] M. Eune, W. Kim, and E. J. Son, *Eur. Phys. J. C* **71**, 1840 (2011); E. J. Son and W. Kim, *Mod. Phys. Lett. A* **26**, 2755 (2011).