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Motion of a mirror under infinitely fluctuating quantum vacuum stress

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The actual value of the quantum vacuum energy density is generally regarded as irrelevant in nongravitational physics. However, this paper presents a nongravitational system where this value does have physical significance. The system is a mirror with an internal degree of freedom that interacts with a scalar field. We find that the force exerted on the mirror by the field vacuum undergoes wild fluctuations with a magnitude proportional to the value of the vacuum energy density, which is mathematically infinite. This infinite fluctuating force gives infinite instantaneous acceleration of the mirror. We show that this infinite fluctuating force and infinite instantaneous acceleration make sense because they will not result in infinite fluctuation of the mirror's position. On the contrary, the mirror's fluctuating motion will be confined in a small region due to two special properties of the quantum vacuum: (1) the vacuum friction that resists the mirror's motion and (2) the strong anticorrelation of vacuum fluctuations that constantly changes the direction of the mirror's infinite instantaneous acceleration and thus cancels the effect of infinities to make the fluctuation of the mirror's position finite.

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I. INTRODUCTION

In quantum field theory, the vacuum, which is defined as the state of lowest possible energy, is not really empty. Its energy is not zero but infinite since it is associated with the zero-point fluctuations of an infinite number of quantum harmonic oscillators. On the one hand, it is generally accepted that these zero-point fluctuations really exist in nature [1] since their physical effects can be experimentally observed in various phenomena such as the spontaneous emission [2], the Lamb shift [3], and the Casimir effect [4]. On the other hand, the infinite value of the vacuum energy is generally regarded as irrelevant since experiments measure only energy differences from the ground state. For example, the Casimir effect, which is the small attractive force between two close parallel uncharged conducting plates, happens because the Casimir vacuum energy density decreases as the plates are moved closer, or, in other words, it comes from a difference of vacuum energies and in practical calculations the infinities cancel.

Nevertheless, the quantum vacuum never stops astonishing us [5]. For example, when it comes to gravity, the actual value of energy matters, not only the difference. According to the principle of general relativity, the energy momentum tensor is the source of the gravitational field. So it is expected that the nonzero vacuum energy will contribute to the cosmological constant, which explains the accelerated expansion of the Universe. Unfortunately, as we stated before, the vacuum energy is mathematically infinite without renormalization and thus would cause a huge cosmological constant for a cutoff at the Planck scale, which disagrees with the tiny measured cosmological constant by a factor of 10^{120} [6]. This discrepancy has been called "the worst theoretical prediction in the history of physics" [7]!

It is generally accepted that the actual value of the vacuum energy matters only when taking gravity into account; otherwise one can only measure the energy differences. However, in this paper, we present a nongravitational physical system where the infinities like that of the vacuum energy do matter. The system is a mirror with an internal harmonic oscillator coupled to a real scalar field in $1+1$ dimension. We find that the fluctuations of the force exerted on the mirror by the field are proportional to the infinite value of the quantum vacuum energy of the scalar field. This infinite force fluctuation leads to infinite instantaneous acceleration of the mirror. However, unlike the vacuum catastrophe in the cosmological constant problem, it is shown that this infinite fluctuating force makes sense because it will not result in infinite fluctuation of the mirror's position. On the contrary, the mirror's fluctuating motion will be confined in a small region due to two special properties of the quantum vacuum: the vacuum friction and the strong anticorrelation of vacuum fluctuations. More precisely, this comes about because (1) there exists vacuum friction (also infinite but with much lower order divergence) to resist the mirror's motion and (2) the force is strongly anticorrelated in time and the time average of the force has finite fluctuations. Then, although the instantaneous acceleration is infinite, it also keeps changing direction, which strongly cancels the effect of infinities and makes the fluctuation of the mirror's position finite.

This paper is organized as follows. In Sec. II, we introduce our special mirror model and explain how it works in detail. In Sec. III, we calculate the force acting on

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the mirror by the field and its fluctuation. The infinite fluctuations of this force, which are proportional to the value of the vacuum energy, are given. In Sec. IV, we calculate the fluctuation of the time average of the force, and find a finite result, which is an indication that our mirror's fluctuating motion under the infinitely fluctuating force might be finite. In Sec. V, we examine the frictional force acting on the mirror due to radiation reaction. In Sec. VI, we derive the mirror's equation of motion. In Sec. VII, we calculate the fluctuating motion of the mirror and show that it is confined to a small region. In Sec. VIII, we compare our mirror's fluctuating motion with Brownian motion and indicate the intrinsic differences between them. In Sec. IX, we discuss our results and compare them with other related works.

Units are chosen throughout such that $c = \hbar = 1$.

II. OUR MIRROR MODEL

A mirror is an object that reflects light. In classical electrodynamics, light waves incident on a material induce small oscillations of the individual particles, for example, electrons in glass, causing each particle to radiate a small secondary wave. All these waves add up together to give reflected and refracted waves. We shall study the case of a mirror that is interacting with a massless scalar field. One often uses a perfectly reflecting boundary as a mirror model, i.e., the mirror reflects all wave modes with arbitrarily high frequencies, by imposing the boundary condition that the scalar field vanishes on the surface of the mirror [Fulling and Davies [8], Eq. (2.3); Berrell and Davies [9], Eq. (4.43)]:

$$\phi[t, X(t)] = 0, \tag{1}$$

where $X(t)$ is the trajectory of the mirror. However, a realistic mirror becomes transparent gradually for high frequency wave modes. Some authors [10,11] add an artificial frequency cutoff by assuming that modes of the quantum field ϕ with frequencies higher than a specific value are unaffected by the mirror. In this paper, we will not adopt this model. Instead, we will adopt a mirror model in which the transparency for high frequency wave modes appears in a natural way.

In our model, the oscillating particle inside the mirror is an harmonic oscillator with a natural frequency Ω. We consider a $1 + 1$ dimensional static mirror with an internal dynamic degree of freedom q coupled to a scalar field ϕ. The mirror is located at the position $x = 0$ in the space of the scalar field. The total action is given by

$$\begin{aligned} S = &\frac{1}{2}\iint \left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right) dt dx \\ &+ \frac{1}{2}\int \left(\left(\frac{dq}{dt}\right)^2 - \Omega^2 q^2\right) dt + \epsilon\int \frac{d\phi(t,0)}{dt} q(t) dt, \end{aligned} \tag{2}$$

where ϵ is the coupling constant. Here it is necessary to point out that the harmonic oscillator q is not oscillating "in space"; it is an "internal" degree of freedom, i.e., a zero-dimensional quantum field inside the mirror.

Varying the action (2) with respect to ϕ and q leads to the Heisenberg equations of motion for the field ϕ and the internal degree of freedom q:

$$\ddot{\phi} - \phi'' = -\epsilon \dot{q} \delta(x), \tag{3}$$

$$\ddot{q} + \Omega^2 q = \epsilon \dot{\phi}(t, 0), \tag{4}$$

where the dot $\dot{}$ denotes the time derivative and the prime $'$ denotes the spatial derivative. The solution of (3) is of the following form:

$$\phi(t, x) = \phi_0(t, x) - \frac{\epsilon}{2} q(t - |x|), \tag{5}$$

where $\phi_0(t, x)$ is the solution of the homogeneous equation

$$\ddot{\phi}_0 - \phi_0'' = 0. \tag{6}$$

One can easily check that (5) is the solution by noticing that

$$\begin{aligned} q''(t - |x|) &= -\frac{d}{dx}(\dot{q}(t - |x|)\mathrm{sgn}(x)) \\ &= \ddot{q}(t - |x|) - 2\dot{q}(t)\delta(x), \end{aligned} \tag{7}$$

where the sign function $\mathrm{sgn}(x)$ is defined as

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Substituting (5) into the equation of motion for the internal oscillator (4) gives

$$\ddot{q} + \frac{\epsilon^2}{2}\dot{q} + \Omega^2 q = \epsilon \dot{\phi}_0(t, 0). \tag{8}$$

This is exactly an equation of motion for a driven damped harmonic oscillator with the natural frequency Ω, the damping coefficient $\frac{\epsilon^2}{2}$, and the driving force $\epsilon\dot{\phi}_0(t, 0)$.

In order to give a clear picture about how the mirror works, we divide the incoming field ϕ_0 into the right moving part and the left moving part:

$$\phi_0 = \phi_0^R + \phi_0^L, \tag{9}$$

where ϕ_0^R is the form of $f(t - x)$ and ϕ_0^L is the form of $g(t + x)$ according to d'Alembert's solution. This solution has the properties

$$\dot{\phi}_0^R = -\phi_0'^R, \tag{10}$$

$$\dot{\phi}_0^L = \phi_0'^L, \tag{11}$$

which are useful in our later calculations. Since (3) and (4) are liner equations, the internal degree of freedom q can also be divided into two parts correspondingly:

$$q = q^R + q^L, \tag{12}$$

and the pairs (ϕ_0^R, q^R) and (ϕ_0^L, q^L) both obey the same equations, (6) and (8). The solution (5) gives us a picture about how the mirror reflects waves. As shown in Fig. 1, the right moving wave $\phi_0^R(t,x)$ is incident on the mirror from left. The mirror reflects a wave $-\frac{\epsilon}{2}q^R(t+x)$ to the left and lets a wave $\phi_0^R - \frac{\epsilon}{2}q^R(t-x)$ pass through to the right. The mirror reflects the left moving wave ϕ_0^L in exactly the same way by just doing a "mirror reflection" in Fig. 1.

Next, to understand the working mechanism of the mirror in detail, we analyze the energy flows during the reflection process by using the following formula for energy flux:

$$T^{01}(t,x) = -\{\dot{\phi}(t,x)\phi'(t,x)\}, \tag{13}$$

where T^{01} is the time-space component of the type (2,0) stress-energy tensor of the field ϕ and the curly brackets $\{\}$ represent the symmetrization operation, which is defined as

$$\{AB\} = \frac{1}{2}(AB + BA), \tag{14}$$

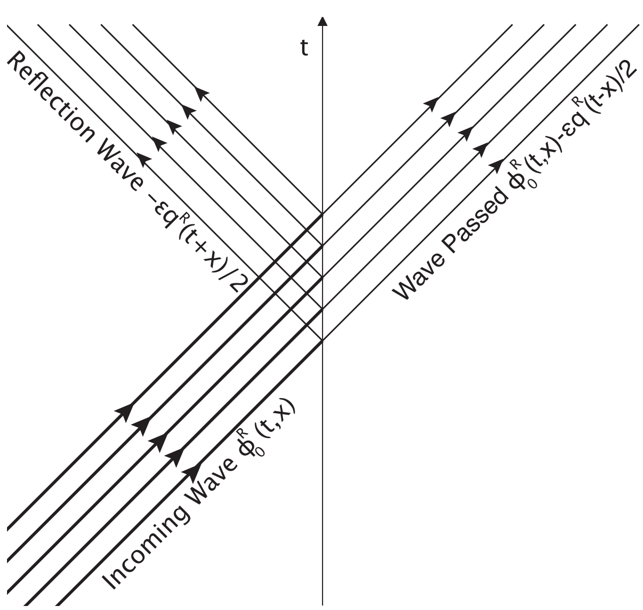

FIG. 1. A figure shows how our mirror works for the right moving wave ϕ_0^R. When it is incident on the mirror from left, it induces oscillations of the internal harmonic oscillator. Then the oscillator radiated a secondary wave $q^R(t-|x|)$ in both direction equally. For the left moving wave ϕ_0^L, the mirror works exactly the same way due to symmetry.

for any two operators A and B. This is irrelevant for classical quantities but will be important later for quantum operators. For simplicity, we first consider the case that only the right moving wave ϕ_0^R exists. Then the energy flux near the left side of the mirror is

$$\begin{aligned}&\lim_{x\to 0^-}\left\{-\left(\phi_0^R(t,x)-\frac{\epsilon}{2}q^R(t+x)\right)^{\cdot}\cdot\left(\phi_0^R(t,x)-\frac{\epsilon}{2}q^R(t+x)\right)'\right\}\\&\quad=\left\{(-\dot{\phi}_0^R\phi'^R_0(t,0))-\left(\frac{\epsilon^2}{4}(\dot{q}^R)^2(t)\right)\right\},\end{aligned} \tag{15}$$

where we have used (10) to eliminate the interference terms. The energy flux near the right side of the mirror is

$$\begin{aligned}&\lim_{x\to 0^+}\left\{-\left(\phi_0^R(t,x)-\frac{\epsilon}{2}q^R(t-x)\right)^{\cdot}\cdot\left(\phi_0^R(t,x)-\frac{\epsilon}{2}q^R(t-x)\right)'\right\}\\&\quad=\left\{(-\dot{\phi}_0^R\phi'^R_0(t,0)-\epsilon\dot{\phi}_0^R(t,0)\dot{q}^R(t))+\left(\frac{\epsilon^2}{4}(\dot{q}^R)^2(t)\right)\right\},\end{aligned} \tag{16}$$

where we have again used (10). The first term $-\dot{\phi}_0^R\phi'^R_0$ inside the parentheses of (15) represents energy that impinges on the mirror from the left per unit of time. The first term, $-\dot{\phi}_0^R\phi'^R_0 - \epsilon\dot{\phi}_0^R\dot{q}^R$, inside the parentheses of (16) represents energy that directly passed through the mirror per unit of time. The second term, $\frac{\epsilon^2}{4}(\dot{q}^R)^2$, inside the parentheses of (15) represents energy radiated to the left per unit of time by the internal harmonic oscillator, which creates the reflective power of the mirror. The same term, $\frac{\epsilon^2}{4}(\dot{q}^R)^2$, inside the parentheses of (16) represents energy radiated to the right per unit of time by the internal harmonic oscillator. The radiated energy to the left and to the right per unit of time add together to give the total radiating power $\frac{\epsilon^2}{2}(\dot{q}^R)^2$. This radiating power is pumped from the incoming wave ϕ_0^R with the pumping power $\epsilon\dot{\phi}_0^R\dot{q}^R$, which is just the difference between the incoming energy flux $-\dot{\phi}_0^R\phi'^R_0$ and the energy flux directly passed through the mirror $-\dot{\phi}_0^R\phi'^R_0 - \epsilon\dot{\phi}_0^R\dot{q}^R$. This reflection process is illustrated in Fig. 2.

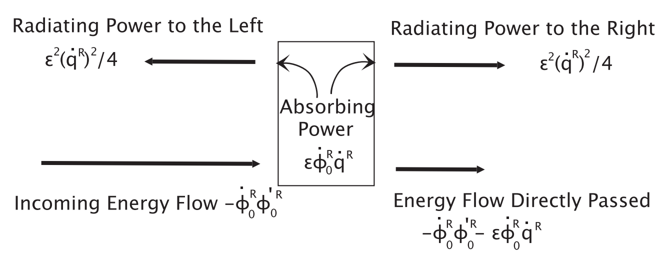

FIG. 2. A figure shows that the incoming filed ϕ_0 is impinging on the mirror. Part of the field energy is absorbed by the internal harmonic oscillator with absorbing power $\epsilon\dot{\phi}_0^R\dot{q}^R$. At the same time, it is radiating energy out with the total power $\frac{\epsilon^2}{2}(\dot{q}^R)^2$.

The reason why the mirror works this way is clear. In fact, note that the internal harmonic oscillator behaves according to the driven damped harmonic oscillation equation (8). From this equation, we notice that the pumping power $\epsilon\dot{\phi}_0^R\dot{q}^R$, which is the "driving force" $\epsilon\dot{\phi}_0^R$ times the "internal velocity" $\dot{q}^R$, is exactly the absorbing power from the external driving force. This absorbing power is dissipating by the damping force due to the radiation. The dissipated power is the "damping force" $\frac{\epsilon^2}{2}\dot{q}^R$ times the internal velocity $\dot{q}^R$, which is exactly equal to the total radiating power $\frac{\epsilon^2}{2}(\dot{q}^R)^2$. So the energy radiated acts as damping on the internal harmonic oscillator.

In summary, the working mechanism of the mirror is that when the wave incidents on the mirror, part of its energy is used to drive the oscillations of the internal harmonic oscillator; at the same time, the internal harmonic oscillator radiates the absorbed energy out equally to both directions. That energy radiated back forms the reflected waves.

The mirror works the same way when considering that the incoming field ϕ_0 contains both the right moving part ϕ_0^R and the left moving part ϕ_0^L. Similar calculations show that the energy flux near the left side of the mirror is

$$\begin{aligned}\Big\{&(-\dot{\phi}_0^R\phi'^R_0(t,0)) - (\dot{\phi}_0^L\phi'^L_0(t,0) - \epsilon\dot{\phi}_0^L(t,0)\dot{q}(t))\\ &-\left(\frac{\epsilon^2}{4}\dot{q}^2(t)\right)\Big\}.\end{aligned} \tag{17}$$

The energy flux near the right side of the mirror is

$$\begin{aligned}\Big\{&-(\dot{\phi}_0^L\phi'^L_0(t,0)) + (-\dot{\phi}_0^R\phi'^R_0(t,0) - \epsilon\dot{\phi}_0^R(t,0)\dot{q}(t))\\ &+\left(\frac{\epsilon^2}{4}\dot{q}^2(t)\right)\Big\}.\end{aligned} \tag{18}$$

The interpretations of the above expressions are similar. We illustrate them in Fig. 3.

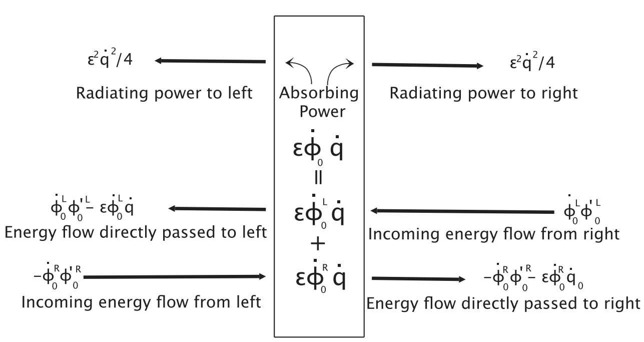

FIG. 3. A figure shows that the incoming field ϕ_0 is impinging on the mirror from both directions. Part of the field energy is absorbed by the internal harmonic oscillator with the total absorbing power $\epsilon\dot{\phi}_0\dot{q}$. At the same time, it is radiating energy out with the total power $\frac{\epsilon^2}{2}\dot{q}^2$.

Next, let us quantize the mirror system. We first go back to the equation of motion (8) to analyze the motion of the internal harmonic oscillator q in detail. In our model, the mirror started to interact with the scalar field since $t = -\infty$. So q's initial oscillation has been completely dissipated due to the friction term $\frac{\epsilon^2}{2}\dot{q}$, and the solution of (8) is then fully determined by the driving force $\epsilon\dot{\phi}_0(t,0)$:

$$q(t) = \frac{1}{\omega}\int_{-\infty}^{t} e^{-a(t-t')}\sin(\omega(t-t'))\epsilon\dot{\phi}_0(t',0)dt', \tag{19}$$

where $a = \frac{\epsilon^2}{4}$ and $\omega = \sqrt{\Omega^2 - \frac{\epsilon^4}{16}}$ is the damped angular frequency. We can quantize the mirror system by expanding ϕ_0 in terms of the sum of standard annihilation and creation operators a_k and $a_k^\dagger$:

$$\phi_0(t,x) = \int_{-\infty}^{+\infty}\frac{dk}{\sqrt{4\pi|k|}}(a_k e^{-i(|k|t-kx)} + a_k^\dagger e^{i(|k|t-kx)}), \tag{20}$$

where the integration over k from $-\infty$ to 0 represents the left moving modes ϕ_0^L and from 0 to $+\infty$ the right moving modes ϕ_0^R. Inserting the above expansion (20) into (19) gives

$$\begin{aligned}q(t) = -i\epsilon\int_{-\infty}^{+\infty}\sqrt{\frac{|k|}{4\pi}}\Bigg(&\frac{a_k e^{-i|k|t}}{-k^2 - \frac{i}{2}\epsilon^2|k| + \Omega^2}\\ &-\frac{a_k^\dagger e^{i|k|t}}{-k^2 + \frac{i}{2}\epsilon^2|k| + \Omega^2}\Bigg)dk.\end{aligned} \tag{21}$$

If we evaluate the average radiating power $\langle\frac{\epsilon^2}{2}\dot{q}^2\rangle$ between frequencies k and $k + \Delta k$ when the system is in vacuum state, which is defined as

$$a_k|0\rangle = 0, \tag{22}$$

for any $k \in (-\infty, +\infty)$, we can see that

$$\langle p(k)\rangle\Delta k = \frac{\epsilon^4}{4\pi}\frac{k^3}{(k^2-\Omega^2)^2 + \frac{\epsilon^4}{4}k^2}\Delta k \to 0, \tag{23}$$

as $k \to +\infty$. Thus, our mirror becomes transparent for high frequency modes. To see more clearly how this transparency property appears, we substitute the annihilation and creation operators a_k and $a_k^\dagger$ in (20) by the position and momentum operators x_k and p_k:

$$a_k = \sqrt{\frac{|k|}{2}}\left(x_k + i\frac{p_k}{|k|}\right), \quad a_k^\dagger = \sqrt{\frac{|k|}{2}}\left(x_k - i\frac{p_k}{|k|}\right). \tag{24}$$

Then the driving force can be expressed as

$$\epsilon\dot{\phi}_0(t,0) = \epsilon\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{2\pi}}|k|\cdot\left(-x_k\sin(|k|t) + \frac{p_k}{|k|}\cos(|k|t)\right), \tag{25}$$

which is a sum of an infinite number of harmonic oscillation modes with different angular frequencies $|k|$. Plugging (25) into (19) we see that each such mode with a specific frequency $|k|$ drives the motion of q independently since there are no correlations between them. Driven by these independent incoming modes, the damped harmonic oscillator q would be excited and eventually settled down to a steady oscillation state, which is also a sum of an infinite number of harmonic oscillations with different frequencies and amplitudes:

$$q(t) = \epsilon\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{2\pi}}\frac{1}{[(\Omega^2-k^2)^2+\frac{\epsilon^4}{4}k^2]^{1/2}}|k| \cdot\left(-x_k\sin(|k|t-\alpha_k) + \frac{p_k}{|k|}\cos(|k|t-\alpha_k)\right), \tag{26}$$

where $\alpha_k = \arctan(\frac{\epsilon^2|k|}{2(\Omega^2-k^2)})$ is the phase lag. Comparing the integrands of the driving force (25) and the internal driven damped harmonic oscillator (26), we observe that except for the phase lag α_k, the only difference is the factor $\frac{1}{[(\Omega^2-k^2)^2+\frac{\epsilon^4}{4}k^2]^{1/2}}$ in the latter expression. This factor shows how the mirror becomes transparent for high frequency wave modes. It is just the amplitude response of a damped harmonic oscillator with the natural frequency Ω and damping coefficient $\frac{\epsilon^2}{2}$ driven by a unit oscillating force with the frequency k when it reaches the final steady state. As shown in Fig. 4, the internal oscillator has almost no response for high frequency driving modes. It is this insensitivity that causes the mirror's transparency for high frequency wave modes.

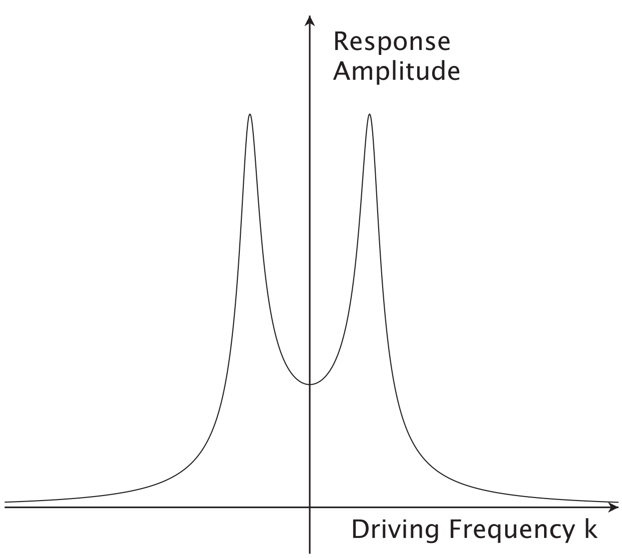

FIG. 4. Steady state variation of amplitude with driving frequency. This graph shows that the internal harmonic oscillator responds weakly to high frequency driven modes.

III. THE FORCE ON THE MIRROR AND ITS INFINITE FLUCTUATION

In the last section, we introduced our mirror model, which is transparent for high frequency wave modes. In this section we will study the force acting on the mirror and its fluctuation when the system is in the vacuum state, which is defined in (22).

Due to quantum fluctuations, there is a net fluctuating force acting on the mirror by the field. As we see from the last section, when the left moving field ϕ_0^L and the right moving field ϕ_0^R incident on the mirror, part of the field energy is absorbed by the internal harmonic oscillator and then the oscillator radiates them out. In this process, the mirror is receiving and sending momentum. On average, the momentum received and sent is symmetric for both sides and thus there is no net force acting on the mirror. However, due to quantum fluctuations, the symmetry between left and right sides can be broken. In other words, sometimes the mirror absorbs more momentum from one side than the other, which produces a net force on the mirror.

The standard definition of the force is

$$F(t) = \lim_{x\to 0^+}(T^{11}(x_-) - T^{11}(x_+)), \tag{27}$$

where $x_+ = (t,x)$ and $x_- = (t,-x)$ $(x \geqslant 0)$ are two spacetime points that are symmetrically located on the two sides of the mirror and T^{11} is the space-space component of stress-energy tensor of type (2,0) of the field ϕ:

$$T^{11}(t,x) = \frac{1}{2}(\dot{\phi}^2(t,x) + \phi'^2(t,x)). \tag{28}$$

From (5), we can get the time and space derivatives of the field ϕ,

$$\dot{\phi}(t,x) = \dot{\phi}_0(t,x) - \frac{\epsilon}{2}\dot{q}(t-|x|), \tag{29}$$

$$\phi'(t,x) = \phi_0'(t,x) + \frac{\epsilon}{2}\dot{q}(t-|x|)\mathrm{sgn}(x). \tag{30}$$

Inserting (28), (29), and (30) into (27) and noticing that when x approaches 0, terms of $\dot{\phi}^2(x_-) - \dot{\phi}^2(x_+)$ go to zero due to continuity of $\dot{\phi}$, we have

$$F(t) = \{-\epsilon\phi_0'(t,0)\dot{q}(t)\}, \tag{31}$$

where the curly brackets $\{\}$ are the symmetrization operation defined in (14). This result is easy to understand.

In fact, decomposing ϕ_0 in the above expression by the sum of the left moving modes ϕ_0^L and the right moving modes ϕ_0^R (9), and using the properties (10) and (11), the above expression becomes

$$F(t) = \{\epsilon\dot{\phi}_0^R(t,0)\dot{q}(t) - \epsilon\dot{\phi}_0^L(t,0)\dot{q}(t)\}. \tag{32}$$

From Fig. 3 we know that $\epsilon\dot{\phi}_0^R\dot{q}$ and $\epsilon\dot{\phi}_0^L\dot{q}$ are energies absorbed per unit of time by the mirror from left and from right, respectively. Since the field ϕ_0 is massless, the energy and momentum are the same up to a sign. Thus, the above formula is just a manifestation that the force is a sum of momenta absorbed but is also a difference between the energy absorbed from two directions.

From the field expansion (20) and the solution of the damped oscillator (21), it is easy to get

$$\langle\phi_0'(t,0)\dot{q}(t)\rangle = -i\frac{\epsilon}{4\pi}\int_{-\infty}^{+\infty}\frac{k|k|}{-k^2+\frac{i}{2}\epsilon^2|k|+\Omega^2}dk = 0. \tag{33}$$

Thus, the expectation value of the force

$$\langle F(t)\rangle \equiv 0. \tag{34}$$

This result is what we expected due to the symmetry of the scalar field: on average, the mirror absorbs equal amounts of momentum from both sides. Next, we calculate the fluctuation of this force in the vacuum state, which is defined as

$$\sigma_F(t) = \langle F^2(t)\rangle - \langle F(t)\rangle^2. \tag{35}$$

Inserting (31) into the above equation gives

$$\begin{aligned}\sigma_F(t) = \frac{\epsilon^2}{4}(&\langle\phi_0'(t,0)\dot{q}(t)\phi_0'(t,0)\dot{q}(t)\rangle + \langle\phi_0'(t,0)\dot{q}(t)\dot{q}(t)\phi_0'(t,0)\rangle + \langle\dot{q}(t)\phi_0'(t,0)\phi_0'(t,0)\dot{q}(t)\rangle \\ &+ \langle\dot{q}(t)\phi_0'(t,0)\dot{q}(t)\phi_0'(t,0)\rangle - \langle\phi_0'(t,0)\dot{q}(t)\rangle^2 - \langle\dot{q}(t)\phi_0'(t,0)\rangle^2 - 2\langle\phi_0'(t,0)\dot{q}(t)\rangle\langle\dot{q}(t)\phi_0'(t,0)\rangle).\end{aligned} \tag{36}$$

We can use Wick's theorem to simplify the above equation. In the case we are considering, for example, the first term in the above equation can be expanded as

$$\begin{aligned}\langle\phi_0'(t,0)\dot{q}(t)\phi_0'(t,0)\dot{q}(t)\rangle = &\langle\phi_0'(t,0)\dot{q}(t)\rangle\langle\phi_0'(t,0)\dot{q}(t)\rangle \\ &+ \langle\phi_0'(t,0)\phi_0'(t,0)\rangle\langle\dot{q}(t)\dot{q}(t)\rangle \\ &+ \langle\phi_0'(t,0)\dot{q}(t)\rangle\langle\dot{q}(t)\phi_0'(t,0)\rangle.\end{aligned} \tag{37}$$

One might note that the last two lines of (36) can be deleted because of (33). We keep them there because they can also be canceled exactly by Wick's expansion of the first four lines. After these cancellations we arrive at

$$\sigma_F(t) = \epsilon^2(\langle\phi_0'(t,0)^2\rangle\langle\dot{q}(t)^2\rangle + \langle\phi_0'(t,0)\dot{q}(t)\rangle\langle\dot{q}(t)\phi_0'(t,0)\rangle). \tag{38}$$

The second term of the above equation is just zero [see Eq. (33)]. Also note that, in the $1+1$ dimension, the term $\langle\phi_0'^2\rangle = \frac{1}{2}(\langle\dot{\phi}_0{}^2\rangle + \langle\phi_0'^2\rangle) = \langle T_{00}\rangle = \langle T_{11}\rangle$, where $\langle T_{00}\rangle$ is the expectation value of vacuum energy density and $\langle T_{11}\rangle$ is the expectation value of vacuum stress. So we obtain our final result for the fluctuation of force acting on the mirror:

$$\sigma_F(t) = \epsilon^2\langle\dot{q}(t)^2\rangle\langle T_{00}\rangle, \tag{39}$$

which is proportional to the product of logarithmically divergent internal kinetic energy,

$$\langle\dot{q}^2\rangle = \frac{\epsilon^2}{4\pi}\int_{-\infty}^{+\infty}dk\frac{|k|^3}{(k^2-\Omega^2)^2+\frac{\epsilon^4}{4}k^2}, \tag{40}$$

and the k^2 divergent vacuum energy density,

$$\langle T_{00}\rangle = \frac{1}{4\pi}\int_{-\infty}^{+\infty}|k|dk = \infty. \tag{41}$$

Here we see that the infinite value of the vacuum energy density does have physical significance. It enters the expression (39) to characterize the fluctuation of the force acting on the mirror. Note that there is no gravitational interaction included in our mirror system. Therefore, this is an example of a nongravitational system where it is not the energy difference from the vacuum but the actual value of the vacuum energy that has physical significance.

As shown in (41), the infinity that appears in the value of the vacuum energy density is an ultraviolet divergence, i.e., it comes from the arbitrarily high frequency field modes. It is interesting that although the mirror is not sensitive to the high frequency field modes, the infinite value of the vacuum energy density still enters our expression (39).

Infinite quantities are usually regarded as unphysical and some regularizations and renormalizations are needed. So it seems that the infinite value of the vacuum energy density $\langle T_{00}\rangle$ in (39) does not make sense, which is similar to what happened in the cosmological constant problem. However, it will be shown in the following sections that this infinite value does make sense because of two special properties of the quantum vacuum: the vacuum friction and the strong anticorrelation of vacuum fluctuations. In other words, the fluctuation of the force acting on the mirror at an instant of time is indeed infinite, but the mirror's position does not undergo infinite fluctuation. On the contrary, its fluctuating motion will be confined in a small region.

IV. THE FINITE FLUCTUATION OF AVERAGE FORCE

Before allowing the mirror to start moving due to the fluctuating force acting on it, we first calculate the fluctuation of the time average of the force acting on the static mirror. The first discussion of the average force fluctuation was given by Barton [12,13]. The reasons to do this are (1) the force only determines the instantaneous acceleration of the mirror while the mirror's position is determined by the force integrated over time, i.e., it is determined by the time accumulation of the force. So we hope we can get some insight by first studying the fluctuation of the average force because the average is a kind of time accumulation; and (2) any apparatus measuring the force cannot respond instantaneously. What the apparatus really measured was not the force at an instant of time but the average in a small time interval. If we finally get a finite result for the fluctuation of the average force, there is the possibility that the fluctuating motion of the mirror is finite.

We will use the Gaussian function $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(t'-t)^2}{2\sigma^2}}$ to define the time average of the force as

$$\bar{F}(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} F(t')e^{-\frac{(t'-t)^2}{2\sigma^2}}dt'. \tag{42}$$

Its fluctuation is defined as

$$\sigma_{\bar{F}}(t) = \langle \bar{F}(t)^2\rangle - \langle \bar{F}(t)\rangle^2. \tag{43}$$

Inserting (42) into the above definition gives

$$\sigma_{\bar{F}(t)} = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{(t_1-t)^2+(t_2-t)^2}{2\sigma^2}} \cdot \mathrm{Corr}(F(t_1),F(t_2))dt_1 dt_2, \tag{44}$$

where

$$\mathrm{Corr}(F(t_1),F(t_2)) = [\langle F(t_1)F(t_2)\rangle - \langle F(t_1)\rangle\langle F(t_2)\rangle] \tag{45}$$

is the correlation function between forces F at times t_1 and t_2. Next, let us calculate the correlation function (45). Plugging (31) into the definition (45) gives

$$\begin{aligned}\mathrm{Corr}(F(t_1),F(t_2)) = \frac{\epsilon^2}{4}(&\langle\phi_0'(t_1,0)\dot{q}(t_1)\phi_0'(t_2,0)\dot{q}(t_2)\rangle + \langle\phi_0'(t_1,0)\dot{q}(t_1)\dot{q}(t_2)\phi_0'(t_2,0)\rangle + \langle\dot{q}(t_1)\phi_0'(t_1,0)\phi_0'(t_2,0)\dot{q}(t_2)\rangle\\ &+ \langle\dot{q}(t_1)\phi_0'(t_1,0)\dot{q}(t_2)\phi_0'(t_2,0)\rangle - \langle\phi_0'(t_1,0)\dot{q}(t_1)\rangle\langle\phi_0'(t_2,0)\dot{q}(t_2)\rangle - \langle\phi_0'(t_1,0)\dot{q}(t_1)\rangle\langle\dot{q}(t_2)\phi_0'(t_2,0)\rangle\\ &- \langle\dot{q}(t_1)\phi_0'(t_1,0)\rangle\langle\phi_0'(t_2,0)\dot{q}(t_2)\rangle - \langle\dot{q}(t_1)\phi_0'(t_1,0)\rangle\langle\dot{q}(t_2)\phi_0'(t_2,0)\rangle).\end{aligned} \tag{46}$$

Similar to the calculation of fluctuation of the force σ_F, we employ Wick's theorem to reduce the products of four operators to the sum of products of pairs of operators to simplify the above equation. For example, the first term can be expanded as

$$\begin{aligned}&\langle\phi_0'(t_1,0)\dot{q}(t_1)\phi_0'(t_2,0)\dot{q}(t_2)\rangle\\ &\quad= \langle\phi_0'(t_1,0)\dot{q}(t_1)\rangle\langle\phi_0'(t_2,0)\dot{q}(t_2)\rangle\\ &\qquad+ \langle\phi_0'(t_1,0)\phi_0'(t_2,0)\rangle\langle\dot{q}(t_1)\dot{q}(t_2)\rangle\\ &\qquad+ \langle\phi_0'(t_1,0)\dot{q}(t_2)\rangle\langle\dot{q}(t_1)\phi_0'(t_2,0)\rangle.\end{aligned} \tag{47}$$

Applying Wick's theorem in (46) gives

$$\begin{aligned}\mathrm{Corr}(F(t_1),F(t_2)) = \epsilon^2(&\langle\phi_0'(t_1,0)\phi_0'(t_2,0)\rangle\langle\dot{q}(t_1)\dot{q}(t_2)\rangle\\ &+ \langle\phi_0'(t_1,0)\dot{q}(t_2)\rangle\langle\dot{q}(t_1)\phi_0'(t_2,0)\rangle).\end{aligned} \tag{48}$$

From (20) and (21) we can easily obtain

$$\langle\phi_0'(t_1,0)\phi_0'(t_2,0)\rangle = \frac{1}{4\pi}\int_{-\infty}^{+\infty}|k|e^{-i|k|(t_1-t_2)}dk, \tag{49}$$

$$\langle\dot{q}(t_1)\dot{q}(t_2)\rangle = \frac{\epsilon^2}{4\pi}\int_{-\infty}^{+\infty}\frac{|k|^3}{(k^2-\Omega^2)^2+\frac{\epsilon^4}{4}k^2}e^{-i|k|(t_1-t_2)}dk, \tag{50}$$

$$\langle\phi_0'(t_1,0)\dot{q}(t_2)\rangle = -i\frac{\epsilon}{4\pi}\int_{-\infty}^{+\infty}\frac{k|k|}{-k^2+\frac{i}{2}\epsilon^2|k|+\Omega^2} \times e^{-i|k|(t_1-t_2)}dk. \tag{51}$$

Thus, we reach an expression for the correlation function,

$$\mathrm{Corr}(F(t_1),F(t_2)) = \frac{\epsilon^4}{16\pi^2}\left(\int_{-\infty}^{+\infty}|k|e^{-i|k|(t_1-t_2)}dk\cdot\int_{-\infty}^{+\infty}\frac{|k'|^3}{(k'^2-\Omega^2)^2+\frac{\epsilon^4}{4}k'^2}e^{-i|k'|(t_1-t_2)}dk' + \int_{-\infty}^{+\infty}\frac{k|k|}{-k^2+\frac{i}{2}\epsilon^2|k|+\Omega^2}e^{-i|k|(t_1-t_2)}dk\cdot\int_{-\infty}^{+\infty}\frac{k'|k'|}{-k'^2-\frac{i}{2}\epsilon^2|k'|+\Omega^2}e^{-i|k'|(t_1-t_2)}dk'\right). \tag{52}$$

Plugging (52) into (44) and changing the order of integration gives

$$\begin{aligned}\sigma_{\bar{F}(t)} &= \frac{\epsilon^4}{16\pi^2}\cdot\frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|k|\cdot\frac{|k'|^3}{(k'^2-\Omega^2)^2+\frac{\epsilon^4}{4}k'^2}dkdk'\int_{-\infty}^{+\infty}e^{-\frac{(t_1-t)^2}{2\sigma^2}-i(|k|+|k'|)t_1}dt_1\int_{-\infty}^{+\infty}e^{-\frac{(t_2-t)^2}{2\sigma^2}+i(|k|+|k'|)t_2}dt_2\\ &= \frac{\epsilon^4}{16\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{|k||k'|^3}{(k'^2-\Omega^2)^2+\frac{\epsilon^4}{4}k'^2}\cdot e^{-\sigma^2(|k|+|k'|)^2}dkdk'\\ &\le \frac{\epsilon^4}{16\pi^2}\int_{-\infty}^{+\infty}|k|e^{-\sigma^2k^2}dk\cdot\int_{-\infty}^{+\infty}\frac{|k'|^3}{(k'^2-\Omega^2)^2+\frac{\epsilon^4}{4}k'^2}e^{-\sigma^2k'^2}dk' < +\infty.\end{aligned} \tag{53}$$

Thus, we get a finite result for the fluctuation of the time-averaged force. The finiteness of the fluctuation of the force average is closely related to the strong anticorrelation property of the vacuum fluctuations. Detailed analysis of this property will be given in Sec. VIII.

V. THE VACUUM FRICTION: DAMPING FORCE WHEN THE MIRROR STARTS TO MOVE

In this section we allow the mirror to begin to move. We are interested in the question of how the mirror will move if we release it at time $t=0$. One might naively think that the mirror's position will fluctuate infinitely under the infinite fluctuating force, although such a result must be unphysical. However, as we stated in the beginning of the last section, the force can only determine the instantaneous acceleration of the mirror, while the position of the mirror is determined by the time integration of the force. We see from the last section that the fluctuation of average force is finite. This gives a hope that the fluctuation of the position of the mirror, which is driven by the force exerted on it, might be finite. To calculate this position fluctuation, i.e., the mean-squared displacement, we need to figure out the equation of motion of the mirror.

P. C. W. Davies has suggested that the quantum vacuum may in certain circumstances be regarded as a type of fluid medium exhibiting friction [14]. We expect that when our mirror starts to move, it will experience a frictional force damping its motion. This force is important in constructing the equation of motion. This section will give the detailed analysis about this force, and the construction of the equation of motion will be given in the next section.

Unlike in the previous sections where we held the mirror fixed at the location $x=0$, in this section we specify the mirror move along a generic trajectory and investigate the damping force acting on it by the field.

Now let us calculate the damping force in detail. Consider the mirror is moving along a generic trajectory $x=X(t(\tau))$, where τ is the proper time associated with this trajectory:

$$t(\tau) = \int_0^\tau \gamma(t(\tau'))d\tau', \tag{54}$$

where $\gamma(t) = \frac{1}{\sqrt{1-\dot{X}^2}}$ is the Lorentz factor and $\dot{}$ denotes the derivative with respect to the coordinate time t as before, i.e., $\dot{X}(t) = \frac{dX(t)}{dt}$. The action of the moving mirror is

$$\begin{aligned}S &= \frac{1}{2}\iint\left(\left(\frac{\partial\phi}{\partial t}\right)^2-\left(\frac{\partial\phi}{\partial x}\right)^2\right)dtdx\\ &\quad+\frac{1}{2}\int\left(\left(\frac{dq}{d\tau}\right)^2-\Omega^2q^2\right)d\tau\\ &\quad+\epsilon\int\frac{d\phi}{d\tau}(t(\tau),X(t(\tau)))q(t(\tau))d\tau.\end{aligned} \tag{55}$$

Note that X and q are different things. X is the mirror's position, which is moving "in space," while q is the mirror's "internal" degree of freedom, which is *not* oscillating "in space."

The equations of motion for the field ϕ and the internal harmonic oscillator q now become

$$\ddot{\phi}-\phi'' = -\epsilon\dot{q}\delta(x-X(t)), \tag{56}$$

$$\frac{d^2q}{d\tau^2}+\Omega^2q^2 = \epsilon\frac{d\phi}{d\tau}(t(\tau),X(t(\tau))). \tag{57}$$

Similar to the static mirror case, the solution of (56) is of the following form:

$$\phi(t,x) = \phi_0(t,x) - \frac{\epsilon}{2}q(t'), \tag{58}$$

where the retarded time t' is determined by the following equation:

$$t - t' = |x - X(t')|. \tag{59}$$

Substituting (58) into the equation of motion for the internal harmonic oscillator (57) gives

$$\frac{d^2q}{d\tau^2} + \frac{\epsilon^2}{2}\frac{dq}{d\tau} + \Omega^2 q = \epsilon\frac{d\phi_0}{d\tau}(t(\tau), X(t(\tau))). \tag{60}$$

Similar to solution (19) for the static mirror, the solution of the above equation of motion (60) of the internal driven damped harmonic oscillator is

$$q(t(\tau)) = \frac{1}{\omega}\int_{-\infty}^{\tau} e^{-a(\tau-\tau')}\sin(\omega(\tau-\tau')) \cdot \epsilon\frac{d\phi_0}{d\tau'}(t(\tau'), X(t(\tau')))d\tau', \tag{61}$$

where $a = \frac{\epsilon^2}{4}$ and $\omega = \sqrt{\Omega^2 - \frac{\epsilon^4}{16}}$ are the same as those in (19). One key difference of the solution (61) from the static case (19) is that when the mirror moves, the driving force changes, which could result in the deviation of the q's motion from its steady oscillation state (21) or (26).

Here we consider the force in the mirror's instantaneous rest frame. In this frame, the force acting on each side of the mirror by the field is in the form of $T_{\mu\nu}x^\mu x^\nu$, where $x^\mu = \gamma(\dot{X}, 1)$ is a unit space-like vector that is orthogonal to the four velocity of the mirror. Thus, the force in the moving mirror's instantaneous rest frame is defined as

$$\begin{aligned} F(t) &= \lim_{x\to 0^+}(T_{\mu\nu}(x_-)x^\mu x^\nu - T_{\mu\nu}(x_+)x^\mu x^\nu) \\ &= \gamma^2\lim_{x\to 0^+}((T_{00}(x_-) - T_{00}(x_+))\dot{X}^2 \\ &\quad + 2(T_{01}(x_-) - T_{01}(x_+))\dot{X} + (T_{11}(x_-) - T_{11}(x_+))), \end{aligned} \tag{62}$$

where $x_- = (t, X(t) - x)$ and $x_+ = (t, X(t) + x)$ $(x \geqslant 0)$ are two spacetime points that are symmetrically located on the two sides of the mirror. T_{00}, T_{01}, and T_{11} are components of the stress-energy tensor of the type (0,2), which in (t, x) coordinates are defined as

$$T_{00} = \frac{1}{2}(\dot{\phi}^2(t,x) + \phi'^2(t,x)), \tag{63}$$

$$T_{01} = \frac{1}{2}(\dot{\phi}(t,x)\phi'(t,x) + \phi'(t,x)\dot{\phi}(t,x)), \tag{64}$$

$$T_{11} = \frac{1}{2}(\dot{\phi}^2(t,x) + \phi'^2(t,x)). \tag{65}$$

From (58) and (59) we can get the time and space derivatives of the field ϕ:

$$\dot{\phi}(t,x) = \begin{cases} \dot{\phi}_0(t,x) - \frac{\epsilon}{2}(\frac{\dot{q}(t')}{1+\dot{X}(t')}), & \text{if } x < X(t') \\ \dot{\phi}_0(t,x) - \frac{\epsilon}{2}(\frac{\dot{q}(t')}{1-\dot{X}(t')}), & \text{if } x > X(t') \end{cases} \tag{66}$$

$$\phi'(t,x) = \begin{cases} \phi'_0(t,x) - \frac{\epsilon}{2}(\frac{\dot{q}(t')}{1+\dot{X}(t')}), & \text{if } x < X(t') \\ \phi'_0(t,x) + \frac{\epsilon}{2}(\frac{\dot{q}(t')}{1-\dot{X}(t')}), & \text{if } x > X(t'). \end{cases} \tag{67}$$

Similar to the static mirror case, we can first insert (66) and (67) into (63), (64), and (65) to express the stress-energy tensor components in terms of ϕ_0 and q, then plug these expressions into (62) to get the force. The result is

$$F(t) = -\epsilon\gamma^2\{\dot{q}(t)(\phi'_0(t, X(t)) + \dot{X}\dot{\phi}_0(t, X(t)))\}, \tag{68}$$

where the curly brackets {} represents the symmetrization operation (14) as before. This formula can be understood as the following: remember that the force we are calculating is evaluated in the mirror's instantaneous rest frame, which should have the same form as the static mirror case (31) when expressed in terms of its own instantaneous rest frame coordinates (t', x') (see Fig. 6):

$$F(t') = -\epsilon\left\{\frac{\partial\phi_0(t',0)}{\partial x'}\cdot\frac{dq(t')}{dt'}\right\}. \tag{69}$$

Changing the above expression (69) from the (t', x') coordinate system to the laboratory coordinate system (t, x) leads to exactly the force expression (68), which is expressed in terms of laboratory frame coordinates.

Next, we can insert the expression (20) for the incident wave ϕ_0 and the expression (61) for the internal degree of freedom q into (68) to get the mean motional force exerted on the mirror by the field when the mirror is moving along a generic trajectory $x = X(t(\tau))$. The result is

$$\begin{aligned} \langle F(t(\tau))\rangle = &-\frac{1}{2}\frac{\epsilon^2}{4\pi}\gamma(t(\tau))\frac{1}{\omega}\int_{-\infty}^{+\infty}dk\int_{-\infty}^{\tau}d\tau'\gamma(t(\tau'))[-k(1+\dot{X}(t(\tau))\dot{X}(t(\tau'))) + |k|(\dot{X}(t(\tau)) + \dot{X}(t(\tau')))] \\ &\times[-a\sin(\omega(\tau-\tau')) + \omega\cos(\omega(\tau-\tau'))]\exp\left(-a(\tau-\tau') + i\left[|k|\int_{\tau'}^{\tau}\gamma(t(\tau''))d\tau'' - k(X(t(\tau)) - X(t(\tau')))\right]\right) + \text{c.c.} \end{aligned} \tag{70}$$

The above expression for the force is quite complicated. However, remember that what we are interested in is the damping force when the mirror starts to move due to the quantum fluctuations of the field after we release it. So let us consider a motion in which the mirror initially stays at the origin for a long time and then starts to move with constant acceleration α along the following trajectory when $t \geq 0$ (as shown in Fig. 5):

$$\begin{cases} t = \frac{1}{\alpha}\sinh(\alpha\tau), \\ x = \frac{1}{\alpha}[\cosh(\alpha\tau) - 1], \end{cases} \tag{71}$$

where τ is the proper time of the trajectory as before. Plugging this trajectory into (70), we obtain that, when the velocity changes from 0 to $v \approx \alpha\tau \ll 1$, the mean motional force is

$$\langle F \rangle = \left[-\frac{\epsilon^4}{4\pi} \int_0^{+\infty} \frac{k^3 dk}{(k^2 - \Omega^2)^2 + \frac{\epsilon^4}{4} k^2} \right] v. \tag{72}$$

The above formula shows that the quantum vacuum does serve as a fluid medium in the sense that our mirror, if it initially stays at rest, would experience a friction force when it starts to move.

Here, we emphasize that the v in the above formula (72) should be understood as the velocity change due to the acceleration from the mirror's original rest frame in which the internal harmonic oscillator was in a steady state equilibrium. For the constantly moving mirror trajectory $X(t) \equiv vt$, the formula (72) dose not apply and the general motional force expression (70) gives zero force. This zero force is just the requirement of Lorentz invariance.

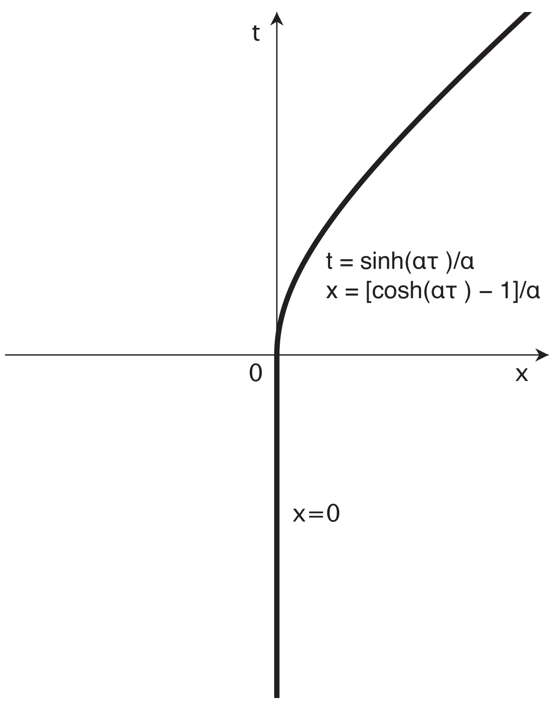

FIG. 5. Trajectory for a mirror that initially stays at rest and then starts to move with constant acceleration α at $t = 0$.

In fact, the friction force (72) arises from the Doppler shift of the vacuum modes due to the changing velocity of the mirror. Referring back to Fig. 3, the mirror absorbs both the left moving field modes (wave number $k < 0$) and the right moving field modes (wave number $k > 0$). The rate at which the mirror absorbs energy from a mode depends on the internal velocity of the internal oscillator $\dot{q}$. Because of the linearity of the system and the lack of any correlations between these modes, the only component of the oscillator motion that is important is the motion in q induced by that same mode earlier in time. If the mirror starts to move, any mode now will be Doppler shifted with respected to that same mode earlier—modes from the trailing side are redshifted and the other blueshifted. Because only the internal motion caused by that mode earlier is connected with the mode now, only that mode will feed or extract energy from that component of its motion. Thus, different amounts of energy (and thus of momentum) will be absorbed from the two directions. The emitted momenta, on the other hand, are always balanced between the two sides and supply no force to the mirror (see the top line in Fig. 3).

For example, if the mirror moves to the right with velocity v, the right moving modes with frequency $|k|$ will be redshifted to the frequency $(\frac{1-v}{1+v})^{1/2}|k|$ and the left moving modes with the same frequency will be blueshifted to the frequency $(\frac{1+v}{1-v})^{1/2}|k|$. So the symmetry between the left moving modes and right moving modes is broken. This asymmetry will result in the force imbalance since the mirror will absorb more momentum from left moving modes than from right moving modes, which gives a net force to the left to resist the mirror's motion. To understand this in detail, let us investigate the following trajectory (as shown in Fig. 6):

$$X(t) = \begin{cases} 0, & \text{if } t < 0 \\ vt, & \text{if } t \geq 0. \end{cases} \tag{73}$$

For this trajectory, the mirror is initially static until it starts to move with constant velocity at time $t = 0$. Direct calculation using the general mean motional force formula (70) shows that when $t \geq 0$, the friction force is

$$\begin{aligned} \langle F(t(\tau)) \rangle = & -\frac{1}{2}\frac{\epsilon^2}{4\pi}\frac{1}{\omega} e^{-\alpha\tau} \int_0^{+\infty} k \left[\left(\frac{1+v}{1-v} \right)^{\frac{1}{2}} e^{ik(\frac{1+v}{1-v})^{\frac{1}{2}}\tau} \right. \\ & \left. - \left(\frac{1-v}{1+v} \right)^{\frac{1}{2}} e^{ik(\frac{1-v}{1+v})^{\frac{1}{2}}\tau} \right] \\ & \cdot \frac{-ik\omega\cos(\omega\tau) - (\Omega^2 - \frac{i}{4}\epsilon^2 k)\sin(\omega\tau)}{\Omega^2 - k^2 - \frac{i}{2}\epsilon^2 k} dk + \text{c.c.} \end{aligned} \tag{74}$$

Note that the exponential factor $e^{-a\tau}$ appearing in the above expression implies that after a long time the force would

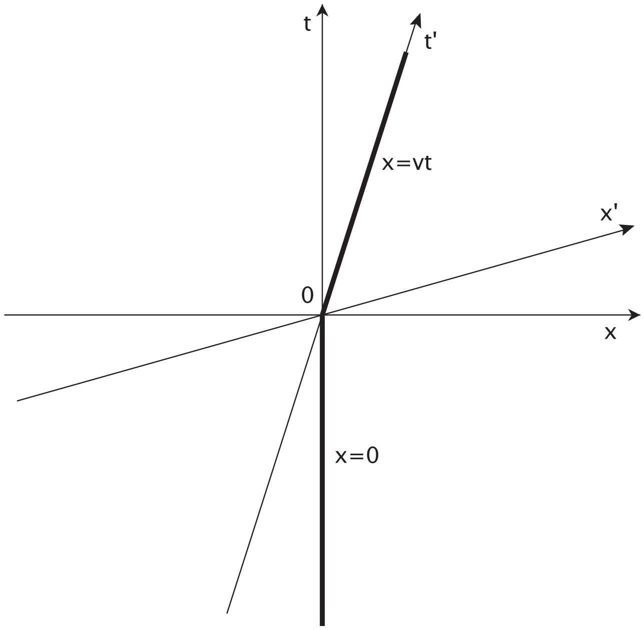

FIG. 6. Trajectory for a mirror that initially stays at rest and then jumps to move with a constant velocity v at time $t = 0$.

decrease to zero. This is just the requirement of Lorentz invariance, since after long times the mirror's memory would fade away and it would not remember what it did long before and thus can be regarded as a moving mirror with constant velocity v, which should experience zero friction force.

What is interesting is that the force at the time $t = 0$ is

$$\langle F(0)\rangle = -\frac{\epsilon^4}{8\pi}\left[\left(\frac{1+v}{1-v}\right)^{\frac{1}{2}} - \left(\frac{1-v}{1+v}\right)^{\frac{1}{2}}\right] \cdot \int_0^{+\infty} \frac{k^3 dk}{(\Omega^2 - k^2)^2 + \frac{1}{4}\epsilon^4 k^2}, \tag{75}$$

which is in agreement with our previous result (72) at the small velocity approximation. Here, the two factors $(\frac{1+v}{1-v})^{\frac{1}{2}}$ and $(\frac{1-v}{1+v})^{\frac{1}{2}}$ are exactly relativistic Doppler shift factors for an observer moving toward or away from a light source with velocity v.

Next, we reproduce the above result (75) by a different method to reveal the role played by the Doppler effect. We will consider everything in the mirror's instantaneous rest frame.

For the trajectory (73), when $t < 0$, the mirror's rest frame is the (t, x) coordinate system and the field ϕ_0 is expanded as the sum of positive frequency modes $\frac{e^{-i(|k|t-kx)}}{\sqrt{4\pi|k|}}$ with coefficients a_k and negative frequency modes $\frac{e^{+i(|k|t-kx)}}{\sqrt{4\pi|k|}}$ with coefficients $a_k^\dagger$ [see (20)]. When $t \geq 0$, the mirror's rest frame is the (t', x') coordinate system [see Fig. (6)] and the same field ϕ_0 is expanded as

$$\phi_0(t', x') = \int_{-\infty}^{+\infty} \frac{dk'}{\sqrt{4\pi|k'|}} (b_{k'} e^{-i(|k'|t' - k'x')} + b_{k'}^\dagger e^{i(|k'|t' - k'x')}), \tag{76}$$

where the wave numbers k' in the (t', x') coordinate system are Doppler shifted from the wave numbers k in the (t, x) system to

$$k' = \begin{cases} (\frac{1+v}{1-v})^{1/2} k, & \text{when} \quad k < 0 \\ (\frac{1-v}{1+v})^{1/2} k, & \text{when} \quad k > 0, \end{cases} \tag{77}$$

and correspondingly, the operator coefficients $b_{k'}$ and a_k are related by

$$b_{k'} = \begin{cases} (\frac{1-v}{1+v})^{1/4} a_k, & \text{when} \quad k < 0 \\ (\frac{1+v}{1-v})^{1/4} a_k, & \text{when} \quad k > 0. \end{cases} \tag{78}$$

Now expand the solution of the internal harmonic oscillator (61) in terms of the new operators $b_{k'}$ and $b_{k'}^\dagger$. For the trajectory (73) we are considering, the oscillation pattern right before $\tau = 0$ is

$$q(\tau) = q^L(\tau) + q^R(\tau), \tag{79}$$

where

$$q^L(\tau) = -i\epsilon \int_{-\infty}^{0} dk' \left(\frac{1-v}{1+v}\right)^{1/2} \sqrt{\frac{|k'|}{4\pi}} \cdot \left(\frac{b_{k'} e^{-i(\frac{1-v}{1+v})^{1/2}|k'|\tau}}{-(\frac{1-v}{1+v})k'^2 - \frac{i}{2}\epsilon^2 (\frac{1-v}{1+v})^{1/2}|k'| + \Omega^2} - \frac{b_{k'}^\dagger e^{i(\frac{1-v}{1+v})^{1/2}|k'|\tau}}{-(\frac{1-v}{1+v})k'^2 + \frac{i}{2}\epsilon^2 (\frac{1-v}{1+v})^{1/2}|k'| + \Omega^2} \right) \tag{80}$$

and

$$q^R(\tau) = -i\epsilon \int_{0}^{+\infty} dk' \left(\frac{1+v}{1-v}\right)^{1/2} \sqrt{\frac{|k'|}{4\pi}} \cdot \left(\frac{b_{k'} e^{-i(\frac{1+v}{1-v})^{1/2}|k'|\tau}}{-(\frac{1+v}{1-v})k'^2 - \frac{i}{2}\epsilon^2 (\frac{1+v}{1-v})^{1/2}|k'| + \Omega^2} - \frac{b_{k'}^\dagger e^{i(\frac{1+v}{1-v})^{1/2}|k'|\tau}}{-(\frac{1+v}{1-v})k'^2 + \frac{i}{2}\epsilon^2 (\frac{1+v}{1-v})^{1/2}|k'| + \Omega^2} \right). \tag{81}$$

Unlike the oscillation (21), after the mirror did an instant jump in velocity at $\tau = 0$, the mirror sees that the field modes are Doppler shifted and the oscillation of q above is no longer a steady state relative to the field in the new (t', x') frame. Driven by these Doppler shifted modes, the oscillation will change and eventually settle down to a new steady state, with the same frequencies as the driving field modes. Once it reached the final steady state again, the frictional force would again become zero, as predicted in

(74). However, during this process, there will be imbalance between the absorbed momentum from the left and the right, which gives the nonzero friction force. In fact, in the mirror's frame, the force (68) would reduce to (31) or (32). And the average momentum absorbed per unit of time from right and from left at the jumping point $\tau = 0$ is

$$\left\langle \left\{ \epsilon \frac{d\phi_0^L}{d\tau} \frac{dq^L}{d\tau} \right\} \right\rangle = \frac{\epsilon^4}{8\pi} \left(\frac{1+v}{1-v} \right)^{\frac{1}{2}} \int_0^{+\infty} \frac{k^3 dk}{(\Omega^2 - k^2)^2 + \frac{1}{4}\epsilon^4 k^2}, \tag{82}$$

and

$$\left\langle \left\{ \epsilon \frac{d\phi_0^R}{d\tau} \frac{dq^R}{d\tau} \right\} \right\rangle = \frac{\epsilon^4}{8\pi} \left(\frac{1-v}{1+v} \right)^{\frac{1}{2}} \int_0^{+\infty} \frac{k^3 dk}{(\Omega^2 - k^2)^2 + \frac{1}{4}\epsilon^4 k^2}, \tag{83}$$

where the curly brackets $\{\}$ denote the symmetrization operation as defined in (14) and we have dropped the terms $\langle \epsilon \frac{d\phi_0^L}{d\tau} \frac{dq^R}{d\tau} \rangle$ and $\langle \epsilon \frac{d\phi_0^R}{d\tau} \frac{dq^L}{d\tau} \rangle$ since they are zero. Note that the force (75) is exactly the difference [15] of (82) and (83). Thus, we can come to the conclusion that if the mirror starts to move from zero velocity and has already acquired a velocity, for example, to the right, it would absorb more momentum from the right per unit of time than from the left. The difference is determined by the Doppler shift factors. It is this imbalance in the absorbed momentum from different directions that leads to the nonzero frictional force.

An important lesson we learn from the above analyses is that the nonzero frictional force happens only when the oscillation of the internal harmonic oscillator has deviated from the steady state. That's why we emphasized after the formula (72) that the v should be understood as the velocity change due to the acceleration from the mirror's original rest frame in which the internal harmonic oscillator was in a steady state equilibrium.

One might also worry about the logarithmically divergent proportional constant in the force expression (72) for the damping force. It does not matter because the damping force is not an observable quantity. The physically observable quantity is the motion of the mirror that depends on the time average of the force, which is proved to be finite in the last section, or the movement of the mirror under the influence of this divergent force. It turns out in the following sections that the effective mass of the mirror is also logarithmically divergent, which exactly cancels the divergence of the damping force to give a finite value of the damping ratio.

VI. THE MIRROR'S EQUATION OF MOTION

Unlike in the last section, where we specified the trajectory the mirror moved along, in this section we release the mirror and let it move freely under the fluctuating force exerting on it by the field. To do this, we add an extra term (the first one) in the action (55) such that

$$\begin{aligned} S = &- M \int d\tau + \frac{1}{2} \iint \left(\left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 \right) dt dx \\ &+ \frac{1}{2} \int \left(\left(\frac{dq}{d\tau} \right)^2 - \Omega^2 q^2 \right) d\tau \\ &+ \epsilon \int \frac{d\phi}{d\tau} (t(\tau), X(t(\tau))) q(t(\tau)) d\tau, \end{aligned} \tag{84}$$

where M is the mirror's bare mass. One can derive the mirror's equation of motion directly from this action (see Appendix A). However, to express the equation of motion in terms of the force in the mirror's instantaneous rest frame we derived in the last section, we choose another way—first deriving the stress-energy tensor of the whole system and applying the continuity equation $\partial_\nu T^{\mu\nu} = 0$ to obtain the equation of motion.

The stress-energy tensor of the whole system is [see (B9) in Appendix B]

$$T^{00} = \frac{1}{2} (\dot{\phi}^2 + \phi'^2) + \gamma M_{\text{eff}} \delta(x - X(t)), \tag{85}$$

$$T^{01} = T^{10} = -\{\dot{\phi}\phi'\} + \gamma M_{\text{eff}} \dot{X} \delta(x - X(t)), \tag{86}$$

$$T^{11} = \frac{1}{2} (\dot{\phi}^2 + \phi'^2) + \gamma M_{\text{eff}} \dot{X}^2 \delta(x - X(t)), \tag{87}$$

where the effective mass includes the mirror's bare mass M and the energy of the internal harmonic oscillator [see (B10) in Appendix B]:

$$M_{\text{eff}} = M + \frac{1}{2} \left(\frac{dq}{d\tau} \right)^2 + \frac{1}{2} \Omega^2 q^2. \tag{88}$$

Next, we apply the continuity equation $\partial_\nu T^{\mu\nu} = 0$ to the above stress-energy tensor. Also using the equation of motion of the field (56), we obtain the equation of energy conservation (for the case $\mu = 0$),

$$\frac{d}{dt} (\gamma M_{\text{eff}}) = \epsilon \{ \dot{q}(t) \dot{\phi}(t, X(t)) \}, \tag{89}$$

and the equation of momentum conservation (for the case $\mu = 1$),

$$\frac{d}{dt} (\gamma M_{\text{eff}} \dot{X}) = -\epsilon \{ \dot{q}(t) \phi'(t, X(t)) \}. \tag{90}$$

Note that the term $-\epsilon\{\dot{q}(t)\phi'(t, X(t))\}$ in the above equation (90) represents the force exerted on the mirror in laboratory frame, which is different from the force in the mirror's instantaneous rest frame (68). We can derive the equation of motion in the mirror's instantaneous rest frame from (89) and (90) by performing a Lorentz boost and thus prove that (68) does serve as the force in the mirror's instantaneous rest frame. First, we let

$$\begin{pmatrix} E \\ P \end{pmatrix} = \begin{pmatrix} \gamma M_{\mathrm{eff}} \\ \gamma M_{\mathrm{eff}} \dot{X} \end{pmatrix} \tag{91}$$

be the energy momentum vector of the mirror in the laboratory frame (t, x) (as shown in Fig. 6). Assuming that at some moment the mirror's instantaneous rest frame is (t', x') (as shown in Fig. 6), i.e., that the frame (t', x') is moving with velocity $\dot{X}$ with respect to the frame (t, x), the energy momentum vector $\begin{pmatrix} E' \\ P' \end{pmatrix}$ in the (t', x') frame is related to the energy momentum vector $\begin{pmatrix} E \\ P \end{pmatrix}$ in the (t, x) frame by the Lorentz boost:

$$\begin{pmatrix} E' \\ P' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\dot{X} \\ -\gamma\dot{X} & \gamma \end{pmatrix} \begin{pmatrix} E \\ P \end{pmatrix}. \tag{92}$$

Differentiating P' in (92) with respect to the mirror's proper time t' and using (89), (90), and (91) yields

$$\begin{aligned} \frac{dP'}{dt'} &= \gamma \frac{dP'}{dt} \\ &= \gamma\left(-\gamma\dot{X}\frac{dE}{dt} + \gamma\frac{dP}{dt}\right) \\ &= -\epsilon\gamma^2\{\dot{q}(t)(\phi'(t, X(t)) + \dot{X}\dot{\phi}(t, X(t)))\}. \end{aligned} \tag{93}$$

Note that in the usual mathematical sense the field ϕ is not differentiable on the mirror's path $(t, X(t))$ since $\dot{\phi}$ and ϕ' have jump discontinuities there [see (66) and (67)]. For this type of discontinuity, the values of $\dot{\phi}$ and ϕ' at $(t, X(t))$ are not defined and may have any value. However, it is natural to define the derivative as the average of the left derivative and right derivative, i.e.,

$$\dot{\phi}(t, X(t)) = \lim_{x\to 0^+} \frac{\dot{\phi}(t, X(t) + x) + \dot{\phi}(t, X(t) - x)}{2}, \tag{94}$$

$$\phi'(t, X(t)) = \lim_{x\to 0^+} \frac{\phi'(t, X(t) + x) + \phi'(t, X(t) - x)}{2}. \tag{95}$$

Applying the above definition to (66) and (67) yields

$$\phi'(t, X(t)) + \dot{X}\dot{\phi}(t, X(t)) = \phi_0'(t, X(t)) + \dot{X}\dot{\phi}_0(t, X(t)). \tag{96}$$

Thus, we can replace the ϕ in (93) by ϕ_0 to obtain the mirror's equation of motion in its instantaneous rest frame:

$$\frac{dP'}{dt'} = -\epsilon\gamma^2\{\dot{q}(t)(\phi_0'(t, X(t)) + \dot{X}\dot{\phi}_0(t, X(t)))\}. \tag{97}$$

Note that the right-hand side of the above equation exactly agree with the force expression (68) that we derived in the last section by a different method.

To analyze the fluctuating motion of the mirror, we next express the mirror's equation of motion in the laboratory frame in terms of the force in its instantaneous rest frame by simple manipulations of the energy momentum conservation equations (89) and (90):

$$\begin{aligned} \gamma M_{\mathrm{eff}} \frac{d^2X}{dt^2} &= -\dot{X}\frac{d}{dt}(\gamma M_{\mathrm{eff}}) - \epsilon\{\dot{q}(t)\phi'(t, X(t))\} \\ &= -\epsilon\{\dot{q}(t)(\phi'(t, X(t)) + \dot{X}\dot{\phi}(t, X(t)))\} \\ &= \frac{1}{\gamma^2}(-\epsilon\gamma^2\{\dot{q}(t)(\phi_0'(t, X(t)) + \dot{X}\dot{\phi}_0(t, X(t)))\}), \end{aligned} \tag{98}$$

where we have used (96) to replace ϕ by ϕ_0 in the last line of the above equation. Note that the expression inside the parentheses of the last line is exactly the force F (68) in the mirror's instantaneous rest frame that we derived in the last section; thus, we reach the following equation of motion that relates the mirror's acceleration with the force F in the mirror's instantaneous rest frame:

$$\gamma^3 M_{\mathrm{eff}} \frac{d^2X}{dt^2} = F. \tag{99}$$

We will analyze the fluctuating motion of the mirror using the above equation (99).

VII. CONFINED FLUCTUATING MOTION OF THE MIRROR

The situation we are considering is one in which we first hold the mirror fixed for a long time and then release it at time $t = 0$. The mirror's position will then start to fluctuate due to the quantum fluctuating force acting on it. We assume that the time scale of the period of the fluctuating motion is small enough that the oscillations of the internal harmonic oscillator would approximately stay in the steady state relative to the laboratory frame. For simplicity, we also assume that the velocity of the mirror is small; then we can neglect the γ^3 term in (99) and the equation of motion becomes

$$M_{\mathrm{eff}} \frac{d^2X}{dt^2} = F. \tag{100}$$

We can rewrite the equation of motion (100) as

$$\frac{d^2X}{dt^2} - \frac{\langle F\rangle}{M_{\mathrm{eff}}} = \frac{F - \langle F\rangle}{M_{\mathrm{eff}}}. \tag{101}$$

The numerator of the term on the right-hand side of the above equation is the deviation of the force from its mean value. We assume that the mirror will fluctuate near the position $x = 0$. In this approximation, we can use the static force expression (31), i.e., the force when the mirror is staying at the origin, to substitute into the numerator $F - \langle F \rangle$. In the following we will denote the static force by F_0 to avoid confusion with the moving force F.

Also note that the factor $\langle \dot{q}^2 \rangle$ in the expression (39) for the fluctuation of the static force F_0 is logarithmic divergence. And from (41) we know that the vacuum energy density factor $\langle T_{00} \rangle$ is k^2 divergent, so the fluctuation of F_0 is $k^2 \ln k$ divergent, while the fluctuation of effective mass, which contains the divergent term $\dot{q}^2$, is only $\ln k$ divergent. This implies that the fluctuation of the mirror's position and velocity is mainly determined by the fluctuation of force. So we can further use the mean value of the effective mass $\langle M_{\mathrm{eff}} \rangle$ to substitute M_{eff} to simplify the above equation (101):

$$\frac{dv}{dt} + \beta v = \frac{F_0}{\langle M_{\mathrm{eff}} \rangle}, \tag{102}$$

where $v = \frac{dX}{dt}$ is the mirror's velocity and

$$\beta = \left(\frac{\epsilon^4}{4\pi} \int_0^{+\infty} dk \frac{k^3}{(k^2 - \Omega^2)^2 + \frac{\epsilon^4}{4} k^2} \right) / \langle M_{\mathrm{eff}} \rangle \tag{103}$$

is the damping coefficient; and we have substituted for $\langle F \rangle$ by equation (72). Both the numerator and denominator of (103) contain divergent integrals over k. To make the meaning of $\frac{\infty}{\infty}$ type quantities precise, we first truncate both the integrals by the same high frequency cutoff $k = \Lambda$, and then take Λ to infinity. Then the logarithmic divergence of the damping force is magically canceled by the logarithmic divergence of the mirror's effective mass, and we get that

$$\beta = \epsilon^2. \tag{104}$$

Equation (102) is a Langevin-type equation. The solution for the velocity in this equation with initial condition $v(0) = 0$ is

$$v(t) = \frac{1}{\langle M_{\mathrm{eff}} \rangle} e^{-\beta t} \int_0^t dt' e^{\beta t'} F_0(t'). \tag{105}$$

Then the fluctuation of the velocity is

$$\begin{aligned} \sigma_v(t) &= \langle v(t)^2 \rangle - \langle v(t) \rangle^2 \\ &= \frac{1}{\langle M_{\mathrm{eff}} \rangle^2} e^{-2\beta t} \int_0^t \int_0^t dt_1 dt_2 e^{\beta(t_1 + t_2)} \\ &\quad \cdot \mathrm{Corr}(F_0(t_1), F_0(t_2)). \end{aligned} \tag{106}$$

Inserting (52) into the above expression, we get

$$\begin{aligned} \sigma_v(t) = &\frac{\epsilon^4}{16\pi^2 \langle M_{\mathrm{eff}} \rangle^2} \int_{-\infty}^{+\infty} dk|k| \\ &\cdot \int_{-\infty}^{+\infty} dk' \frac{|k'|^3}{(k'^2 - \Omega^2)^2 + \frac{\epsilon^4}{4} k'^2} \cdot \frac{1}{\beta^2 + (|k| + |k'|)^2} \\ &\cdot (1 - 2e^{-\beta t} \cos(|k| + |k'|)t + e^{-2\beta t}). \end{aligned} \tag{107}$$

What we are interested is the large time behavior of the mirror, so now we let t be large enough such that $\beta t \gg 1$; then in such a limit the mirror would be in equilibrium with the quantum scalar field. In this limit, the above expression reduces to

$$\begin{aligned} \sigma_v(t) = &\frac{\epsilon^4}{16\pi^2 \langle M_{\mathrm{eff}} \rangle^2} \int_{-\infty}^{+\infty} dk|k| \int_{-\infty}^{+\infty} dk' \\ &\cdot \frac{|k'|^3}{(k'^2 - \Omega^2)^2 + \frac{\epsilon^4}{4} k'^2} \cdot \frac{1}{\beta^2 + (|k| + |k'|)^2}. \end{aligned} \tag{108}$$

The k' integral in (108) is convergent and is proportional to

$$\begin{aligned} &\int_{-\infty}^{+\infty} dk' \frac{|k'|^3}{(k'^2 - \Omega^2)^2 + \frac{\epsilon^4}{4} k'^2} \cdot \frac{1}{\beta^2 + (|k| + |k'|)^2} \\ &\sim \frac{2 \ln k}{k^2}, \quad \text{as } k \to +\infty. \end{aligned} \tag{109}$$

Therefore, the whole integral over k, which is divergent, is proportional to

$$4 \int^{\Lambda} dk \frac{\ln k}{k} \sim 2(\ln \Lambda)^2, \quad \text{as } \Lambda \to +\infty. \tag{110}$$

According to (40), we have the prefactor

$$\frac{\epsilon^4}{16\pi^2 \langle M_{\mathrm{eff}} \rangle^2} \sim 1/(\ln \Lambda)^2, \quad \text{as } \Lambda \to +\infty. \tag{111}$$

Therefore, after taking the limit $\Lambda \to +\infty$, we obtain

$$\sigma_v = 2, \tag{112}$$

which means that the standard deviation of the velocity is $\sqrt{2}$ times the light speed! This result is clearly unphysical. It results from our unphysical approximation scheme, namely the small velocity assumption we made in the beginning. In fact, when the velocity of the mirror becomes large, the damping force βv would not be linear in velocity and the small velocity approximation is not valid any more. More precisely, from (75) we know that the damping coefficient is velocity dependent:

$$\beta = \frac{1}{2v}\left(\left(\frac{1+v}{1-v}\right)^{1/2} - \left(\frac{1-v}{1+v}\right)^{1/2}\right)\epsilon^2$$
$$= \left(1 + \frac{v^2}{2} + \frac{3v^4}{8} + \frac{5v^6}{16} + \cdots\right)\epsilon^2, \quad (113)$$

which reduces to (104) at a small velocity approximation. Therefore, when the mirror's velocity approaches 1, the damping coefficient would go to infinity to make sure that the mirror's velocity never reaches the light speed 1. If we further fully consider the relativistic effect, the increased mirror's "relativistic mass" would just make the result even smaller [see the γ^3 factor in (99)]. Therefore, we can confidently conclude that

$$\sigma_v < 1, \quad (114)$$

which means the mirror's velocity will oscillate wildly due to the fluctuation of the quantum field vacuum. However, we will see that this wild oscillation is confined in a small region, that is, the mirror does not diffuse like a Brownian particle.

To prove this, let us calculate the mean-squared displacement of the mirror. Strictly speaking, we need to solve the relativistic equation of motion of the mirror. But the relativistic calculation is too messy. Fortunately, we can continue using the nonrelativistic Newtonian equation (102) to calculate the mean-squared displacement. The result is not the true answer but an upper bound of the true answer because when we replace the Newtonian equation (102) by the relativistic equation of motion (99), the mirror would become heavier [due to the γ^3 factor in (99)] and the damping force would become stronger [see (113)].

Now let us perform the calculation. Integrating (105) with time, we obtain the solution of the position of the mirror for the initial condition $X(0) = 0$ and $v(0) = 0$:

$$X(t) = \frac{1}{\langle M_{\text{eff}}\rangle}\int_0^t dt' e^{-\beta t'}\int_0^{t'} dt'' e^{\beta t''} F_0(t''). \quad (115)$$

Then the mean-squared displacement of the mirror is given by

$$\sigma_X(t) = \langle X(t)^2\rangle - \langle X(t)\rangle^2$$
$$= \frac{1}{\langle M_{\text{eff}}\rangle^2}\int_0^t dt_1 e^{-\beta t_1}\int_0^{t_1} dt_2 e^{\beta t_2}$$
$$\cdot \int_0^t dt_3 e^{-\beta t_3}\int_0^{t_3} dt_4 e^{\beta t_4}\text{Corr}(F_0(t_2), F_0(t_4)). \quad (116)$$

Inserting (52) into the above expression, we get

$$\sigma_X(t) = \frac{\epsilon^4}{16\pi^2\langle M_{\text{eff}}\rangle^2}\int_{-\infty}^{+\infty} dk|k|\int_{-\infty}^{+\infty} dk'$$
$$\cdot \frac{|k'|^3}{(k'^2-\Omega^2)^2 + \frac{\epsilon^4}{4}k'^2}\cdot\frac{1}{\beta^2 + (|k|+|k'|)^2}$$
$$\cdot\left[\frac{1}{\beta^2}(1-e^{-\beta t})^2 + \frac{4\sin^2(\frac{|k|+|k'|}{2}t)}{(|k|+|k'|)^2}\right.$$
$$\left.-\frac{1}{\beta}(1-e^{-\beta t})\frac{2\sin(|k|+|k'|)t}{|k|+|k'|}\right]. \quad (117)$$

The double integral of the last two terms over k and k' is convergent, but the effective mass in the denominator is divergent. So the last two terms give no contribution to the mean-squared displacement when we take the limit, and the above expression reduces to

$$\sigma_X(t) = \left(\frac{\epsilon^4}{16\pi^2\langle M_{\text{eff}}\rangle^2}\int_{-\infty}^{+\infty} dk|k|\int_{-\infty}^{+\infty} dk'\right.$$
$$\left.\cdot \frac{|k'|^3}{(k'^2-\Omega^2)^2 + \frac{\epsilon^4}{4}k'^2}\cdot\frac{1}{\beta^2 + (|k|+|k'|)^2}\right)$$
$$\times\left[\frac{1}{\beta^2}(1-e^{-\beta t})^2\right]. \quad (118)$$

Note that the expression inside the parentheses is just equation (108), the mean-squared velocity σ_v, which is less than 1 as we concluded in (114). Thus, we can further conclude that the mean-squared displacement

$$\sigma_X(t) < \frac{1}{\beta^2}(1-e^{-\beta t})^2, \quad (119)$$

or, equivalently, the standard deviation of the mirror's position, grows with time as

$$\Delta X(t) < \frac{1}{\beta}(1-e^{-\beta t}), \quad (120)$$

where β is the damping coefficient. When t is small, we have

$$\Delta X(t) < t, \quad (121)$$

which implies that just after we release the mirror from rest, it starts to diffuse with almost the speed of light! However, as time grows, i.e., when $t \to +\infty$, we always have

$$\Delta X(t) < \frac{1}{\beta}, \quad (122)$$

which means that the diffusion of the mirror does not continue to increase and its fluctuating motion is confined in the small region $(-\frac{1}{\beta}, \frac{1}{\beta})$! The length of this region is inversely proportional to the damping coefficient, which is physically reasonable because stronger damping would resist the mirror's motion and thus reduce the size of its fluctuating region.

Note also that the damping coefficient β is related to the coupling constant ϵ by Eq. (104) or, more precisely, by Eq. (113), which implies that the stronger the coupling, the higher the damping and thus the smaller the range of the fluctuating motion. One might be suspicious of this result since it means that when the coupling ϵ goes to 0, the fluctuating range would go to infinity. But if the coupling constant $\epsilon = 0$, i.e., there is no interaction between the field and the mirror at all, the mirror should not do any fluctuating motion. It should just sit at the location $x = 0$. However, this is not a contradiction but a manifestation of the discontinuity of the expression (117) of the mean-squared displacement at $\epsilon = 0$. In fact, if $\epsilon = 0$, the effective mass M_{eff} reduces to the mirror's finite bare mass M and thus the prefactor $\frac{\epsilon^4}{16\pi^2 \langle M_{\text{eff}} \rangle^2}$ is just 0, which makes sure that our whole expression (117) is 0. So our result does satisfy the "no interaction implies no fluctuation" requirement.

VIII. DIFFERENCE WITH BROWNIAN MOTION: THE STRONGLY ANTICORRELATION NATURE OF QUANTUM VACUUM FLUCTUATIONS

We have concluded in the last section that the fluctuating motion of our mirror would be confined in the small region $(-\frac{1}{\beta}, \frac{1}{\beta})$. To better understand the underlying physical mechanism, we compare the fluctuating motion of our mirror with a Brownian particle.

Consider a one-dimensional Brownian particle whose motion is also described by a Langevin-type equation:

$$\frac{dv}{dt} + \beta v = \frac{F_B}{m}, \tag{123}$$

where m is the mass of the Brownian particle, β is the damping coefficient, and F_B is the stochastic fluctuating force. The only nontrivial difference between the above equation of motion (123) for the Brownian particle and the equation of motion (102) for our mirror is the different stochastic property of the driven force F_B and F_0.

For the Brownian particle, the force F_B is usually assumed to have a Gaussian probability distribution with a correlation function,

$$\text{Corr}(F_B(t_1), F_B(t_2)) = C\delta(t_1 - t_2), \tag{124}$$

where C is a constant characterizing the strength of the force. The δ-function form of the correlation is an approximation. It means that the force at the time t_1 is completely uncorrelated with the force at any other time t_2. For the motion of a "macroscopic" particle at a much larger time scale compared with the collision time of the molecules, the δ correlation becomes exact.

However, the force correlation function (52) for our mirror is quite different. There are two terms in (52); each of them is a product of two integrals. The first integral of the first term $\int_{-\infty}^{+\infty} |k| e^{-i|k|(t_1-t_2)} dk$ does not converge under the usual definition of the improper integral. However, we can make it converge by analytic continuation, i.e., redefine the integral as

$$\begin{aligned} f_1(\Delta t) &= \int_{-\infty}^{+\infty} |k| e^{-i|k|(t_1-t_2)} dk \\ &= \lim_{\eta\to 0^+} \int_{-\infty}^{+\infty} |k| e^{-i|k|(t_1-t_2-i\eta)} dk = -\frac{2}{\Delta t^2}, \end{aligned} \tag{125}$$

where $\Delta t = t_1 - t_2$. The second integral of the first term

$$f_2(\Delta t) = \int_{-\infty}^{+\infty} \frac{|k'|^3 e^{-i|k'|(t_1-t_2)}}{(k'^2 - \Omega^2)^2 + \frac{\epsilon^4}{4} k'^2} dk' \tag{126}$$

conditionally converges to a finite positive value under the usual definition of the improper integral. Further, when $\Delta t \to 0$, $f_2(\Delta t)$ logarithmically diverges to $+\infty$. Thus, the first term $f_1 f_2 \to -\infty$ when $\Delta t \to 0$. The second term contains another two integrals, each of which approaches to 0 when $\Delta t \to 0$. Therefore, due to continuity, the correlation function is always negative for a small enough Δt, and its absolute value can be arbitrarily large, i.e., the force has strong anticorrelation at a small time scale. This strong anticorrelation implies that if the force at some time t_1 is in the positive x direction, after some very short time Δt, the force would be in the negative x direction. On average, the infinite fluctuations of force at different times are strongly canceled. This is why although the force fluctuation at any specific instant is infinite, we still obtained the finite fluctuation of the force average in Sec. IV. Here, it is necessary to point out that Ford and Roman [16] have also noted and discussed this kind of anticorrelation property of the Minkowski vacuum. Unlike our direct calculations above, they used a sampling function with a characteristic width a to smear out the singularities. The anticorrelations we obtained above agree with their result in the limit as a approaches zero. In addition, Parkinson and Ford investigated a related anticorrelation effect in [17].

The fluctuating motion of the Brownian particle and of the motion of our mirror are different under these two different stochastic fluctuating forces. In particular, the mean-squared displacement for the Brownian particle grows linearly with time:

$$\sigma_X(t) \sim \frac{C}{\beta^2 m^2} t \quad \text{when } t \to +\infty, \tag{127}$$

which is different from the bounded fluctuating motion of our mirror [see (122)]. In other words, the Brownian particles would exhibit diffusion while our mirror would be confined in a small region.

IX. DISCUSSIONS AND CONCLUSIONS

We have seen that in our nongravitational mirror system, the value of quantum vacuum energy does have physical significance in how the fluctuations in the force on the mirror depend on this value. It provides an infinite fluctuating force acting on the mirror and gives infinite instantaneous acceleration of the mirror. Astonishingly, this infinity makes sense in that, under this fluctuating force, the mirror's fluctuating motion would not diverge but would be confined in a small region due to the special properties of vacuum friction and the anticorrelation of quantum vacuum fluctuations.

It is clear from the calculations that our mirror does not exhibit Brownian motion and thus no diffusion happens. Gour and Sriramkumar [10] also studied a mirror interacting with the quantum vacuum using the mirror model (1) with an artificial high frequency cutoff. However, they concluded that the mirror would experience Brownian motion and thus exhibit diffusion. This conclusion is based on the assumption that "The stochastic force is completely independent of the position of the Brownian particle" (page 20 of [10]). This assumption is intrinsically equivalent to the Brownian motion correlation condition (124) that we have discussed in the last section. So it is not surprising that this assumption leads to their Brownian motion conclusion. It can be shown by direct calculations that the correlation between the position and the stochastic force is not zero but is highly anticorrelated. Following a similar procedure as we did in this paper, it is not difficult to reproduce the result of a bounded fluctuating motion of the mirror without diffusion.

Jaekel and Reynaud [11] also discussed this issue using an approach based on fluctuation-dissipation theorems. They concluded that a mirror coupled to the Minkowski vacuum would exhibit diffusion that is characterized by a logarithmically increasing behavior at long times. In addition, Ford etc. [18–20] investigated fluctuating motions of a particle or a mirror in modified quantum vacuums other than the Minkowski vacuum, such as in the presence of boundaries [18,19] and in Robertson-Walker spacetimes [20]. They also obtained the similar logarithmically increasing quantum diffusion results.

Let us comment on the differences of our results from the works of all of the authors above. The differences mainly come from the fact that (I) we are using different mirror models and thus (II) different methods of handling infinities or singularities. Concretely speaking, the above authors are using the perfectly reflecting mirror model (1), which is point like without any internal structure, by simply imposing a boundary condition. However, a realistic mirror must have some internal structures interacting with the photon field. Our mirror is still point like but with an internal structure: an internal harmonic oscillator that makes it work like a real mirror.

This intrinsic difference results in distinct methods of handling infinities. It is well known that treating particles as point like can result in divergences even in classical field theory, so it is not surprising that they would lead to divergences or singularities. In particular, the authors of [10] and [11] had to treat the infinities by introducing an artificial high frequency cutoff and their results are cutoff dependent; the authors of [18–20] regularized the singularities in the correlation functions by an integration by parts procedure. However, it is not clear what is the correct way to regularize these singular correlation functions to obtain finite results in the point-defined limit of ordinary quantum field theory. Unphysical results such as the "negative" fluctuations were obtained using covariant point separation regularization [21]. Similar negative mean-squared velocity and position fluctuations were also obtained in [18–20] by a nonrigorous integration by parts procedure, even though the authors interpreted these results as decreases of uncertainties in the position and velocity of quantum particles.

Our mirror model avoids these problems since the infinities disappear naturally even when we take the high frequency cutoff Λ to infinity. More precisely, when calculating the fluctuation of the mirror's position, the divergence of the mirror's instantaneous acceleration, which comes from the divergent vacuum energy density, is canceled by vacuum friction and strongly anticorrelated vacuum fluctuations. We are not directly dealing with the actual value of the vacuum energy density, but we find that the infinite value is acceptable in our nongravitational mirror system in the sense that this infinity only results in a finite observable effect. Whether or not the infinite naive expectation value of T_{00} has direct physical effects or could be eliminated by renormalization of the cosmological constant or whether a detailed treatment of the effects of this infinity on the gravitational field could also disappear if one concentrated on observable quantities will be the subject of further work.

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APPENDIX A: DERIVATION OF THE MIRROR'S EQUATION OF MOTION BY DIRECTLY VARYING $X(t)$

We first rewrite the action (84) as

$$S = \frac{1}{2}\iint \left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right) dtdx + \int \left(-M + \frac{1}{2}\frac{\dot{q}^2}{1-\dot{X}^2} - \frac{1}{2}\Omega^2 q^2\right)\sqrt{1-\dot{X}^2}dt + \epsilon\int d(q\phi) - \epsilon\int \dot{q}\phi(t,X(t))dt. \tag{A1}$$

Varying the above action with respect to the mirror's position $X(t)$ yields

$$\delta S = \int \frac{\dot{X}\,\delta\dot{X}}{\sqrt{1-\dot{X}^2}}\left(M + \frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + \frac{1}{2}\Omega^2 q^2\right)dt - \epsilon\int \dot{q}\phi'(t,X(t))\delta X dt = -\int \delta X\left[\frac{d}{dt}(\gamma M_{\mathrm{eff}}\dot{X}) + \epsilon\dot{q}\phi'(t,X(t))\right]dt. \tag{A2}$$

Letting $\delta S = 0$, we obtain exactly the same equation of motion (90).

APPENDIX B: DERIVATION OF THE STRESS-ENERGY TENSOR

The stress-energy tensor can be determined by the functional derivative of the total action S of the system with respect to the background metric $g_{\mu\nu}$:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{B1}$$

To start, let us rewrite the action (84) in a generic background metric $g_{\mu\nu}$ as follows:

$$S = -\frac{1}{2}\iint \sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi dtdx - M\int\sqrt{-g_{\mu\nu}(t,X(t))dX^\mu dX^\nu} + \frac{1}{2}\int\left[\left(\frac{dq}{d\tau}\right)^2 - \Omega^2 q^2\right]d\tau + \epsilon\int\frac{d\phi}{d\tau}(t(\tau),X(t(\tau)))q(t(\tau))d\tau, \tag{B2}$$

where τ is the proper time along the mirror trajectory that is related to the global time coordinate t by

$$d\tau = \sqrt{-g_{\mu\nu}(t,x)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}dt \tag{B3}$$

and the last three terms in (B2) are integrated along the mirror trajectory. Here we are using the sign convention $(-,+)$. To obtain the functional derivative, we first change the variable τ to the global time coordinate t by using (B3) and then transform the first two single integrals in the action (B2) into double integrals, i.e., extend the domain of integration from the line $x = X(t)$ to the whole spacetime by inserting Dirac delta functions:

$$S = -\frac{1}{2}\iint\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi dtdx - M\iint\sqrt{-g_{\mu\nu}(t,x)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}\delta(x-X(t))dtdx + \frac{1}{2}\iint\left[\frac{1}{\sqrt{-g_{\mu\nu}(t,x)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}}\left(\frac{dq}{dt}\right)^2 - \Omega^2q^2\sqrt{-g_{\mu\nu}(t,x)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}\right]\cdot\delta(x-X(t))dtdx + \epsilon\int q(t)d\phi(t,X(t)). \tag{B4}$$

Varying the above action with respect to $g_{\mu\nu}$ gives

$$\delta S = -\frac{1}{2}\iint\delta(\sqrt{-g})g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi dtdx - \frac{1}{2}\iint\sqrt{-g}(\delta g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi dtdx - M\iint\frac{(\delta g_{\mu\nu})\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}{2\sqrt{-g_{\mu\nu}(t,x)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}}\delta(x-X(t))dtdx + \frac{1}{2}\iint\left[\frac{1}{-g_{\mu\nu}(t,x)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}\left(\frac{dq}{dt}\right)^2 - \Omega^2q^2\right]\cdot\frac{(\delta g_{\mu\nu})\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}{2\sqrt{-g_{\mu\nu}(t,x)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}}\delta(x-X(t))dtdx. \tag{B5}$$

Also, we have

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}, \quad \delta g^{\mu\nu} = -g^{\mu\lambda}g^{\nu\rho}\delta g_{\lambda\rho}. \tag{B6}$$

Plugging the above two relations into (B5), we get

$$\delta S = \frac{1}{2}\iint\sqrt{-g}\left(\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}g^{\lambda\rho}\partial_\lambda\phi\partial_\rho\phi + \frac{1}{\sqrt{-g}}\left(M + \frac{1}{2}\cdot\frac{1}{-g_{\mu\nu}\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}\left(\frac{dq}{dt}\right)^2 + \frac{1}{2}\Omega q^2\right)\cdot\frac{\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}{\sqrt{-g_{\mu\nu}\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}}\delta(x-X(t))\right)\delta g_{\mu\nu}dtdx. \tag{B7}$$

Therefore, the stress-energy tensor of the whole system is

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \\ &= \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}g^{\lambda\rho}\partial_\lambda\phi\partial_\rho\phi \\ &\quad + \frac{1}{\sqrt{-g}}\left(M + \frac{1}{2}\cdot\frac{1}{-g_{\mu\nu}\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}\left(\frac{dq}{dt}\right)^2 + \frac{1}{2}\Omega q^2\right) \\ &\quad \cdot \frac{\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}{\sqrt{-g_{\mu\nu}\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}}\delta(x - X(t)). \end{aligned} \tag{B8}$$

For the case we are considering, the background metric is flat, i.e., $g_{\mu\nu} = \eta_{\mu\nu}$; then the above expression becomes

$$\begin{aligned} T^{\mu\nu} &= \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\rho}\partial_\lambda\phi\partial_\rho\phi \\ &\quad + M_{\mathrm{eff}}\frac{\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}{\sqrt{1-\dot{X}^2}}\delta(x - X(t)), \end{aligned} \tag{B9}$$

where the effective mass is

$$M_{\mathrm{eff}} = M + \frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + \frac{1}{2}\Omega^2 q^2. \tag{B10}$$

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