PHYSICAL REVIEW D **89,** 085006 (2014)

Theory of $U(1)$ gauged Q-balls revisited

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(Received 11 November 2013; published 3 April 2014)

In this paper, the main properties of $(3+1)$-dimensional $U(1)$ gauged Q balls are examined. In particular, it is shown that the relation $\frac{dE}{dQ}=\omega$ holds for such gauged Q balls in the general case. As a consequence, it is shown that the well-known estimate for the maximal charge of stable gauged Q balls was derived by means of an inconsistent procedure and cannot be considered as correct. A simple method for obtaining the main characteristics of gauged Q balls using only the nongauged background solution for the scalar field in the case, when the backreaction of the gauge field on the scalar field is small and the linearized theory can be used, is proposed. The criteria of applicability of the linearized theory, which do not reduce to the demand of the smallness of the coupling constant, are established. Some interesting properties of gauged Q balls, as well as the advantages of the proposed method, are demonstrated by the example of two models, admitting, in the linear approximation in the perturbations, exact analytic solutions for gauged Q balls.

DOI: 10.1103/PhysRevD.89.085006 PACS numbers: 11.27.+d, 04.20.Jb

I. INTRODUCTION

Nontopological solitons in a theory of a complex scalar field with global $U(1)$ symmetry, proposed in Ref. [1] and known as Q balls [2], are widely discussed in the literature. A simplest generalization of Q balls to the gauged case, i.e., from the global $U(1)$ symmetry to the gauge $U(1)$ symmetry, is straightforward. Although the existence of gauged Q balls was put in question in the well-known paper of Ref. [2], there are some papers devoted to this subject. The most known paper is Ref. [3], in which gauged Q balls were examined analytically and numerically (for simplicity, from here on, we call $U(1)$ gauged Q balls "gauged Q balls," unless otherwise stated). To our knowledge, for the first time, an analysis of what is now called gauged Q balls was made in Ref. [4]. In this remarkable paper, not only were the conditions for the existence of such Q balls derived, but also the case of small coupling of the gauge field to the scalar field was discussed, the corresponding linearized equations of motion were obtained, and even an approximate solution to these equations was found. One can also recall Ref. [5], in which gauged nontopological soliton solutions were obtained numerically. The scalar field part of the model of Ref. [5] coincides with the two-fields model proposed in Ref. [6]; that is why the gauged nontopological soliton solution found in Ref. [5] is not a gauged Q ball in the sense of Coleman's definition of Q balls [2], although it is of the same kind. Another interesting paper is Ref. [7], in which not only numerical but also approximate analytic solutions for gauged Q balls and Q shells were obtained. For a certain class of scalar field potentials and for a sufficiently weak interaction between the scalar field and the $U(1)$ gauge field, the existence of gauged Q balls was proven in Refs. [8,9] in a mathematically rigorous way. A solution for a gauged Q ball in such a case of a weak coupling of the $U(1)$ gauge field to the scalar field (i.e., an exact solution in the linear approximation above the background solution) was recently found in Ref. [10] in the gauged version of the model proposed in Ref. [11]; the solution for the case in which the coupling in this model is not weak was studied in Ref. [10] numerically.

Meanwhile, we think that there is a lack of understanding of the physical properties of $U(1)$ gauged Q balls, so in this paper, we present some results concerning both the general properties of gauged Q balls and the particular case of the small backreaction of the gauge field. The paper is organized as follows. In Sec. II, we present the general setup and introduce the notations that will be used throughout the paper. In Sec. III, we study the main properties of gauged Q balls; in particular, we prove that the relation $\frac{dE}{dQ}=\omega$ holds for *any* gauged Q ball. We also discuss different issues concerning the stability of such Q balls and show that the well-known statement about the existence of a maximal charge of stable gauged Q balls, which was made in Ref. [3], is incorrect. In Sec. IV, we thoroughly examine the case in which the backreaction of the gauge field on the scalar field is small. We propose a very useful method for studying the gauged Q ball properties *without* solving the whole system of linearized equations of motion for the fields. The resulting compact formulas allow us to simplify the calculations considerably. We show that the small parameter of the theory does not coincide with e^2 in the general case (here, e is the coupling constant of the gauge field to the scalar field), which implies that the fulfillment of the condition $e^2 \ll 1$ does not guarantee that the linearized theory can be used for calculations. These results are illustrated by examples of two models providing, in the linear approximation in the perturbations, exact analytic solutions for gauged Q balls.

II. SETUP

We consider the action, describing the simplest $U(1)$ gauge-invariant four-dimensional scalar field theory, in the form

$$S = \int d^4x \Big((\partial^\mu \phi^* - ieA^\mu \phi^*)(\partial_\mu \phi + ieA_\mu \phi) - V(\phi^*\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Big) \quad (1)$$

and take the standard spherically symmetric ansatz for the fields describing a gauged Q ball,

$$\phi(t,\vec{x}) = e^{i\omega t} f(r), \quad f(r)|_{r\to\infty} \to 0, \quad \frac{df(r)}{dr}\Big|_{r=0} = 0, \quad (2)$$

$$A_0(t,\vec{x}) = A_0(r), \quad A_0(r)|_{r\to\infty} \to 0, \quad \frac{dA_0(r)}{dr}\Big|_{r=0} = 0, \quad (3)$$

$$A_i(t,\vec{x}) \equiv 0, \quad (4)$$

where $r = \sqrt{\vec{x}^2}$ and $f(r)$, $A_0(r)$ are real functions. We suppose that the function $f(r)$ has no nodes and $f(0) > 0$.

Taking into account Eqs. (2)–(4), we can use the effective action

$$S_{\text{eff}} = \int d^3x \Big((\omega+g)^2 f^2 - \partial_i f \partial_i f - V(f) + \frac{1}{2e^2}\partial_i g \partial_i g \Big), \quad (5)$$

where $g = eA_0$, $V(f) = V(\phi^*\phi)$ [Eq. (2) implies that $\phi^*\phi = f^2$], instead of Eq. (1).[1] For the scalar field potential, the conditions

$$V(0) = 0, \qquad \frac{dV}{df}\Big|_{f=0} = 0 \quad (6)$$

are supposed to fulfill. It should be noted that gauged Q balls in theories with $V(f) \equiv 0$ or $V(f) = M^2 f^2$ do not exist; see, for example, Ref. [4].

The equations of motion, following from effective action (5), take the form

$$2e^2(\omega + g)f^2 = \Delta g, \quad (7)$$

$$2(\omega + g)^2 f + 2\Delta f - \frac{dV}{df} = 0, \quad (8)$$

where $\Delta = \sum_{i=1}^3 \partial_i \partial_i$. We define the charge of a gauged Q ball as

$$Q = 2\int d^3x (\omega + g) f^2. \quad (9)$$

Note that the physical charge is

$$Q_{\text{phys}} = eQ, \quad (10)$$

but for convenience, below we will use the charge Q defined by Eq. (9), not Q_{phys}.

It was shown in Ref. [3] that for a gauged Q ball solution the sign of $\omega + g$ always coincides with the sign of ω. Because of the symmetry $\omega \to -\omega$, $g \to -g$ of the equations of motion, without loss of generality, for simplicity we can consider $\omega \geq 0$. In this case, according to Eq. (9), for $\omega > 0$ we get $Q > 0$. As it was shown in Ref. [4], $g \equiv 0$ for $\omega = 0$ and only a purely static solution for the scalar field can exist, so our choice $\omega \geq 0$ implies $Q \geq 0$.

The energy of a gauged Q ball at rest is defined by

$$E = \int d^3x \Big((\omega + g)^2 f^2 + \partial_i f \partial_i f + V(f) + \frac{1}{2e^2}\partial_i g \partial_i g \Big). \quad (11)$$

III. SOME GENERAL PROPERTIES OF $U(1)$ GAUGED Q BALLS

A. $\frac{dE}{dQ}$ for gauged Q balls

It is well known that for ordinary (nongauged) Q balls the relation $\frac{dE}{dQ} = \omega$ holds. We have failed to find any note about the validity of this or an analogous relation for Abelian gauged Q balls in the literature on this subject. So, below, we present a simple proof of the fact that for gauged Q balls of form of Eqs. (2)–(4) in a theory described by action (1) the relation $\frac{dE}{dQ} = \omega$ also holds.

It is reasonable to suppose that the only parameter, which characterizes the charge and the energy for given $V(f)$ and e, is ω. Thus, differentiating the energy (11) with respect to ω, we get

[1] Though the Q ball solution is supposed to be spherically symmetric, sometimes it is convenient to keep the coordinates x^i and the corresponding volume element d^3x, especially in the calculations for which the spherical symmetry of the fields is not required.

$$\frac{dE}{d\omega} = \int \left(2\frac{d(\omega+g)}{d\omega}(\omega+g)f^2 + 2(\omega+g)^2 f\frac{df}{d\omega} + 2\partial_i f\partial_i \frac{df}{d\omega} + \frac{dV}{df}\frac{df}{d\omega} + \frac{1}{e^2}\partial_i g\partial_i \frac{dg}{d\omega}\right) d^3x = \int \left(2\frac{d(\omega+g)}{d\omega}(\omega+g)f^2 + 2(\omega+g)^2 f\frac{df}{d\omega} + \left(-2\Delta f + \frac{dV}{df}\right)\frac{df}{d\omega} + \frac{1}{e^2}\partial_i g\partial_i \frac{dg}{d\omega}\right) d^3x$$
$$= \int \left((\omega+g)\left(2\frac{d(\omega+g)}{d\omega}f^2 + 4(\omega+g)f\frac{df}{d\omega}\right) + \frac{1}{e^2}\partial_i g\partial_i \frac{dg}{d\omega}\right) d^3x, \tag{12}$$

where we have used Eq. (8). For convenience, let us use the notation $q = 2(\omega+g)f^2$. Equation (12) can be rewritten as

$$\frac{dE}{d\omega} = \int \left((\omega+g)\frac{dq}{d\omega} + \frac{1}{e^2}\partial_i g\partial_i \frac{dg}{d\omega}\right) d^3x = \omega\frac{dQ}{d\omega} + \int \left(g\frac{dq}{d\omega} + \frac{1}{e^2}\partial_i g\partial_i \frac{dg}{d\omega}\right) d^3x, \tag{13}$$

where, according to Eq. (9), $Q = \int q d^3x$. Equation (7) implies that $\frac{dq}{d\omega} = \frac{1}{e^2}\Delta\frac{dg}{d\omega}$. Substituting it into Eq. (13), we arrive at

$$\frac{dE}{d\omega} = \omega\frac{dQ}{d\omega} + \frac{1}{e^2}\int \left(g\Delta\frac{dg}{d\omega} + \partial_i g\partial_i \frac{dg}{d\omega}\right) d^3x. \tag{14}$$

The integral in Eq. (14) is equal to zero, which can be easily seen by performing integration by parts (since $g|_{r\to\infty} \sim \frac{Q}{r}$ is assumed for gauged Q balls and consequently $\frac{dg}{d\omega}|_{r\to\infty} \sim \frac{dQ/d\omega}{r}$, the surface term, arising when an integration by parts is performed, obviously vanishes). Thus, we get $\frac{dE}{d\omega} = \omega\frac{dQ}{d\omega}$, which leads to

$$\frac{dE}{dQ} = \omega \tag{15}$$

for $\frac{dQ}{d\omega} \neq 0$. We stress that the fulfillment of Eq. (15) is the general property inherent to any $U(1)$ gauged Q ball. As for the points at which $\frac{dQ}{d\omega} = 0$ (and, consequently, $\frac{dE}{d\omega} = 0$), they correspond to the cusps on the $E(Q)$ diagram (like the cusps in Figs. 1 and 4 of Sec. IV), indicating the existence of a (locally) minimal or (locally) maximal charge. The cusps also separate different branches of the $E(Q)$ dependence.

B. Stability of gauged Q balls

In the absence of interactions with fermions, there are three types of stability of Q balls. They are the stability with respect to decay into free particles (i.e., quantum mechanical stability), the stability against decay into Q balls with smaller charges (i.e., against fission), and the classical stability (the stability with respect to small perturbations of fields). Here, we will not consider the classical stability, because, in the general case, its consistent study for gauged Q balls is a rather complicated task and lies beyond the scope of this paper. Here, we will consider only the quantum mechanical stability and stability against fission.

1. *Quantum mechanical stability and maximal charge of gauged Q-balls*

We start with the stability with respect to decay into free scalar particles. Suppose that there exist free particles of mass $M = \sqrt{\frac{1}{2}\frac{d^2V}{df^2}|_{f\equiv 0}}$ in the theory at hand [all the reasonings presented below are based on the assumption that there are no extra fields except those in action (1); otherwise, the situation can be more complicated]. In this case, the criterion for Q ball stability looks very simple,

$$E(Q) < MQ, \tag{16}$$

where $E(Q)$ is the energy of a Q ball with the charge Q.

It is necessary to note that, as it was shown in Refs. [3,4], the inequality

$$\omega < M$$

should hold for a Q ball in such a theory. Indeed, the existence of free scalar particles of mass M implies that the relevant scalar field part of the action has the form

$$S_{\text{scalar}} \approx \int d^4x(\partial^\mu\phi^*\partial_\mu\phi - M^2\phi^*\phi) \tag{17}$$

for small values of $\phi^*\phi$. The latter means that for any Q ball in this theory, satisfying Eqs. (2) and (3), inequality $\omega < M$ must hold; otherwise, the corresponding solution to Eq. (8) does not fall off at infinity rapidly enough to ensure the finiteness of the Q ball charge and energy.

An interesting observation is that gauged Q balls (as well as nongauged Q balls) cannot *emit* free scalar particles; they can only *decay* into such particles. Indeed, since the charge of a free particle in the theory at hand is $Q_p = 1$, for a Q ball, we get

$$E(Q+N)=E(Q)+\int_Q^{Q+N}\frac{dE}{d\tilde{Q}}d\tilde{Q} < E(Q)+M\int_Q^{Q+N}d\tilde{Q}=E(Q)+MN, \quad (18)$$

where N stands for the number of emitted particles. We see that the emission of $N > 0$ free scalar particles is energetically forbidden.[2] One can easily show that an analogous emission of scalar antiparticles (i.e., particles with $Q_p = -1$) is also energetically forbidden.

It this connection, we would like to comment on the method of derivation of the maximal charge of stable gauged Q balls, presented in Ref. [3]. Although the corresponding estimates for the maximal charge were obtained within the particular model of Ref. [3], they are used in many papers concerning gauged Q balls. It is stated in Ref. [3] that for a charge Q, such that for a Q ball of this charge the inequality $\frac{dE}{dQ} > M$ (in our notations) holds, it is energetically favorable to have a Q ball with the charge Q_{max} and $Q - Q_{\mathrm{max}}$ free scalar particles. The maximal charge Q_{max} is defined as a solution to equation $\frac{dE}{dQ} = M$. As it was shown above, for any gauged Q ball in a theory with $\frac{dV(\phi^*\phi)}{d(\phi^*\phi)}|_{\phi^*\phi=0} = M^2 > 0$, the inequality $\frac{dE}{dQ} = \omega < M$ holds, and Q balls with $\frac{dE}{dQ} \geq M$ can never exist (contrary to what was stated in Ref. [3]). If any approximate solution leads to the existence of gauged Q balls with $\frac{dE}{dQ} \geq M$ in a theory admitting the existence of free particles of mass M, such an approximate solution is not valid. Thus, the procedure used in Ref. [3] for estimating the value of the maximal charge of stable gauged Q balls contradicts the main properties of gauged Q balls and cannot be considered as correct, as well as the consequent statement about the existence of the maximal charge.[3]

Of course, stable gauged Q balls with maximal charges may exist. As in the nongauged case (see, for example, Refs. [13,14]), the existence of the maximal charge in the gauged case can be determined by the form of the scalar field potential or by the values of the model parameters, and such a (locally) maximal charge corresponds to a cusp in the $E(Q)$ dependence. Explicit examples, which will be presented in Sec. IV, clearly demonstrate it. Meanwhile, there are many models with the charge of an absolutely stable nongauged Q ball (classically stable, quantum mechanically stable, and stable against fission) not bound from above; see, for example, Refs. [13,15]. So, we do not see any evident physical reason why it should not be so for gauged Q balls. In this connection, we would like to comment on the common belief that the Coulomb repulsion makes a gauged Q ball with some large charge unstable. Indeed, the repulsion due to the gauge field exists. But let us look at Eq. (7), which can be rewritten as

$$\Delta g - 2e^2 f^2 g = 2e^2 \omega f^2. \quad (19)$$

This equation implies that the gauge field inside a Q ball is effectively massive, which dilutes the strength of the repulsion considerably in comparison with the Coulomb long-range repulsion. Thus, although its influence on the stability of gauged Q balls cannot be ignored, it should be reconsidered more accurately.

2. Stability against fission

Now, we turn to examining the stability against fission. For ordinary (nongauged) Q balls, the corresponding stability criterion takes the form

$$d^2E/dQ^2 < 0. \quad (20)$$

If $E(0) = 0$, then Eq. (20) clearly leads to

$$E(Q_1) + E(Q_2) > E(Q_1 + Q_2), \quad (21)$$

which implies that Q ball fission is energetically forbidden. But in many models, the $d^2E/dQ^2 < 0$ branches of the $E(Q)$ dependence are such that there exists a minimal charge $Q_{\mathrm{min}} \neq 0$: $E(Q_{\mathrm{min}}) = E_{\mathrm{min}} \neq 0$. In this case, one can try to redefine the function $E(Q)$ in the region $[0, Q_{\mathrm{min}}]$ in order to get a continuous and differentiable auxiliary function $E_{\mathrm{aux}}(Q)$: $E_{\mathrm{aux}}(0) = 0$, $E_{\mathrm{aux}}(Q)$ is a monotonically increasing function for $Q > 0$, $d^2E_{\mathrm{aux}}(Q)/dQ^2 < 0$, and $E_{\mathrm{aux}}(Q) = E(Q)$ for $Q \geq Q_{\mathrm{min}}$. If it is possible to construct such a function $E_{\mathrm{aux}}(Q)$, then inequality (21) is valid for Q_1, $Q_2 \geq 0$ and, consequently, for Q_1, $Q_2 \geq Q_{\mathrm{min}}$. Of course, such reasonings apply for gauged Q balls too.

It was shown in Ref. [13] that the necessary condition for the existence of the function $E_{\mathrm{aux}}(Q)$ is $\frac{E(Q_{\mathrm{min}})}{Q_{\mathrm{min}}} > \omega_{\mathrm{min}} = \frac{dE}{dQ}|_{Q=Q_{\mathrm{min}}}$. For nongauged Q balls, this relation always holds because the equality

[2]This reasoning works only for Q balls from *the same* branch of the $E(Q)$ dependence, transitions between Q balls from different branches (like those in Fig. 4 of Sec. IV) with the emission of free scalar particles, or/and antiparticles and vector particles (photons) are not energetically forbidden in the general case.

[3]In Ref. [12] it was observed that Q balls cannot emit free particles of mass M if the condition $\frac{dE}{dQ} = \omega < M$ is fulfilled.

$$E = \omega Q + \frac{2}{3}\int d^3x \partial_i f \partial_i f \quad (22)$$

holds for nongauged Q balls, leading to $E(Q_{\min}) = \omega_{\min}Q_{\min} + \frac{2}{3}\int d^3x\partial_i f\partial_i f > \omega_{\min}Q_{\min}$.

For $U(1)$ gauged Q balls, one obtains [3]

$$E = \omega Q + \int d^3x\left(\frac{2}{3}\partial_i f\partial_i f - \frac{1}{3e^2}\partial_i g\partial_i g\right). \quad (23)$$

This relation can be easily derived by applying the scale transformation technique of Ref. [16] to effective action (5), substituting the result into Eq. (11) [to exclude $V(f)$] and using equation of motion (7). One sees that, contrary to the case of ordinary Q balls (22), the integral in Eq. (23) is not positive definite. Thus, at this stage, we cannot make any conclusion about the stability against fission for gauged Q balls with $d^2E/dQ^2 < 0$ in the general case. Of course, one may simply check that $\frac{E(Q_{\min})}{Q_{\min}} > \omega_{\min}$ for a particular gauged Q ball solution. For example, if there exist free scalar particles of mass M in the theory under consideration and the part of the $E(Q)$ dependence with $d^2E/dQ^2 < 0$, which we are interested in, starts at $Q = Q_{\min} \neq 0$ and $E(Q_{\min}) \geq MQ_{\min}$ (as we will see below, it holds at least for one example that will be discussed in our paper later; see also Ref. [5] where this situation is realized), then Q balls from this branch are stable against fission. Indeed, $\omega_{\min} < M \leq \frac{E(Q_{\min})}{Q_{\min}}$; the latter means that, according to Ref. [13], we always can construct an auxiliary function $E_{\text{aux}}(Q)$ possessing the properties presented above. An explicit example of the function $E_{\text{aux}}(Q)$ can be found in Appendix A.

Here, we discuss another possibility. Suppose we have gauged Q ball solutions in a model with $e = e_x$ (without loss of generality, we consider $e_x > 0$) for the region of frequencies ω we are interested in. Let us also suppose that there exist nongauged Q ball solutions in this model with $e = 0$ for the same region of frequencies. In this case, one may assume that gauged Q ball solutions exist for any $0 < e < e_x$. This assumption is not obvious, and we cannot justify it in a mathematically rigorous way for the general case; meanwhile, as we will see explicitly in the next section, it is valid at least when the backreaction of the gauge field is small (this fact was also proven in Ref. [8] in a mathematically rigorous way for a certain class of the scalar field potentials). If the conditions presented above are fulfilled, then for a gauged Q ball in a theory with $e = e_x$, the inequality

$$E > \omega Q$$

holds. To show it, let us take the energy (11) and differentiate it with respect to the coupling constant e while keeping ω fixed. Performing calculations analogous to those made in Eqs. (12) and (13), we arrive at

$$\frac{dE}{de} = \omega\frac{dQ}{de} + \int\left(g\frac{dq}{de} - \frac{1}{e^3}\partial_i g\partial_i g + \frac{1}{e^2}\partial_i g\partial_i\frac{dg}{de}\right)d^3x. \quad (24)$$

Equation (7) implies that $\frac{dq}{de} = \frac{1}{e^2}\Delta\frac{dg}{de} - \frac{2q}{e}$; substituting it into Eq. (24), we obtain

$$\frac{dE}{de} = \omega\frac{dQ}{de} + \int\left(\frac{1}{e^2}g\Delta\frac{dg}{de} - \frac{2}{e}gq - \frac{1}{e^3}\partial_i g\partial_i g + \frac{1}{e^2}\partial_i g\partial_i\frac{dg}{de}\right)d^3x. \quad (25)$$

Substituting Eq. (7) into Eq. (25) and performing integrations by parts in the resulting integral (since $g|_{r\to\infty} \sim \frac{Q}{r}$ is assumed for gauged Q balls and consequently $\frac{dg}{de}|_{r\to\infty} \sim \frac{dQ/de}{r}$, the corresponding surface term vanishes again), we get

$$\frac{dE}{de} = \omega\frac{dQ}{de} + \frac{1}{e^3}\int \partial_i g\partial_i g d^3x. \quad (26)$$

Since e is supposed to be nonnegative and $g \sim e^2$ for $e \to 0$, we have

$$\frac{dE}{de} \geq \omega\frac{dQ}{de}. \quad (27)$$

Now, we integrate Eq. (27) in e from $e = 0$ to $e = e_x$ and obtain

$$E(Q(e_x), e_x) - \omega Q(e_x) > E(Q(0), 0) - \omega Q(0). \quad (28)$$

$Q(0)$ and $E(Q(0), 0)$ stand for the nongauged Q ball, for which $E(Q(0), 0) - \omega Q(0) > 0$ holds for any ω, leading to

$$E(Q(e_x), e_x) - \omega Q(e_x) > 0. \quad (29)$$

The latter means that $\frac{E(Q_{\min})}{Q_{\min}} > \omega_{\min}$ for such a gauged Q ball, which implies that, according to the reasonings presented above, the function $E_{\text{aux}}(Q)$ can be constructed and the stability against fission for Q balls corresponding to the $\frac{d^2E}{dQ^2} < 0$ branch of the $E(Q)$ dependence can be established.

It is clear that the gauged Q ball with $\frac{d^2E}{dQ^2} < 0$ cannot decay into a Q ball from the same branch with $\frac{d^2E}{dQ^2} < 0$ and an anti-Q ball (i.e., a Q ball with $\omega < 0$ and $Q < 0$). Indeed, for a gauged Q ball, we have

$$E(Q_2) = E(Q_2 + Q_1) - \int_{Q_2}^{Q_2+Q_1} \frac{dE}{d\tilde{Q}} d\tilde{Q} = E(Q_2 + Q_1) - \int_{Q_2}^{Q_2+Q_1} \omega(\tilde{Q}) d\tilde{Q}$$
$$< E(Q_2 + Q_1) - \int_{Q_2}^{Q_2+Q_1} \omega(\tilde{Q}) d\tilde{Q} + E(-Q_1) < E(Q_2 + Q_1) + E(-Q_1),$$

where $Q_1 > 0$ and $Q_2 > 0$, which means that such a decay is energetically forbidden.

IV. GAUGED Q BALLS WITH SMALL BACKREACTION OF THE GAUGE FIELD

A. Linearized equations of motion

It seems that it is very difficult, or even impossible, to find a model providing an exact analytic solution for a gauged Q ball in the general case. Meanwhile, if the backreaction of the gauge field is supposed to be small [$|g(r)| \ll \omega$, $|f(r) - f_0(r)| \ll f_0(r)$, where $f_0(r) = f_0(r, \omega)$ is a nongauged Q ball solution in the case $e = 0$], one can try to use the linear approximation in $g(r)$ and $\varphi(r) = f(r) - f_0(r)$ above the nongauged background solution, which simplifies the analysis. In this case, Eqs. (7) and (8) can be reduced to the form

$$\Delta g - 2e^2 \omega f_0^2 = 0, \tag{30}$$

$$\Delta\varphi + \omega^2 \varphi + 2\omega g f_0 - \frac{1}{2} \frac{d^2V}{df^2}\bigg|_{f=f_0} \varphi = 0, \tag{31}$$

where f_0 is defined as a solution to the equation

$$\omega^2 f_0 + \Delta f_0 - \frac{1}{2} \frac{dV}{df}\bigg|_{f=f_0} = 0 \tag{32}$$

and the condition $|\varphi(r)| \ll f_0(r)$ is supposed to hold for any r.[4] To our knowledge, for the first time, the system of equations (30) and (31) with (32) was analyzed in Ref. [4], in which the coupling constant e was assumed to be small. Note that here we do not put any restrictions on the possible values of e; we only assume that the fields g and φ are suppressed by the small factor proportional to e^2. Although this factor is proportional to e^2, it does not coincide with e^2 in the general case. This implies that the linearized theory above the background solution f_0, described by Eqs. (30) and (31), cannot be used when only $e^2 \ll 1$ holds—the fields g and φ should also remain small compared to ω and f_0, respectively. We will discuss this issue in detail later; see also a simple justification of this fact in Appendix B.

[4] In fact, this condition is too stringent and, as we will see later, can be relaxed.

B. Charge and the energy of gauged Q balls

Linearizing the charge (9) and the energy (11) with respect to the background solution $f_0(r)$, performing integration by parts, and using the linearized equations of motion, we arrive at

$$Q = Q_0 + \Delta Q = Q_0 + 4\pi \int_0^\infty dr r^2 (2g f_0^2 + 4\omega f_0 \varphi), \tag{33}$$

$$E = E_0 + \Delta E = E_0 + 4\pi\omega \int_0^\infty dr r^2 (g f_0^2 + 4\omega f_0 \varphi), \tag{34}$$

where Q_0 and E_0 are defined by Eqs. (9) and (11) with the background solution $f_0(r)$ for the scalar field and with $g \equiv 0$.

Now, let us calculate ΔQ and ΔE. To this end, let us take equation (32) and differentiate it with respect to ω. We get

$$2\omega f_0 + \omega^2 \frac{df_0}{d\omega} + \Delta \frac{df_0}{d\omega} - \frac{1}{2} \frac{d^2V}{df^2}\bigg|_{f=f_0} \frac{df_0}{d\omega} = 0. \tag{35}$$

Now we take Eq. (31), multiply it by $\frac{df_0}{d\omega}$, integrate over the spatial volume, and perform integration by parts in the term containing Δ. We get

$$\int \left(\varphi \left(\Delta \frac{df_0}{d\omega} + \omega^2 \frac{df_0}{d\omega} - \frac{1}{2} \frac{d^2V}{df^2}\bigg|_{f=f_0} \frac{df_0}{d\omega} \right) + 2\omega g f_0 \frac{df_0}{d\omega} \right) d^3x = 0. \tag{36}$$

Substituting Eq. (35) into Eq. (36), we arrive at

$$\omega \int \left(g f_0 \frac{df_0}{d\omega} - \varphi f_0 \right) d^3x = 0. \tag{37}$$

Now, let us consider the charge (33). According to Eq. (37),

$$\begin{aligned} \Delta Q &= 4\pi \int_0^\infty dr r^2 (2g f_0^2 + 4\omega f_0 \varphi) \\ &= 4\pi \int_0^\infty dr r^2 \left(2g f_0^2 + 4\omega g f_0 \frac{df_0}{d\omega} \right) \\ &= 4\pi \int_0^\infty dr r^2 g \frac{dq}{d\omega}, \end{aligned} \tag{38}$$

where q is defined as $q = 2\omega f_0^2$. The last integral can be transformed as

$$\int d^3xg\frac{dq}{d\omega} = \frac{d}{d\omega}\int d^3xgq - \int d^3xq\frac{dg}{d\omega} = \frac{d}{d\omega}\int d^3xgq - \int d^3x\frac{1}{e^2}\Delta g\frac{dg}{d\omega}$$
$$= \frac{d}{d\omega}\int d^3xgq - \int d^3x\frac{1}{e^2}g\Delta\frac{dg}{d\omega} = \frac{d}{d\omega}\int d^3xgq - \int d^3xg\frac{dq}{d\omega},$$

where we have used Eq. (30) and the relation $\Delta\frac{dg}{d\omega} = e^2\frac{dq}{d\omega}$, which follows from Eq. (30). Thus, we get

$$\int d^3xg\frac{dq}{d\omega} = \frac{1}{2}\frac{d}{d\omega}\int d^3xgq. \tag{39}$$

Let us define

$$I = \frac{1}{2}\int d^3xgq. \tag{40}$$

Then, from Eqs. (38) and (39), we get

$$\Delta Q = \frac{dI}{d\omega}. \tag{41}$$

Now, it is easy to show that

$$\Delta E = \omega\Delta Q - I = \omega\frac{dI}{d\omega} - I. \tag{42}$$

With the help of Eq. (30), the integral I can be expressed in the form

$$I = -\frac{1}{2e^2}\int d^3x\partial_i g\partial_i g, \tag{43}$$

which is nothing but the energy of the gauge field taken with the minus sign [see Eq. (11)].

Relations (41) and (42) allow us to check explicitly the validity of Eq. (15) in the linearized theory. Indeed,

$$\frac{d(E_0+\Delta E)}{d(Q_0+\Delta Q)} = \frac{\frac{d(E_0+\Delta E)}{d\omega}}{\frac{d(Q_0+\Delta Q)}{d\omega}} = \frac{\omega\frac{dQ_0}{d\omega} + \omega\frac{d^2I}{d\omega^2}}{\frac{dQ_0}{d\omega} + \frac{d^2I}{d\omega^2}} = \omega. \tag{44}$$

Now, we turn to the calculation of the integral I. First, we take Eq. (30). Given a background solution f_0, the spherically symmetric solution to Eq. (30) such that $g|_{r\to\infty} \to 0$, $\frac{dg}{dr}|_{r=0} = 0$ takes the form [4]

$$g = g(r) = -e^2\int_r^\infty q(y)ydy - e^2\frac{1}{r}\int_0^r q(y)y^2dy. \tag{45}$$

Substituting it into Eq. (40), we get

$$I = -2\pi e^2\left(\int_0^\infty q(r)r^2\int_r^\infty q(y)ydydr + \int_0^\infty q(r)r\int_0^r q(y)y^2dydr\right). \tag{46}$$

By performing integration by parts, it is easy to show that

$$\int_0^\infty q(r)r^2\int_r^\infty q(y)ydydr = \int_0^\infty q(r)r\int_0^r q(y)y^2dydr. \tag{47}$$

Thus, we arrive at

$$\frac{I}{4\pi} = -4e^2\omega^2\int_0^\infty f_0^2(r)r\int_0^r f_0^2(y)y^2dydr, \tag{48}$$

where we have used $q = 2\omega f_0^2$. Equivalently, Eq. (48) can be rewritten as

$$\frac{I}{4\pi} = -2e^2\omega^2\int_0^\infty\left(\int_0^r f_0^2(y)y^2dy\right)^2\frac{1}{r^2}dr. \tag{49}$$

From Eqs. (41), (42), and (48), we see that the charge and the energy of a gauged Q ball through the terms linear in e^2 can be calculated using only the background solution f_0 for the nongauged Q ball. The corresponding formulas look like

$$Q(\omega) = Q_0(\omega) + \frac{dI(\omega)}{d\omega}, \tag{50}$$

$$E(\omega) = E_0(\omega) + \omega\frac{dI(\omega)}{d\omega} - I(\omega), \tag{51}$$

$$I(\omega) = -16\pi e^2\omega^2\int_0^\infty f_0^2(r,\omega)r\int_0^r f_0^2(y,\omega)y^2dydr \tag{52}$$

where $Q_0(\omega)$ and $E_0(\omega)$ are the charge and the energy of the nongauged Q ball, respectively. Thus, to examine the main properties of gauged Q balls in a theory with a small parameter (proportional to e^2) standing for the backreaction of the gauge field, it is not necessary to solve explicitly the linearized differential equation (31), which is a rather complicated task and can be made only numerically in the general case. Instead of this, one can simply take the corresponding nongauged background solution $f_0(r,\omega)$, evaluate the double integral in Eq. (48) (numerically, in the general case) to get the function $I(\omega)$, and calculate the corresponding $E(Q)$ dependence.[5] We remind the reader that for obtaining Eqs. (50)–(52) we have used only the

[5]As we will see below, in most cases, the energy of a gauged Q ball at a given charge can be calculated using the formula that is even simpler than those in Eqs. (50) and (51).

supposition that φ and g are *exact* solutions to linearized equations of motion; the restriction $e^2 \ll 1$ has not been used.

C. Validity criteria for the linear approximation

It is clear that in a nonlinear theory the linear approximation above a background solution is valid if the corrections are much smaller than the background solution itself. In our case, this suggests that the relations

$$|g(r)| \ll \omega, \tag{53}$$

$$|\varphi(r)| \ll f_0(r) \tag{54}$$

should be fulfilled for any r. We start with the first relation. Equation (45) implies that $\frac{dg}{dr} \geq 0$ for any r and $g|_{r\to\infty} \to 0$ [of course, if the corresponding integrals in Eq. (45) converge, which is exactly our case]. This means that $|g(r)| \leq |g(0)|$ for any r. The value of $g(0)$ can be easily obtained from Eq. (45) and takes the form

$$g(0) = -2e^2\omega \int_0^\infty f_0^2(y)ydy. \tag{55}$$

In this case, Eq. (53) can be rewritten as

$$2e^2 \int_0^\infty f_0^2(y)ydy \ll 1. \tag{56}$$

This inequality implies that the natural small parameter of the theory is not simply e^2. Indeed, in principle, it is possible that even for a very small value of e^2 the integral $\int_0^\infty f_0^2(y)ydy$ is large enough and inequality (56) is not fulfilled and vice versa. As we will see below, the parameter $\frac{|g(0)|}{\omega}$ plays an important role in the estimation of the small parameter of the theory.

Now we turn to Eq. (54). As will be shown below by particular examples, it is quite possible that $\frac{|\varphi(r)|}{f_0(r)}$ grows with r (this happens in both models, which will be studied below; see also Appendix C, in which it is shown explicitly for a certain wide class of the scalar field potentials). Formally, the linear approximation breaks down at large r in such a case. So, there arises a question: is it possible to use Eqs. (50)–(52), which were obtained in the linear approximation, when it breaks down, though at large r? The answer is yes, and below we will justify why it is so.

To start with, let us suppose that there exists an exact solution $f(r)$ to Eqs. (7) and (8) for a given ω, as well as a solution f_0 to Eq. (32) with the same ω as the one in f. Now, we take Eqs. (9) and (11) and write the exact equations

$$\triangle Q = Q - Q_0 = \int d^3x(2gf_0^2 + 4\omega(f-f_0)f_0) + \int d^3x(2(\omega+g)(f-f_0)^2 + 4g(f-f_0)f_0), \tag{57}$$

$$\triangle E = E - E_0 = \int d^3x(\omega g f_0^2 + 4\omega^2(f-f_0)f_0) + \int d^3x\left(V(f) - V(f_0) - (f-f_0)\frac{dV}{df}\bigg|_{f=f_0}\right)$$
$$+ \int d^3x(\omega^2(f-f_0)^2 + 2\omega g(f-f_0)f_0 + \omega g(f-f_0)^2 + \partial_i(f-f_0)\partial_i(f-f_0)), \tag{58}$$

where we have used Eqs. (7) and (32) while performing integrations by parts. The functions f_0 and f are supposed to fall off rapidly at large r, so that $f - f_0$ also falls off rapidly. Now, let us assume that in the inner region $r \leq \hat{R}$, from which the main contribution to Q_0 and E_0 comes, $|g| \ll \omega$ and $|\varphi| = |f - f_0| \ll f_0$. The radius $\hat{R}$ can be defined as

$$\int_{\hat{R}}^\infty f_0^2 r^2 dr = \epsilon \int_0^{\hat{R}} f_0^2 r^2 dr, \tag{59}$$

with $\epsilon \ll 1$ and $\hat{R} = \hat{R}(\omega)$. In this case, in the inner region $r \leq \hat{R}$, the last integrals in the rhs of Eqs. (57) and (58) can be neglected in comparison with those containing only the linear terms in g and φ, whereas the second integral in the rhs of Eq. (58) is equal to zero in this approximation. The outer region $r > \hat{R}$ is supposed to be chosen such that the fields f and f_0 have very small absolute values inside it; see Eq. (59). In this case, even though φ can be of the order of f_0 or larger, due to negligibly small *absolute* values of the fields f and f_0, we have [of course, we suppose that $V(f)$ and $V(f_0)$ are also negligibly small in this area]

$$\triangle Q_{\text{outer}} \ll \triangle Q_{\text{inner}}, \qquad \triangle E_{\text{outer}} \ll \triangle E_{\text{inner}},$$

which leads to

$$\triangle Q \approx 4\pi \int_0^{\hat{R}} (2gf_0^2 + 4\omega\varphi f_0)r^2 dr$$
$$\approx 4\pi \int_0^\infty (2gf_0^2 + 4\omega\varphi f_0)r^2 dr, \tag{60}$$

$$\triangle E \approx 4\pi\omega \int_0^{\hat{R}} (gf_0^2 + 4\omega\varphi f_0)r^2 dr$$
$$\approx 4\pi\omega \int_0^\infty (gf_0^2 + 4\omega\varphi f_0)r^2 dr \tag{61}$$

with a good accuracy. Of course, Eqs. (50)–(52) are valid with the same accuracy if the linearized theory works only in the inner region. This is enough for all practical purposes.

The problem is how to check that the condition $|\varphi(r)| \ll f_0(r)$ is valid in the inner region and the contribution of the outer region is negligibly small. The only fully consistent way to do it is to solve Eq. (31) with a particular $f_0(r,\omega)$. For a fixed ω, this can be done numerically in the general case, the situation is more complicated if one has to analyze a rather wide range of ω: a search for solutions to Eq. (31) and the subsequent calculation of $\frac{|\varphi(r)|}{f_0(r)}$ for $r < \hat{R}$ may take quite a long time. But we think that at least an estimate of the maximal value of $\frac{|\varphi(r)|}{f_0(r)}$ for $r < \hat{R}$ can be made without solving Eq. (31).

Below, we will propose two parameters, which can be useful for such an estimate. To find the first parameter, we notice that if $|\varphi(r)| \ll f_0(r)$ then

$$\left|\int_0^{\hat{R}} \varphi f_0 r^2 dr\right| \ll \left|\int_0^{\hat{R}} f_0^2 r^2 dr\right|. \tag{62}$$

The opposite is not correct. Indeed, if φ changes its sign at some $r > 0$ (this is exactly the situation realized in model 2, which will be presented below), then it is possible that the integral in the lhs of Eq. (62) is equal to zero for nonzero φ, which does not provide any estimate. Meanwhile, inequality (62) may be useful taken together with other parameters, which will be discussed later. For $r > \hat{R}$, the absolute values of the field φ are negligibly small in the outer region, so we can rewrite inequality (62) as

$$\left|\int_0^{\infty} \varphi f_0 r^2 dr\right| \ll \left|\int_0^{\infty} f_0^2 r^2 dr\right|. \tag{63}$$

In other words, the fulfillment of the latter inequality implies that the contribution of the fields from the outer region, where $|\varphi|$ can be of the order of f_0 and larger, is negligibly small in comparison with the main contribution of the inner region. Now, multiplying Eq. (63) by $4\pi\omega^2$ and using Eqs. (60) and (61) [or Eqs. (33) and (34)], we can rewrite Eq. (63) as

$$\frac{|2\triangle E - \omega\triangle Q|}{2\omega Q_0} \ll 1. \tag{64}$$

It can be rewritten in the explicit form as

$$\left|\frac{1}{4\omega}\frac{dI}{d\omega} - \frac{I}{2\omega^2}\right| \ll 4\pi\int_0^{\infty} f_0^2 r^2 dr. \tag{65}$$

Now, we turn to the second parameter. To find it, we take the inhomogeneous equation (31) and rewrite it as

$$\Delta\varphi + \omega^2\varphi - \frac{1}{2}\frac{d^2V}{df^2}\bigg|_{f=f_0}\varphi = -2\omega g(r)f_0(r) \leq - 2\omega g(0)f_0(r), \tag{66}$$

where we have used the fact that $g(r) < 0$ and $|g(r)| \leq |g(0)|$ for any r. Equation (66) suggests that, at least for an estimation of φ, one can consider the simplified equation

$$\Delta\hat{\varphi} + \omega^2\hat{\varphi} - \frac{1}{2}\frac{d^2V}{df^2}\bigg|_{f=f_0}\hat{\varphi} + 2\omega g(0)f_0(r) = 0 \tag{67}$$

instead of Eq. (31). We think that the difference between φ and $\hat{\varphi}$ should be of the order of φ, which is not critical for the estimate. But according to Eq. (35), Eq. (67) can be solved exactly—its solution has the form

$$\hat{\varphi} = g(0)\frac{df_0}{d\omega}. \tag{68}$$

Thus, instead of $\frac{|\varphi(r)|}{f_0(r)}$, we can try to estimate $\frac{|\hat{\varphi}(r)|}{f_0(r)}$, for which

$$\left|\frac{g(0)}{f_0(r)}\frac{df_0(r)}{d\omega}\right| \ll 1 \tag{69}$$

should hold. Note that the new parameter in Eq. (69) is proportional to the first parameter $\frac{|g(0)|}{\omega}$. We think that in order to get better estimates one should calculate Eq. (69) at several different points of r for a given ω.

The fulfillment of Eq. (56) together with Eqs. (64) and (69) suggests, although it does not ensure, that the linear approximation is valid for g and φ in the inner region, whereas the outer region does not make any significant contribution, and the linearized theory indeed can be used for a description of gauged Q ball.[6] Note that, as will be shown below by an explicit example (model 2), the fulfillment of Eq. (56) does not imply the fulfillment of Eqs. (64) and (69) and vice versa. Thus, in the general case, one should estimate *all* the parameters presented above while analyzing the question about the applicability of Eqs. (50)–(52). It is possible simply to define the function

$$\alpha(\omega) = \max_i\left\{\frac{|g(0)|}{\omega}, \frac{|2\triangle E - \omega\triangle Q|}{2\omega Q_0}, \left|\frac{g(0)}{f_0(r_i)}\frac{df_0(r_i)}{d\omega}\right|\right\} \tag{70}$$

and consider it as the natural small parameter depending on ω, for which

$$\alpha(\omega) \ll 1 \tag{71}$$

[6]Of course, the breakdown of the linear approximation at large r does not mean that a solution to nonlinear equations (7) and (8) does not exist—it simply means that the linear approximation does not describe the Q ball properly far away from its center.

should hold. We emphasize that for calculating $\alpha(\omega)$ only the background solution $f_0(r,\omega)$ is necessary. Because of the dependence of the parameter $\alpha(\omega)$ on ω, one can say that it "runs" with ω. It is obvious that $\alpha(\omega) \sim e^2$, but, as we will see below, the smallness of e^2 does not guarantee the fulfillment of Eq. (71).

D. Comparison of gauged and nongauged Q balls

Now, let us compare some properties of gauged [obtained in the linear approximation in $\alpha(\omega)$] and nongauged Q balls. We start with comparing the energies of Q balls at a given charge Q. For a gauged Q ball, we have $Q = Q_0(\omega_1) + \triangle Q(\omega_1)$, whereas for a nongauged Q ball, $Q = Q_0(\omega_2)$. From $Q_0(\omega_1) + \triangle Q(\omega_1) = Q_0(\omega_2)$, in the linear approximation in $\alpha(\omega)$, we obtain

$$\triangle Q(\omega_1) = (\omega_2 - \omega_1)\frac{dQ_0}{d\omega}\bigg|_{\omega=\omega_1}. \tag{72}$$

Now, let us compare the energies of gauged and nongauged Q balls with the same charges. We get

$$\begin{aligned} E(\omega_1) - E_0(\omega_2) &= E_0(\omega_1) + \triangle E(\omega_1) - E_0(\omega_2) \approx \triangle E(\omega_1) - (\omega_2 - \omega_1)\frac{dE_0}{d\omega}\bigg|_{\omega=\omega_1} \\ &= \triangle E(\omega_1) - \frac{\frac{dE_0}{d\omega}|_{\omega=\omega_1}}{\frac{dQ_0}{d\omega}|_{\omega=\omega_1}}\triangle Q(\omega_1) = \triangle E(\omega_1) - \omega_1 \triangle Q(\omega_1), \end{aligned} \tag{73}$$

where we have used Eq. (72) and the relation $\frac{dE_0}{dQ_0} = \omega$. But according to Eqs. (41), (42), and (43),

$$\triangle E(\omega_1) - \omega_1 \triangle Q(\omega_1) = -I(\omega_1) = \frac{1}{2e^2}\int d^3x \partial_i g \partial_i g, \tag{74}$$

which is always positive for $g \not\equiv 0$. Thus, for any charge $Q > 0$, the energy of a gauged Q ball is larger than the energy of the corresponding nongauged Q ball with the same charge that is, of course, the expected result. It is interesting to note that Eq. (74) also follows from Eq. (26). Indeed, $\triangle Q \sim e^2$ and $\triangle E \sim e^2$, leading to $\frac{dQ}{de} = \frac{2\triangle Q}{e}$ and $\frac{dE}{de} = \frac{2\triangle E}{e}$. Substituting the latter relations into Eq. (26), we get Eq. (74).

Now, we turn to examining another property of Q balls—cusps on the $E(Q)$ diagrams. Such cusps, which are a consequence of the existence of (locally) minimal or/and (locally) maximal charges, exist on $E(Q)$ diagrams in many models of nongauged Q balls. The origin of the cusps is the following: for a (locally) minimal or a (locally) maximal charge, we have $\frac{dQ}{d\omega}|_{\omega=\hat{\omega}} = 0$ with $\hat{\omega} > 0$, whereas from $\frac{dE}{d\omega} = \omega\frac{dQ}{d\omega}$, it follows that $\frac{dE}{d\omega}|_{\omega=\hat{\omega}} = 0$, which leads to the appearance of a cusp at the point $Q_m = Q(\hat{\omega})$. Of course, analogous cusps are expected in the gauged case also. For example, one can recall the model of Ref. [5], in which the function $E(Q)$ was drawn with the help of gauged nontopological soliton solutions, which were obtained by solving numerically the corresponding exact equations of motion, although for rather small values of the expansion parameter (the cusp is clearly seen on the $E(Q)$ diagram presented in Ref. [5]). Below, we will obtain relations between the charges, corresponding to the cusps, in the gauged and nongauged cases.

The position of a cusp in the gauged case is defined by $\frac{d(Q_0+\triangle Q)}{d\omega}|_{\omega=\hat{\omega}_1} = 0$, whereas for the nongauged case, it is defined by $\frac{dQ_0}{d\omega}|_{\omega=\hat{\omega}_2} = 0$. The difference between the charges in the linear order in $\alpha(\omega)$ is

$$\begin{aligned} Q(\hat{\omega}_1) - Q_0(\hat{\omega}_2) &= Q_0(\hat{\omega}_1) + \triangle Q(\hat{\omega}_1) - Q_0(\hat{\omega}_2) \\ &\approx \triangle Q(\hat{\omega}_1) + (\hat{\omega}_1 - \hat{\omega}_2)\frac{dQ_0}{d\omega}\bigg|_{\omega=\hat{\omega}_2} \\ &= \triangle Q(\hat{\omega}_1) \approx \triangle Q(\hat{\omega}_2). \end{aligned} \tag{75}$$

We see that in the linear approximation in $\alpha(\omega)$ the difference between the charges corresponding to the cusps in the gauged and nongauged cases is defined by the value of $\triangle Q$ at ω corresponding to the cusp in the nongauged case. As we will see below using explicit examples, the difference can be positive, negative, or even zero.

One makes an interesting observation from Eqs. (73), (74), and (75). Since $I(\omega_1) \approx I(\omega_2)$ in the linear order in $\alpha(\omega)$, for ω_2 which is not very close to $\hat{\omega}_2$, one has $E(\omega_1) = E_0(\omega_2) - I(\omega_2)$. Suppose that we have a nongauged Q ball with the charge Q_x and the energy $E_0(Q_x)$. Then, the energy of the corresponding gauged Q ball with the same *charge* Q_x (not with the same ω) is simply

$$E(Q_x) = E_0(Q_x) - I(\omega)|_{\omega=Q_0^{-1}(Q_x)}, \tag{76}$$

where $-I(\omega)|_{\omega=Q_0^{-1}(Q_x)}$ is just the energy of the gauge field produced by the nonpointlike charge Q_x [recall Eq. (43)]. Near the cusps, this formula must be used very carefully: at first, it is necessary to check that for a given charge Q_x of the nongauged Q ball the corresponding gauged Q ball really exists [see Eq. (75)] and that Eq. (72) results in $\omega_2 - \omega_1 \sim \alpha(\omega_2)$. If it is not so, one should use Eqs. (50)

and (51) instead of Eq. (76). But it is clear that Eq. (76) can be used for most values of the Q ball charge.

E. Explicit examples of gauged Q balls

In the general case, a straightforward numerical evaluation of the function $I(\omega)$ for a given background solution $f_0(r,\omega)$ may take quite a long time. So, to illustrate how the general results, presented above, can be used for calculations, we choose two models with very simple background Q ball solutions $f_0(r,\omega)$. The simplicity of the background solutions allows us not only to find the function $I(\omega)$ analytically in both cases but also to obtain exact analytic solutions to the system of linearized equations (30), (31).

1. Model 1

Let us consider the model proposed in Ref. [11] with the potential (in our notations)

$$V(\phi^*\phi) = -\mu^2\phi^*\phi\ln(\beta^2\phi^*\phi), \tag{77}$$

where μ and β are the model parameters. The spherically symmetric background (nongauged) solution for the Q ball in this model takes the form

$$f_0(r) = \mu\xi \mathrm{e}^{-\frac{\omega^2}{2\mu^2}}\mathrm{e}^{-\frac{\mu^2r^2}{2}}, \tag{78}$$

where $0 \le \omega < \infty$ and $\xi = \frac{\mathrm{e}}{\beta\mu}$. The charge and the energy of the Q ball looks like

$$Q_0 = 2\pi^{\frac{3}{2}}\xi^2\frac{\omega}{\mu}\mathrm{e}^{-\frac{\omega^2}{\mu^2}}, \tag{79}$$

$$E_0 = 2\pi^{\frac{3}{2}}\xi^2\mu\left(\frac{\omega^2}{\mu^2}+\frac{1}{2}\right)\mathrm{e}^{-\frac{\omega^2}{\mu^2}}. \tag{80}$$

For additional details concerning nongauged Q balls in the model with potential (77), see Ref. [14], in which this model was thoroughly investigated.

The integral in Eq. (52) can be easily calculated analytically for the background solution defined by Eq. (78). The result looks like

$$\frac{I}{4\pi} = -\mu e^2\frac{\sqrt{\pi}}{4\sqrt{2}}\xi^4\left(\frac{\omega}{\mu}\right)^2\mathrm{e}^{-\frac{2\omega^2}{\mu^2}}. \tag{81}$$

We see that Eq. (81) has a very simple form. The corrections $\triangle Q$ and $\triangle E$ can also be calculated analytically, and for the charge and the energy of the gauged Q ball, we get

$$Q = Q_0 + \triangle Q = 2\pi^{\frac{3}{2}}\xi^2(\tilde{Q}_0 + e^2\xi^2\triangle\tilde{Q}) = 2\pi^{\frac{3}{2}}\xi^2\tilde{Q}, \tag{82}$$

$$E = E_0 + \triangle E = \mu 2\pi^{\frac{3}{2}}\xi^2(\tilde{E}_0 + e^2\xi^2\triangle\tilde{E}) = \mu 2\pi^{\frac{3}{2}}\xi^2\tilde{E}, \tag{83}$$

with

$$\tilde{Q}_0 = \tilde{\omega}\mathrm{e}^{-\tilde{\omega}^2}, \tag{84}$$

$$\tilde{E}_0 = \left(\tilde{\omega}^2+\frac{1}{2}\right)\mathrm{e}^{-\tilde{\omega}^2}, \tag{85}$$

$$\triangle\tilde{Q} = \left(\sqrt{2}\tilde{\omega}^3 - \frac{\tilde{\omega}}{\sqrt{2}}\right)\mathrm{e}^{-2\tilde{\omega}^2}, \tag{86}$$

$$\triangle\tilde{E} = \left(\sqrt{2}\tilde{\omega}^4 - \frac{\tilde{\omega}^2}{2\sqrt{2}}\right)\mathrm{e}^{-2\tilde{\omega}^2}, \tag{87}$$

where $\tilde{\omega} = \frac{\omega}{\mu}$. We also define the parameter

$$\alpha_1 = e^2\xi^2, \tag{88}$$

which will be used below. In Fig. 1, one can see an example of the $E(Q)$ diagram for the gauged Q ball in this model. This diagram was plotted using Eqs. (50) and (51). We see from Fig. 1 that the energy of the gauged Q ball is larger than the energy of the corresponding nongauged Q ball with the same charge, as it was shown in the previous section. One sees that there is a cusp on the $E(Q)$ diagram for the gauged case as well as for the nongauged case. There are maximal charges, which correspond to these cusps. For the nongauged case, the charge is maximal at $\tilde{\omega} = \frac{1}{\sqrt{2}}$. From Eq. (86), it follows that $\triangle Q|_{\tilde{\omega}=\frac{1}{\sqrt{2}}} = 0$, which means that the values of the maximal charge in the gauged and nongauged cases coincide through the linear order in α_1 [this also implies that one can use Eq. (76) instead of Eqs. (50) and (51) for plotting the $E(Q)$ diagram, presented in Fig. 1, with the same accuracy]. It follows from Fig. 1 that on the lower branch of the $E(Q)$ diagram $\frac{d^2E}{dQ^2} < 0$ and $E(0) = 0$ (the latter corresponds to $\omega \to \infty$), which means

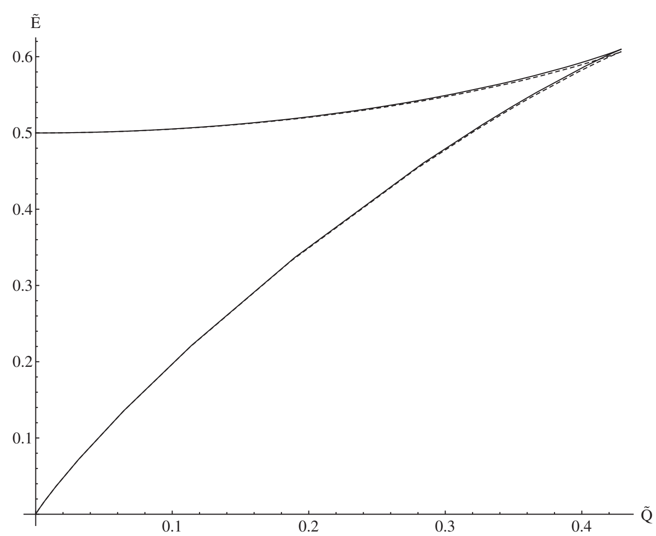

FIG. 1. $E(Q)$ for the gauged (solid line) and nongauged (dashed line) cases. Here, $\alpha_1 = 0.05$ and $0 \le \tilde{\omega} \le 10$.

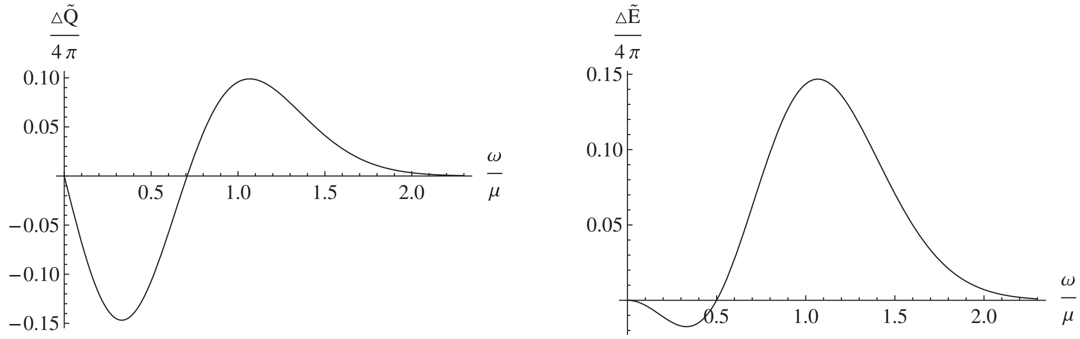

FIG. 2. $\triangle\tilde{Q}$ (left plot) and $\triangle\tilde{E}$ (right plot) for $0 \le \tilde{\omega} \le 2.3$.

that gauged Q balls from this branch are stable against fission.

In Fig. 2, the plots of corrections $\triangle\tilde{Q}$ and $\triangle\tilde{E}$ are presented. One sees from these plots that $\triangle Q$ and $\triangle E$ can be negative or positive for a given ω (although the energy of gauged Q ball is always larger than the energy of the corresponding nongauged Q ball with the same charge).

The parameter $\frac{|2\triangle E-\omega\triangle Q|}{2\omega Q_0}$ in Eq. (64), which is necessary for checking the applicability of linear approximation, can be obtained directly from Eqs. (84), (86), and (87). It is not difficult to show that it can be estimated as

$$\frac{|2\triangle E-\omega\triangle Q|}{2\omega Q_0} = \frac{\alpha_1}{\sqrt{2}}\tilde{\omega}^2 \mathrm{e}^{-\tilde{\omega}^2} \le \frac{1}{\sqrt{2}\mathrm{e}}\alpha_1. \qquad (89)$$

As for the parameter $\frac{|g(0)|}{\omega}$ in Eq. (56), it can also be calculated analytically for Eq. (78) and takes the form

$$\frac{|g(0)|}{\omega} = \alpha_1 \mathrm{e}^{-\tilde{\omega}^2} \le \alpha_1. \qquad (90)$$

And finally, the parameter (69) does not depend on r in this model and has the form

$$\left|\frac{g(0)}{f_0(r)}\frac{df_0(r)}{d\omega}\right| = \alpha_1\tilde{\omega}^2 \mathrm{e}^{-\tilde{\omega}^2} \le \frac{1}{\mathrm{e}}\alpha_1. \qquad (91)$$

We see that in the model under consideration all the parameters $\frac{|g(0)|}{\omega}$, $\frac{|2\triangle E-\omega\triangle Q|}{2\omega Q_0}$, and $|\frac{g(0)}{f_0(r)}\frac{df_0(r)}{d\omega}|$ can be easily estimated. The most stringent ω-independent restriction on $e\xi$ comes from Eq. (90) and looks very simple:

$$e^2\xi^2 = \alpha_1 \ll 1.$$

Note that for large $\tilde{\omega}$ Eq. (70) gives

$$\alpha(\omega) = e^2\xi^2\tilde{\omega}^2 \mathrm{e}^{-\tilde{\omega}^2}.$$

It means that, with a fixed $e\xi$, the larger $\tilde{\omega}$ is, the smaller the parameter $\alpha(\omega)$ is. In other words, the larger $\tilde{\omega}$ is, the larger the maximal value of $e\xi$, for which the linearized theory can be used with this $\tilde{\omega}$, is. Nevertheless, one can consider α_1 as an ω-independent small parameter for this model, which can be useful in certain cases.

The restriction $\alpha_1 \ll 1$ clearly shows that the fulfillment of $e^2 \ll 1$ is not sufficient to ensure the validity of the linear approximation. Indeed, even for a very small value of e^2, the parameter ξ, which is defined by the parameters of the scalar field potential, can be rather large to make the use of the linear approximation impossible (this fact was previously observed in Ref. [10]).

For completeness, below we present the explicit solution for the fields g and φ in this model. It satisfies the conditions $\frac{dg}{dr}|_{r=0} = 0$, $g|_{r\to\infty} = 0$, $\frac{d\varphi}{dr}|_{r=0} = 0$, and $\varphi|_{r\to\infty} = 0$ and can be factorized into terms containing ω and r. For the first time, this exact, in the linear approximation, solution was obtained in Ref. [10], and in our notations, it has the form

$$g(r) = \mu\alpha_1\Phi_g(\omega)F_g(r), \qquad (92)$$

$$\varphi(r) = \mu\alpha_1\xi\Phi_\varphi(\omega)F_\varphi(r), \qquad (93)$$

where

$$\Phi_g(\omega) = \frac{\sqrt{\pi}}{2}\frac{\omega}{\mu}\mathrm{e}^{-\frac{\omega^2}{\mu^2}}, \qquad (94)$$

$$F_g(r) = -\frac{1}{\mu r}\mathrm{erf}(\mu r), \qquad (95)$$

$$\Phi_\varphi(\omega) = \sqrt{\pi}\left(\frac{\omega}{\mu}\right)^2 \mathrm{e}^{-\frac{3\omega^2}{2\mu^2}}, \qquad (96)$$

$$F_\varphi(r) = \mathrm{e}^{-\frac{3\mu^2r^2}{2}}\left(\frac{1}{4\sqrt{\pi}} + \frac{1}{4}\mathrm{e}^{\mu^2r^2}\left(\mu r + \frac{1}{2\mu r}\right)\mathrm{erf}(\mu r)\right). \qquad (97)$$

Here $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}dt$.

The explicit solution, presented above, allows us to estimate how $\frac{|\varphi(r)|}{f_0(r)}$ grows with r. In Fig. 3, the function $\frac{F_\varphi(r)}{\mathrm{e}^{-\frac{\mu^2r^2}{2}}}$ is presented, clearly indicating the growth. One sees from this plot that, for example, for a given ω the value of

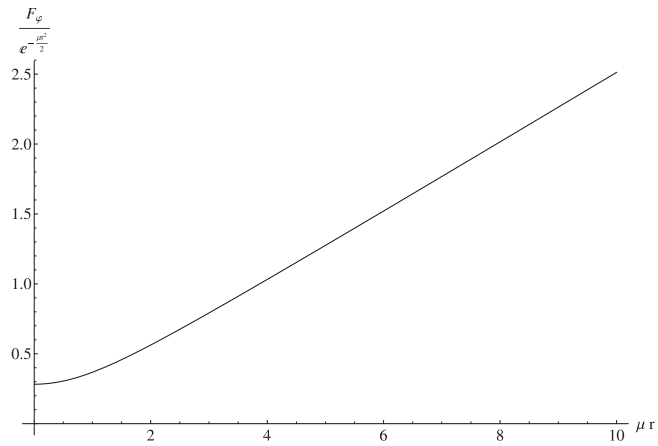

FIG. 3. The function $\frac{F_\varphi(r)}{e^{-\frac{\mu^2 r^2}{2}}}$.

$\frac{\varphi(r)}{f_0(r)}|_{\mu r=10}$ is approximately five times larger than the value of $\frac{\varphi(r)}{f_0(r)}|_{\mu r=2}$. Meanwhile, the absolute values of the fields φ and f_0 are proportional to the factor of the order of e^{-50} at $\mu r = 10$, which is extremely small. This confirms that if, with an appropriate choice of α_1, the linear approximation is valid in the inner region of this gauged Q ball (which can be defined, for example, as $0 \leq \mu r \leq 10$), Eqs. (81)–(87) are also valid.

2. Model 2

Now, we consider the model with a piecewise parabolic potential, which was proposed in Ref. [1] and thoroughly examined in Ref. [13].[7] The piecewise scalar field potential in this model has the form

$$V(\phi^*\phi) = M^2\phi^*\phi\theta\left(1-\frac{\phi^*\phi}{v^2}\right) + (m^2\phi^*\phi + v^2(M^2-m^2))\theta\left(\frac{\phi^*\phi}{v^2}-1\right), \quad (98)$$

where $M^2 > 0$, $M^2 > m^2$, and θ is the Heaviside step function with the convention $\theta(0) = \frac{1}{2}$. The background solution for the Q ball in this model takes the form

$$f_0(r < R) = f_0^<(r) = v\frac{R\sin(\sqrt{\omega^2 - m^2}r)}{r\sin(\sqrt{\omega^2 - m^2}R)}, \quad (99)$$

$$f_0(r > R) = f_0^>(r) = v\frac{Re^{-\sqrt{M^2-\omega^2}r}}{re^{-\sqrt{M^2-\omega^2}R}}, \quad (100)$$

where R is defined as

$$R = R(\omega) = \frac{1}{\sqrt{\omega^2 - m^2}}\left(\pi - \arctan\left(\frac{\sqrt{\omega^2 - m^2}}{\sqrt{M^2 - \omega^2}}\right)\right). \quad (101)$$

The charge and the energy of the Q ball looks like

$$Q_0 = 4\pi R^2\omega v^2\left(\frac{(M^2 - m^2)(R\sqrt{M^2 - \omega^2} + 1)}{(\omega^2 - m^2)\sqrt{M^2 - \omega^2}}\right), \quad (102)$$

$$E_0 = \omega Q_0 + 4\pi\frac{R^3 v^2(M^2 - m^2)}{3}. \quad (103)$$

For additional details concerning nongauged Q balls in the model with potential (98), see Ref. [13].

As in the previous case, the integral in Eq. (52) can be calculated analytically for the background solution defined by Eqs. (99) and (100). The result looks like

$$\frac{I}{4\pi} = e^2\omega^2\left[a^4\left(\frac{\sin(2\sqrt{\omega^2 - m^2}R)}{2\sqrt{\omega^2 - m^2}} - R + \frac{\mathrm{Si}(2\sqrt{\omega^2 - m^2}R)}{2\sqrt{\omega^2 - m^2}} - \frac{\mathrm{Si}(4\sqrt{\omega^2 - m^2}R)}{4\sqrt{\omega^2 - m^2}}\right) - 4b^2\left(a^2\left(\frac{R}{2} - \frac{\sin(2\sqrt{\omega^2 - m^2}R)}{4\sqrt{\omega^2 - m^2}}\right) + \frac{b^2 e^{-2\sqrt{M^2-\omega^2}R}}{2\sqrt{M^2 - \omega^2}}\right)\mathrm{E}_1(2\sqrt{M^2 - \omega^2}R) + \frac{2b^4}{\sqrt{M^2 - \omega^2}}\mathrm{E}_1(4\sqrt{M^2 - \omega^2}R)\right], \quad (104)$$

[7] Another model with a piecewise parabolic potential (it was also proposed in Ref. [1]), admitting a rather simple solution, was discussed in detail in Ref. [15]. The model in Ref. [13] provides a simpler solution [especially for $R(\omega)$, which, contrary to the case of Ref. [15], has a very simple analytic form (101)], which appears to be more useful for illustrative purposes and numerical analysis.

where

$$\mathrm{Si}(y) = \int_0^y \frac{\sin(t)}{t} dt, \tag{105}$$

$$\mathrm{E}_1(y) = \int_y^\infty \frac{\mathrm{e}^{-t}}{t} dt \tag{106}$$

and

$$a = a(\omega) = \frac{vR}{\sin(\sqrt{\omega^2 - m^2}R)}, \tag{107}$$

$$b = b(\omega) = \frac{vR}{\mathrm{e}^{-\sqrt{M^2-\omega^2}R}}. \tag{108}$$

We see that Eq. (104) has a much more complicated form than the corresponding result for the previous model. In principle, with the help of Eqs. (101), (107), and (108), the derivative $\frac{dI}{d\omega}$ can also be calculated analytically, although we derived it numerically for obtaining the $E(Q)$ dependence.

To perform numerical calculations, one should pass to dimensionless variables. The most natural choice for the scale parameter in this model is the mass parameter M. Thus, we choose the dimensionless variables $\tilde{\omega} = \frac{\omega}{M}$ and $\tilde{r} = Mr$. The background scalar field takes the form

$$f_0(\omega, r) = v\tilde{f}_0(\tilde{\omega}, \tilde{r}). \tag{109}$$

It is not difficult to show that the charge and the energy can be represented as

$$Q = Q_0 + \Delta Q = \frac{v^2}{M^2}\left(\tilde{Q}_0 + \frac{e^2v^2}{M^2}\Delta\tilde{Q}\right), \tag{110}$$

$$E = E_0 + \Delta E = \frac{v^2}{M}\left(\tilde{E}_0 + \frac{e^2v^2}{M^2}\Delta\tilde{E}\right), \tag{111}$$

with $\tilde{Q}_0$, $\tilde{E}_0$, $\Delta\tilde{Q}$, and $\Delta\tilde{E}$ being dimensionless functions depending on $\tilde{\omega}$ and $\frac{m^2}{M^2}$ only. This suggests that the parameter

$$\alpha_2 = \frac{e^2v^2}{M^2} \tag{112}$$

in this model is such that $\alpha(\omega) \sim \alpha_2$. It is confirmed by the fact that, as can be shown from Eqs. (30) and (31) using Eqs. (98) and (109), solutions for the fields g and φ can be expressed in the form

$$g = M\alpha_2\tilde{g}, \tag{113}$$

$$\varphi = v\alpha_2\tilde{\varphi}. \tag{114}$$

where the dimensionless functions $\tilde{g}$ and $\tilde{\varphi}$ depend only on $\tilde{\omega}$, and $\frac{m^2}{M^2}$ and $\tilde{r}$ and do not depend on v and e.

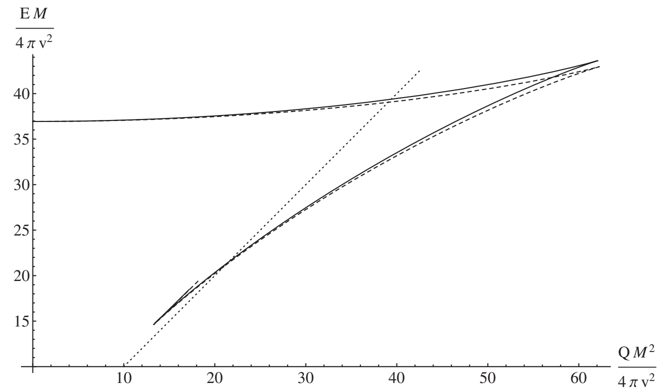

FIG. 4. $E(Q)$ for the gauged (solid line) and nongauged (dashed line) cases. The dotted line stands for free scalar particles of mass M at rest. Here, $m^2 < 0$, $\frac{|m|}{M} = 0.6$, $\alpha_2 = 0.001$, and $0 \leq \tilde{\omega} \leq 0.99$.

As in the previous case, Eq. (112) clearly shows that the linear approximation can be used if not only $e^2 \ll 1$ holds but if $\alpha(\omega) \ll 1$ holds, too. Indeed, even for a very small value of e^2, the relation $\frac{v^2}{M^2}$ can be large enough to make the use of the linear approximation impossible (as we will see below, for $\alpha_2 = 0.001$, the linear approximation does not work well enough for all values of ω, whereas such value of α_2 can be obtained by choosing $v = M$ and $e^2 = 0.001$, which looks small enough). On the other hand, for larger values of e^2, the value of $\frac{v^2}{M^2}$ can be chosen to be rather small to make $\alpha(\omega) \ll 1$.

For a numerical analysis, we choose the case $m^2 < 0$, which is, in our opinion, the most interesting for illustrative purposes. In Fig. 4, one can see an example of the $E(Q)$ diagram for the gauged Q ball in our model. This diagram was plotted using Eqs. (50) and (51).

Let us discus the properties of the gauged Q balls at hand. Again, we see from Fig. 4 that the energy of the gauged Q ball is larger than the energy of the corresponding nongauged Q ball for the same values of charge. One also sees that there are two cusps on the $E(Q)$ diagram for the gauged case as well as for the nongauged case. There are locally minimal charges Q^{min} and locally maximal charges Q^{max}, which correspond to these cusps. For the nongauged case with $\frac{|m|}{M} = 0.6$, the charge is locally maximal at $\tilde{\omega} \approx 0.2846$, whereas it is locally minimal at $\tilde{\omega} \approx 0.9426$. We calculated numerically the values of ΔQ for these values of $\tilde{\omega}$. According to Eq. (75), we have

$$\frac{M^2}{4\pi v^2}(Q^{\mathrm{max}} - Q_0^{\mathrm{max}}) \approx -0.421\alpha_2,$$

$$\frac{M^2}{4\pi v^2}(Q^{\mathrm{min}} - Q_0^{\mathrm{min}}) \approx 0.139\alpha_2.$$

Of course, the difference between Q^{max} and Q_0^{max} cannot be seen in Fig. 4 by the naked eye because of the small value of the parameter α_2. Contrary to the case of the previous model, here $Q^{\mathrm{max}} < Q_0^{\mathrm{max}}$ and $Q^{\mathrm{min}} > Q_0^{\mathrm{min}}$. The latter

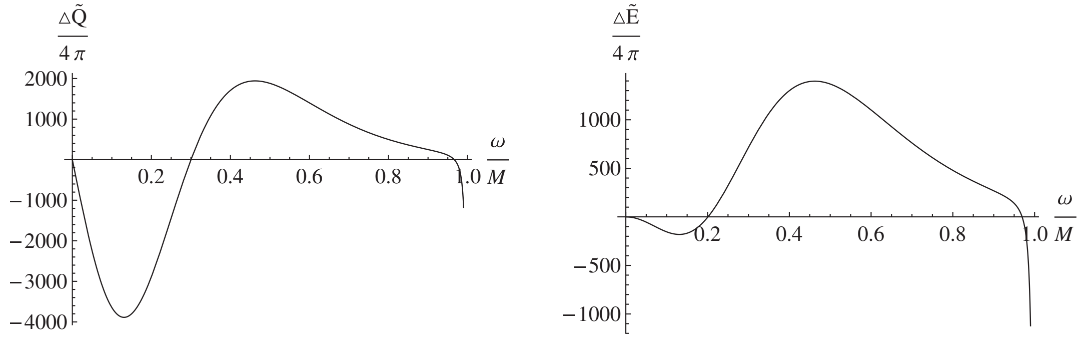

FIG. 5. $\triangle\tilde{Q}$ (left plot) and $\triangle\tilde{E}$ (right plot) for $m^2 < 0$, $\frac{|m|}{M} = 0.6$ and $0 \leq \tilde{\omega} \leq 0.99$.

relations are not universal even in the model under consideration. For example, for $\frac{|m|}{M} = 1.3$,

$$\frac{M^2}{4\pi v^2}(Q^{\max} - Q_0^{\max}) \approx 0.0009\alpha_2,$$
$$\frac{M^2}{4\pi v^2}(Q^{\min} - Q_0^{\min}) \approx 0.0056\alpha_2,$$

i.e., now $Q^{\max} > Q_0^{\max}$.

We also present the plots of $\triangle\tilde{Q}$ and $\triangle\tilde{E}$; see Fig. 5. Again, one sees from these plots that the corrections $\triangle Q$ and $\triangle E$ can be negative or positive for a given ω.

The plots of the parameters $\frac{|g(0)|}{\omega}$, $\eta = \frac{|2\triangle E - \omega\triangle Q|}{2\omega Q_0}$, and $\rho(r_i) = \frac{|g(0)|}{f_0(r_i)}\frac{df_0(r_i)}{d\omega}$, which are necessary for checking the validity of the linear approximation, are presented in Fig. 6. All these parameters were calculated numerically. We calculated $\rho(r)$ at two points: the first one is defined by $f_0(R_{\mathrm{e}}, \omega) = \mathrm{e}^{-1} f_0(0, \omega)$, whereas the second point, R_ϵ, is defined by Eq. (59) with $\epsilon = 10^{-2}$ and corresponds to the radius of the inner region. Both R_{e} and R_ϵ depend on ω; see Appendix D for details concerning the calculation of R_{e} and R_ϵ.

We see from Fig. 6 that all the parameters depend on ω in different ways. This explicit example confirms that in order to check the validity of the linear approximation in the general case (in which the dependence of these parameters on ω is very complicated or can not be obtained analytically) it is better to estimate all the parameters $\frac{|g(0)|}{\omega}$, $\eta = \frac{|2\triangle E - \omega\triangle Q|}{2\omega Q_0}$, and $\rho(r_i) = \frac{|g(0)|}{f_0(r_i)}\frac{df_0(r_i)}{d\omega}$ or to calculate the function $\alpha(\omega) = \max_i\{\frac{|g(0)|}{\omega}, \eta, |\rho(r_i)|\}$, defined by Eq. (70), which is presented in Fig. 7 for the set of the model parameters chosen above. Figures 6 and 7 demonstrate that, although the use of parameters like α_2 as ω-independent small parameters can be convenient for calculations and for rough estimates, they cannot replace the natural small parameters $\alpha(\omega)$ in the general case.

Of course, the smaller α_2 is, the wider the region (or regions) of frequencies ω, in which the linear approximation works, is. Meanwhile, the chosen set of the parameters, for which Figs. 6 and 7 were plotted, is very useful for illustrative purposes. Based on these reasons, as well as to make the differences between the gauged and nongauged cases visible by the naked eye, we keep Fig. 4 as it is, although, according to Fig. 7, the linear approximation works well enough only in the vicinity of $\tilde{\omega} \approx 0.85$ for $\alpha_2 = 0.001$.

Now, let us turn to the discussion of stability of gauged Q balls in this model. We will focus on the lowest branch in Fig. 4, for which $\frac{d^2E}{dQ^2} < 0$. The results of Sec. III imply that Q balls from the lowest branch are stable against fission.

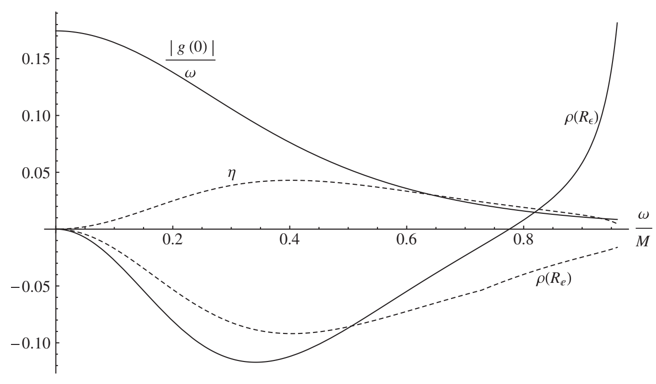

FIG. 6. $\frac{|g(0)|}{\omega}$, η and ρ for $m^2 < 0$, $\frac{|m|}{M} = 0.6$, $\alpha_2 = 0.001$, and $0 \leq \tilde{\omega} \leq 0.96$.

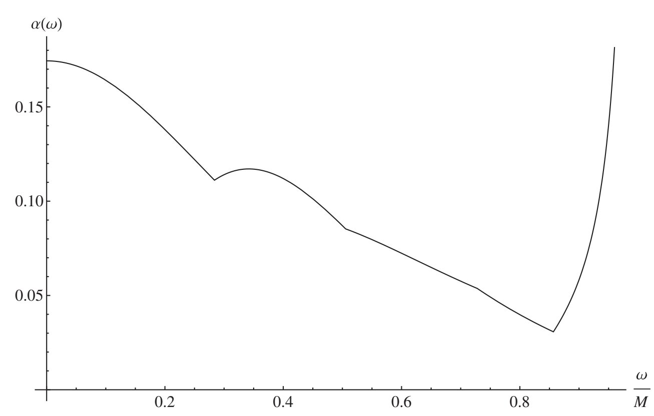

FIG. 7. $\alpha(\omega)$ for $m^2 < 0$, $\frac{|m|}{M} = 0.6$, $\alpha_2 = 0.001$, and $0 \leq \tilde{\omega} \leq 0.96$.

We also see that the part of this lowest branch lies below the line $E = MQ$ standing for free scalar particles of mass M. This means that, at least in the absence of fermions, Q balls from this part of the branch are stable with respect to decay into free particles.

The last type of stability, which could be discussed here, is the classical stability. Of course, we may wonder that the main properties of gauged and nongauged Q balls are similar at least in the case in which the parameter $\alpha(\omega)$ is rather small. So, if we suppose that the classical stability criterion for ordinary Q balls [6], which states that Q balls for which $\frac{dQ}{d\omega} = (\frac{d^2E}{dQ^2})^{-1} < 0$ (for $Q > 0$, $\omega > 0$) holds are classically stable, is valid for gauged Q balls, then the gauged Q balls from the lowest branch are also classically stable. In such a case, the part of the lowest branch, which lies below the line $E = MQ$, consists of absolutely stable Q balls. But we can not justify that the classical stability criterion for ordinary Q balls works for gauged Q balls, too, so we consider the Q balls from the part of the lowest branch, which lies below the $E = MQ$ line, as stable against fission and against decay into free scalar particles only. According to Fig. 4, for the given values of the model parameters ($m^2 < 0$, $\frac{|m|}{M} = 0.6$, $\alpha_2 = 0.001$), such Q balls have charges in the range $21.5 \lesssim \frac{M^2}{4\pi v^2} Q \lesssim 61.8$ and energies in the range $21.5 \lesssim \frac{M}{4\pi v^2} E \lesssim 43.5$.

As for the previous model, for completeness, below we present an explicit solution for the fields g and φ in this model. The solution for the field g can be obtained directly from Eq. (45) with Eqs. (99) and (100), whereas solution for the field φ was obtained by means of the method of variation of parameters. We present it for the simplest case $m = 0$. This solution, satisfying $\frac{dg}{dr}|_{r=0} = 0$, $g|_{r\to\infty} = 0$, $\frac{d\varphi}{dr}|_{r=0} = 0$, $\varphi|_{r\to\infty} = 0$, and the nonstandard matching conditions at $r = R$ [because $\frac{d^2V}{df^2}|_{f=v} \sim \delta(r - R)$], has the form for the gauge field

$$g(r < R) = g_<(r) = C_1\left(\ln(\omega r) - \mathrm{Ci}(2\omega r) + \frac{\sin(2\omega r)}{2\omega r}\right) + C_2, \tag{115}$$

$$g(r > R) = g_>(r) = \frac{C_3}{r} + C_4\left(\frac{e^{-2\sqrt{M^2-\omega^2}r}}{2\sqrt{M^2-\omega^2}r} - E_1(2\sqrt{M^2-\omega^2}r)\right), \tag{116}$$

where

$$\mathrm{Ci}(y) = -\int_y^\infty \frac{\cos(t)}{t}dt, \tag{117}$$

$$C_1 = C_1(\omega) = e^2v^2\omega R^2\frac{1}{\sin^2(\omega R)}, \tag{118}$$

$$C_2 = C_2(\omega) = -e^2v^2\omega R^2\left(2e^{2\sqrt{M^2-\omega^2}R}E_1(2\sqrt{M^2-\omega^2}R) + \frac{-\mathrm{Ci}(2\omega R) + \ln(\omega R) + 1}{\sin^2(\omega R)}\right), \tag{119}$$

$$C_3 = C_3(\omega) = -e^2v^2\omega R^2\left(\frac{M^2}{\omega^2\sqrt{M^2-\omega^2}} + \frac{R}{\sin^2(\omega R)}\right), \tag{120}$$

$$C_4 = C_4(\omega) = e^2v^2\omega R^2(2e^{2\sqrt{M^2-\omega^2}R}), \tag{121}$$

and for the scalar field

$$\varphi(r < R) = B\frac{\sin(\omega r)}{r} + \frac{\sin(\omega r)}{\omega r}\int_0^r G_<(t)\cos(\omega t)dt - \frac{\cos(\omega r)}{\omega r}\int_0^r G_<(t)\sin(\omega t)dt, \tag{122}$$

$$\varphi(r>R) = A\frac{e^{-\sqrt{M^2-\omega^2}r}}{r} - \frac{e^{\sqrt{M^2-\omega^2}r}}{2\sqrt{M^2-\omega^2}r}\int_r^\infty G_>(t)e^{-\sqrt{M^2-\omega^2}t}dt - \frac{e^{-\sqrt{M^2-\omega^2}r}}{2\sqrt{M^2-\omega^2}r}\int_R^r G_>(t)e^{\sqrt{M^2-\omega^2}t}dt, \tag{123}$$

where

$$G_<(r) = -2\omega r g_<(r) f_0^<(r), \tag{124}$$

$$G_>(r) = -2\omega r g_>(r) f_0^>(r), \tag{125}$$

$$B = B(\omega) = \frac{1}{D}F_1\frac{e^{\sqrt{M^2-\omega^2}R}}{\sin(\omega R)} - \frac{F_2}{\omega} + \frac{F_3}{\omega^2 R}, \tag{126}$$

$$A = A(\omega) = \frac{e^{\sqrt{M^2-\omega^2}R}}{D}\left(F_1 e^{\sqrt{M^2-\omega^2}R}\left(1 + \frac{D}{2\sqrt{M^2-\omega^2}}\right) + F_3\frac{M^2\sin(\omega R)}{\omega^2}\right), \tag{127}$$

$$D = D(\omega) = \frac{M^2R}{1 + R\sqrt{M^2-\omega^2}}, \tag{128}$$

$$F_1 = F_1(\omega) = \int_R^\infty G_>(t)e^{-\sqrt{M^2-\omega^2}t}dt, \tag{129}$$

$$F_2 = F_2(\omega) = \int_0^R G_<(t)\cos(\omega t)dt, \tag{130}$$

$$F_3 = F_3(\omega) = \int_0^R G_<(t)\sin(\omega t)dt. \tag{131}$$

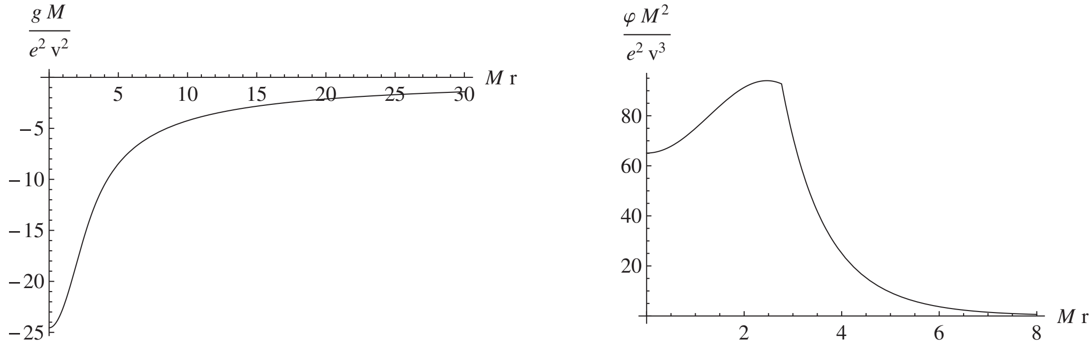

FIG. 8. Solutions for the fields $g(r)$ (left plot) and $\varphi(r)$ (right plot). Here, $m = 0$ and $\tilde{\omega} = 0.8$.

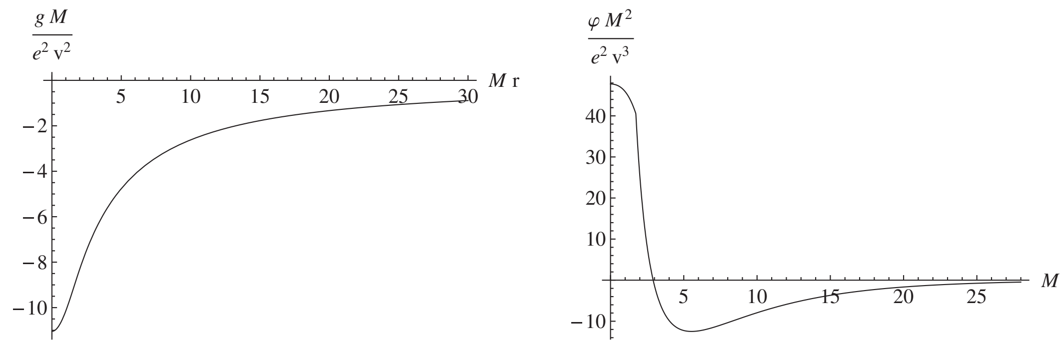

FIG. 9. Solutions for the fields $g(r)$ (left plot) and $\varphi(r)$ (right plot). Here, $m = 0$ and $\tilde{\omega} = 0.99$.

We see that, even for the very simple background solution (99), (100), the solution for g and φ appears to be complicated. Contrary to the case of model 1, the solution (115)–(131) cannot be factorized. It should be mentioned that the double integrals in Eqs. (115)–(131) [the functions $\mathrm{Ci}(y)$ and $\mathrm{E}_1(y)$ have integral representations themselves] in principle can be transformed to the form containing only integrals of one variable by performing integration by parts (corresponding calculations are straightforward, though rather tedious); this is possible only because of the simplicity of the background solution (99), (100).

The plots of this solution for the fields g and φ are presented in Figs. 8 and 9. One can see the breaks on the curves corresponding to the field φ. This is an artifact of linearization in the theory with potential (98), which also contains a break [recall that $\frac{d^2V}{df^2}|_{f=v} \sim \delta(r-R)$]. Of course, the break in potential (98) can be regularized, leading to the smooth behavior of φ at $r = R$.

One also sees that for different values of $\tilde{\omega}$ solutions for the field φ have different form: in the first case, φ increases at small r, whereas in the second case, it decreases at small r starting from $r = 0$. It should be noted that, contrary to what we have in model 1 [see Eq. (97)], in both cases, the solution for the field φ crosses the axis Mr and then tends to zero from below. It is clearly seen in Fig. 9; as for the case presented in Fig. 8, the solution crosses the axis Mr at $Mr \approx 17.38$, which is out of range of the presented plot. Solutions for other values of the model parameters have the form similar either to the solution presented in Fig. 8 or to the solution presented in Fig. 9.

Finally, we would like to note that, as it can be checked explicitly, the relation $\frac{|\varphi(r)|}{f_0(r)}$ grows logarithmically with r in this model (see also Appendix C). But, again, this growth is very slow relative to the exponential fall of f_0 and φ, and, analogously to the previous case, this confirms that with an appropriate choice of the model parameters (including ω) and with $\alpha(\omega) \ll 1$ the use of the linear approximation is fully justified.

V. CONCLUSION

In the present paper, we studied some general properties of $U(1)$ gauged Q balls. In particular, we showed that, as in the case of ordinary nongauged Q balls, the relation $\frac{dE}{dQ} = \omega$ also holds for $U(1)$ gauged Q balls. Based on this result and using the fact that $\omega < M$ in theories admitting the existence of free scalar particles of mass M, we demonstrated that the statement about the existence of the maximal charge of stable gauged Q balls, presented in Ref. [3], was obtained by means of the erroneous inequality $\frac{dE}{dQ} \geq M$ and thus cannot be considered as correct.

We also presented a powerful method for analyzing gauged Q balls in the case in which the backreaction of the gauge field on the scalar field is small. Provided a nongauged (background) Q ball solution $f_0(r, \omega)$, for a given value of the coupling constant e, the strength of the backreaction of the gauge field can be estimated by calculating the parameter $\alpha(\omega)$ defined by Eq. (70), which depends on ω and the background solution $f_0(r,\omega)$ only. This parameter is proportional to e^2 in general but does not coincide with it. We have shown that our results can be used not if only the inequality $e^2 \ll 1$ holds but if the overall parameter $\alpha(\omega)$ is also rather small to ensure the validity of the linear approximation [in principle, the smallness of $\alpha(\omega)$ does not exclude the cases in which e is not small]. The main parameters of gauged Q balls in such a theory—the charge and the energy—can also be calculated using the background solution $f_0(r,\omega)$ only [using Eqs. (50) and (51) or the even simpler Eq. (76)], whereas an explicit solution to the system of linearized equations of motion is not necessary at all.

The obtained results were illustrated by the examples of two exactly solvable models proving the efficiency of the proposed method—indeed, even for the very simple background solution (99), (100), the explicit analytic solution for gauged Q ball (115)–(131), which was obtained in the linear approximation in φ and g, appears to be rather complicated and its derivation (at least for the field φ) is more bulky than the analytical evaluation of integral (48) for Eqs. (99) and (100). Without Eqs. (50)–(52) or Eq. (76), evaluation of Eqs. (33) and (34) does not seem to be a simple task, taking into account the necessity to solve numerically (in the general case) the differential equation (31) to get a solution for the field φ. Obviously, it is a much more complicated task than the evaluation of the double integral in Eq. (48), even with a background nongauged solution $f_0(r,\omega)$ obtained numerically.

We hope that the results presented in this paper can be useful for the future research in this area.

ACKNOWLEDGMENTS

The authors are grateful to M. V. Libanov, V. A. Rubakov, S. V. Troitsky, and I. P. Volobuev for valuable discussions. The work was supported by RFBR Grant No. 14-02-31384. The work of E. Y. N. was supported in part by Grant No. NS-2835.2014.2 of the President of Russian Federation and by RFBR Grant No. 13-02-01127a. The work of M. N. S. was supported in part by Grant No. NS-3042.2014.2 of the President of Russian Federation and by RFBR Grant No. 12-02-93108-CNRSL-a.

APPENDIX A

Let $\frac{E(Q_{\min})}{Q_{\min}} = \hat{\omega} > \omega_{\min} = \frac{dE}{dQ}|_{Q=Q_{\min}}$. In this case, the function $E_{\mathrm{aux}}(Q)$ can be chosen as

$$E_{\mathrm{aux}}(Q) = (2\hat{\omega} - \omega_{\min})Q + \frac{Q^2}{Q_{\min}}(\omega_{\min} - \hat{\omega}), \quad Q < Q_{\min},$$
$$E_{\mathrm{aux}}(Q) = E(Q), \qquad Q \geq Q_{\min}.$$

One can check that

$$E_{\mathrm{aux}}(0) = 0,$$
$$E_{\mathrm{aux}}(Q_{\min}) = E(Q_{\min}),$$
$$\left.\frac{dE_{\mathrm{aux}}(Q)}{dQ}\right|_{Q=Q_{\min}} = \left.\frac{dE(Q)}{dQ}\right|_{Q=Q_{\min}},$$
$$\frac{dE_{\mathrm{aux}}(Q)}{dQ} = 2(\hat{\omega} - \omega_{\min})\left(1 - \frac{Q}{Q_{\min}}\right) + \omega_{\min} > 0 \quad \text{for } Q \leq Q_{\min},$$
$$\frac{d^2E_{\mathrm{aux}}(Q)}{dQ^2} = \frac{2}{Q_{\min}}(\omega_{\min} - \hat{\omega}) < 0;$$

i.e., all the necessary conditions are fulfilled. Since $E_{\mathrm{aux}}(0) = 0$, inequality (21) holds for Q_1, $Q_2 \geq 0$ and, consequently, for Q_1, $Q_2 \geq Q_{\min}$. For more details, see Ref. [13].

APPENDIX B

Let us consider Eqs. (30) and (31) and represent the coupling constant e in Eq. (30) as $e = \gamma e'$, where $\gamma > 0$ is a constant. Let us define $f_0' = \gamma f_0$, $\varphi' = \gamma\varphi$. With these notations, Eqs. (30) and (31) can be rewritten as

$$\Delta g - 2e'^2\omega f_0'^2 = 0, \tag{B1}$$

$$\Delta\varphi' + \omega^2\varphi' + 2\omega g f_0' - \frac{1}{2}\left.\frac{d^2V'(f')}{df'^2}\right|_{f'=f_0'}\varphi' = 0, \tag{B2}$$

where $V'(f') = \gamma^2 V(\frac{f'}{\gamma})$. Equations (B1) and (B2) have the same form as Eqs. (30) and (31), but now with the coupling constant e' instead of e and with the scalar field potential that differs from the one in Eq. (31). Meanwhile, in fact, the system of equations remains the same—we have only changed the variables. This simple argumentation shows that not only the coupling constant e defines whether the linear approximation can be used, but it is the coupling constant *together* with the parameters of the scalar field potential.

APPENDIX C

Let us show that $\frac{|\varphi(r)|}{f_0(r)}$ grows logarithmically with r for potentials satisfying

$$\left.\frac{dV}{df}\right|_{f=0} = 0, \qquad \frac{1}{2}\left.\frac{d^2V}{df^2}\right|_{f=0} = M^2.$$

It is clear that for $\sqrt{M^2 - \omega^2}r \gg 1$ the background solution in such a model has the form $f_0(r) \sim \frac{e^{-\sqrt{M^2-\omega^2}r}}{r}$, whereas $g(r) \sim \frac{1}{r}$. So, Eq. (31) can be written as

$$\Delta\varphi + (\omega^2 - M^2)\varphi = C\frac{e^{-\sqrt{M^2-\omega^2}r}}{r^2}, \tag{C1}$$

for $\sqrt{M^2-\omega^2}r \gg 1$, where C is a constant [in fact, $C = C(\omega)$, but it is not important for the present calculation]. Now, we define $\psi = r\varphi$ and rewrite Eq. (C1) as

$$\frac{d^2\psi}{dr^2} + (\omega^2 - M^2)\psi = C\frac{e^{-\sqrt{M^2-\omega^2}r}}{r}. \tag{C2}$$

Equation (C2) can easily be solved by means, say, of the method of variation of parameters. Its solution [such that $\psi(r)|_{r\to\infty} \to 0$] takes the form

$$\psi = -\frac{C}{2\sqrt{M^2-\omega^2}}\left(e^{-\sqrt{M^2-\omega^2}r}\ln(\xi r) + e^{\sqrt{M^2-\omega^2}r}\int_r^\infty \frac{e^{-2\sqrt{M^2-\omega^2}z}}{z}dz\right), \tag{C3}$$

where ξ is a constant. Recalling that $\varphi = \frac{\psi}{r}$, in the leading order, we get

$$\left.\frac{|\varphi(r)|}{f_0(r)}\right|_{r\to\infty} \sim \ln(\xi r).$$

APPENDIX D

The radius $R_e = R_e(\omega)$ is defined by the relation $f_0(R_e) = e^{-1}f_0(0)$. Using Eqs. (99), (100), and (101), it is not difficult to show that R_e satisfies the following equations:

$$\frac{\sqrt{\omega^2-m^2}R_e}{\sin(\sqrt{\omega^2-m^2}R_e)} = e, \quad \text{for } R_e < R, \tag{D1}$$

$$\frac{\sqrt{M^2-m^2}R_e e^{\sqrt{M^2-\omega^2}R_e}}{e^{\sqrt{M^2-\omega^2}R}} = e, \quad \text{for } R_e \geq R. \tag{D2}$$

At first, we solved numerically Eq. (D1) for a given ω. The result satisfying $R_e < R = R(\omega)$ was accepted; the result satisfying $R_e > R = R(\omega)$ was rejected, and the solution to Eq. (D2) was accepted as the value of R_e.

The coordinate R_ϵ is defined by Eq. (59). Suppose that $R_\epsilon = R_\epsilon(\omega) > R(\omega)$ for $m^2 < 0$, $\frac{|m|}{M} = 0.6$, and $\epsilon = 10^{-2}$. In this case, Eq. (59) gives

$$R_\epsilon = R + \frac{1}{2\sqrt{M^2-\omega^2}}\ln\left(\frac{(\epsilon+1)(\omega^2-m^2)}{\epsilon(M^2-m^2)(1+R\sqrt{M^2-\omega^2})}\right). \tag{D3}$$

It can be checked that the equality $R_\epsilon(\omega) > R(\omega)$ indeed holds for the chosen set of the parameters.

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