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Factorization of the $S^3/\mathbb{Z}_n$ partition function

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We investigate $S^3/\mathbb{Z}_n$ partition function of three-dimensional $\mathcal{N}=2$ supersymmetric field theories. In a gauge theory the partition function is the sum of the contributions of sectors specified by holonomies, and we should carefully choose the relative signs among the contributions. We argue that the factorization to holomorphic blocks is a useful criterion to determine the signs and propose a formula for them. We show that the orbifold partition function of a general nongauge theory is correctly factorized provided that we take appropriate relative signs. We also present a few examples of gauge theories. We point out that the sign factor for the orbifold partition function is closely related to a similar sign factor in the lens space index and the three-dimensional index.

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I. INTRODUCTION

In this paper, we discuss a rather technical issue concerning the sign of the partition function of three-dimensional $\mathcal{N}=2$ supersymmetric theories. It is defined by the path integral

$$Z_{\mathcal{M}}=\int\mathcal{D}\Phi e^{-S},\tag{1}$$

where $\mathcal{M}$ is the background manifold and Φ collectively represents all dynamical fields in the theory. The overall factor is often ignored because it does not affect some observables such as correlation functions. However, in recent progress of supersymmetric field theories, the partition function itself plays an important role. For example, we can determine the superconformal R-charge of a three-dimensional $\mathcal{N}=2$ theory at infrared fixed point by maximizing the real part of the free energy $F=-\log Z_{S^3}$ [1,2].

In order to compute the partition function including the overall factor unambiguously through the path integral (1) we need to fix the measure of the path integral carefully. A convenient way to do this is to exploit the fact that $Z_{\mathcal{M}}$ is obtained from the supersymmetric index of a three-dimensional $\mathcal{N}=1$ theory as the small radius limit. Let us consider the case of the S^3 partition function. To obtain the three-dimensional theory, we start from a four-dimensional $\mathcal{N}=1$ theory in the background $S^3\times S^1$. We define S^3 by

$$|z_1|^2+|z_2|^2=1,\qquad z_1,z_2\in\mathbb{C}.\tag{2}$$

If we regard the S^1 direction as a time, the path integral of the four-dimensional theory is interpreted as the index

$$I(p_1,p_2,z_a)=\mathrm{tr}[(-1)^F q^{D-\frac{R}{2}}(q^{-1}p_1)^{J_1+\frac{R}{2}}(q^{-1}p_2)^{J_2+\frac{R}{2}}z_a^{F_a}],\tag{3}$$

where F, R, D, and F_a are the fermion number, the $U(1)_R$ charge, the dilatation, and flavor charges, respectively. J_1 and J_2 are the angular momenta rotating z_1 and z_2, respectively. The exponent of q in (3),

$$D-J_1-J_2-\frac{3}{2}R=\{Q,Q^\dagger\},\tag{4}$$

is exact with respect to a supercharge Q, and (3) is independent of the variable q. Unlike the partition function Z_{S^3}, there is a natural normalization of I; in the trace over the Hilbert space, every gauge-invariant state contributes to the index by weight 1, and there is no ambiguity of the normalization except for the signature. The S^3 partition function is obtained as the $\beta\to 0$ limit of the index. In nonsupersymmetric theories the small radius limit may in general diverge. However, in the reduction from a four-dimensional $\mathcal{N}=1$ theory to a three-dimensional $\mathcal{N}=2$ theory that we consider here, we can obtain a finite result due to the cancellation between bosonic and fermionic contributions.[1]

Therefore, once we obtain the four-dimensional index, we can unambiguously obtain the partition function by the small radius limit [3–5].

$$Z_{S^3}(b,\mu_a)=\lim_{\beta\to 0}I(p_i=e^{-\beta\omega_i},z_a=e^{-\beta\mu_a}),\qquad b=\sqrt{\frac{\omega_1}{\omega_2}}.\tag{5}$$

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[1]For this cancellation we should carefully include the zero-point contribution, which is often neglected.

ω_i and μ_a are interpreted in the three-dimensional theory as squashing parameters and real mass parameters. The partition function depends on ω_i through the single parameter b [5–11].

The ambiguity in the signature is due to the ambiguity in the statistics of the vacuum state. The statistics of states in the Hilbert space is fixed once that of the vacuum state is specified. However, there is no general rule to fix it, and we need an additional criterion to fix the overall sign.

For some use of $Z_{\mathcal{M}}$, like F maximization, we only need the absolute value of $Z_{\mathcal{M}}$, and one may think that the sign ambiguity is not important. However, if the theory has multiple sectors, we should sum up their contributions, and we need to fix the relative signs among them. This is the case when we consider a gauge theory on a manifold with a nontrivial fundamental group. In such a case there are degenerate vacua labeled by holonomies associated with nontrivial cycles. In this paper we focus on the orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$ defined from $\boldsymbol{S}^3$ in (2) by the identification

$$(z_1, z_2) \sim (\omega z_1, \omega^{-1} z_2) \quad \omega = e^{\frac{2\pi i}{n}}. \tag{6}$$

The fundamental group of this manifold is $\mathbb{Z}_n$, and vacua are labeled by $\mathbb{Z}_n$-valued holonomies $h_a^{(\mathrm{dyn})}$ associated with dynamical $U(1)_a$ gauge symmetries as well as continuous moduli parameters $\mu_a^{(\mathrm{dyn})}$. It is also possible to introduce mass parameters $\mu_a^{(\mathrm{ext})}$ and nontrivial holonomies $h_a^{(\mathrm{ext})}$ for global $U(1)_a$ symmetries. (We label both gauge and global symmetries by a.) The partition function is obtained by summing up the contribution from sectors with different $h_a^{(\mathrm{dyn})}$,

$$\begin{aligned} &Z_{\boldsymbol{S}^3/\mathbb{Z}_n}(b, \mu_a^{(\mathrm{ext})}, h_a^{(\mathrm{ext})}) \\ &\quad = \sum_{h_a^{(\mathrm{dyn})}} f(h) \int d\mu^{(\mathrm{dyn})} e^{-S^{\mathrm{cl}}_{S^3/\mathbb{Z}_n}} Z^{\text{1-loop}}_{\boldsymbol{S}^3/\mathbb{Z}_n}(b, \mu_a, h_a), \end{aligned} \tag{7}$$

where $f(h)$ is the sign factor that we would like to determine, and the explicit form of the integrand will be given in the next section.

A formula for the orbifold partition function has actually already been given in [12]. They derive the formula in two ways. One is the orbifold projection from the $\boldsymbol{S}^3$ partition function, and the other is reduction from the lens space index, which is obtained from the $\boldsymbol{S}^3 \times \boldsymbol{S}^1$ index by the orbifold projection. In both derivations, they do not take account of the possible emergence of nontrivial sign factors. The formula has been used for some applications, and works well. However, in some cases, we need to introduce extra sign factors. For example, it is demonstrated in [13] for a few examples of dual pairs that the matching of the orbifold partition function of dual theories requires nontrivial sign factors.

A similar problem of relative weight also arises in the instanton sum in the $\boldsymbol{S}^4$ partition function. It would be instructive to understand how we can determine the relative weights in that case before we explain our strategy for the $\boldsymbol{S}^3/\mathbb{Z}_n$ partition function.

Let us consider an $\mathcal{N} = 2$ supersymmetric gauge theory on $\boldsymbol{S}^4$. By equivariant localization, we can localize the dynamics of the theory at two poles of $\boldsymbol{S}^4$, and the partition function is written as [14,15]

$$Z = \sum_{k_N, k_S = 0}^{\infty} f(k_N, k_S) Z_N(k_N) Z_S(k_S), \tag{8}$$

where k_N and k_S are, respectively, the instanton number at the north pole and the anti-instanton number at the south pole. (Precisely, we need to perform the integral over the Coulomb branch parametrized by constant scalar fields. Here, we focus only on the instanton sum, and consider the contribution from a specific point in the Coulomb branch.) We introduced the unknown phase factor $f(k_N, k_S)$. This phase factor is strongly restricted by assuming the locality of the theory [16]. If we assume the locality, the path integral for the localized modes at the two poles should be performed independently, and thus the partition function is factorized into contributions from the poles;

$$Z = \sum_{k_N = 0}^{\infty} g(k_N) Z_N(k_N) \sum_{k_S = 0}^{\infty} h(k_S) Z_S(k_S), \tag{9}$$

where $g(k_N)$ and $h(k_S)$ are unknown phase factors. Now we use the fact that disconnected components of the configuration space of the theory are labeled only by the total instanton number $k_N - k_S$. This means that two configurations labeled by $(k_N^{(1)}, k_S^{(1)})$ and $(k_N^{(2)}, k_S^{(2)})$ are in the same component of the configuration space if $k_N^{(1)} - k_S^{(1)} = k_N^{(2)} - k_S^{(2)}$. We can interpolate them by continuous deformation and the relative phase between the contributions from them can, in principle, be determined unambiguously by the continuity of the action functional. Here, let us assume, for simplicity, that there are no relative phases. Namely, $g(k_N + n) h(k_S + n)$ does not depend on n. Then the relation

$$\frac{g(k_N + 1)}{g(k_N)} = \frac{h(k_S)}{h(k_S + 1)} \tag{10}$$

holds. The left-(right-)hand side of this equation is independent of k_S (k_N), and (10) is a constant independent of both k_N and k_S. Let c be the constant. We obtain

$$g(k_N) = g(0) c^{k_N}, \qquad h(k_S) = h(0) c^{-k_S}, \tag{11}$$

and the total partition function is

$$Z = f(0,0) \sum_{k_N, k_S = 0}^{\infty} c^{k_N - k_S} Z_N(k_N) Z_S(k_S). \tag{12}$$

Now we have determined the phase factor except the overall phase $f(0,0)$ and the constant c. The factor $c^{k_N - k_S}$ can be identified with the contribution of the topological θ term.

In this way, the relative phases for instanton sectors of a four-dimensional gauge theory can be fixed up to a few parameters by the factorization of the partition function. We take the same strategy to determine the relative signs among holonomy sectors of the orbifold partition function. Actually, it is known that the S^3 and $S^2 \times S^1$ partition functions are factorized into factors so-called holomorphic blocks [17–21], and a similar factorization is expected for the orbifold. Each block is identified with the vortex partition function on a solid torus [18,20]. In the following, we determine the relative signs of holonomy sectors by requiring the factorization of the orbifold partition function.

II. ORBIFOLD PARTITION FUNCTION

A. Naive projection

Let us first summarize how the formula for the orbifold partition function $Z_{S^3/\mathbb{Z}_n}$ is obtained from the S^3 partition function by naive $\mathbb{Z}_n$ orbifold projection [12].

We consider a theory with general gauge group G and matter representation R. At a generic point in the Coulomb branch, the gauge group G is broken into its Cartan subgroup H. Let V_a, W_α, and Φ_i denote the vector multiplets for the Cartan part, W-bosons, and chiral multiplets, respectively. For later convenience, we include external vector multiplets for global $U(1)_a$ symmetries in V_a. For distinction, we denote dynamical and nondynamical components by $V_a^{(\mathrm{dyn})}$ and $V_a^{(\mathrm{ext})}$, respectively. The scalar components $\mu_a^{(\mathrm{dyn})}$ for the dynamical vector multiplets parametrize the Coulomb branch while those for external vector multiplets, $\mu_a^{(\mathrm{ext})}$, are real mass parameters.

By localization, we can reduce the path integral of an $\mathcal{N}=2$ supersymmetric theory on S^3 into a finite dimensional matrix integral with the integrand consisting of the classical and one-loop factors:

$$Z_{S^3}(\mu^{(\mathrm{ext})}) = \int d\mu^{(\mathrm{dyn})} e^{-S^{\mathrm{cl}}_{S^3}(\mu)} Z^{\text{1-loop}}_{S^3}(\mu). \tag{13}$$

The integration measure is given by

$$\int d\mu^{(\mathrm{dyn})} \equiv \frac{1}{|W|}\prod_a \int_{-\infty}^{\infty} d\mu_a^{(\mathrm{dyn})}, \tag{14}$$

where a runs over the dynamical part of V_a, and $|W|$ is the order of the Weyl group of the gauge group. The one-loop factor is the product of the contributions of W_α and Φ_i

$$Z^{\text{1-loop}}_{S^3} = \frac{1}{\prod_\alpha s_b(\mu_\alpha + \frac{iQ}{2})}\frac{1}{\prod_i s_b(\mu_i - \frac{iQ}{2}(1-\Delta_i))}, \tag{15}$$

where $Q = b + b^{-1}$. Δ_i is the Weyl weight of the scalar component of a chiral multiplet Φ_i. μ_α and μ_i are the scalar components of the vector multiplets coupling to W_α and Φ_i;

$$\mu_\alpha = q_{\alpha a}\mu_a, \qquad \mu_i = q_{ia}\mu_a, \tag{16}$$

where $q_{\alpha a}$ and q_{ia} are the $U(1)_a$ charge of W-boson W_α and the chiral multiplet Φ_i, respectively. It is convenient to include R-charge in the charge matrix. We define $q_{\alpha 0}$ and q_{i0} as the R-charges of the fermions in the multiplets W_α and Φ_i,

$$q_{\alpha 0} = 1, \qquad q_{i0} = \Delta_i - 1, \tag{17}$$

and we set the corresponding scalar parameter by

$$\mu_0 \equiv \mu_R = \frac{iQ}{2}. \tag{18}$$

Including the contribution of the $U(1)_R$ symmetry, we define

$$\hat{\mu}_\alpha = \mu_\alpha + \frac{iQ}{2}, \qquad \hat{\mu}_i = \mu_i - \frac{iQ}{2}(1-\Delta_i). \tag{19}$$

Then the one-loop factor (15) is simply rewritten as

$$Z^{\text{1-loop}}_{S^3} = \frac{1}{\prod_I s_b(\hat{\mu}_I)}, \tag{20}$$

where I runs over both W-bosons and chiral multiplets. $s_b(z)$ is the double sine function, and can be expressed as an infinite product corresponding to the spherical harmonic expansion on S^3. The contribution of a multiplet I is

$$\frac{1}{s_b(\hat{\mu}_I)} = \prod_{p,q=0}^{\infty} \frac{b(q+\frac{1}{2}) + b^{-1}(p+\frac{1}{2}) + i\hat{\mu}_I}{b(p+\frac{1}{2}) + b^{-1}(q+\frac{1}{2}) - i\hat{\mu}_I}. \tag{21}$$

The denominator and the numerator come from the bosonic and the fermionic modes with angular momenta $(J_1, J_2) = (p,q)$, respectively.

The classical factor exists when the theory has Chern-Simons terms. If the action contains the Chern-Simons term

$$\frac{ik_{ab}}{4\pi}\int A_a dA_b, \tag{22}$$

the scalar field quadratic terms in the supersymmetric completion of (22) give the classical factor

$$e^{-S^{\mathrm{cl}}_{S^3}(\mu)} = e^{-\pi i k_{ab}\mu_a\mu_b}. \tag{23}$$

Let us move on to the orbifold partition function $Z_{S^3/\mathbb{Z}_n}$. The orbifold $S^3/\mathbb{Z}_n$ is defined from the S^3 in (2) by the identification (6). The vacua are parametrized by the scalar field μ_a and holonomies

$$h_a = \frac{n}{2\pi}\oint_\gamma A_a, \tag{24}$$

where γ is a nontrivial loop in $S^3/\mathbb{Z}_n$ generating the fundamental group. h_a is quantized to be an integer, and h_a and $h_a + n$ are identified because they are transformed to each other by a large gauge transformation. We define $h_I = \{h_\alpha, h_i\}$, holonomies coupling to W_α and Φ_i, in a similar way to (16):

$$h_I = \sum_a q_{Ia} h_a. \tag{25}$$

We can turn on nontrivial holonomies for global symmetries as well as the dynamical gauge symmetries. Note that we will not turn on the holonomy for the R-symmetry because it breaks supersymmetry. (For a more general lens space $L(p,q)$, we need to turn on nontrivial $U(1)_R$ holonomy to preserve supersymmetry.)

After the orbifold projection, only modes compatible with the identification (6) contribute to the one-loop factor. The condition for $\mathbb{Z}_n$ invariance for modes of multiplet I is

$$p - q = h_I \mod n, \tag{26}$$

and the one-loop factor is obtained by restricting the product over p and q in (21) by the condition (26). We define the function s_{b,h_I} to express the contribution of each multiplet by

$$\frac{1}{s_{b,h_I}(\hat{\mu}_I)} = \prod_{(p,q)\in\Lambda_{[h_I]}} \frac{b(q+\frac{1}{2}) + b^{-1}(p+\frac{1}{2}) + i\hat{\mu}_I}{b(p+\frac{1}{2}) + b^{-1}(q+\frac{1}{2}) - i\hat{\mu}_I}, \tag{27}$$

where $\Lambda_{[h]}$ is the set consisting of (p,q) satisfying the condition (26):

$$\Lambda_{[h]} = \{(p,q) | p, q \geq 0, p - q = h \mod n\}. \tag{28}$$

The one-loop factor (21) is replaced by

$$Z^{\text{1-loop}}_{S^3/\mathbb{Z}_n} = \prod_I \frac{1}{s_{b,h_I}(\hat{\mu}_I)}. \tag{29}$$

For the classical part, the $\mathbb{Z}_n$ orbifolding gives rise to the extra $1/n$ factor in the action, and the Chern-Simons term gives the holonomy dependent phase [22–24][2]:

[2]We find a slightly different formula $e^{\frac{\pi i}{n}k_{ab}h_a h_b}$ for the holonomy dependent phase in the literature. This is, however, not gauge invariant even for integer Chern-Simons levels. A simple derivation of the holonomy dependent phase in (30) is given in Appendix A 1.

$$e^{-S^{\text{cl}}_{S^3/\mathbb{Z}_n}(\mu,h)} = e^{-\frac{\pi i}{n}k_{ab}\mu_a\mu_b} e^{-\frac{\pi i k_{ab}}{n}(n-1)h_a h_b}. \tag{30}$$

The integration measure is

$$\int d\mu^{(\text{dyn})} = \frac{1}{|W|}\prod_a \int_{-\infty}^{\infty} \frac{d\mu_a^{(\text{dyn})}}{n}. \tag{31}$$

This is normalized by using the relation to the lens space index [12]. Combining these factors, we obtain the formula (7).

Before ending this section, we would like to comment on a subtlety lurking in the formula (7). For the gauge invariance of the factor (30), the Chern-Simons levels k_{ab} must be integers. This is, however, not always the case. If the theory has parity anomaly, some components of the bare Chern-Simons level should be half odd integers to cancel the anomaly. In this case, the factor (30) itself is not gauge invariant. Namely, it may change its sign under a large gauge transformation that shifts h_a by $h_a \to h_a + nc_a$ ($c_a \in \mathbb{Z}$). Of course, this is not an essential problem. For the consistency, we only need the gauge invariance of the whole integrand in (7) including the one-loop contribution. In the following, we propose a general formula for the sign factor $f(h)$, which will give the sign of the partition function for each holonomy sector in a gauge-invariant way.

B. Projection operator

As we mentioned in the introduction, we use the factorization to holomorphic blocks to determine the sign factor. For $\mathcal{M} = S^3$ and $S^2 \times S^1$, it is known that the partition function is written in terms of holomorphic blocks by [17–21]

$$Z_\mathcal{M} = \sum_A B^A(x_a, q) B^A(\tilde{x}_a, \tilde{q}). \tag{32}$$

In the case of S^3, the variables x_a, q, $\tilde{x}_a$, and $\tilde{q}$ are given by

$$q = e^{2\pi i b^2}, \quad x_a = e^{2\pi b\mu_a}, \quad \tilde{q} = e^{2\pi i b^{-2}}, \quad \tilde{x}_a = e^{2\pi b^{-1}\mu_a}. \tag{33}$$

This factorization is also expected for the orbifold partition function with a different definition for the variables x_a, q, $\tilde{x}_a$, and $\tilde{q}$. This factorization is naturally interpreted in Higgs branch localization, in which the index A labels Higgs vacua and the blocks are identified with the vortex partition functions [18,20].

In order to obtain a factorized form of the orbifold partition function, it is convenient to rewrite s_{b,h_I} in (27) as

$$\frac{1}{s_{b,h_I}(\hat{\mu}_I)} = \mathcal{P}_{(\hat{\mu}_I,h_I)} \frac{1}{s_b(\hat{\mu}_I)}, \tag{34}$$

where $\mathcal{P}_{(z,h)}$ with $z \in \mathbb{C}$ and $h \in \mathbb{Z}_n$ is the operator acting on a function of z defined by

$$\mathcal{P}_{(z,h)} f(z) = \prod_{(k,l)\in L_h} f(z_{k,l}),$$
$$z_{k,l} \equiv \frac{z + ib(k - \frac{n-1}{2}) + ib^{-1}(l - \frac{n-1}{2})}{n}, \quad (35)$$

where

$$L_h = \{(k,l) | 0 \le k, l < n, k - l = h \bmod n\}. \quad (36)$$

An advantage of rewriting (27) with this operator is that the operator preserves the factorized form of the function. Namely, if a function $f(z)$ is the product of two functions $g(z)$ and $h(z)$, the relation $\mathcal{P}_{(z,h)} f(z) = (\mathcal{P}_{(z,h)} g(z))(\mathcal{P}_{(z,h)} h(z))$ holds. Therefore, if the S^3 partition function $Z_{S^3}(\hat{\mu})$ of a theory is factorized into holomorphic blocks as in (32), and if the orbifold partition function is obtained from $Z_{S^3}(\hat{\mu})$ by applying the operator $\mathcal{P}_{(\hat{\mu},h)}$, we can immediately obtain the factorized form of the orbifold partition function by applying $\mathcal{P}_{(\hat{\mu},h)}$ to the holomorphic blocks for S^3.

The operator $\mathcal{P}_{(z,h)}$ is defined to simplify the expression of the $\mathbb{Z}_n$ projection of the one-loop factor, and it is *a priori* not guaranteed that it correctly reproduces the classical factor $e^{-S^{\rm cl}_{S^3/\mathbb{Z}_n}}$ in the orbifold partition function. Interestingly, up to the sign factor which we have not fixed yet, it reproduces the classical factor (30) in the orbifold partition function from (23) for S^3. Let us consider a Chern-Simons term with factorized Chern-Simons level $k_{ab} = \kappa c_a c_b$:

$$\frac{i\kappa c_a c_b}{4\pi} \int A_a dA_b. \quad (37)$$

For S^3, this gives the classical factor

$$e^{-S^{\rm cl}_{S^3}} = e^{-\kappa\pi i \hat{\mu}^2}, \qquad \hat{\mu} = c_a \mu_a. \quad (38)$$

Applying the operator $\mathcal{P}_{(\hat{\mu},h)}$ on this function, we obtain

$$\mathcal{P}_{(\hat{\mu},h)} e^{-\kappa\pi i\hat{\mu}^2} = e^{\frac{\pi i\kappa}{n}\frac{(b+b^{-1})^2}{12}(n^2-1)} \times \exp\left[-\frac{\pi i\kappa}{n}(\hat{\mu}^2 + [h](n-[h]))\right], \quad (39)$$

where $h = c_a h_a$ and $[h]$ denotes the smallest non-negative integer in $h + n\mathbb{Z}$. (We have not yet assumed that κ is an integer.) We rewrite this as

$$\mathcal{P}_{(\hat{\mu},h)} e^{-\kappa\pi i\hat{\mu}^2} = (b\text{-dependent factor})\rho(h)^{2\kappa} \times \exp\left[-\frac{\pi i\kappa}{n}(\hat{\mu}^2 + (n-1)h^2)\right]. \quad (40)$$

In this paper, the factorization is used simply as a criterion for the correct choice of sign factors, and we are not interested in the prefactor depending only on b. The exponential factor is nothing but the classical factor in (30), and $\rho(h)$ is

$$\rho(h) = e^{\frac{\pi i}{2n}[h](n-[h])} e^{-\frac{\pi i}{2n}(n-1)h^2}. \quad (41)$$

This function always takes $+1$ or -1 depending on h. As we will show shortly, we can compose the sign factor $f(h)$ by using $\rho(h)$.

C. Factorization and sign factor

Let us consider a general nongauge theory on S^3. As is pointed out in [20], it is convenient to decompose the Chern-Simons level into the part canceling the parity anomaly and the remaining part:

$$k_{ab} = \sum_\alpha \kappa_\alpha c_{\alpha a} c_{\alpha b} - \frac{1}{2}\sum_I q_{Ia} q_{Ib}. \quad (42)$$

κ_α and $c_{\alpha a}$ in the first term are integers, and the second term is the fractional contribution that cancels the parity anomaly. With this decomposition, we rewrite the partition function as

$$Z_{S^3}(\mu) = e^{-\pi i k_{ab}\mu_a\mu_b} \frac{1}{\prod_I s_b(\hat{\mu}_I)} = \prod_\alpha (e^{-\pi i\mu_\alpha^2})^{\kappa_\alpha} \prod_I Z_\Delta(\hat{\mu}_I), \quad (43)$$

where $Z_\Delta(\hat{\mu})$ is the partition function of the "tetrahedron theory,"[25] and is given by

$$Z_\Delta(\hat{\mu}) = \frac{e^{\frac{\pi i}{2}\hat{\mu}^2}}{s_b(\hat{\mu})}. \quad (44)$$

The two kinds of factors in the product (43) are known to be factorized into holomorphic blocks [20]:

$$Z_\Delta(\mu) = e^{-\pi i\frac{b^2+b^{-2}}{24}} B_\Delta(x,q) B_\Delta(\tilde{x},\tilde{q}), \quad (45)$$

$$e^{-\pi i\mu^2} = e^{\pi i\frac{b^2+b^{-2}}{12}} B_{\rm CS}(x;q) B_{\rm CS}(\tilde{x};\tilde{q}). \quad (46)$$

The blocks are given by

$$B_\Delta(x;q) = (qx^{-1};q),$$
$$B_{\rm CS}(x;q) = \frac{1}{(-q^{\frac{1}{2}}x;q)(-q^{\frac{1}{2}}x^{-1};q)}, \quad (47)$$

where $(x;q)$ is the q-Pochhammer symbol

$$(x;q) = \prod_{k=0}^{\infty}(1 - xq^k). \tag{48}$$

An important feature of the factorization is that the information of the background manifold is encoded in the definition of the arguments of holomorphic blocks. For S^3 they are given by (33), and the blocks for the orbifold should be given by the same functions with different arguments. We can confirm this by computing the holomorphic blocks for the orbifold by applying the operator $\mathcal{P}_{(z,h)}$ to the holomorphic blocks for S^3. Indeed, we can easily show

$$\begin{aligned}\mathcal{P}_{(\mu,h)}B_\Delta(x;q) &= B_\Delta(x';q'),\\ \mathcal{P}_{(\mu,h)}B_\Delta(\tilde{x};\tilde{q}) &= B_\Delta(\tilde{x}';\tilde{q}'),\\ \mathcal{P}_{(\mu,h)}B_{\mathrm{CS}}(x;q) &= B_{\mathrm{CS}}(x',q'),\\ \mathcal{P}_{(\mu,h)}B_{\mathrm{CS}}(\tilde{x};\tilde{q}) &= B_{\mathrm{CS}}(\tilde{x}',\tilde{q}'),\end{aligned} \tag{49}$$

where the variables for the orbifold are

$$q' = \omega q^{\frac{1}{n}}, \qquad x' = \omega^h x^{\frac{1}{n}}, \qquad \tilde{q}' = \omega \tilde{q}^{\frac{1}{n}}, \qquad \tilde{x}' = \omega^{-h}\tilde{x}^{\frac{1}{n}}. \tag{50}$$

The arguments above guarantee that we obtain the orbifold partition function that is correctly factorized into the holomorphic blocks by simply applying the projection operator to the factors in the S^3 partition function (43):

$$\begin{aligned}Z_{S^3/\mathbb{Z}_n}(\mu,h) &\propto \prod_\alpha (\mathcal{P}_{(\mu_\alpha,h_\alpha)}e^{-\pi i\mu_\alpha^2})^{\kappa_\alpha}\prod_I \mathcal{P}_{(\hat{\mu}_I,h_I)}Z_\Delta(\hat{\mu}_I)\\ &\propto \prod_\alpha e^{\frac{\pi i}{n}\kappa_\alpha(\hat{\mu}_\alpha^2+(n-1)h_\alpha^2)}\prod_I \frac{\rho(h_I)e^{\frac{\pi i}{2n}(\hat{\mu}_I^2+(n-1)h_I^2)}}{s_{b,h_I}(\hat{\mu}_I)}\\ &= e^{-S^{\mathrm{cl}}_{S^3/\mathbb{Z}_n}(\mu,h)}\prod_I \frac{\rho(h_I)}{s_{b,h_I}(\hat{\mu}_I)}.\end{aligned} \tag{51}$$

"$\propto$" means the ignorance of prefactors depending only on b. By comparing this to (7), we obtain

$$f(h) = \prod_I \rho(h_I). \tag{52}$$

(We cannot fix the overall sign factor independent of h, which we are not interested in.) We can absorb the sign factor by the redefinition of the orbifold double sine function

$$s^{\mathrm{imp}}_{b,h}(z) = \rho(h)s_{b,h}(z), \tag{53}$$

and then we can present the orbifold partition function in the same form with the original one.

$$Z_{S^3/\mathbb{Z}_n}(\mu,h) = e^{-S^{\mathrm{cl}}_{S^3/\mathbb{Z}_n}(\mu,h)}\frac{1}{\prod_I s^{\mathrm{imp}}_{b,h_I}(\hat{\mu}_I)}. \tag{54}$$

In [13], a similar improvement of the orbifold double sine function $\hat{s}_{b,h}(z) = \sigma_h s_{b,h}(z)$ is proposed for odd $n = 2m+1$. The extra sign factor σ_h is related to $\rho(h)$ through

$$\sigma_h = (-)^{mh}\rho(h). \tag{55}$$

In [13], only theories without parity anomaly are considered. This means that the charge assignment q_{Ia} for every $U(1)$ gauge symmetry satisfies $\sum_I q_{Ia} \in 2\mathbb{Z}$. In such a case, the difference of σ_h and $\rho(h)$ does not affect the partition function. In the case of even n, [13] did not succeed in finding such an improvement. The reason is that in [13] the sign factor σ_h is assumed to be a periodic function of h with period n. The function $\rho(h)$ does not satisfy this condition. It may change its sign under the shift $h \to h+n$.

$$\frac{\rho(h+n)}{\rho(h)} = (-1)^{(n-1)h+\frac{n(n-1)}{2}}. \tag{56}$$

This means that the improved function $s^{\mathrm{imp}}_{b,h}$ may change its sign in the large gauge transformation $h \to h+n$. This, however, does not cause any problem. If the parity anomaly arising in the one-loop part is correctly canceled by the bare Chern-Simons term, the partition function (54) is invariant under the shift $h \to h+n$.

D. Gauge theories

In the previous subsection we gave a prescription to fix the relative signs among holonomy sectors. It is implemented by the redefinition of the orbifold double sine function (53). The orbifold partition function of an arbitrary nongauge theory with this improvement is correctly factorized into holomorphic blocks.

We expect that this improvement works for gauge theories, too. Namely, the orbifold partition function,

$$\begin{aligned}&Z_{S^3/\mathbb{Z}_n}(\mu^{(\mathrm{ext})},h^{(\mathrm{ext})})\\ &\quad= \sum_{h^{(\mathrm{dyn})}}\int d\mu^{(\mathrm{dyn})}e^{-S^{\mathrm{cl}}_{S^3/\mathbb{Z}_n}(\mu,h)}\frac{1}{\prod_I s^{\mathrm{imp}}_{b,h}(\hat{\mu}_I)},\end{aligned} \tag{57}$$

for a gauge theory is correctly factorized into holomorphic blocks as in (32). Unfortunately, we have not succeeded in proving this for an arbitrary gauge theory. Here, we consider two examples of gauge theories, supersymmetric quantum electrodynamics (SQED) with $N_f = 1$, and an $su(2)$ Chern-Simons theory with an adjoint chiral multiplet.

1. SQED

As the first example, let us consider SQED with one flavor $(q, \tilde{q})$. This theory has four $U(1)$ symmetries. One is a gauge symmetry $U(1)_G$, and the others are global symmetries. See Table I for charge assignments. $U(1)_R$ is an R-symmetry. $U(1)_A$ is a flavor symmetry acting on q and $\tilde{q}$ with charge $+1$. $U(1)_V$ is the topological symmetry, and the corresponding external gauge field A_V couples to the $U(1)_G$ flux dA_G through the Chern-Simons term

$$\frac{1}{2\pi}\int A_V dA_G. \tag{58}$$

The partition function is

$$Z^{S^3/\mathbb{Z}_n}_{\mathrm{SQED}} = \sum_{h_G=0}^{n-1}\int \frac{e^{\frac{2\pi i}{n}\mu_V\mu_G}e^{\frac{2\pi i}{n}h_V h_G}}{s^{\mathrm{imp}}_{b,h_q}(\hat{\mu}_q)s^{\mathrm{imp}}_{b,h_{\tilde{q}}}(\hat{\mu}_{\tilde{q}})}\frac{d\mu_G}{n}, \tag{59}$$

where $\hat{\mu}_I$ and h_I are defined by

$$\hat{\mu}_q = (\Delta-1)\frac{iQ}{2}+\mu_A+\mu_G,$$
$$\hat{\mu}_{\tilde{q}} = (\Delta-1)\frac{iQ}{2}+\mu_A-\mu_G. \tag{60}$$

$$h_q = h_A + h_G, \qquad h_{\tilde{q}} = h_A - h_G. \tag{61}$$

Let us confirm that this orbifold partition function is factorized into holomorphic blocks. Although it would not be difficult to directly prove the factorization by performing the integral by using the residue theorem, we take another way. In [13], it is confirmed that the orbifold partition function of the SQED with an appropriate choice of the sign factor coincides with that of the XYZ model, the system consisting of three chiral multiplets X, Y, and Z interacting through the superpotential $W = XYZ$. Because the XYZ model is a nongauge theory and we have already proved the factorization of nongauge theories, this duality relation guarantees the factorization of the partition function of the SQED.

The charge assignments for the XYZ model is also shown in Table I, and the partition function is given by

TABLE I. The charge assignments in the SQED and XYZ model, a model consisting of chiral multiplets X, Y, and Z, which is dual to the SQED. For $U(1)_R$ symmetry the charges of the fermion components are shown.

	q	$\tilde{q}$	X	Y	Z
$U(1)_R$	$\Delta-1$	$\Delta-1$	$-\Delta$	$-\Delta$	$2\Delta-1$
$U(1)_G$	1	-1	⋯	⋯	⋯
$U(1)_V$	0	0	1	-1	0
$U(1)_A$	1	1	-1	-1	2

$$Z^{S^3/\mathbb{Z}_n}_{\mathrm{XYZ}} = \frac{1}{s^{\mathrm{imp}}_{b,h_X}(\hat{\mu}_X)s^{\mathrm{imp}}_{b,h_Y}(\hat{\mu}_Y)s^{\mathrm{imp}}_{b,h_Z}(\hat{\mu}_Z)}, \tag{62}$$

with the parameters

$$\hat{\mu}_X = -\Delta\frac{iQ}{2}-\mu_A+\mu_V, \qquad \hat{\mu}_Y = -\Delta\frac{iQ}{2}-\mu_A-\mu_V,$$
$$\hat{\mu}_Z = (2\Delta-1)\frac{iQ}{2}+2\mu_A, \tag{63}$$

and

$$h_X = -h_A + h_V, \qquad h_Y = -h_A - h_V, \qquad h_Z = 2h_A. \tag{64}$$

It is easy to numerically check the coincidence of (59) and (62). If we use the result of [13], what we have to do to confirm the relation $Z_{\mathrm{XYZ}} = Z_{\mathrm{SQED}}$ is to show the sign factor determined in [13] by requiring $Z_{\mathrm{XYZ}} = Z_{\mathrm{SQED}}$ to be the same as the extra sign factor introduced by replacing $s_{b,h}$ by $s^{\mathrm{imp}}_{b,h}$. Indeed, the product of the five sign factors corresponding to the five double sine functions in Z_{SQED} and Z_{XYZ} coincides with the factor given in [13].

$$\sigma(h_G, h_V, h_A) = \rho(h_q)\rho(h_{\tilde{q}})\rho(h_X)\rho(h_Y)\rho(h_Z). \tag{65}$$

2. $su(2)$ gauge theory

Next, as a simple example of non-Abelian gauge theory, we consider an $su(2)_1$ gauge theory coupled to an adjoint chiral multiplet Φ. (We use $su(2)$ instead of $SU(2)$ to emphasize that we do not specify the global structure of the gauge group.) Jafferis and Yin [26] proposed that this theory is dual to the theory consisting of a single chiral multiplet $X \sim \mathrm{tr}\Phi^2$. If we can show the matching of the partition functions for the dual pair, the factorization of the $su(2)$ theory is guaranteed as the previous example.

In general, we cannot completely specify a gauge theory only by local information. There may be different theories distinguished by global structure of the gauge group. In [27] importance of such distinction in four-dimensional supersymmetric gauge theories is pointed out, and it is investigated how such theories are related to each other by Seiberg duality. The duality is checked in [28] by matching the lens space index, which is sensitive to the global structure of the gauge group. Similar aspects in three-dimensional gauge theories are studied in [29].

The $su(2)$ theory we discuss here is also an example of such a theory. It has only an adjoint chiral multiplet as a matter field, and no elementary fields are transformed by the center of $SU(2)$. Therefore, precisely speaking, there are two choices of the gauge group, $SU(2)$ and $SO(3)$. Although it is an important problem to clarify how the different choices of the gauge group affect the duality and the factorization, we will not do any detailed analysis here.

We only present the result of a preliminary analysis based on the numerical computation of the partition function.

The symmetries and the charge assignments for the dual theories are shown in Table II. In the $su(2)$ theory, the action contains the Chern-Simons term

$$\frac{i}{4\pi}\int \mathrm{tr}_{\mathrm{fund}}\left(A_G dA_G - \frac{2i}{3}A_G^3\right) \tag{66}$$

for the dynamical gauge field, and

$$\frac{(-3/2)i}{4\pi}\int ((\Delta-1)A_R + A_A)d((\Delta-1)A_R + A_A) + \frac{i}{4\pi}\int A_R dA_R \tag{67}$$

for the external $U(1)_R$ and $U(1)_A$ gauge fields. On the other hand, the chiral free theory contains the Chern-Simons term

$$\frac{(-1/2)i}{4\pi}\int ((2\Delta-1)A_R + 2A_A)d((2\Delta-1)A_R + 2A_A) \tag{68}$$

for the external gauge fields.

We define the holonomy parameter h for the dynamical $su(2)$ gauge group by

TABLE II. The symmetries and the charge assignments in Jafferis-Yin duality. The $U(1)_R$ charges in the table are those for fermion component of the multiplets.

	Φ	X
$SU(2)_G$	adj	$\cdots$
$U(1)_R$	$\Delta - 1$	$2\Delta - 1$
$U(1)_A$	1	2

$$U = \exp\left(i\oint_\gamma A\right) = \begin{pmatrix} e^{\frac{\pi i h}{n}} & 0 \\ 0 & e^{-\frac{\pi i h}{n}} \end{pmatrix}. \tag{69}$$

If the gauge group is $SU(2)$, U^n must be the unit matrix, and the holonomy is quantized by $h \in 2\mathbb{Z}$, while for the $SO(3)$ gauge group h can be an arbitrary integer. The periodicity of h also depends on the global structure of the gauge group and the background manifold, and there are two possibilities, $h \sim h + 2n$ or $h \sim h + n$. Here, we will not argue which of these possibilities should be adopted. We simply compute the partition functions for all the holonomy sectors labeled by $h = 0, 1, \ldots, 2n-1$ and infer from the duality which sectors should be summed up.

The orbifold partition function of each holonomy sector of the $su(2)$ theory is

$$Z_{su(2)}^{S^3/\mathbb{Z}_n}(h, h_A) = \int \frac{d\mu_G}{2n} e^{-\frac{\pi i}{2n}\mu_G^2} e^{-\frac{\pi i}{2n}(n-1)h^2} e^{\frac{\pi i}{n}[\frac{3}{2}\{(\Delta-1)\frac{iQ}{2}+\mu_A\}^2+\frac{Q^2}{4}]} e^{\frac{3\pi i}{2n}(n-1)h_A^2} \times \frac{1}{s_{b,h_{\phi_+}}^{\mathrm{imp}}(\hat{\mu}_{\phi_+}) s_{b,h_{\phi_0}}^{\mathrm{imp}}(\hat{\mu}_{\phi_0}) s_{b,h_{\phi_-}}^{\mathrm{imp}}(\hat{\mu}_{\phi_-}) s_{b,h_{W^+}}^{\mathrm{imp}}(\hat{\mu}_{W^+}) s_{b,h_{W^-}}^{\mathrm{imp}}(\hat{\mu}_{W^-})}, \tag{70}$$

where the parameters are given by

$$\hat{\mu}_{\phi_+} = (\Delta-1)\frac{iQ}{2} + \mu_A + \mu_G,$$
$$\hat{\mu}_{\phi_0} = (\Delta-1)\frac{iQ}{2} + \mu_A,$$
$$\hat{\mu}_{\phi_-} = (\Delta-1)\frac{iQ}{2} + \mu_A - \mu_G,$$
$$\hat{\mu}_{W^+} = \frac{iQ}{2} + \mu_G, \qquad \hat{\mu}_{W^-} = \frac{iQ}{2} - \mu_G, \tag{71}$$

$$h_{\phi_+} = h_A + h, \qquad h_{\phi_0} = h_A, \qquad h_{\phi_-} = h_A - h,$$
$$h_{W^+} = h, \qquad h_{W^-} = -h. \tag{72}$$

The orbifold partition function of the chiral free theory is

$$Z_X^{S^3/\mathbb{Z}_n}(h_A) = e^{\frac{\pi i}{2n}\{(2\Delta-1)\frac{iQ}{2}+2\mu_A\}^2} e^{\frac{2\pi i}{n}(n-1)h_A^2} \frac{1}{s_{b,h_X}^{\mathrm{imp}}(\hat{\mu}_X)}, \tag{73}$$

where the parameters are given by

$$\hat{\mu}_X = (2\Delta-1)\frac{iQ}{2} + 2\mu_A, \qquad h_X = 2h_A. \tag{74}$$

Numerical results are divided to four cases:

(i) In the case of $n \in 4\mathbb{Z}$, only the even sector coincides up to a constant factor.

$$\sum_{h=0,2,\ldots,2n-2} Z_{su(2)}^{S^3/\mathbb{Z}_n}(h, h_A) = 2Z_X^{S^3/\mathbb{Z}_n}(h_A). \tag{75}$$

(ii) In the case of $n \in 4\mathbb{Z}+1$, both even and odd sectors coincide up to a constant factor.

$$\sum_{h=0,2,\ldots,2n-2} Z_{su(2)}^{S^3/\mathbb{Z}_n}(h, h_A) = \sum_{h=1,3,\ldots,2n-1} Z_{su(2)}^{S^3/\mathbb{Z}_n}(h, h_A) = \sqrt{2}e^{\frac{\pi i}{4}} Z_X^{S^3/\mathbb{Z}_n}(h_A). \tag{76}$$

(iii) In the case of $n \in 4\mathbb{Z}+2$, only the odd sector coincides up to a constant factor.

$$\sum_{h=1,3,\ldots,2n-1} Z_{su(2)}^{S^3/\mathbb{Z}_n}(h,h_A) = -2iZ_X^{S^3/\mathbb{Z}_n}(h_A). \tag{77}$$

(iv) In the case of $n \in 4\mathbb{Z}+3$, both even and odd sectors coincide with a different constant factor.

$$\sum_{h=0,2,\ldots,2n-2} Z_{su(2)}^{S^3/\mathbb{Z}_n}(h,h_A) = \sqrt{2}e^{\frac{\pi i}{4}}Z_X^{S^3/\mathbb{Z}_n}(h_A), \tag{78}$$

$$\sum_{h=1,3,\ldots,2n-1} Z_{su(2)}^{S^3/\mathbb{Z}_n}(h,h_A) = -\sqrt{2}e^{\frac{\pi i}{4}}Z_X^{S^3/\mathbb{Z}_n}(h_A). \tag{79}$$

In all cases, we do not need additional sign factor if we use the improved function $s_{b,h}^{\mathrm{imp}}$. These results strongly suggest that we should take the following holonomy sectors to sum up:

(i) $n \in 4\mathbb{Z}$: $h = 0, 2, \ldots, 2n-2$.
(ii) $n \in 4\mathbb{Z}+1$: $h = 0, 2, \ldots, 2n-2$ or $h = 1, 3, \ldots, 2n-1$.
(iii) $n \in 4\mathbb{Z}+2$: $h = 1, 3, \ldots, 2n-1$.
(iv) $n \in 4\mathbb{Z}+3$: $h = 0, 2, \ldots, 2n-2$ or $h = 1, 3, \ldots, 2n-1$.

Unfortunately, we have no clear explanation for these nontrivial choices of the holonomy sectors. We hope we can return to this problem in future work.

III. THE SIGN FACTOR TO THE LENS SPACE INDEX AND THE THREE-DIMENSIONAL INDEX

In this section we discuss the effect of the sign factor $f(h)$ derived in Sec. II B to the four-dimensional lens space index [12] and three-dimensional superconformal index [30–32]. Since the orbifold partition function is related to the lens space index as well as the three-dimensional superconformal index through dimensional reductions, the same or similar sign factor should appear in those indices. Let us first summarize the lens space index and its reductions and then discuss the sign factors.

The lens space index can be obtained by the orbifold projection of the four-dimensional index on $S^3 \times S^1$. The projection is performed along the Hopf fiber direction of the S^3 of the $S^3 \times S^1$, and it is realized by leaving the modes compatible with the identification (6). Since the rotation along the Hopf fiber direction is characterized by $(J_1 - J_2)$, the projection is realized by inserting the operator

$$\frac{1}{n}\sum_{m=0}^{n-1} e^{2\pi i\frac{m}{n}(J_1-J_2)} \tag{80}$$

to the index (3). Namely, the lens space index I_n is

$$I_n(p_1,p_2,z_a) = \frac{1}{n}\sum_{m=0}^{n-1} I(e^{\frac{2\pi i}{n}m}p_1, e^{-\frac{2\pi i}{n}m}p_2, e^{-\frac{2\pi i}{n}h_a}z_a), \tag{81}$$

where we introduced the holonomies for global symmetries h_a.

Let us focus on a theory consisting of a single chiral multiplet Φ whose charge for a $U(1)$ global symmetry is 1. The four-dimensional index on $S^3 \times S^1$ defined by (3) for the theory can be rewritten as follows [33]:

$$I_\Phi(p_1,p_2,z) = \mathrm{Pexp}'[I_\Phi^{\mathrm{sp}}] \tag{82}$$

$$I_\Phi^{\mathrm{sp}}(p_1,p_2,z) = \sum_{i,j=0}^{\infty}(zp_1^ip_2^j - z^{-1}p_1^{j+1}p_2^{i+1}) \tag{83}$$

$$= \frac{z}{1-p_1p_2}\left(\frac{1}{1-p_1}+\frac{p_2}{1-p_2}\right) - \frac{p_1p_2z^{-1}}{1-p_1p_2}\left(\frac{p_1}{1-p_1}+\frac{1}{1-p_2}\right), \tag{84}$$

where z is a fugacity for the global symmetry, and the prime of the plethystic exponential denotes that it includes the zero-point contributions; $\mathrm{Pexp}'[x] = e^{x/2}\mathrm{Pexp}[x]$. In this form the insertion of the operator (80) is equivalent to leaving the modes invariant under the condition

$$i - j = h \mod n \tag{85}$$

in (83). Then the lens space index for this theory can be written as follows:

$$I_{n,\Phi}(z,h) = \mathrm{Pexp}'[I_{n,\Phi}^{\mathrm{sp}}(z,h)] \tag{86}$$

$$I_{n,\Phi}^{\mathrm{sp}}(z,h) = \frac{z}{1-p_1p_2}\left(\frac{p_1^{[h]}}{1-p_1^n}+\frac{p_2^{n-[h]}}{1-p_2^n}\right) - \frac{p_1p_2z^{-1}}{1-p_1p_2}\left(\frac{p_1^{n-[h]}}{1-p_1^n}+\frac{p_2^{[h]}}{1-p_2^n}\right). \tag{87}$$

Let us now consider the effect of the sign factor (41). The orbifold partition function is derived in the small radius limit (5) of the lens space index. Since the sign factor is independent of the radius β, it can be uplifted to the lens space index, and we define the improved index as

$$I_n^{\mathrm{imp}}(z,h) = \left(\prod_I \rho(h_I)\right) I_n(z,h). \tag{88}$$

For the theory considered above, the improved lens space index is written as

$$I_{n,\Phi}^{\mathrm{imp}}(z,h) = e^{-\frac{i\pi}{2}h(1-h)}\mathrm{Pexp}'[I_{n,\Phi}^{\mathrm{imp,sp}}(z,h)] \tag{89}$$

$$I^{\mathrm{imp,sp}}_{n,\Phi}(z,h) = \frac{z}{1-p_1p_2}\left(\frac{p_1^h}{1-p_1^n}+\frac{p_2^{n-h}}{1-p_2^n}\right) - \frac{p_1p_2z^{-1}}{1-p_1p_2}\left(\frac{p_1^{n-h}}{1-p_1^n}+\frac{p_2^h}{1-p_2^n}\right). \quad (90)$$

Note that $[h]$ in $I_{n,\Phi}$ is now replaced by h. Although both $I_{n,\Phi}$ and $\rho(h)$ include $[h]$ the combination of them (88) is analytic in terms of h.

In order to illustrate the effect of the phase factor to the three-dimensional index, we first describe the three-dimensional index for the tetrahedron theory, which is written as follows:

$$I^{\mathrm{3d}}_{\Delta} = (-1)^{-\frac{1}{2}|h|}((-q^{\frac{1}{2}})^{\frac{|h|-h}{2}}z^{-\frac{|h|-h}{2}})\mathrm{Pexp}\left[\frac{q^{\frac{|h|}{2}}z}{1-q} - \frac{q^{1+\frac{|h|}{2}}z^{-1}}{1-q}\right]. \quad (91)$$

It is noticed in [19] that the three-dimensional index can be written in an analytic form by adding a simple phase factor:

$$I^{\mathrm{3d,imp}}_{\Delta} = i^{|h|}I^{\mathrm{3d}}_{\Delta} = \mathrm{Pexp}\left[\frac{q^{\frac{h}{2}}z}{1-q} - \frac{q^{1+\frac{h}{2}}z^{-1}}{1-q}\right]. \quad (92)$$

The phase factor is needed for the three-dimensional index to be factorized. As is pointed out in [12], the three-dimensional index is obtained from the four-dimensional lens space index by taking the $n\to\infty$ limit. When we take this limit, h is kept finite and is identified with the magnetic flux in $\boldsymbol{S}^2\times\boldsymbol{S}^1$. In this limit the sign factor (41) can be rewritten as

$$\rho(h) = e^{\frac{\pi i}{2}(|h|-h^2)}. \quad (93)$$

The contribution of the classical factor in (30) in the large n limit becomes

$$e^{-\pi i k_{ab}h_a h_b}. \quad (94)$$

As the tetrahedron theory has a Chern-Simons term with level $-1/2$, its contribution combined with the extra sign factor (93) is

$$e^{\frac{\pi i}{2}h^2}\times e^{\frac{\pi i}{2}(|h|-h^2)} = i^{|h|}. \quad (95)$$

This is precisely the factor introduced in (91) to improve the three-dimensional index. Note that the improved three-dimensional index is analytic in terms of h, which is inherited from the analytic structure of the improved four-dimensional index.

IV. CONCLUSIONS

We focused on the sign of the orbifold partition function. We proposed a formula that systematically determines the relative signs among holonomy sectors. We use the factorization to the holomorphic blocks as a criterion to determine correct signs. For nongauge theories we proved that the partition function with signs determined by the formula is correctly factorized into the holomorphic blocks. In the case of gauge theories, as a simple example, we considered two theories: SQED with $N_f=1$ and $su(2)$ gauge theory with an adjoint flavor. For the former we confirmed the factorization of the orbifold partition function. In the case of the $su(2)$ theory, we have not fully understood the summation over the holonomy sectors. We guessed which holonomy sector contributes to the partition function with the help of the duality to a nongauge theory, which is known as Jafferis-Yin duality. We found that we have to choose an appropriate subset of the holonomy sectors to obtain the partition function consistent to the duality. The formula for the sign factor gives the correct relative phases for the contributing sectors. Therefore, up to the subtlety for the choice of the holonomy sector, which is probably related to the global aspects of the gauge bundle on the orbifold, the formula seems to give appropriate signs.

We also discussed the sign factors in the lens space index of the four-dimensional theories and the index for three-dimensional theories. The formulas for these indices have been known and contain nontrivial sign factors, which are often implicit in the literature. We showed that they are closely related to the sign factor for the orbifold partition function.

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APPENDIX

1. Classical contribution of Chern-Simons terms

Let us compute the Chern-Simons term

$$S_{\mathrm{CS}} = \frac{ik}{4\pi}\int_{\mathcal{M}} AdA = \frac{ik}{4\pi}\int_X F\wedge F \quad (\mathrm{A1})$$

for a flat $U(1)$ connection. X is a manifold whose boundary is $\mathcal{M} = S^3/\mathbb{Z}_n$. To compute $e^{-S_{\rm CS}}$ unambiguously for an arbitrary integer k, X should be a spin manifold [34].

We represent $\mathcal{M} = S^3/\mathbb{Z}_n$ as the Hopf fibration over $B = S^2$. Let us represent B as the boundary of the solid cylinder C: $r \leq 1,\ 0 \leq z \leq n+1$, where (r, ϕ, z) is the cylindrical coordinate system in three-dimensional space. We divide $\mathcal{M}$ into $n+1$ regions: $M_0, M_1, \ldots, M_n$. The ℓth region M_ℓ is defined by $\ell \leq z \leq \ell+1$. Let $0 \leq \psi_\ell < 2\pi$ be the fiber coordinate in the region M_ℓ. On the boundary between adjacent regions, the fiber coordinates are related by

$$\psi_\ell|_{z=\ell} = (\psi_{\ell+1} + \phi)|_{z=\ell}. \tag{A2}$$

We want to realize a flat connection with the holonomy

$$h = \frac{n}{2\pi} \oint_\gamma A, \tag{A3}$$

where γ is a cycle along an S^1 fiber. Let us consider the following gauge potential in M_ℓ:

$$A_\ell|_{M_\ell} = \frac{h}{n} d\psi_\ell + c_\ell d\phi. \tag{A4}$$

For the gauge field on the top ($z = n+1$) and the bottom ($z = 0$) of the cylinder to be smooth and flat, we need to set

$$c_0 = c_n = 0. \tag{A5}$$

The gauge field jumps on the boundaries by

$$A_\ell - A_{\ell-1}|_{z=\ell} = \left(c_\ell - c_{\ell-1} - \frac{h}{n}\right) d\phi. \tag{A6}$$

For this to be a gauge symmetry, the coefficients c_ℓ should satisfy

$$c_\ell - c_{\ell-1} - \frac{h}{n} \in \mathbb{Z}. \tag{A7}$$

To compute the Chern-Simons action for this gauge potential, we need to define the manifold X and to extend the gauge potential into X. We adopt an X that is a topologically n-centered Taub-NUT. We represent X as an S^1 fibration over C with n centers placed on the axis at $z = 1, 2, \ldots, n$. X is also divided into $n+1$ regions $X_0, X_1, \ldots, X_n$ at $z = 1, 2, \ldots, n$. We take the following ansatz for the extension of the gauge field in the ℓth region:

$$A_\ell = \left[\frac{h}{n} - B(z)(1 - f(r))\right] d\psi_\ell + c_\ell f(r) d\phi, \tag{A8}$$

where $B(z)$ and $f(r)$ are continuous functions satisfying

$$B(0) = B(n+1) = 0, \qquad f(0) = 0, \qquad f(1) = 1. \tag{A9}$$

These conditions guarantee that (A8) coincides to (A4) on $\mathcal{M}$.

Let us think about the smoothness of the gauge field at the center at $z = \ell$. The center is located on the boundary between M_ℓ and $M_{\ell-1}$. In each region the gauge field near the center is given by

$$A_\ell = b_\ell d\psi_\ell, \quad A_{\ell-1} = b_\ell d\psi_{\ell-1}, \quad b_\ell = \frac{h}{n} - B(\ell). \tag{A10}$$

If we assume

$$b_\ell \in \mathbb{Z}, \tag{A11}$$

these gauge field can be eliminated by the gauge transformation

$$\begin{aligned} A_\ell &\to A'_\ell = A_\ell - b_\ell d\psi_\ell, \\ A_\ell &\to A'_{\ell-1} = A_{\ell-1} - b_\ell d\psi_{\ell-1}, \end{aligned} \tag{A12}$$

and then the gauge field is smooth at the center. For the gauge potential after the gauge transformation (A12), the jump of the gauge field on the boundary is

$$A'_\ell - A'_{\ell-1}|_{z=\ell} = (c_\ell - c_{\ell-1} - B(\ell)) f(r) d\phi, \tag{A13}$$

and A'_ℓ and $A'_{\ell-1}$ are smoothly connected if

$$c_\ell - c_{\ell-1} = B(\ell) \quad (1 \leq \ell \leq n). \tag{A14}$$

Note that the condition (A11) follows from this and (A7).

By using relations we obtained above, we can easily compute the integral of the instanton density over X_ℓ.

$$\begin{aligned} \int_{X_\ell} F_\ell \wedge F_\ell &= 8\pi^2 c_\ell [B(z)]_\ell^{\ell+1} \left[f(r) - \frac{1}{2} f(r)^2\right]_0^1 \\ &= 4\pi^2 c_\ell (c_{\ell+1} - 2c_\ell - c_{\ell-1}). \end{aligned} \tag{A15}$$

This depends on the constants c_ℓ that satisfy the conditions (A5) and (A7). Although there are infinitely many solutions to these conditions and the integral (A15) depends on the choice of a solution, the classical action is uniquely determined up to the unphysical shift by $2\pi i\mathbb{Z}$.

$$\begin{aligned} S &= \frac{ik}{4\pi} \int_X F \wedge F = \pi i k \sum_{\ell=1}^{n-1} c_\ell (c_{\ell+1} - 2c_\ell + c_{\ell-1}) \\ &= \pi i k \frac{1-n}{n} h^2 \mod 2\pi i. \end{aligned} \tag{A16}$$

We can easily extend this result to a general Chern-Simons term for multiple Abelian gauge fields,

$$S = \frac{ik_{ab}}{4\pi} \int_{S^3/\mathbb{Z}_n} A_a dA_b = \pi i \frac{1-n}{n} k_{ab} h_a h_b \mod 2\pi i. \tag{A17}$$

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