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Vacuum polarization of the quantized massive fields in Friedman-Robertson-Walker spacetime

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The stress-energy tensor of the quantized massive fields in a spatially open, flat, and closed Friedman-Robertson-Walker universe is constructed using the adiabatic regularization (for the scalar field) and the Schwinger-DeWitt approach (for the scalar, spinor, and vector fields). It is shown that the stress-energy tensor calculated in the sixth adiabatic order coincides with the result obtained from the regularized effective action, constructed from the heat kernel coefficient a_3. The behavior of the tensor is examined in the power-law cosmological models, and the semiclassical Einstein field equations are solved exactly in a few physically interesting cases, such as the generalized Starobinsky models.

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I. INTRODUCTION

As is well known the physical content of the quantum field theory in curved background is encoded in its most important observable—the renormalized stress-energy tensor, T_{ab}, evaluated in a suitably chosen state. Such a tensor serves as the right-hand side of the semiclassical Einstein field equations, "the source term," allowing in principle to determine the evolution of the system unless the (expected) quantum gravity effect become dominant. This is its principal role. On the other hand, the renormalized stress-energy tensor of the quantized field(s) is interesting in its own right, as it is crucial in analysis of the particle production, vacuum polarization, energy conditions and quantum inequalities. Unfortunately, because of the intrinsic difficulties of the problem, the exact stress-energy tensor of the standard test fields is, despite the enormous literature on the subject, known only for simple geometries of high symmetry. It is therefore unavoidable that in order to construct a more realistic T_{ab} that depends functionally on the wide class of metrics one has to invent some approximations. Having in mind further applications it seems that the most fruitful approach consists in constructing the (regularized) effective action, W_R, of the quantized fields. By construction it depends functionally on the metric and the stress-energy tensor can be calculated in a standard way.

In its most general form, the effective action is expected to be a nonlocal functional describing both particle creation and the vacuum polarization effects. Such a general form of the effective action of the quantized fields in curved background is unknown. However, when the mass of the field is sufficiently large, the creation of the real particles is negligible and the effective action can be approximated by its local, purely geometric part. This condition can be made more precise by studying the ratio of the Compton length associated with the field, λ_C, and the characteristic radius of curvature, L. It is expected that when $\lambda_C/L \ll 1$, the vacuum polarization effects dominate and W_R is given by the Schwinger-DeWitt expansion [1–3]

$$W_R = \frac{1}{32\pi^2}\sum_{n=3}^{\infty}\frac{(n-3)!}{(m^2)^{n-2}}\int d^4x g^{1/2}\mathrm{Tr}a_n, \tag{1}$$

where m is the mass of the field, a_n are the Hadamard-DeWitt coefficients and Tr is a supertrace. The Schwinger-DeWitt expansion has been used in a number of cases ranging from the black hole physics to cosmology. (See e.g., Refs. [4–11] and the references cited therein).

The Schwinger-DeWitt approach certainly does not exhaust the list of useful approximations. Indeed, there is another very powerful method, namely the adiabatic regularization, which is especially well suited for the study of the quantized fields in cosmology and what is of great importance to us here, the stress-energy tensor in the Friedman-Robertson-Walker (FRW) universe with a scale factor $a(t)$ can be calculated within the framework of this approximation. This (slightly more restrictive) approach to the construction of the renormalized stress-energy tensor consists in summing (integrating) the adiabatic approximations of the mode functions and their derivatives and regularizing the thus obtained divergent expressions [12–22]. This approach can be made more precise by introducing the so-called slowness parameter and expanding the mode functions and the stress-energy tensor in its inverse powers [23]. In four dimensions the first three terms of the expansion of the stress-energy tensor are divergent and the regularization consists in subtracting of the infinite terms up to fourth adiabatic order. It is expected that when $\dot{a}/a$, $\ddot{a}/a\ldots \ll (m^2a^2+k^2)^{1/2}$, where a dot denotes differentiation with respect to the conformal time and $0\leq k<\infty$, this procedure yields reasonable results [24].

In this paper, which extends the results of Refs. [18,25] to the spatially curved FRW spacetime, we shall employ both approaches. We construct the first order approximation to the regularized stress-energy tensor of the massive scalar scalar field in a large mass limit in the FRW spacetime within the framework of the adiabatic method. On the other hand, we shall show that identical results can be obtained using the Schwinger-DeWitt approach. Moreover, using the latter approach we shall calculate the stress-energy tensor of the massive spinor and vector fields. Since the main emphasis is put on the derivation of the regularized stress-energy tensor itself, we shall restrict ourselves to the semiclassical Einstein field equations which can be treated analytically and construct a family of the self-consistent solutions that additionally satisfy some simplifying assumptions.

The paper is organized as follows. The detailed calculations of the stress-energy tensor of the quantized massive scalar field in the FRW cosmology within the framework of the adiabatic approximation are presented in Sec. II. Some of the intermediate (but important) calculations are relegated to the Appendix. In Sec. III we calculate the stress-energy tensor of the massive spinor, scalar and vector field with the aid of the Schwinger-DeWitt method and demonstrate the equality of the tensors obtained using both methods. The results presented in Secs. II and III comprise the core results of this paper. In Sec. IV we solve the semiclassical Einstein field equations in a few exemplary, physically motivated cases. The last section contains the discussion and the final remarks. Throughout the paper the natural units are chosen, and we follow the Misner, Thorne and Wheeler conventions [26].

II. ADIABATIC APPROXIMATION OF THE STRESS-ENERGY TENSOR

In this section we will be concerned with the neutral scalar massive field satisfying the covariant Klein-Gordon equation

$$-\Box\phi+(m^2+\xi R)\phi=0, \tag{2}$$

in the Friedman-Robertson-Walker (FRW) spacetime, where R is the curvature scalar, ξ is the coupling parameter and m is the mass of the field. The line element describing FRW geometry can be written as

$$ds^2=-dt^2+a^2(t)d\sigma^2 \tag{3}$$

with $d\sigma^2=h_{ij}dx^idx^j$, where h_{ij} is the metric tensor on the maximally symmetric three-dimensional spaces $\mathcal{S}^3$, $\mathcal{E}^3$ or $\mathcal{H}^3$, and the two particularly useful representations of $d\sigma^2$ can be written in the form

$$d\sigma^2=\frac{dr^2}{1-Kr^2}+r^2d\Omega^2 \tag{4}$$

and

$$d\sigma^2=d\chi^2+f^2(\chi)d\Omega^2, \tag{5}$$

where $d\Omega^2$ is the metric on a unit sphere. Possible forms of the function $f(\chi)$ as well as the admissible values of K are given respectively by

$$f(\chi)=\begin{cases}\sin\chi & \text{for } \mathcal{S}^3\\ \chi & \text{for } \mathcal{E}^3\\ \sinh\chi & \text{for } \mathcal{H}^3\end{cases} \tag{6}$$

and

$$K=\begin{cases}1 & \text{for } \mathcal{S}^3\\ 0 & \text{for } \mathcal{E}^3\\ -1 & \text{for } \mathcal{H}^3.\end{cases} \tag{7}$$

It is advantageous to introduce the new time coordinate, η, defined as

$$\frac{d\eta}{dt}=a^{-1}(t) \tag{8}$$

and to consider the transformed line element

$$ds^2=a^2(\eta)(-d\eta^2+d\sigma^2). \tag{9}$$

Now, making use of the (formal) transformation,

$$r\to ir \qquad a(\eta)\to ia(\eta), \tag{10}$$

or equivalently,

$$\chi\to i\chi \qquad a(\eta)\to ia(\eta), \tag{11}$$

one obtains the metric with the signature $(+,+,+,+)$, i.e., the Euclidean version of the line element. It should be noted that for $\mathcal{S}^3$ and $\mathcal{H}^3$, the transformations (10) and (11) effectively change the type of geometry, what is seen from the Table I, where the left column describes the Lorentzian geometries, whereas in the right column we have listed their Euclidean counterparts after the transformation.

The stress-energy tensor of the scalar field satisfying (2) can be written as

$$T_{ab}=\nabla_a\phi\nabla_b\phi-\frac{1}{2}g_{ab}(\nabla_c\phi\nabla^c\phi+m^2\phi^2-2\xi\nabla_c\nabla^c\phi^2)+\xi G_{ab}\phi^2-\xi\nabla_a\nabla_b\phi^2, \tag{12}$$

TABLE I. The Lorentzian and Euclidean version of the FRW metric. The right column is obtained from the left by means of transformation (10) and (11).

σ	Lorentzian	Euclidean
K	1	-1
	0	0
	-1	1
$f(\chi)$	$\sin\chi$	$i\sinh\chi$
	χ	$i\chi$
	$\sinh\chi$	$i\sin\chi$

where G_{ab} is the Einstein tensor. In the FRW spacetime described by the line element (9) the (η,η) component of the tensor T_{ab} assumes the form

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + 6\xi H\phi\dot{\phi} - \left(2\xi - \frac{1}{2}\right)h^{ij}\partial_i\phi\partial_j\phi - 2\xi\phi\Delta^{(3)}\phi + 3\xi H^2\phi^2 + 3\xi K\phi^2 + \frac{1}{2}m^2a^2\phi^2, \tag{13}$$

where $H = \dot{a}/a$ and $\Delta^{(3)}$ is the Laplacian associated with the tensor h_{ij}. Similarly, the trace of the tensor may be written in the form

$$T_a^a = \frac{1}{a^2}(1-6\xi)(\dot{\phi}^2 - h^{ij}\partial_i\phi\partial_j\phi) + 2(3\xi-1)m^2\phi^2 + \frac{1}{a^2}6\xi(6\xi-1)\left(K + \frac{\ddot{a}}{a}\right)\phi^2. \tag{14}$$

It is expected that in a state that reflects the symmetries of the maximally symmetric three-dimensional space, the component T_{00} should be independent of spatial coordinates, what in practice means that one can perform suitable averaging [13]. Making use of the relation

$$\left(\int\sqrt{h}d\sigma\right)^{-1}\int h^{ij}\partial_i\phi\partial_j\phi\sqrt{h}d\sigma = -\phi\Delta^{(3)}\phi, \tag{15}$$

one gets

$$T_{00} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi\Delta^{(3)}\phi + 6\xi H\phi\dot{\phi} + 3\xi H^2\phi^2 + 3\xi K\phi^2 + \frac{1}{2}m^2a^2\phi^2, \tag{16}$$

where, for notational simplicity, the braces denoting averaging have been omitted. Similar calculations carried out for the trace of the tensor T_{ab} yield

$$T_a^a = \frac{1}{a^2}\left\{(1-6\xi)[\dot{\phi}^2 + \phi\Delta^{(3)}\phi] + 2(3\xi-1)m^2a^2\phi^2 + 6\xi(6\xi-1)\left(K + \frac{\ddot{a}}{a}\right)\phi^2\right\}. \tag{17}$$

Numerous analyses (for a comprehensive list relevant references see Ref. [24]) show that for the Laplace-Beltrami operator $\Delta^{(3)}$ on a three-dimensional homogeneous space, one has

$$\Delta^{(3)}\phi = -(k^2 - K)\phi, \tag{18}$$

where $k = 1, 2, 3, \ldots$ for $K = 1$ and $0 \le k < \infty$ for $K = -1$ or 0.

One expects that T_{ab} should satisfy the covariant conservation equation, $\nabla_a T^{ab} = 0$, that for the problem on hand reduces to

$$T_x^x = T_0^0 + \frac{a}{3\dot{a}}\dot{T}_0^0, \tag{19}$$

where

$$T_x^x \equiv T_1^1 = T_2^2 = T_3^3. \tag{20}$$

It follows then that, because of the spatial symmetries, it suffices to calculate the energy density of the field, $\rho = -T_0^0$, as the remaining components can be obtained with practically no effort from (19) and (20). Other methods may, therefore, serve as a useful check of the calculations.

Now, let us introduce the function $\mu(x)$ defined as

$$\phi(x) = \frac{\mu(x)}{a(\eta)}. \tag{21}$$

Equations (2), (16) and (17) have, respectively, the form

$$\ddot{\mu} + [k^2 + m^2a^2]\mu + \left[\frac{\ddot{a}}{a}(6\xi-1) + (6\xi-1)K\right]\mu = 0, \tag{22}$$

$$T_{00} = \frac{1}{2a^2}[(1-6\xi)H^2\mu^2 + (6\xi-1)K\mu^2 + k^2\mu^2 + m^2a^2\mu^2 + 2H(6\xi-1)\mu\dot{\mu} + \dot{\mu}^2] \tag{23}$$

and

$$T_a^a = \frac{1}{a^4}\left[(6\xi-1)(2H\mu\dot{\mu} - \dot{\mu}^2 - H^2\mu^2) + (k^2-K)\mu^2(6\xi-1) + 2(3\xi-1)m^2a^2\mu^2 + 6\xi(6\xi-1)\left(\frac{\ddot{a}}{a} + K\right)\mu^2\right]. \tag{24}$$

One can decompose the field ϕ into modes

$$\mu(x) = \int d\Omega(\mathbf{k})[\mu_k(\eta)\mathfrak{Y}_\mathbf{k}(\mathbf{x})a_\mathbf{k} + \mu_k^*(\eta)\mathfrak{Y}_\mathbf{k}^*(\mathbf{x})a_\mathbf{k}^\dagger], \tag{25}$$

where the functions $\mathfrak{Y}_\mathbf{k}(\mathbf{x})$ satisfy

$$\Delta^{(3)}\mathfrak{Y}_\mathbf{k}(\mathbf{x}) = -(k^2 - K)\mathfrak{Y}_\mathbf{k}(\mathbf{x}), \tag{26}$$

with k defined as in (18). The measure (depending on spatial curvature) is defined as in Appendix A of Ref. [12] (see also Ref. [24] for a comprehensive list of references).

Making use of the standard commutation relations of the field operator and the conjugate momentum one obtains the relations for the operators $a_{\mathbf{k}}$ and $a^{\dagger}_{\mathbf{k}}$

$$[a_{\mathbf{k}}, a_{k'}] = [a^{\dagger}_{\mathbf{k}}, a^{\dagger}_{k'}] = 0,$$
$$[a_{\mathbf{k}}, a^{\dagger}_{k'}] = \delta(\mathbf{k}, k'), \tag{27}$$

where $\delta(\mathbf{k}, k')$ satisfies

$$\int d\Omega(\mathbf{k})\delta(\mathbf{k}, k')f(\mathbf{k}) = f(k'). \tag{28}$$

The ground state of the field is defined as

$$a_{\mathbf{k}}|0>= 0 \tag{29}$$

and the canonical commutators of ϕ and π lead to commutation relations of $a_{\mathbf{k}}$ and $a^{\dagger}_{\mathbf{k}}$ provided the Wronskian condition

$$\mu_k\dot{\mu}^*_k - \dot{\mu}_k\mu^*_k = i \tag{30}$$

is satisfied. Finally, with the aid of the formula

$$\int d\Omega(\mathbf{k})|\mathfrak{Y}_{\mathbf{k}}(\mathbf{x})|^2 f(\mathbf{k}) = \frac{1}{2\pi^2}\int d\omega(k)f(k), \tag{31}$$

where

$$\int d\omega(k) = \begin{cases} \int_0^\infty dk k^2 & \text{if } K = 0, -1, \\ \sum_{k=1}^\infty k^2 & \text{if } K = 1, \end{cases} \tag{32}$$

one can calculate the mean values of the components of the stress-energy tensor. For the (η, η) component one has

$$T_{00} = \frac{1}{4\pi a^2}\int d\omega(k)(\dot{\mu}_k\dot{\mu}^*_k + H^2\mu_k\mu^*_k - H\mu_k\dot{\mu}^*_k - H\dot{\mu}_k\mu^*_k$$
$$+ k^2\mu_k\mu^*_k - 6H^2\xi\mu_k\mu^*_k + 6K\xi\mu_k\mu^*_k - K\mu_k\mu^*_k$$
$$+ m^2a^2\mu_k\mu^*_k + 6H\xi\dot{\mu}_k\mu^*_k + 6H\xi\mu_k\dot{\mu}^*_k), \tag{33}$$

and the whole procedure reduces to construction of the mode functions μ_k and performing appropriate integrations (summations).

In the adiabatic method one postulates that the solutions of Eq. (22) can be written as

$$\mu_k = \frac{1}{(2W_k)^{1/2}}e^{-i\int W_k d\eta}. \tag{34}$$

Such a form automatically guarantees that the Wronskian condition (30) is satisfied. Now, substituting (34) into (22), one obtains

$$W^2_k = k^2 + m^2a^2 + (6\xi - 1)\frac{\ddot{a}}{a} + (6\xi - 1)K + \frac{3}{4}\frac{\dot{W}^2_k}{W^2_k} - \frac{1}{2}\frac{\ddot{W}_k}{W_k}, \tag{35}$$

and the problem amounts to finding the solutions of the above equation. In the adiabatic calculations one solves Eq. (35) iteratively, taking the first two terms in its right-hand side as the zeroth-order solution

$$W^{(0)}_k = (k^2 + m^2a^2)^{1/2} \equiv \beta^{1/2}. \tag{36}$$

It should be noted that the parameter K which characterizes the curvature of the spatial geometry is of the second adiabatic order. The zeroth-order solution is then substituted into the right-hand side of Eq. (35), giving the second-order approximation

$$W^{(2)}_k = -\frac{m^2a\ddot{a}}{4\beta^{3/2}} - \frac{\ddot{a}}{2\sqrt{\beta}a} + \frac{3\xi\ddot{a}}{\sqrt{\beta}a} + \frac{5m^4a^2\dot{a}^2}{8\beta^{5/2}} - \frac{m^2\dot{a}^2}{4\beta^{3/2}}$$
$$- \frac{K}{2\sqrt{\beta}} + \frac{3K\xi}{\sqrt{\beta}}, \tag{37}$$

and this procedure may be repeated as many times as needed. As the result one obtains the chain of approximate solutions of Eq. (35) of ascending complexity. To simplify calculations and to determine the actual order of the constructed terms in complicated series expansions it is useful to introduce the parameter ϵ (which will be set to 1 at final stage of the calculation):

$$\frac{d}{d\eta} \to \varepsilon\frac{d}{d\eta} \quad \text{and} \quad K \to \varepsilon^2 K. \tag{38}$$

To put it in other words, the number of "time" derivatives as well as the power of K serve as the expansion parameter. In the problem at hand, we need the sixth-order approximation, and since the resulting formulas are rather lengthy we have relegated them to the Appendix. It should be noted that although the form of $W^{(i)}_k$ functions is not unique and depends on the simplification strategy, the resulting regularized tensor is unique.

Having computed the mode functions we can go a step further and construct the stress-energy tensor of the quantized massive field. This, however, is divergent and the standard procedure is to subtract the first three adiabatic terms, i.e., the zeroth, second and fourth, from the full sixth-order expression. It is important to subtract all the terms of the given order; otherwise, there would be a problem with the unique aspects of the final solution [23]. Now, inserting the thus obtained approximate mode functions $\mu_k(\eta)$ into the general formula (33), expanding the resulting expression in powers of ε and subsequently performing the regularization one obtains

$$T_{00}^{(R)} = \frac{1}{4\pi a^2}\int d\omega(k) t_{00}^R, \tag{39}$$

where $T_{00}^{(R)}$ is the regularized component of the stress-energy tensor and t_{00}^R is the regularized integrand of (33), which (in our representation) consists of 518 terms. In what follows we shall omit the superscript R as all the components of the stress-energy tensor that we encounter are already regularized.

The calculation of T_{00} is rather involved and for obvious reasons it will not be presented here, fortunately the final result is surprisingly simple. The integrations over k for $K = 0$ and $K = -1$ present no problem. On the other hand the sums over k for $K = 1$ cannot be evaluated analytically in a simple way. The usual approach is to use the Abel-Plana formula, i.e., to change summations to integrations. (See Refs. [27,28]). Unfortunately, one of the integrals cannot be computed analytically and we are unable to obtain a compact final result. In what follows we shall concentrate on $K = -1$ and $K = 0$ geometries and postpone the $K = 1$ case until the next section, which is devoted to the Schwinger-DeWitt approach.

Let us return to the stress-energy tensor for $K \leq 0$. Simple integrations term after term give

$$T_0^0 = \frac{1}{4\pi^2 a^4}(\tau_0 + K\tau_1 + K^2\tau_2 + K^3\tau_3), \tag{40}$$

where

$$\begin{aligned}\tau_0 = {}& -\frac{3\xi^2 a^{(3)2}}{8m^2a^4} + \frac{3\xi a^{(3)2}}{20m^2a^4} - \frac{17a^{(3)2}}{1120m^2a^4} + \frac{9\xi^3\ddot{a}^3}{m^2a^5} - \frac{27\xi^2\ddot{a}^3}{4m^2a^5} + \frac{33\xi\ddot{a}^3}{20m^2a^5} - \frac{673\ddot{a}^3}{5040m^2a^5} + \frac{7\xi\dot{a}^6}{20m^2a^8} - \frac{3\xi^2 a^{(5)}\dot{a}}{4m^2a^4} + \frac{3\xi a^{(5)}\dot{a}}{10m^2a^4}\\ & - \frac{17a^{(5)}\dot{a}}{560m^2a^4} + \frac{3\xi^2a^{(4)}\ddot{a}}{4m^2a^4} - \frac{3\xi a^{(4)}\ddot{a}}{10m^2a^4} + \frac{17a^{(4)}\ddot{a}}{560m^2a^4} + \frac{6\xi^2a^{(4)}\dot{a}^2}{m^2a^5} - \frac{39\xi^2a^{(3)}\dot{a}^3}{2m^2a^6} + \frac{79\xi a^{(3)}\dot{a}^3}{10m^2a^6} - \frac{683a^{(3)}\dot{a}^3}{840m^2a^6} + \frac{135\xi^3\dot{a}^2\ddot{a}^2}{2m^2a^6}\\ & + \frac{27\xi^2\dot{a}^4\ddot{a}}{m^2a^7} - \frac{423\xi^2\dot{a}^2\ddot{a}^2}{8m^2a^6} - \frac{57\xi\dot{a}^4\ddot{a}}{5m^2a^7}, + \frac{173\dot{a}^4\ddot{a}}{140m^2a^7} - \frac{1237\dot{a}^2\ddot{a}^2}{1120m^2a^6} - \frac{27\xi^3a^{(3)}\dot{a}\,\ddot{a}}{m^2a^5} + \frac{81\xi^2a^{(3)}\dot{a}\,\ddot{a}}{4m^2a^5} - \frac{99\xi a^{(3)}\dot{a}\,\ddot{a}}{20m^2a^5}\\ & + \frac{673a^{(3)}\dot{a}\,\ddot{a}}{1680m^2a^5} - \frac{12\xi a^{(4)}\dot{a}^2}{5m^2a^5} + \frac{17a^{(4)}\dot{a}^2}{70m^2a^5} + \frac{533\xi\dot{a}^2\ddot{a}^2}{40m^2a^6} - \frac{211\dot{a}^6}{2520m^2a^8},\end{aligned} \tag{41}$$

$$\begin{aligned}\tau_1 = {}& \frac{27\xi^3\ddot{a}^2}{2m^2a^4} - \frac{33\xi^2\ddot{a}^2}{4m^2a^4} + \frac{67\xi\ddot{a}^2}{40m^2a^4} - \frac{9\ddot{a}^2}{80m^2a^4} + \frac{15\xi^2\dot{a}^4}{2m^2a^6} - \frac{57\xi\dot{a}^4}{20m^2a^6} + \frac{4\dot{a}^4}{15m^2a^6} - \frac{27\xi^3a^{(3)}\dot{a}}{m^2a^4} - \frac{67\xi a^{(3)}\dot{a}}{20m^2a^4} + \frac{9a^{(3)}\dot{a}}{40m^2a^4} + \frac{108\xi^3\dot{a}^2\ddot{a}}{m^2a^5}\\ & - \frac{66\xi^2\dot{a}^2\ddot{a}}{m^2a^5} + \frac{67\xi\dot{a}^2\ddot{a}}{5m^2a^5} - \frac{9\dot{a}^2\ddot{a}}{10m^2a^5} + \frac{33\xi^2a^{(3)}\dot{a}}{2m^2a^4},\end{aligned} \tag{42}$$

$$\tau_2 = \frac{81\xi^3\dot{a}^2}{2m^2a^4} - \frac{87\xi^2\dot{a}^2}{4m^2a^4} + \frac{31\xi\dot{a}^2}{8m^2a^4} - \frac{11\dot{a}^2}{48m^2a^4} \tag{43}$$

and

$$\tau_3 = -\frac{9\xi^3}{2m^2a^2} + \frac{9\xi^2}{4m^2a^2} - \frac{3\xi}{8m^2a^2} + \frac{1}{48m^2a^2}. \tag{44}$$

It can be easily shown that for the spatially flat FRW spacetime the resulting tensor reduces to the tensor calculated in Refs. [18,25]. To the best of our knowledge, the results for $K = -1$ are new. On the other, hand the case $K = 1$ can be treated numerically. Finally, making use of Eq. (19) one easily gets the spatial component of the stress-energy tensor. Equivalently, one can start with the trace equation and repeat the steps that lead to Eq. (40). Both approaches give the same result and to avoid proliferation of lengthy equations, the spatial components of the tensor (for scalar, spinor and vector field) will not be presented here. We postpone further analysis of the stress-energy tensor (40)–(44) until Sec. IV and concentrate on the Schwinger-DeWitt approach.

III. THE STRESS-ENERGY TENSOR IN SCHWINGER-DEWITT APPROACH

In the four dimensions, the first-order approximation to the one-loop effective action of the quantized massive field in a large mass limit, W_R, is the (integrated) linear combination of eight curvature invariants of background dimensionality six, constructed form the Riemann tensor and its covariant derivatives. However, in the Weyl-flat spacetimes, where the Weyl tensor C_{abcd} identically vanishes, the action can be further simplified. Since our main task in this section is to construct the stress-energy tensor in a FRW universe, we shall present the effective action in this simplified form. For general results the reader is referred to Refs. [8,9].

Now, besides the scalar field satisfying Eq. (2) let us consider the spinor field, $\phi^{(1/2)}$, and the vector field , $\phi_a^{(1)}$, described, respectively, by

$$(\gamma^a \nabla_a + m)\phi^{(1/2)} = 0 \tag{45}$$

and

$$(\delta^a_b \Box - \nabla_b \nabla^a - R^a_b - \delta^a_b m^2)\phi^{(1)}_a = 0, \tag{46}$$

where γ^a are Dirac matrices and all symbols have their usual meaning. The simplified Schwinger-DeWitt approximation of the one-loop W_R for the massive scalar, spinor and vector fields can be written as [25]

$$W_R \overset{C=0}{=} \frac{1}{192\pi^2 m^2} \int d^4x g^{1/2} (a_1^{(s)} R \Box R + a_2^{(s)} R_{ab} \Box R^{ab} + a_3^{(s)} R^3 + a_4^{(s)} R R_{ab} R^{ab} + a_5^{(s)} R_a^b R_b^c R_c^a), \tag{47}$$

where the spin-dependent coefficients are tabulated in Table II. We have eliminated the terms with the Riemann tensor from the action making use of the equations constructed from $C_{abcd} C^{cd}_{ij} C^{ijab}$, $R C_{abcd} C^{abcd}$ and $C_{abcd} \Box C^{abcd}$. For the chosen class of geometries the functional derivatives of such (integrated) invariants vanish.

It should be noted that the general action can be used in any geometry, provided the Compton length associated with the mass of the field is much smaller than the characteristic radius of the curvature. Consequently, using the Schwinger-DeWitt method one can construct the stress-energy tensor in FRW spacetime not only for the quantized fields of various spins, but also for a nonvanishing value of the parameter K. There is another subtlety that should be discussed in more detail: the effective action has been derived with the assumption that the metric is positively defined. It means that one should perform suitable transformation, calculate the stress-energy tensor, and, finally, transform it back to the physical space. From Table I, one concludes that the Euclidean metric with $K = -1$ is related by the transformations (10) and (11) to the Lorentzian $K = 1$ metric and, similarly, the Euclidean metric with $K = 1$ is related by the same transformation with the Lorentzian $K = -1$ case. Fortunately, the common wisdom, which can be verified in the present problem, is that computing the stress-energy tensor for the positively defined metric and transforming it back to the Lorentzian case, one obtains precisely the same results as if the calculations had been performed in the physical metric from the very beginning.

The stress-energy tensor can be calculated from the standard relation

$$T^{ab} = \frac{2}{g^{1/2}} \frac{\delta}{\delta g_{ab}} W_R. \tag{48}$$

Making use of the results of Refs. [8,9] one obtains the desired tensor. The thus constructed stress-energy tensor of the quantized massive scalar field is identical with the tensor obtained in the previous section [see Eqs. (40)–(44)] and will not be repeated here. It should be noted, however, that now the tensor (40)–(44) can be calculated also for $K = 1$. The results for the spinor and vector fields are listed below.

The stress energy tensor of the spinor field has a simple form

$$T_0^{R0} = \frac{1}{4\pi^2 a^4} (\tau_0^{(1/2)} + K \tau_1^{(1/2)}), \tag{49}$$

where

$$\tau_0^{(1/2)} = -\frac{137\dot{a}^6}{2016 a^8 m^2} + \frac{5\dot{a}^4 \ddot{a}}{28 a^7 m^2} - \frac{11\dot{a}^3 a^{(3)}}{168 a^6 m^2} - \frac{25\dot{a}^2 \ddot{a}^2}{336 a^6 m^2} + \frac{\dot{a}^2 a^{(4)}}{70 a^5 m^2} + \frac{\dot{a}\, \ddot{a}\, a^{(3)}}{42 a^5 m^2} - \frac{\ddot{a}^3}{126 a^5 m^2} - \frac{\dot{a} a^{(5)}}{560 a^4 m^2} + \frac{\ddot{a} a^{(4)}}{560 a^4 m^2} - \frac{a^{(3)2}}{1120 a^4 m^2} \tag{50}$$

and

$$\tau_1^{(1/2)} = -\frac{7\dot{a}^4}{480 a^6 m^2} + \frac{\dot{a}^2 \ddot{a}}{60 a^5 m^2} - \frac{\dot{a} a^{(3)}}{240 a^4 m^2} + \frac{\ddot{a}^2}{480 a^4 m^2}. \tag{51}$$

For the vector field one has

$$T_0^{R0} = \frac{1}{4\pi^2 a^4} (\tau_0^{(1)} + K \tau_1^{(1)} + K^2 \tau_2^{(1)} + K^3 \tau_3^{(1)}), \tag{52}$$

where

TABLE II. The spin-dependent coefficients of the simplified effective action W_R.

	$\alpha_1^{(s)}$	$\alpha_2^{(s)}$	$\alpha_3^{(s)}$	$\alpha_4^{(s)}$	$\alpha_5^{(s)}$
$s = 0$	$\frac{1}{40}[1 + 4\xi(5\xi - 2)]$	$-\frac{1}{70}$	$\frac{23}{4536} - \frac{\xi}{180}[13 + 90\xi(2\xi - 1)]$	$-\frac{1}{84} - \frac{\xi}{30}$	$\frac{2}{63}$
$s = \frac{1}{2}$	$\frac{7}{360}$	$-\frac{23}{420}$	$\frac{211}{22680}$	$-\frac{19}{280}$	$\frac{151}{1260}$
$s = 1$	$\frac{29}{360}$	$-\frac{22}{105}$	$\frac{73}{7560}$	$-\frac{19}{140}$	$\frac{38}{105}$

$$\tau_0^{(1)} = -\frac{449\dot{a}^6}{840a^8m^2} + \frac{211\dot{a}^4\ddot{a}}{140a^7m^2} - \frac{481\dot{a}^3a^{(3)}}{840a^6m^2} - \frac{2327\dot{a}^2\ddot{a}^2}{3360a^6m^2} + \frac{9\dot{a}^2a^{(4)}}{70a^5m^2} + \frac{127\dot{a}\,\ddot{a}\,a^{(3)}}{560a^5m^2} - \frac{127\ddot{a}^3}{1680a^5m^2} - \frac{9\dot{a}a^{(5)}}{560a^4m^2} + \frac{9\ddot{a}a^{(4)}}{560a^4m^2} - \frac{9a^{(3)2}}{1120a^4m^2}, \quad (53)$$

$$\tau_1^{(1)} = -\frac{\dot{a}^4}{20a^6m^2} + \frac{\dot{a}^2\ddot{a}}{30a^5m^2} - \frac{\dot{a}a^{(3)}}{120a^4m^2} + \frac{\ddot{a}^2}{240a^4m^2}, \quad (54)$$

$$\tau_2^{(1)} = \frac{\dot{a}^2}{48a^4m^2} \quad (55)$$

and

$$\tau_3^{(1)} = \frac{1}{48a^2m^2}. \quad (56)$$

Once again we stress that the results obtained within the framework of the Schwinger-DeWitt approximation should be valid in any spacetime provided the Compton length associated with a field is smaller than the characteristic curvature of the spacetime geometry. Having established that for the scalar field propagating in the FRW ($K \leq 0$) spacetimes the Schwinger-DeWitt approach gives the same result as the adiabatic method, the natural question is whether it is also true for higher spins. In our earlier work [25] we reported on calculations of the regularized stress-energy tensor of the massive spinor and vector fields in spatially flat ($K = 0$) cosmologies. We extended the results presented in Ref. [29] and calculated the renormalized tensor T_{ab} of the scalar, spinor and vector field. It can be shown that for such geometries the tensors obtained in any of the methods discussed above are identical. We hypothesize that the same is true for $K = -1$ case.

IV. SEMICLASSICAL EINSTEIN FIELD EQUATIONS

Although quite simple in its basic concepts, the stress-energy tensor is not very easy to comprehend. At this point we can ask an important question: Having the stress-energy tensor of the quantized massive fields at our disposal, how do we use it? This question is even more important as the semiclassical equations involve higher-order derivatives. Of course, the detailed analysis of the equations requires numerical analysis that is beyond the scope of this paper. Here we shall concentrate on a few problems that can be treated analytically.

We shall start with the stress-energy tensor itself and to gain some insight into its nature let us analyze the power-law cosmological models with a scale factor given by

$$a(t) = \left(\frac{t}{t_0}\right)^q \quad (57)$$

and with $K = 0$. Simple manipulations give

$$T_x^x(t) = \frac{q-2}{q} T_t^t(t), \quad (58)$$

or with a natural interpretation of the components of the stress-energy tensor

$$p(t) = \frac{2-q}{q}\rho(t), \quad (59)$$

where

$$\rho_{\xi=0}^{(0)} = \frac{H^6}{20160q^4\pi^2m^2} \times (3060 - 7498q + 105q^2 + 5328q^3 + 370q^4), \quad (60)$$

$$\rho_{\xi=1/6}^{(0)} = -\frac{H^6}{5040q^4\pi^2m^2} \times (-30 + 79q - 42q^2 - 9q^3 + 2q^4), \quad (61)$$

$$\rho^{(1/2)} = -\frac{H^6}{40320q^4\pi^2m^2} \times (-360 + 878q - 441q^2 - 108q^3 + 31q^4) \quad (62)$$

and

$$\rho^{(1)} = -\frac{H^6}{6720q^4\pi^2m^2} \times (-540 + 1282q - 455q^2 - 372q^3 + 50q^4) \quad (63)$$

for the scalar, spinor and vector fields, respectively. Here $H(t)$ is the Hubble parameter. First, observe that for $q > 2$ the energy density and the pressure have opposite signs and the coefficient of proportionality, say α, in the equation of state satisfies $-1 < \alpha$. Further, depending on spin (and the parameter ξ) there are the regions where energy density is negative. Since all roots of Eqs (60)–(63) are different and real, for each spin one has five regions with the property that any two adjacent regions have the opposite sign of the energy density.

Having constructed the stress-energy tensor of the quantized massive fields one can attempt to solve the semiclassical Einstein field equations with the right-hand side being a sum of the quantum and the classical part. Usually, the semiclassical equations are studied in their simplest form, i.e., with the assumption that the renormalized ("observed") coupling parameters κ_1, κ_2 in the quadratic part of the total gravitational action,

$$S_q = \int d^4x\sqrt{-g}(\kappa_1 R_{ab}R^{ab} + \kappa_2 R^2), \quad (64)$$

identically vanish. However, it is a well-known fact that the quadratic terms (or more generally four-derivative theory) may lead to many interesting solutions and phenomena (see e.g., [30–36]) and in this section we will show that this is indeed the case. It should be noted, however, that for the Weyl-flat ($C_{abcd} = 0$) line element, the functional derivatives of the quadratic curvature invariants are not independent. In this case (and only in this case) one can add the term [34]

$$\kappa_3 H^{(3)}_{ab} = \kappa_3 \left(-\frac{1}{12} R^2 g_{ab} + R^{cd} R_{cadb} \right) \tag{65}$$

to the left-hand side of the semiclassical equations without spoiling the logic of the quadratic theory. It should be emphasized that the tensor (65) is conserved only in the conformally flat spacetimes. For the general parameters κ_1, κ_2 and κ_3 the semiclassical Einstein field equations assume the standard form [34]

$$G_{ab}[g] + \kappa_1 H^{(1)}_{ab} + \kappa_2 H^{(2)}_{ab} + \kappa_3 H^{(3)}_{ab} + \Lambda g_{ab} = 8\pi T^{(\mathrm{total})}_{ab}[g], \tag{66}$$

where $T^{(\mathrm{total})}_{ab} = T^{(\mathrm{class})}_{ab} + T_{ab}$; i.e., the right-hand side of (66) is a sum of the classical and quantum stress-energy tensor and

$$H^{(2)}_{ab} = -\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int d^4x \sqrt{-g} R^2, \tag{67}$$

$$H^{(1)}_{ab} = -\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int d^4x \sqrt{-g} R^{ij} R_{ij}. \tag{68}$$

Moreover, since

$$H^{(2)}_{ab} - 3H^{(1)}_{ab} = 0, \tag{69}$$

one can put, say, $\kappa_1 = 0$ and consider simplified system. Before proceeding further, let us observe that the quantum part of the total stress-energy tensor can be made large simply by taking large number of the quantum fields. Indeed, for N fields of a given spin, s with masses m_i the lowest order approximation to the renormalized effective action is still of the form (47) with

$$\frac{1}{m^2} \to \sum_{i=1}^{N} \frac{1}{m_i^2}. \tag{70}$$

For the Friedman-Robertson-Walker metric the semiclassical equations with the right- hand side given by the renormalized stress-energy tensor in a large mass limit are analytically intractable, and consequently one has to refer to approximations or accept some additional, physically motivated assumptions imposed on the scale factor. To this end, let us consider the function $a(t)$ which also satisfies the additional relation [31,37]

$$a'(t) = (c_1 a^n(t) - K)^{1/2}, \tag{71}$$

where t is the "ordinary" time coordinate related to η by Eq. (8), c_1 is a constant that should be determined and a prime denotes differentiation with respect to t. On general grounds, one expects that $n = 2$ and the problem at hand reduces to substitutions of

$$a^{(2i+1)}(t) = c_1^i (c_1 a(t)^2 - K)^{1/2} \tag{72}$$

and

$$a^{(2i)} = c_1^i a(t), \tag{73}$$

($i = 0, 1, 2, \ldots$) into the semiclassical equations. Let us return to our constraint equation (71) and list the temporal evolution scenarios of the model that are admissible. Simple analysis shows that if $c_1 > 0$, one has

$$a(t) = c_1^{-1/2} \cosh(c_1^{1/2}(t - t_0)), \tag{74}$$

$$a(t) = c_1^{-1/2} \sinh(c_1^{1/2}(t - t_0)) \tag{75}$$

and

$$a(t) = a(t_0) \exp(c_1^{1/2}(t - t_0)), \tag{76}$$

for $K = 1$, $K = -1$ and $K = 0$, respectively. On the other hand, for $c_1 < 0$ one has only one solution,

$$a(t) = |c_1|^{-1/2} \sin(|c_1|^{1/2}(t - t_0)), \tag{77}$$

with $K = -1$. Finally, for $c_1 = 0$, one has a static universe with the constant scale factor.

With the condition (71) being satisfied, one has further simplification of the semiclassical equations as both $H^{(1)}_{ab}$ and $H^{(2)}_{ab}$ vanish. Upon calculating the stress-energy tensor in the (t, r, θ, ϕ) coordinates from the very beginning or transforming the equations (66) to the standard coordinates and making use of (71), one obtains

$$-3c_1 + 3\tau c_1^3 - 3\kappa_3 c_1^2 + \Lambda = -8\pi\rho \tag{78}$$

and

$$-3c_1 + 3\tau c_1^3 - 3\kappa_3 c_1^2 + \Lambda = 8\pi p, \tag{79}$$

where

$$\tau = \frac{1}{3\pi m^2} \begin{cases} \frac{37}{252} - \frac{29}{10}\xi + 18\xi^2 - 36\xi^3, & \text{for } s = 0 \\ -\frac{31}{5040}, & \text{for } s = 1/2 \\ -\frac{5}{84}, & \text{for } s = 1. \end{cases} \tag{80}$$

The T_x^x component can be calculated from the covariant divergence equation, which has the form (19) with the dot (differentiation with respect to η) substituted by a prime. It should be noted that that the stress-energy tensor depends on the spin of the field but is independent of the parameter K. The type of the spatial geometry enters the result through the constraint equation (71). Inspection of Eqs. (78) and (79) shows that $\rho = -p$, and, consequently, the evolution of the model is described by one independent equation with the effective cosmological term $\Lambda_{\rm eff} = \Lambda + 8\pi\rho$. In the absence of $\Lambda_{\rm eff}$ and with $\kappa_3 = 0$ one has either the trivial solution $c_1 = 0$ or $c_1^2 = 1/\tau$. It can be demonstrated that for the massive scalar field with the arbitrary curvature coupling, the quantity τ is positive for $\xi < \xi_{\rm crit} = 0.1023$. Consequently, there is no solution for the conformally coupled ($\xi = 1/6$) scalar field. Restricting to the physical choices of ξ one concludes that the self-consistent solution is possible for the minimal coupling ($\xi = 0$). Similarly, there are no self-consistent solutions of the simplified semiclassical equations for the quantized massive spinor and massive vector fields. This behavior is identical to that observed previously in the spatially flat FRW universe [25].

Now, (still with $\Lambda_{\rm eff} = 0$) let us allow for a small but otherwise general coefficient κ_3 and look for a positive solution. For $\tau > 0$ one has

$$c_1 = \frac{\kappa_3 + \sqrt{4\tau + \kappa_3^2}}{2\tau} \tag{81}$$

for any κ_3. Similarly, for $\tau < 0$ one has either (81) or

$$c_1 = \frac{\kappa_3 - \sqrt{4\tau + \kappa_3^2}}{2\tau}, \tag{82}$$

provided $\kappa_3 < 0$ and $4\tau + \kappa_3^2 > 0$.

Finally, since the general solutions of the full semiclassical equation (78) is not particularly illuminating, we shall construct its approximation for small $\Lambda_{\rm eff}$ and vanishing κ_3. Simple manipulations give

$$c_1 \approx \frac{\Lambda_{\rm eff}}{3}\left(1 + \frac{\Lambda_{\rm eff}}{9}\tau\right), \tag{83}$$

$$c_1 \approx \frac{1}{\sqrt{\tau}} - \frac{\Lambda_{\rm eff}}{3} \tag{84}$$

and

$$c_1 \approx -\left(\frac{1}{\sqrt{\tau}} + \frac{\Lambda_{\rm eff}}{3}\right). \tag{85}$$

The approximate solution (83) is valid for any τ whereas (84) and (85) require $\tau > 0$. The above solutions properly combined with the formulas (74)–(77) describing temporal evolution of the scale factor yield quite rich family of sef-consistent solutions of the semiclassical Einstein field equations.

Let us return to Eqs. (66) with $\Lambda = 0$ and $\rho = 0$. Putting $v = a^2 a'^2 + K a^2$ and restricting to the minimally coupled scalar field, one gets

$$\begin{aligned} v = {}& -\kappa_3 \frac{v^2}{a^4} - \tilde{\kappa}\left(\frac{3Kv'}{a} - \frac{2vv'}{a^3} - \frac{v'^2}{4a^2} - Kv'' + \frac{vv''}{a^2}\right) + \frac{1}{1680\pi m^2}\left(\frac{844v^3}{9a^8} - \frac{692v^2v'}{a^7} - \frac{320Kv^2}{3a^6} + \frac{1366v^2v''}{3a^6} + \frac{1237vv'^2}{4a^6}\right. \\ & + \frac{3034Kvv'}{3a^5} - \frac{136v^{(3)}v^2}{a^5} + \frac{673v'^3}{36a^5} - \frac{1081vv'v''}{6a^5} - \frac{2182Kvv''}{3a^4} - \frac{4577Kv'^2}{12a^4} + \frac{17v^{(4)}v^2}{a^4} + \frac{51vv''^2}{4a^4} + \frac{17v^{(3)}vv'}{a^4} - \frac{17v'^2v''}{4a^4} \\ & \left. - \frac{255K^2v'}{a^3} + \frac{238Kv^{(3)}v}{a^3} + \frac{1183Kv'v''}{6a^3} + \frac{255K^2v''}{a^2} - \frac{34Kv^{(4)}v}{a^2} - \frac{51Kv''^2}{4a^2} - \frac{17Kv^{(3)}v'}{a^2} + 17K^2v^{(4)} - \frac{102K^2v^{(3)}}{a}\right), \end{aligned} \tag{86}$$

where $\tilde{\kappa} = 2(\kappa_1 + 3\kappa_2)$ and a prime denotes differentiation of v with respect to the scale factor a. With the quantum fields absent, one obtains the equation in the form considered previously by Starobinsky [31], and from Eq. (78) one has $c_1 = -1/\kappa_3$ with $\kappa_3 < 0$. On the other hand, with the quantum fields present, simple calculation shows that

$$v(a) = c_1 a^4 \tag{87}$$

solves Eq. (86), provided c_1 satisfies a simplified version of Eq. (78). Now, in order to analyze the stability of the solutions, let

$$v(a) = c_1 a^4 (1 + \Delta(a)) \tag{88}$$

with $|\Delta| \ll 1$. Retaining only the linear terms in Δ, one obtains a fourth-order differential equation, which is intractable and requires numerical assistance. Since the numerical analysis of the general backreaction problem is beyond the scope of this paper we shall look for simplifications. First, let us restrict to the minimally coupled scalar field in the spatially flat Universe. It can be easily shown that $c_1 = 6m\sqrt{21\pi/37}$ and

$$\frac{1480\Delta}{153a^4} - \frac{1552\Delta'}{51a^3} + \frac{1142\Delta''}{51a^2} + \frac{12\Delta^{(3)}}{a} + \Delta^{(4)} = 0. \tag{89}$$

This equation can be easily integrated to give

$$\begin{aligned} \Delta(a) = {}& C_1 a^{-3/2-q_1} + C_2 a^{-3/2+q_1} + C_3 a^{-3/2-q_2} \\ & + C_4 a^{-3/2+q_2}, \end{aligned} \tag{90}$$

where $q_1 \approx 1.78$ and $q_2 \approx 3.6$, and C_i are integration constants. We have omitted the standard form of the roots expressed in terms of surds and give their approximate values. Restricting our general solution to the solutions that remain close to (76) in the past one concludes that there are no solution that remains close as $t \to \infty$. This result is quite general and can be extended to include quadratic terms. Indeed, the $\Delta(a)$ function satisfies

$$\sum_{i=0}^{4} p_i \frac{\Delta^{(i)}}{a^{4-i}} = 0, \tag{91}$$

where the numerical coefficients p_i depend on the type of the field and the coupling constants κ_1, κ_2 and κ_3, whereas the $\Delta^{(i)}$ denotes the ith derivative with respect to a. The general solution of the differential equation (91) can always be constructed by solving the fourth-order polynomial equation. It should be noted, however, that for $K \neq 0$ the differential equation is far more complicated and is analytically intractable.

V. FINAL REMARKS

In this paper, which is a generalization of Refs. [18,25], we have constructed the stress-energy tensor of the quantized scalar, spinor and vector fields in a large mass limit in the Friedman-Robertson-Walker spacetime. For the massive scalar field with arbitrary coupling constant ξ, the stress-energy tensor has been calculated within the framework of the adiabatic and Schwinger-DeWitt approximations. In the calculationally more involved case of spinor and vector fields, only the Schwinger-DeWitt method has been used. It is expected, however, that both methods should give the same results also for the $s = 1/2$ and $s = 1$ fields. Since the constructed tensor depends functionally on the scale factor one can attempt to solve the semiclassical Einstein equations. Unfortunately, in general, the resulting higher-order equations are too complicated to be solved analytically and to make them tractable additional simplifying assumptions are needed. Here, after the analysis of the stress-energy tensor in the power-law cosmological models we solved, with the assumption that Eq. (71) with $n = 2$ is satisfied, the semiclassical equations in a self-consistent way. This may be considered as a generalization of the Starobinsky model to include the massive fields.

The results of this paper, with some additional effort, can be extended to the D-dimensional cosmological models ($D > 4$). It is interesting to analyze how the self-consistent solutions depend on D and study their behavior under small perturbations [25,38,39]. The second line of investigations concerns numerical solutions of the semiclassical equations both with and without the classical matter fields (for example along the lines of Refs. [32,33,39]). Finally, we observe that the influence of the quantized field on the evolution of the model can be studied perturbatively. This set of problems, however, lies beyond the scope of this work and we intend to report on such calculations in future publications. We conclude with the remark that both adiabatic and Schwinger-DeWitt approximations are the powerful computational methods and they still are the source of new findings and fresh ideas.

APPENDIX

In this Appendix we list the functions $W_k^{(2i)}$ for $i = 2$ and 3, valid for all three values of the parameter K. The zeroth and the second-order terms is given by (36) and (37), respectively. The fourth-order terms are given by

$$\begin{aligned}
W_k^{(4)} = {}& \frac{m^2 a a^{(4)}}{16\beta^{5/2}} + \frac{a^{(4)}}{8\beta^{3/2} a} - \frac{3\xi a^{(4)}}{4\beta^{3/2} a} - \frac{19 m^4 a^2 \ddot{a}^2}{32\beta^{7/2}} - \frac{3m^2 \ddot{a}^2}{16\beta^{5/2}} + \frac{9m^2 \xi \ddot{a}^2}{4\beta^{5/2}} - \frac{\ddot{a}^2}{4\beta^{3/2} a^2} \\
& - \frac{1105 m^8 a^4 \dot{a}^4}{128\beta^{11/2}} + \frac{221 m^6 a^2 \dot{a}^4}{32\beta^{9/2}} - \frac{19 m^4 \dot{a}^4}{32\beta^{7/2}} - \frac{7m^4 a^2 a^{(3)} \dot{a}}{8\beta^{7/2}} - \frac{3m^2 a^{(3)} \dot{a}}{8\beta^{5/2}} + \frac{15 m^2 \xi a^{(3)} \dot{a}}{4\beta^{5/2}} \\
& + \frac{221 m^6 a^3 \dot{a}^2 \ddot{a}}{32\beta^{9/2}} - \frac{9m^4 a \dot{a}^2 \ddot{a}}{4\beta^{7/2}} - \frac{75 m^4 \xi a \dot{a}^2 \ddot{a}}{8\beta^{7/2}} + \frac{m^2 \dot{a}^2 \ddot{a}}{4\beta^{5/2} a} - \frac{3m^2 \xi \dot{a}^2 \ddot{a}}{2\beta^{5/2} a} + \frac{\dot{a}^2 \ddot{a}}{4\beta^{3/2} a^3} - \frac{3\xi \dot{a}^2 \ddot{a}}{2\beta^{3/2} a^3} \\
& - \frac{9\xi^2 \ddot{a}^2}{2\beta^{3/2} a^2} + \frac{9\xi \ddot{a}^2}{4\beta^{3/2} a^2} - \frac{a^{(3)} \dot{a}}{4\beta^{3/2} a^2} + \frac{3\xi a^{(3)} \dot{a}}{2\beta^{3/2} a^2} - K\left(\frac{3m^2 a \ddot{a}}{8\beta^{5/2}} - \frac{9m^2 \xi a \ddot{a}}{4\beta^{5/2}} + \frac{\ddot{a}}{4\beta^{3/2} a}\right. \\
& \left. + \frac{9\xi^2 \ddot{a}}{\beta^{3/2} a} - \frac{3\xi \ddot{a}}{\beta^{3/2} a} - \frac{25 m^4 a^2 \dot{a}^2}{16\beta^{7/2}} + \frac{75 m^4 \xi a^2 \dot{a}^2}{8\beta^{7/2}} + \frac{3m^2 \dot{a}^2}{8\beta^{5/2}} - \frac{9m^2 \xi \dot{a}^2}{4\beta^{5/2}}\right) - K^2\left(\frac{1}{8\beta^{3/2}} + \frac{9\xi^2}{2\beta^{3/2}} - \frac{3\xi}{2\beta^{3/2}}\right).
\end{aligned} \tag{A1}$$

The sixth-order terms are more complicated and can be written in the form

$$W_k^{(6)} = \overset{(0)}{W}_k + K\overset{(1)}{W}_k + K^2\overset{(2)}{W}_k + K^3\overset{(3)}{W}_k, \tag{A2}$$

where

$$\overset{(0)}{W}_k = \frac{414125a^6\dot{a}^6m^{12}}{1024\beta^{17/2}} - \frac{248475a^4\dot{a}^6m^{10}}{512\beta^{15/2}} - \frac{248475a^5\dot{a}^4\ddot{a}m^{10}}{512\beta^{15/2}} + \frac{34503a^2\dot{a}^6m^8}{256\beta^{13/2}}$$
$$+ \frac{36465\xi a^3\dot{a}^4\ddot{a}m^8}{128\beta^{13/2}} + \frac{107491a^3\dot{a}^4\ddot{a}m^8}{256\beta^{13/2}} + \frac{1055a^4\dot{a}^3a^{(3)}m^8}{16\beta^{13/2}} - \frac{631\dot{a}^6m^6}{128\beta^{11/2}} - \frac{631a^3\ddot{a}^3m^6}{128\beta^{11/2}}$$
$$- \frac{4353a^2\dot{a}^2\ddot{a}^2m^6}{64\beta^{11/2}} - \frac{663\xi a\dot{a}^4\ddot{a}m^6}{8\beta^{11/2}} - \frac{8471a\dot{a}^4\ddot{a}m^6}{128\beta^{11/2}} - \frac{3315\xi a^2\dot{a}^3a^{(3)}m^6}{32\beta^{11/2}} - \frac{479a^2\dot{a}^3a^{(3)}m^6}{16\beta^{11/2}}$$
$$- \frac{1391a^3\dot{a}\,\ddot{a}\,a^{(3)}m^6}{64\beta^{11/2}} - \frac{815a^3\dot{a}^2a^{(4)}m^6}{128\beta^{11/2}} + \frac{399\xi a\ddot{a}^3m^4}{32\beta^{9/2}} + \frac{a\ddot{a}^3m^4}{2\beta^{9/2}} + \frac{1575\xi^2\dot{a}^2\ddot{a}^2m^4}{16\beta^{9/2}}$$
$$+ \frac{919\dot{a}^2\ddot{a}^2m^4}{128\beta^{9/2}} + \frac{69a^2a^{(3)2}m^4}{128\beta^{9/2}} + \frac{399\xi\dot{a}^4\ddot{a}m^4}{32\beta^{9/2}a} - \frac{133\dot{a}^4\ddot{a}m^4}{64\beta^{9/2}a} + \frac{55\dot{a}^3a^{(3)}m^4}{16\beta^{9/2}} + \frac{957\xi a\dot{a}\,\ddot{a}\,a^{(3)}m^4}{16\beta^{9/2}}$$
$$+ \frac{59a\dot{a}\,\ddot{a}\,a^{(3)}m^4}{64\beta^{9/2}} + \frac{663\xi a\dot{a}^2a^{(4)}m^4}{32\beta^{9/2}} - \frac{31a\dot{a}^2a^{(4)}m^4}{64\beta^{9/2}} + \frac{55a^2\ddot{a}a^{(4)}m^4}{64\beta^{9/2}} + \frac{27a^2\dot{a}a^{(5)}m^4}{64\beta^{9/2}}$$
$$+ \frac{51\xi\ddot{a}^3m^2}{8\beta^{7/2}a} - \frac{19\ddot{a}^3m^2}{32\beta^{7/2}a} + \frac{315\xi^2\dot{a}^2\ddot{a}^2m^2}{8\beta^{7/2}a^2} - \frac{393\xi\dot{a}^2\ddot{a}^2m^2}{16\beta^{7/2}a^2} + \frac{3\dot{a}^2\ddot{a}^2m^2}{\beta^{7/2}a^2} - \frac{21\xi a^{(3)2}m^2}{8\beta^{7/2}} + \frac{9a^{(3)2}m^2}{32\beta^{7/2}}$$
$$+ \frac{69\xi\dot{a}^4\ddot{a}m^2}{8\beta^{7/2}a^3} - \frac{23\dot{a}^4\ddot{a}m^2}{16\beta^{7/2}a^3} - \frac{69\xi\dot{a}^3a^{(3)}m^2}{8\beta^{7/2}a^2} + \frac{23\dot{a}^3a^{(3)}m^2}{16\beta^{7/2}a^2} - \frac{225\xi^2\dot{a}\,\ddot{a}\,a^{(3)}m^2}{4\beta^{7/2}a} + \frac{219\xi\dot{a}\,\ddot{a}\,a^{(3)}m^2}{8\beta^{7/2}a}$$
$$- \frac{3\dot{a}\,\ddot{a}\,a^{(3)}m^2}{\beta^{7/2}a} + \frac{69\xi\dot{a}^2a^{(4)}m^2}{16\beta^{7/2}a} - \frac{23\dot{a}^2a^{(4)}m^2}{32\beta^{7/2}a} - \frac{9\xi\ddot{a}a^{(4)}m^2}{2\beta^{7/2}} + \frac{33\ddot{a}a^{(4)}m^2}{64\beta^{7/2}} - \frac{21\xi\dot{a}a^{(5)}m^2}{8\beta^{7/2}}$$
$$- \frac{aa^{(6)}m^2}{64\beta^{7/2}} + \frac{27\xi^3\ddot{a}^3}{2\beta^{5/2}a^3} - \frac{27\xi^2\ddot{a}^3}{2\beta^{5/2}a^3} + \frac{9\xi\ddot{a}^3}{2\beta^{5/2}a^3} - \frac{7\ddot{a}^3}{16\beta^{5/2}a^3} + \frac{153\xi^2\dot{a}^2\ddot{a}^2}{8\beta^{5/2}a^4} - \frac{105\xi\dot{a}^2\ddot{a}^2}{8\beta^{5/2}a^4} + \frac{53\dot{a}^2\ddot{a}^2}{32\beta^{5/2}a^4}$$
$$+ \frac{45\xi^2a^{(3)2}}{8\beta^{5/2}a^2} - \frac{21\xi a^{(3)2}}{8\beta^{5/2}a^2} + \frac{9a^{(3)2}}{32\beta^{5/2}a^2} + \frac{9\xi\dot{a}^4\ddot{a}}{2\beta^{5/2}a^5} - \frac{3\dot{a}^4\ddot{a}}{4\beta^{5/2}a^5} - \frac{9\xi\dot{a}^3a^{(3)}}{2\beta^{5/2}a^4} + \frac{3\dot{a}^3a^{(3)}}{4\beta^{5/2}a^4} - \frac{99\xi^2\dot{a}\,\ddot{a}\,a^{(3)}}{4\beta^{5/2}a^3}$$
$$+ \frac{57\xi\dot{a}\,\ddot{a}\,a^{(3)}}{4\beta^{5/2}a^3} - \frac{27\dot{a}\,\ddot{a}\,a^{(3)}}{16\beta^{5/2}a^3} + \frac{9\xi\dot{a}^2a^{(4)}}{4\beta^{5/2}a^3} - \frac{3\dot{a}^2a^{(4)}}{8\beta^{5/2}a^3} + \frac{27\xi^2\ddot{a}a^{(4)}}{4\beta^{5/2}a^2} - \frac{57\xi\ddot{a}a^{(4)}}{16\beta^{5/2}a^2} + \frac{13\ddot{a}a^{(4)}}{32\beta^{5/2}a^2}$$
$$- \frac{3\xi\dot{a}a^{(5)}}{4\beta^{5/2}a^2} + \frac{\dot{a}a^{(5)}}{8\beta^{5/2}a^2} + \frac{3\xi a^{(6)}}{16\beta^{5/2}a} - \frac{a^{(6)}}{32\beta^{5/2}a} - \frac{5967\xi a^2\dot{a}^2\ddot{a}^2m^6}{32\beta^{11/2}} + \frac{34503a^4\dot{a}^2\ddot{a}^2m^8}{256\beta^{13/2}}$$
$$- \frac{477\xi\dot{a}^2\ddot{a}^2m^4}{32\beta^{9/2}} - \frac{135\xi^2\ddot{a}^3m^2}{8\beta^{7/2}a} + \frac{11\dot{a}a^{(5)}m^2}{32\beta^{7/2}} \tag{A3}$$

$$\overset{(1)}{W}_k = \frac{36465\xi a^4\dot{a}^4m^8}{128\beta^{13/2}} - \frac{12155a^4\dot{a}^4m^8}{256\beta^{13/2}} - \frac{5967\xi a^2\dot{a}^4m^6}{32\beta^{11/2}} + \frac{1989a^2\dot{a}^4m^6}{64\beta^{11/2}} - \frac{5967\xi a^3\dot{a}^2\ddot{a}m^6}{32\beta^{11/2}}$$
$$+ \frac{1989a^3\dot{a}^2\ddot{a}m^6}{64\beta^{11/2}} + \frac{399\xi\dot{a}^4m^4}{32\beta^{9/2}} - \frac{133\dot{a}^4m^4}{64\beta^{9/2}} + \frac{399\xi a^2\ddot{a}^2m^4}{32\beta^{9/2}} - \frac{133a^2\ddot{a}^2m^4}{64\beta^{9/2}} + \frac{1575\xi^2a\dot{a}^2\ddot{a}m^4}{8\beta^{9/2}}$$
$$+ \frac{231\xi a\dot{a}^2\ddot{a}m^4}{16\beta^{9/2}} - \frac{63a\dot{a}^2\ddot{a}m^4}{8\beta^{9/2}} + \frac{147\xi a^2\dot{a}a^{(3)}m^4}{8\beta^{9/2}} - \frac{49a^2\dot{a}a^{(3)}m^4}{16\beta^{9/2}} - \frac{135\xi^2\ddot{a}^2m^2}{4\beta^{7/2}} + \frac{135\xi\ddot{a}^2m^2}{16\beta^{7/2}}$$
$$- \frac{15\ddot{a}^2m^2}{32\beta^{7/2}} + \frac{45\xi^2\dot{a}^2\ddot{a}m^2}{2\beta^{7/2}a} - \frac{15\xi\dot{a}^2\ddot{a}m^2}{2\beta^{7/2}a} + \frac{5\dot{a}^2\ddot{a}m^2}{8\beta^{7/2}a} - \frac{225\xi^2\dot{a}a^{(3)}m^2}{4\beta^{7/2}} + \frac{15\xi\dot{a}a^{(3)}m^2}{\beta^{7/2}} - \frac{15\dot{a}a^{(3)}m^2}{16\beta^{7/2}}$$
$$+ \frac{5aa^{(4)}m^2}{32\beta^{7/2}} + \frac{81\xi^3\ddot{a}^2}{2\beta^{5/2}a^2} - \frac{27\xi^2\ddot{a}^2}{\beta^{5/2}a^2} + \frac{45\xi\ddot{a}^2}{8\beta^{5/2}a^2} - \frac{3\ddot{a}^2}{8\beta^{5/2}a^2} + \frac{27\xi^2\dot{a}^2\ddot{a}}{2\beta^{5/2}a^3} - \frac{9\xi\dot{a}^2\ddot{a}}{2\beta^{5/2}a^3} + \frac{3\dot{a}^2\ddot{a}}{8\beta^{5/2}a^3}$$
$$+ \frac{9\xi\dot{a}a^{(3)}}{2\beta^{5/2}a^2} - \frac{3\dot{a}a^{(3)}}{8\beta^{5/2}a^2} + \frac{27\xi^2a^{(4)}}{4\beta^{5/2}a} - \frac{9\xi a^{(4)}}{4\beta^{5/2}a} + \frac{3a^{(4)}}{16\beta^{5/2}a} - \frac{15\xi aa^{(4)}m^2}{16\beta^{7/2}} - \frac{27\xi^2\dot{a}a^{(3)}}{2\beta^{5/2}a^2} \tag{A4}$$

$$\overset{(2)}{W}_k = -\frac{15m^2a\ddot{a}}{32\beta^{7/2}} - \frac{135m^2\xi^2a\ddot{a}}{8\beta^{7/2}} + \frac{45m^2\xi a\ddot{a}}{8\beta^{7/2}} - \frac{3\ddot{a}}{16\beta^{5/2}a} + \frac{81\xi^3\ddot{a}}{2\beta^{5/2}a} - \frac{81\xi^2\ddot{a}}{4\beta^{5/2}a} + \frac{27\xi\ddot{a}}{8\beta^{5/2}a} + \frac{1575m^4\xi^2a^2\dot{a}^2}{16\beta^{9/2}} - \frac{525m^4\xi a^2\dot{a}^2}{16\beta^{9/2}} - \frac{15m^2\dot{a}^2}{32\beta^{7/2}} - \frac{135m^2\xi^2\dot{a}^2}{8\beta^{7/2}} + \frac{45m^2\xi\dot{a}^2}{8\beta^{7/2}} + \frac{175m^4a^2\dot{a}^2}{64\beta^{9/2}} \tag{A5}$$

and

$$\overset{(3)}{W}_k = -\frac{1}{16\beta^{5/2}} + \frac{27\xi^3}{2\beta^{5/2}} - \frac{27\xi^2}{4\beta^{5/2}} + \frac{9\xi}{8\beta^{5/2}}. \tag{A6}$$

[1] B. S. DeWitt, Phys. Rep. **19**, 295 (1975).
[2] A. Barvinsky and G. Vilkovisky, Phys. Rep. **119**, 1 (1985).
[3] I. Avramidi, Theor. Math. Phys. **79**, 494 (1989).
[4] V. P. Frolov and A. Zelnikov, Phys. Lett. **115B**, 372 (1982).
[5] V. P. Frolov and A. Zelnikov, Phys. Rev. D **29**, 1057 (1984).
[6] L. Kofman and V. Sahni, Phys. Lett. **127B**, 197 (1983).
[7] L. Kofman, V. Sahni, and A. A. Starobinsky, Sov. Phys. JETP **58**, 1090 (1983).
[8] J. Matyjasek, Phys. Rev. D **61**, 124019 (2000).
[9] J. Matyjasek, Phys. Rev. D **63**, 084004 (2001).
[10] J. Matyjasek and D. Tryniecki, Phys. Rev. D **79**, 084017 (2009).
[11] J. Matyjasek, D. Tryniecki, and K. Zwierzchowska, Phys. Rev. D **81**, 124047 (2010).
[12] L. Parker and S. Fulling, Phys. Rev. D **9**, 341 (1974).
[13] S. Fulling, L. Parker, and B. Hu, Phys. Rev. D **10**, 3905 (1974).
[14] S. Fulling and L. Parker, Ann. Phys. (N.Y.) **87**, 176 (1974).
[15] T. Bunch and P. Davies, J. Phys. A **11**, 1315 (1978).
[16] T. Bunch, J. Phys. A **13**, 1297 (1980).
[17] P. R. Anderson and L. Parker, Phys. Rev. D **36**, 2963 (1987).
[18] A. Kaya and M. Tarman, J. Cosmol. Astropart. Phys. 04 (2011) 040.
[19] A. Kaya and M. Tarman, J. Cosmol. Astropart. Phys. 01 (2012) 040.
[20] A. Landete, J. Navarro-Salas, and F. Torrenti, Phys. Rev. D **88**, 061501 (2013).
[21] A. Landete, J. Navarro-Salas, and F. Torrenti, arXiv:1311.4958.
[22] T.-P. Hack, arXiv:1306.3074.
[23] L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge University Press, Cambridge, England, 2009).
[24] A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Energoatomizdat, Moscow, 1988).
[25] J. Matyjasek and P. Sadurski, Phys. Rev. D **88**, 104015 (2013).
[26] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[27] G. Ghika and M. Visinescu, Nuovo Cim. A **46**, 25 (1978).
[28] P. R. Anderson and W. Eaker, Phys. Rev. D **61**, 024003 (1999).
[29] V. Beilin, G. Vereshkov, Y. Grishkan, N. Ivanov, V. Nesterenko *et al.*, Sov. Phys. JETP **51**, 1045 (1980).
[30] K. S. Stelle, Gen. Relativ. Gravit. **9**, 353 (1978).
[31] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[32] P. Anderson, Phys. Rev. D **28**, 271 (1983).
[33] P. Anderson, Phys. Rev. D **29**, 615 (1984).
[34] L. Parker and J. Z. Simon, Phys. Rev. D **47**, 1339 (1993).
[35] S. Capozziello and A. Stabile, Classical Quantum Gravity **26**, 085019 (2009).
[36] A. L. Maroto and I. Shapiro, Phys. Lett. B **414**, 34 (1997).
[37] A. Dobado and A. L. Maroto, Phys. Lett. B **316**, 250 (1993).
[38] A. Pelinson, I. Shapiro, and F. Takakura, Nucl. Phys. **B648**, 417 (2003).
[39] J. F. Koksma and T. Prokopec, Phys. Rev. D **78**, 023508 (2008).